Recall: Let $1<p<\infty$, $\Omega \subset \mathbb{R}^n$ bounded, open and $Q>1$.

If $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ is a Q-minimizer for $\int_\Omega (1+|Dv|^p)$, then for some constant $c = c(n,p,Q)$:

$$(*) \quad \fint_{B_R} |Du|^p \leq c + c\left(\fint_{B_{2R}} |Du|^{\frac{np}{n+p}}\right)^{\frac{n+p}{n}}$$

for all balls $B_{2R} \subset \Omega$.

$\therefore$ $f = |Du|^{\frac{np}{n+p}}$ satisfies an inhomogeneous reverse Hölder inequality on increasing supports: put $q = \frac{n+p}{n} = 1 + \frac{p}{n} > 1$, then

$$(**) \quad \fint_{B_R} f^q \leq c + c\left(\fint_{B_{2R}} f\right)^q$$

for all $B_{2R} \subset \Omega$.

$\boxed{\text{Gehring:}}$ $(**)$ (and $(*)$) self-improves.

Gehring's Lemma

(F.W. Gehring: The L^p-integrability of the partial derivatives of a quasiconformal mapping. Acta Math. 130 (1973), 265–277.)

TH 2 Let $1<p<\infty$, $0<\sigma<1$, $0\leqq\theta<1$ and $K\geqq 0$. Let $\mathbb{B}\subset\mathbb{R}^n$ be an open ball.

Assume $f,g\in L^p(\mathbb{B})^+$ satisfy

$$(2.1)\quad \left(\fint_{\sigma B} f^p\right)^{\frac{1}{p}} \leqq \theta\left(\fint_B f^p\right)^{\frac{1}{p}} + K\fint_B f + \left(\fint_B g^p\right)^{\frac{1}{p}}$$

for all balls $B\subset\mathbb{B}$.

$\boxed{\text{CASE } \theta=0}$ There exists $q_0=q_0(n,p,K)>p$ s.t. for $q\leqq q_0$, $0<r<1$,

$$(*)\quad \left(\fint_{r\mathbb{B}} f^q\right)^{\frac{1}{q}} \leqq \frac{c(n,p)}{r^{\frac{n}{q}}(1-r)^{\frac{n}{p}}}\left[\left(\fint_{\mathbb{B}} f^p\right)^{\frac{1}{p}} + \left(\fint_{\mathbb{B}} g^q\right)^{\frac{1}{q}}\right].$$

In particular, if $g\in L^q(\mathbb{B})$, then $f\in L^q_{loc}(\mathbb{B})$ too.

$\boxed{\text{CASE } 0<\theta<1}$ There exists $q_1=q_1(n,p,K,\theta)>p$ s.t. for $q\leqq q_1$, $0<r<1$, if $f,g\in L^q(\mathbb{B})$, then

$$(**)\quad \left(\fint_{r\mathbb{B}} f^q\right)^{\frac{1}{q}} \leqq \frac{c(n,p,\theta)}{r^{\frac{n}{q}}(1-r)^{\frac{n}{p}}}\left[\left(\fint_{\mathbb{B}} f^p\right)^{\frac{1}{p}} + \cancel{c(n,p,\theta)}\left(\fint_{\mathbb{B}} g^q\right)^{\frac{1}{q}}\right].$$

Proof of Gehring's Lemma by way of maximal inequalities:

Assume (2.1) holds, and $\boxed{\theta > 0}$.

Step 1: Reduction to inequality on $\mathbb{R}^n$.

Put $\delta(x) := \operatorname{dist}(x, \mathbb{R}^n \setminus B)^{\frac{n}{p}}$, $x \in \mathbb{R}^n$.

Define $F = \delta f$, $G = \delta g + \left(\int_B f^p\right)^{\frac{1}{p}} \mathbb{1}_B$.

Then $F, G \in L^p(\mathbb{R}^n)^+$ and $F = G \equiv 0$ on $\mathbb{R}^n \setminus B$.

Assertion. Assume WLOG $\theta \geqslant \frac{1}{4}$. Then

$$(2.2) \quad \left(\fint_{\sigma B} F^p\right)^{\frac{1}{p}} \leq \theta^{\frac{1}{2}} \left(\fint_B F^p\right)^{\frac{1}{p}} + 2K \fint_B F + C \left(\fint_B G^p\right)^{\frac{1}{p}}$$

for all balls $B \subset \mathbb{R}^n$, where

$$C = \frac{c(n,p)}{\sigma^{\frac{n}{p}} (1-\sigma)^{\frac{n}{p}} \left(1 - \theta^{\frac{p}{2n}}\right)^{\frac{n}{p}}}.$$

Proof of (2.2)

WLOG assume $\sigma B \cap \mathbb{B} \neq \emptyset$.

We distinguish two cases.

Case 1 $\quad \lambda B \subset \mathbb{B}$ for some $\lambda > 1$
(to be specified)

Write $\mathbb{B} = B(x_0, R)$, $B = B(x, r)$.

Then $\lambda B \subset \mathbb{B}$ means $|x - x_0| + \lambda r < R$.

Note

$$\max_{\sigma B} \delta = \left(R - |x - x_0| + \sigma r \right)^{\frac{n}{p}}$$

$$\min_{B} \delta = \left(R - |x - x_0| - r \right)^{\frac{n}{p}}$$

so

$$\min_{B} \delta = \underbrace{\left[\frac{R - |x - x_0| - r}{R - |x - x_0| + \sigma r} \right]^{\frac{n}{p}}}_{0 < \mu < 1} \max_{\sigma B} \delta$$

Use $\min_B \delta = \mu \cdot \max_{\sigma B} \delta$ in (2.1):

$$\left(\fint_{\sigma B} F^p\right)^{\frac{1}{p}} \leq \max_{\sigma B} \delta \left(\fint_{\sigma B} f^p\right)^{\frac{1}{p}} \leq$$

$$\mu^{-1} \min_B \delta \left\{ \theta \left(\fint_B f^p\right)^{\frac{1}{p}} + K \fint_B f + \left(\fint_B g^p\right)^{\frac{1}{p}} \right\} \leq$$

$$\mu^{-1}\theta \left(\fint_B F^p\right)^{\frac{1}{p}} + \mu^{-1} K \fint_B F + \mu^{-1} \left(\fint_B G^p\right)^{\frac{1}{p}}.$$

Want $\mu^{-1}\theta < 1$, say $\mu^{-1}\theta \leq \sqrt{\theta}$, then $\mu \geq \sqrt{\theta}$, ie,

$$\mu = \left[\frac{R - |x - x_0| - r}{R - |x - x_0| + \sigma r}\right]^{\frac{n}{p}} \geq \sqrt{\theta}.$$

Here $\dfrac{R - |x - x_0|}{r} > \lambda$, so

$$\mu = \left[\frac{\frac{R - |x - x_0|}{r} - 1}{\frac{R - |x - x_0|}{r} + \sigma}\right]^{\frac{n}{p}} > \left(\frac{\lambda - 1}{\lambda + \sigma}\right)^{\frac{n}{p}},$$

ie $\mu \geq \sqrt{\theta}$ provided $\lambda \geq \dfrac{1 + \sigma \theta^{\frac{p}{2n}}}{1 - \theta^{\frac{p}{2n}}}$.

Choose $\boxed{\lambda = \dfrac{3}{1 - \theta^{\frac{p}{2n}}}}$

<u>Case 2</u> $\lambda B \not\subset \mathbb{B}$

with $\lambda = \dfrac{3}{1-\theta^{\frac{p}{2n}}} > 3$.

Write $\mathbb{B} = B(x_0, R)$, $B = B(x,r)$.

$\lambda B \not\subset \mathbb{B}$ & $\sigma B \cap \mathbb{B} \neq \emptyset$ means

$\lambda r + |x - x_0| > R$ & $|x - x_0| < \sigma r + R$.

$$\max_{\sigma B} \delta < \max_{B} \delta = (R + r - |x-x_0|)^{\frac{n}{p}} \begin{cases} > (1-\sigma)^{\frac{n}{p}} r^{\frac{n}{p}}, \\ < (\lambda-1)^{\frac{n}{p}} r^{\frac{n}{p}}, \end{cases}$$

$$|B \cap \mathbb{B}| \begin{cases} > \omega_n \left(\dfrac{R + r - |x - x_0|}{2}\right)^n & \text{if } B \not\subset \mathbb{B}, \\ = |B| = \omega_n r^n & \text{if } B \subset \mathbb{B}. \end{cases}$$

$\max_B \delta^p < (\lambda-1)^n r^n = \dfrac{(\lambda-1)^n}{\omega_n} |B|$, so

$$\left(⨍_{\sigma B} F^p\right)^{\frac{1}{p}} \leq \left(\max_B \delta^p \frac{1}{\sigma^n |B|} \int_{\mathbb{B}} f^p \right)^{\frac{1}{p}} \leq$$

$$\left(\frac{\lambda - 1}{\omega_n^{\frac{1}{n}} \sigma}\right)^{\frac{n}{p}} \left(\int_{\mathbb{B}} f^p \right)^{\frac{1}{p}}.$$

Now
$$\left(\fint_B G^p\right)^{\frac{1}{p}} \geqq \left(\int_{\mathbb{B}} f^p\right)^{\frac{1}{p}} \left(\frac{|B \cap \mathbb{B}|}{|B|}\right)^{\frac{1}{p}}$$

and
$$\frac{|B \cap \mathbb{B}|}{|B|} > \left(\frac{1-\sigma}{2}\right)^n,$$

so
$$\left(\fint_{\sigma B} F^p\right)^{\frac{1}{p}} \leq \left(\frac{2}{\omega_n^{\frac{1}{n}}} \cdot \frac{\lambda-1}{\sigma(1-\sigma)}\right)^{\frac{n}{p}} \left(\fint_B G^p\right)^{\frac{1}{p}}.$$

The assertion follows. $\square$

The situation is simpler when $\boxed{\theta=0}$.

We get:

$$(2.2') \quad \left(\fint_{\sigma B} F^p\right)^{\frac{1}{p}} \leqq 2K \fint_B F + c\left(\fint_B G^p\right)^{\frac{1}{p}}$$

for all balls $B \subset \mathbb{R}^n$, where

$$c = \frac{c(n,p)}{\sigma^{\frac{n}{p}}(1-\sigma)^{\frac{n}{p}}}.$$

We now focus on the situations:

$\boxed{\theta = 0}$ $\quad 0 < \sigma < 1$, $1 < p < \infty$, $K \geq 0$

and $F, G \in L^p(\mathbb{R}^n)^+$ s.t.

$$(2.3) \qquad \left(\fint_{\sigma B} F^p \right)^{\frac{1}{p}} \leq K \fint_B F + \left(\fint_B G^p \right)^{\frac{1}{p}}$$

for all $B \subset \mathbb{R}^n$

$\boxed{\theta \in (0,1)}$ $\quad 0 < \sigma < 1$, $1 < p < \infty$, $K \geq 0$

and $F, G \in L^p(\mathbb{R}^n)^+$ s.t.

$$(2.4) \qquad \left(\fint_{\sigma B} F^p \right)^{\frac{1}{p}} \leq \theta \left(\fint_B F^p \right)^{\frac{1}{p}} + K \fint_B F + \left(\fint_B G^p \right)^{\frac{1}{p}}$$

for all $B \subset \mathbb{R}^n$.

In the latter we must assume F is integrable to a higher power to reach a conclusion.

Hardy-Littlewood maximal function

When $f \in L^1_{loc}(\mathbb{R}^n)$ define (the centered maximal function)

$$Mf(x) := \sup_{r>0} \fint_{B(x,r)} |f(y)|\,dy \ , \ x \in \mathbb{R}^n.$$

Remark Mf is lower semicontinuous, hence measurable.

TH 3 There exist $c_1 = c_1(n), c_2 = c_2(n) > 0$ s.t. for $1 < p < \infty$ and $f \in L^p(\mathbb{R}^n)$

$$c_1 \int_{\mathbb{R}^n} |f|^p \leqq \frac{p-1}{p} \int_{\mathbb{R}^n} |Mf|^p \leqq c_2 2^p \int_{\mathbb{R}^n} |f|^p .$$

We shall return to the proof of TH 3 later.

Remark When $f \neq 0$ then $Mf \notin L^1(\mathbb{R}^n)$.

Note that (2.4) implies that

$$M(F^p)(x)^{\frac{1}{p}} \leq \theta M(F^p)(x)^{\frac{1}{p}} + K M(F)(x) + M(G^p)(x)^{\frac{1}{p}}$$

for all $x \in \mathbb{R}^n$.

Clearly $M(F^p)(x) < \infty \quad \forall x$, so

$$M(F^p)(x)^{\frac{1}{p}} \leq \frac{K}{1-\theta} M(F)(x) + \frac{1}{1-\theta} M(G^p)(x)^{\frac{1}{p}}.$$

Likewise for (2.3), so we're looking at

$$\boxed{(2.5) \quad M(F^p)^{\frac{1}{p}} \leq K_1 M(F) + M(G_1^p)^{\frac{1}{p}}.}$$

$\boxed{\text{Assume } G_1, F \in L^q(\mathbb{R}^n)^+}$ $\quad q > p > 1$

Step 2:

Use TH 3 :

$$c_1 \int_{\mathbb{R}^n} F^q = c_1 \int_{\mathbb{R}^n} (F^p)^{\frac{q}{p}} \leq \frac{\frac{q}{p} - 1}{\frac{q}{p}} \int_{\mathbb{R}^n} M(F^p)^{\frac{q}{p}}$$

ie

$$c_1 \int_{\mathbb{R}^n} F^q \le \frac{q-p}{q} \int_{\mathbb{R}^n} M(F^p)^{\frac{q}{p}} \le \tag{2.5}$$

$$\frac{q-p}{q} \int_{\mathbb{R}^n} \left(K_1 MF + M(G_1^p)^{\frac{1}{p}} \right)^q \le$$

$$2^{q-1} \frac{q-p}{q} \int_{\mathbb{R}^n} \left(K_1^q (MF)^q + M(G_1^p)^{\frac{q}{p}} \right) \le$$

$$2^{q-1} \frac{q-p}{q} \left[c_2 K_1^q \frac{q}{q-1} 2^q \int_{\mathbb{R}^n} F^q + \frac{q}{q-p} c_2 2^{q/p} \int_{\mathbb{R}^n} G_1^q \right] \le$$

$$c_2 K_1^q 2^{2q-1} \frac{q-p}{q-1} \int_{\mathbb{R}^n} F^q + c_2 2^{2q-1} \int_{\mathbb{R}^n} G_1^q .$$

Because $\int_{\mathbb{R}^n} F^q < \infty$ we get

if

$$c_1 - c_2 K_1^q 2^{2q-1} \frac{q-p}{q-1} > 0$$

ie, eg, if $q \le q_0 = p + \varepsilon(n,p)$

~~we get~~ that

$$\int_{\mathbb{R}^n} F^q \le \tilde{C}(n,p) \int_{\mathbb{R}^n} G_1^q .$$

Recall definitions

$$F = \delta f \,, \quad G_1 = \left(\delta g + \left(\int_{\mathbb{B}} f^p \right)^{\frac{1}{p}} \mathbb{1}_{\mathbb{B}} \right) (1-\theta)^{-1}$$

and note that for $0<r<1$,

$$\min_{r\mathbb{B}} \delta^p = (1-r)^n \frac{|\mathbb{B}|}{\omega_n} \,,$$

$$\max_{r\mathbb{B}} \delta^p = \frac{|\mathbb{B}|}{\omega_n}$$

to conclude with an inequality of type (*) or (**).

In case $\boxed{\theta = 0}$ we <u>don't need</u> to assume $F \in L^q(\mathbb{R}^n)$: (2.3) is stable under mollification — recall:

$$(2.3) \quad \left(\fint_{\sigma B} F^p \right)^{\frac{1}{p}} \leq K \fint_B F + \left(\fint_B G^p \right)^{\frac{1}{p}}$$

for all $B \subset \mathbb{R}^n$.

Let $(\rho_\varepsilon)_{\varepsilon>0}$ be a standard mollifier.

Put $F_\varepsilon = \rho_\varepsilon * F$, $H_\varepsilon = (\rho_\varepsilon * G^p)^{\frac{1}{p}}$.

Then $\left(⨍_{\sigma B} F_\varepsilon^p\right)^{\frac{1}{p}} = \left(⨍_{\sigma B} \left|\int_{\mathbb{R}^n} \rho_\varepsilon(y) F(x-y)\,dy\right|^p dx\right)^{\frac{1}{p}}$

$$\overset{\text{Minkowski}}{\leq} \int_{\mathbb{R}^n} \rho_\varepsilon(y) \left(⨍_{\sigma B} F(x-y)^p\,dx\right)^{\frac{1}{p}} dy$$

$$\overset{(2.3)}{\leq} \int_{\mathbb{R}^n} \rho_\varepsilon(y) \left[K ⨍_B F(x-y)\,dx + \left(⨍_B G(x-y)^p dx\right)^{\frac{1}{p}} \right] dy$$

$$= K ⨍_B F_\varepsilon + \int_{\mathbb{R}^n} \rho_\varepsilon(y) \left(⨍_B G(x-y)^p\,dx\right)^{\frac{1}{p}} dy$$

$$\overset{\text{Jensen}}{\leq} K ⨍_B F_\varepsilon + \left(\int_{\mathbb{R}^n} \rho_\varepsilon(y) ⨍_B G(x-y)^p\,dx\,dy\right)^{\frac{1}{p}}$$

$$= K ⨍_B F_\varepsilon + \left(⨍_B H_\varepsilon^p\right)^{\frac{1}{p}} \quad \text{for all } B \subset \mathbb{R}^n.$$

We may now apply the previous result to F_ε :

For $q \leq q_0 = p + \varepsilon(n,p)$:

$$\int_{\mathbb{R}^n} F_\varepsilon^q \leq \tilde{C}(n,p,K) \int_{\mathbb{R}^n} H_\varepsilon^q .$$

When $G \in L^q(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} H_\varepsilon^q \to \int_{\mathbb{R}^n} G^q$

and so by Fatou's Lemma

$$\int_{\mathbb{R}^n} F^q \leq \tilde{C}(n,p,K) \int_{\mathbb{R}^n} G^q < \infty . \quad \square$$

The above proof is a combination of those of

- T. Iwaniec: The Gehring Lemma. In P. Duren, J. Heinonen, B. Osgood and B. Palka, 'Quasiconformal Mappings and Analysis', Springer, 1998. (see also chapter 14 of [9])

- E.W. Stredulinsky: Higher Integrability from Reverse Hölder Inequalities. Indiana Univ. Math. J. 29, 3 (1980), 408–417.