The Hardy-Littlewood maximal function

There are many variants — we denote for each $f \in L^1_{loc}(\mathbb{R}^n)$,

$$Mf(x) := \sup_{r>0} \fint_{B(x,r)} |f(y)|\,dy \quad , \quad x \in \mathbb{R}^n.$$

Variants include

$$M_1 f(x) = \sup_{B \ni x} \fint_B |f(y)|\,dy \quad \text{`sup over all balls containing } x\text{'}$$

$$M_2 f(x) = \sup_{Q \ni x} \fint_Q |f(y)|\,dy \quad \text{`sup over all cubes containing } x\text{'}$$

A cube Q is a set of form $(a,b)^n$ (= open ball in ℓ^∞-metric of radius $\frac{b-a}{2}$).

Remark For fixed $r>0$, $x \mapsto \fint_{B(x,r)} |f|$ is continuous, hence $x \mapsto Mf(x)$ is lsc (=lower semicontinuous, ie $\{Mf > t\}$ is open for all t), and so Mf is in particular Borel. Likewise for $M_1 f$, $M_2 f$.

Recall from L3&4 :

TH 3: There exist $c_1 = c_1(n), c_2 = c_2(n), c_3 = c_3(n) > 0$ s.t.

$$c_1 c_3^{1-p} \int_{\mathbb{R}^n} |f|^p \, dx \leq \frac{p-1}{p} \int_{\mathbb{R}^n} (Mf)^p \, dx \leq c_2 2^{p-1} \int_{\mathbb{R}^n} |f|^p dx$$

for all $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$.

Remark Correction of constants compared to statement in L3&4, however immaterial ~~for proof of Gehring's Lemma.~~

The result will follow in a standard way from

TH 4: There exist $c_1 = c_1(n)$, $c_2 = c_2(n)$ and $c_3 = c_3(n) > 0$ s.t. for all $f \in L^1(\mathbb{R}^n)$, $t > 0$:

$$c_1 \int_{\{|f| > c_3 t\}} |f| \leq t \lambda(t) \leq c_2 \int_{\{|f| > \frac{t}{2}\}} |f| .$$

Covering Lemma (Vitali or Wiener)

Let $\mathcal{F}$ be a family of open balls (or cubes) in $\mathbb{R}^n$. Assume

$$d := \sup_{B \in \mathcal{F}} \operatorname{diam} B < \infty,$$

and let $k > 3$. There exists a subfamily $\mathcal{F}'$ of $\mathcal{F}$ consisting of disjoint balls, and so each $B \in \mathcal{F}$ is contained in kB' for some $B' \in \mathcal{F}'$. In particular,

$$\bigcup \mathcal{F} \subseteq \bigcup_{B \in \mathcal{F}'} kB.$$

Remark. Lemma would be false if $k \leq 3$, for example when $n = 1$ and

$$\mathcal{F} = \left\{ \left(-\tfrac{1}{i}, 2 - \tfrac{3}{i}\right), \left(\tfrac{3}{j} - 2, \tfrac{1}{j}\right) : i, j \in \mathbb{N} \right\}.$$

$\boxed{\text{Pf:}}$ Put $c := \frac{2}{k-1} < 1$. Decompose $\mathcal{F}$ as $\mathcal{F} = \bigcup_{j=0}^{\infty} \mathcal{F}_j$, where

$$\mathcal{F}_j := \{ B \in \mathcal{F} : c^{j+1} d < \operatorname{diam} B \leq c^j d \}.$$

Let $\mathcal{F}_0'$ be a maximal subfamily consisting of disjoint balls from $\mathcal{F}_0$.

The existence follows from Hausdorff's maximality principle (there is no uniqueness). Let $j \in \mathbb{N}_0$ and assume we have chosen subfamilies $\mathcal{F}_0', \dots, \mathcal{F}_j'$ of $\mathcal{F}_0, \dots, \mathcal{F}_j$, respectively. We take $\mathcal{F}_{j+1}'$ to be a maximal subfamily of $\mathcal{F}_{j+1}$ consisting of disjoint balls that are also disjoint with the balls of $\mathcal{F}_0' \cup \dots \cup \mathcal{F}_j'$. By induction this defines $\mathcal{F}_j' \subset \mathcal{F}_j$ for all $j \in \mathbb{N}_0$. By maximality of $\mathcal{F}_j'$ it follows that each $B \in \mathcal{F}_j$ intersects at least one ball from $\mathcal{F}_0' \cup \dots \cup \mathcal{F}_j'$, say $B' \in \mathcal{F}_i'$. Then

$$\operatorname{diam} B \leq c^j d \leq c^i d = c^{-1} c^{i+1} d < \frac{k-1}{2} \operatorname{diam} B'$$

and as $B \cap B' \neq \emptyset$ we get $B \subset kB'$. $\square$

Lemma Writing $Mf =: M_0 f$, and

$$\lambda_i(t) := |\{x \in \mathbb{R}^n : M_i f(x) > t\}| \quad (t>0)$$

we have

$$\lambda_i(t) \leq \frac{3^n}{t} \int_{\{|f| > \frac{t}{2}\}} |f| \quad (f \in L^1(\mathbb{R}^n))$$

$\boxed{\text{Pf:}}$ We consider $i=0$, the other cases being the same.

Fix $t>0$ and put $S_t = \{x : Mf(x) > t\}$.

For each $x \in S_t$ select $r_x > 0$ s.t.

$\fint_{B(x,r_x)} |f| > t$. Note $r_x < \left\{\frac{t}{\omega_n} \int_{\mathbb{R}^n} |f|\right\}^{\frac{1}{n}} < \infty$

for all $x \in S_t$, so the previous covering lemma applies and yields a subfamily $B(x_j, r_{x_j})$, $j \in \mathbb{N}$, consisting of disjoint balls and so

$$S_t \subseteq \bigcup_{x \in S_t} B(x, r_x) \subseteq \bigcup_{j=1}^{\infty} B(x_j, kr_{x_j}),$$

where $k>3$. Now $|S_t| \leq |\bigcup_1^{\infty} B(x_j, kr_{x_j})|$

$$\leq \sum_1^{\infty} k^n |B(x_j, r_{x_j})| < \sum_1^{\infty} k^n \frac{1}{t} \int_{B(x_j, r_{x_j})} |f| =$$

$$\frac{k^n}{t}\int_{\bigcup B(x_j, r_{x_j})} |f| \leq \frac{k^n}{t}\int_{\mathbb{R}^n} |f| .$$

Now let $k \searrow 3$ to get: $\lambda(t) \leq \frac{3^n}{t}\int_{\mathbb{R}^n} |f|$.

Apply the above to $|f| \mathbb{1}_{\{|f| > \frac{t}{2}\}}$:

$$Mf \leq M(|f| \mathbb{1}_{\{|f|>\frac{t}{2}\}}) + M(|f| \mathbb{1}_{\{|f| \leq \frac{t}{2}\}})$$

$$\leq M(|f| \mathbb{1}_{\{|f|>\frac{t}{2}\}}) + \frac{t}{2},$$

ie

$$\{Mf > t\} \subseteq \{M(|f| \mathbb{1}_{\{|f|>\frac{t}{2}\}}) > \tfrac{t}{2}\},$$

so

$$\lambda(t) \leq |\{M(|f| \mathbb{1}_{\{|f|>\frac{t}{2}\}}) > \tfrac{t}{2}\}| \leq$$

$$\frac{2 \cdot 3^n}{t}\int_{\{|f|>\frac{t}{2}\}} |f| . \quad \square$$

<u>Corollary</u> (Lebesgue's differentiation theorem)

Let $f \in L^1_{loc}(\mathbb{R}^n)$. Then for $\mathcal{L}^n$-a.e. $x \in \mathbb{R}^n$, $\lim_{r \searrow 0} f_{x,r}$ exists in $\mathbb{R}$ and

$$\lim_{r \searrow 0} ⨍_{B(x,r)} |f - f_{x,r}| = 0.$$

<u>Remark</u> Many variants possible!
For example when $f \in L^p_{loc}(\mathbb{R}^n)$ we also have
$$\lim_{r \searrow 0} \fint_{B(x,r)} |f - f_{x,r}|^p = 0 \quad \text{a.e.} \quad (1 \le p < \infty)$$

$\boxed{\text{Pf:}}$ We recall that $C^0_c(\mathbb{R}^n)$ (=space of cont. & compactly supported functions) is dense in $L^1(\mathbb{R}^n)$.

WLOG $f \in L^1(\mathbb{R}^n)$. Let $\varepsilon > 0$ and take $g \in C^0_c(\mathbb{R}^n)$ s.t. $\|f - g\|_{L^1} < \varepsilon$. Clearly,
$\lim_{r \searrow 0} g_{x,r} = g(x)$ and $\lim_{r \searrow 0} \fint_{B(x,r)} |g - g_{x,r}| = 0$
for <u>all</u> x. Now $\overline{\lim_{s,r \searrow 0}} |f_{x,r} - f_{x,s}| \le$

$$2 \overline{\lim_{r \searrow 0}} \fint_{B(x,r)} |f - f_{x,r}| \le 2 \overline{\lim_{r \searrow 0}} \fint_{B(x,r)} \Big(|f - g| + |g - g_{x,r}| + |g_{x,r} - f_{x,r}| \Big) \le 4 \overline{\lim_{r \searrow 0}} \fint_{B(x,r)} |f - g| \le 4 M(f-g)(x)$$

and so for $t > 0$,

$$|\{x : \overline{\lim_{r,s \searrow 0}} |f_{x,s} - f_{x,r}| > t\}| + |\{x : \overline{\lim_{r \searrow 0}} \fint_{B(x,r)} |f - f_{x,r}| > t\}|$$
$$\le 2 |\{x : M(f-g) > \tfrac{t}{4}\}| \le 2 \frac{2 \cdot 3^n}{\frac{t}{4}} \int |f - g| \le \frac{16 \cdot 3^n}{t} \varepsilon . \square$$

Dyadic cubes

By a cube in $\mathbb{R}^n$ we mean a set of the form $(a,b)^n$ — i.e., it's an open cube with sides that are parallel to the coordinate axes.

Fix a cube Q_0. By the dyadic subcubes of Q_0 we mean the cubes obtained by successively splitting each cube into 2^n subcubes of half side-length.

Observe:

- if Q_1, Q_2 are Q_0-dyadic, then either $Q_1 \cap Q_2 = \emptyset$ or $Q_1 \subseteq Q_2$ or $Q_2 \subseteq Q_1$.

- if $\mathcal{F}$ is a family of Q_0-dyadic cubes, then it contains a subfamily $\mathcal{F}'$ consisting of disjoint Q_0-dyadic cubes s.t. $\bigcup \mathcal{F} = \bigcup \mathcal{F}'$.

- almost all $x \in Q_0$ defines a (unique) max descending sequence of Q_0-dyadic cubes $(Q_j)_{j \in \mathbb{N}}$ s.t. $\bigcap Q_j = \{x\}$.

The dyadic cubes in $\mathbb{R}^n$:

$$\mathcal{D} := \bigcup_{k\in\mathbb{Z}} \mathcal{D}_k \,, \quad \text{where}$$

$$\mathcal{D}_k := \{ z + (0,2^k) : z \in 2^k\mathbb{Z}^n \} \quad (k \in \mathbb{Z})$$

Calderón - Zygmund decomposition lemma:

Let Q_0 be a cube in $\mathbb{R}^n$ (or: $Q_0 = \mathbb{R}^n$). For $f \in L^1(Q_0)$ and $t > \fint_{Q_0} |f|$ ($= 0$ when $Q_0 = \mathbb{R}^n$) there exists disjoint cubes $Q_j \subset Q_0$, $j \in \mathbb{N}$ (Q_0- dyadic) s.t.

$$t < \fint_{Q_j} |f| \leq 2^n t \qquad \forall j$$

and $\quad |f(x)| \leq t \quad$ for a.e. $x \in Q_0 \setminus \bigcup_j Q_j$.

$\boxed{\text{Pf:}}$ We consider the case $Q_0 = \mathbb{R}^n$, the other cases being similar. Put

$$L_f := \left\{ x \in \mathbb{R}^n : \begin{array}{l} \text{there exists dyadic cubes } Q_j \\ \text{s.t. } \bigcap_{j=1}^{\infty} Q_j = \{x\} \text{ and } \fint_{Q_j} |f| \to |f(x)| \end{array} \right\},$$

where we have fixed a representative f.

By Lebesgue's differentiation theorem L_f has full measure in $\mathbb{R}^n$: $|\mathbb{R}^n \setminus L_f| = 0$.

Put $E = \{x \in L_f : |f(x)| > t\}$.

Each $x \in E$ is contained in a unique maximal descending sequence of dyadic cubes $Q_k \in \mathcal{D}_k$ with $\bigcap_{k \in \mathbb{Z}} Q_k = \{x\}$.

Since $⨍_{Q_k} |f| \to 0$ as $k \to \infty$ and $⨍_{Q_k} |f| \to |f(x)| > t$ as $k \to -\infty$ there exists a maximal $k(x) \in \mathbb{Z}$ s.t.

$⨍_{Q_{k(x)}} |f| > t$. Then $⨍_{Q_{k(x)+1}} |f| \leq t$, ie

$$t < ⨍_{Q_{k(x)}} |f| \leq \frac{1}{|Q_{k(x)}|} \int_{Q_{k(x)+1}} |f| = 2^n ⨍_{Q_{k(x)+1}} |f| \leq 2^n t.$$

Consequently, E is covered by the family $\mathcal{F} = \{Q_{k(x)} : x \in E\}$, and since the cubes in $\mathcal{F}$ are dyadic there exists a disjoint subfamily $\mathcal{F}'$ with $\bigcup \mathcal{F} = \bigcup \mathcal{F}'$.

Now $\mathcal{F}' = \{Q_j : j \in \mathbb{N}\}$ works, since $E \subset \bigcup \mathcal{F}'$ so $|f| \leq t$ a.e. on $\mathbb{R}^n \setminus \bigcup \mathcal{F}'$. $\square$

<u>Lemma</u>: Let $f \in L^1(\mathbb{R}^n)$ and $t > 0$. Then

$$\int_{\{|f| > \omega_n n^{\frac{n}{2}} t\}} |f| \leq 2^n \omega_n n^{\frac{n}{2}} t \lambda(t),$$

where $\lambda(t) = |\{x : Mf(x) > t\}|$.

$\boxed{\text{Pf:}}$ Fix $t > 0$. Let Q_j, $j \in \mathbb{N}$, be a C-Z decomposition for $|f|$ at level $t\omega_n n^{\frac{n}{2}}$:

Q_j, $j \in \mathbb{N}$, are disjoint dyadic cubes s.t.

$$|f| \leq t\omega_n n^{\frac{n}{2}} \quad \text{a.e. on} \quad \mathbb{R}^n \setminus \bigcup_j Q_j,$$

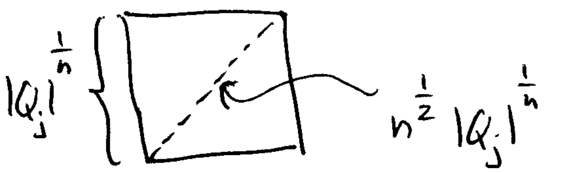
$$t\omega_n n^{\frac{n}{2}} < ⨍_{Q_j} |f| \leq t\, 2^n \omega_n n^{\frac{n}{2}} \quad \forall j.$$

Since $\left|\{|f| > t\omega_n n^{\frac{n}{2}}\} \setminus \bigcup_j Q_j\right| = 0$,

$$\int_{\{|f| > t\omega_n n^{\frac{n}{2}}\}} |f| \leq \int_{\bigcup_j Q_j} |f| = \sum_j |Q_j| ⨍_{Q_j} |f|$$

$$\le t\, 2^n \omega_n n^{\frac{n}{2}} \left|\bigcup_j Q_j\right| .$$

Note: if $x \in Q_j$, then $Q_j \subset B(x, \sqrt{n}\, |Q|^{\frac{1}{n}})$,



Q_j

so

$$Mf(x) \ge \fint_{B(x, \sqrt{n}\, |Q_j|^{\frac{1}{n}})} |f| \ge \frac{1}{\omega_n n^{\frac{n}{2}} |Q_j|} \int_{Q_j} |f| > t ,$$

hence $\{ Mf > t \} \supset \bigcup Q_j$, ie

$$\int_{\{|f| > t\omega_n n^{\frac{n}{2}}\}} |f| \le t\, 2^n \omega_n n^{\frac{n}{2}} \lambda(t) . \quad \square$$

Hence TH 4 holds with

$$c_1 = \frac{1}{2^n \omega_n n^{\frac{n}{2}}}, \quad c_2 = 3^n, \quad c_3 = \omega_n n^{\frac{n}{2}} . \quad \square$$

$\boxed{\text{Pf of TH 3:}}$

$$\lambda(t) = |\{x : Mf(x) > t\}| \quad , \quad t>0.$$

From TH 4:

$$c_1 \int_{|f|>c_3 t} |f| \overset{*}{\leq} t\lambda(t) \overset{**}{\leq} c_2 \int_{|f|>\frac{t}{2}} |f| \quad , \quad \forall t>0.$$

$$\int_{\mathbb{R}^n} (Mf)^p = p\int_0^\infty t^{p-1}\lambda(t)\,dt \overset{*}{\geq}$$

$$c_1 p \int_0^\infty t^{p-2} \int_{|f|>c_3 t} |f|\,dx\,dt \overset{\text{Fubini-Tonelli}}{=}$$

$$c_1 p \int_{\mathbb{R}^n} \int_0^{\frac{|f(x)|}{c_3}} t^{p-2}\,dt\,|f|\,dx = \frac{c_1 p}{p-1}\int_{\mathbb{R}^n} \left(\frac{|f|}{c_3}\right)^{p-1} |f|\,dx$$

$$= \frac{p}{p-1} c_1 c_3^{1-p} \int_{\mathbb{R}^n} |f|^p\,dx.$$

$$\int_{\mathbb{R}^n} (Mf)^p = p\int_0^\infty t^{p-1}\lambda(t)\,dt \overset{**}{\leq} c_2 p \int_0^\infty t^{p-2}\int_{|f|>\frac{t}{2}} |f|\,dx\,dt$$

Fubini-Tonelli

$$= \quad c_2 p \int_{\mathbb{R}^n} \int_0^{2|f(x)|} t^{p-2}\,dt\,|f|\,dx =$$

$$\frac{c_2 p}{p-1} \int_{\mathbb{R}^n} (2|f|)^{p-1} |f| \, dx = c_2 2^{p-1} \frac{p}{p-1} \int_{\mathbb{R}^n} |f|^p,$$

hence

$$c_1 c_3^{1-p} \int_{\mathbb{R}^n} |f|^p \, dx \leq \frac{p-1}{p} \int_{\mathbb{R}^n} (Mf)^p \, dx \leq c_2 2^{p-1} \int_{\mathbb{R}^n} |f|^p dx. \square$$