 # Partial regularity of minimizers II

Background (Discussed in Part I)

Let $F : \mathbb{R}^{N\times n} \to \mathbb{R}$ be a C^2 function s.t.

(*) $\ell|\lambda|^2 \leq F''(z)[\lambda,\lambda] \leq L|\lambda|^2$

for all $z, \lambda \in \mathbb{R}^{N\times n}$, where $0<\ell\leq L<\infty$ are constants.

Given a bounded, open subset $\Omega\subset\mathbb{R}^n$ and $g\in W^{1,2}(\Omega,\mathbb{R}^N)$.

(P) Find $u\in W^{1,2}_g(\Omega,\mathbb{R}^N)$ s.t. $\int_\Omega F(Du)\leq\int_\Omega F(Dv)$ for all $v\in W^{1,2}_g(\Omega,\mathbb{R}^N)$.

(P) is an example of a 'regular variational problem' in the sense of Hilbert. Solutions to (P) are called F-minimizers.

— existence and uniqueness of an F-minimiser $u\in W^{1,2}_g(\Omega,\mathbb{R}^N)$:

- existence by direct method
- uniqueness by strict convexity of F

— regularity?

<u>Scalar case $N=1$</u> (one unknown function)

Ladyženskaya & Ural'seva (based on work by De Giorgi, Nash, Moser):

$$u \in C^{1,\alpha}_{loc}(\Omega) \quad \text{for all} \quad \alpha < 1$$

(and, in fact, if $F \in C^\infty$ then $u \in C^\infty$ too)

<u>Vectorial case $N>1$</u> (more than one unknown function)

Minimizers can have singularities (not diff., not cont., not bounded, ...)

Nečas (1975): not diff., but Lipschitz (in very high dimensions)

Šverák & Yan (2000, 2002):

not Lipschitz (when $n \geqslant 3, N \geqslant 5$)
not bounded (when $n \geqslant 5, N \geqslant 12$)

⋮

However, $u \in W^{2,2}_{loc}(\Omega, \mathbb{R}^N)$ (by the difference-quotient method), and there's partial regularity:

If u is an F-minimizer, and

$$\Sigma_u := \left\{ x \in \Omega : \begin{array}{l} \overline{\lim\limits_{r \searrow 0}} \, |(Du)_{x,r}| < \infty \text{ and} \\ \underline{\lim\limits_{r \searrow 0}} \fint_{B(x,r)} |Du - (Du)_{x,r}|^2 = 0 \end{array} \right\},$$

then $\Omega \setminus \Sigma_u$ is open and u (the precise representative*)) is $C^{1,\alpha}_{loc}$ on $\Omega \setminus \Sigma_u$ for all $\alpha < 1$. (DeGiorgi, Almgren, Miranda, Morrey, Giaquinta & Giusti, Giaquinta & Ivert, ...)

By Lebesgue's differentiation theorem the singular set Σ_u has Lebesgue measure 0.

But we can estimate the size of Σ_u much more efficiently if we use $u \in W^{2,2}_{loc}$ and Hausdorff measures. We'll return to this point later in the course.

*) precise representative of $f \in L^1_{loc}$ is $\tilde{f}(x) := \begin{cases} \lim\limits_{r \searrow 0} f_{x,r} & \text{if limit } \exists \\ 0 & \text{otherwise} \end{cases}$

We intend to discuss the partial regularity of minimizers for more general variational integrals:

$$\mathcal{F}(v) = \int_\Omega F(x, v, Dv), \quad v \in W^{1,2},$$

where

$$F : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R} \text{ is cont., and}$$

satisfies

(H1) $z \mapsto F(x,y,z)$ is C^2, and
$(x,y,z) \mapsto F_{zz}(x,y,z)$ is continuous

(H2) $\ell |\lambda|^2 \leq F_{zz}(x,y,z)[\lambda,\lambda] \leq L|\lambda|^2$

for all $z, \lambda \in \mathbb{R}^{N\times n}$ (and $x \in \Omega, y \in \mathbb{R}^N$)

(H3) $\ell |z|^2 - L \leq F(x,y,z) \leq L(|z|^2+1)$

for all $x \in \Omega$, $y \in \mathbb{R}^N$, $z \in \mathbb{R}^{N\times n}$

(H4) $|F(x_1, y_1, z) - F(x_2, y_2, z)|$

$$\leq L\,\omega_\alpha(|x_1 - x_2| + |y_1 - y_2|)(|z|^2+1)$$

for all $x_1, x_2 \in \Omega$, $y_1, y_2 \in \mathbb{R}^N$, $z \in \mathbb{R}^{N\times n}$,

where $\omega_\alpha(t) := \min\{1, t^\alpha\}$, $t \geq 0$,

$0 < \alpha \leq 1$.

<u>Note</u> (H1) & (H2) : for fixed x, y

$z \mapsto F(x,y,z)$ is a <u>regular variational integrand</u>

— Existence of an F-minimizer $u \in W^{1,2}_g(\Omega, \mathbb{R}^N)$ follows by use of the direct method ((H3) + convexity in z & joint continuity)

— Uniqueness : probably not in general (note: $\mathcal{F}(v)$ isn't convex)

<u>TH 5:</u> Assume (H1–4) and let $u \in W^{1,2}_g(\Omega, \mathbb{R}^N)$ be an F-minimizer. Put

$$\Omega \setminus \Sigma_u := \Big\{ x \in \Omega : \ \overline{\lim_{r \searrow 0}} \big(|(u)_{x,r}| + |(Du)_{x,r}| \big) < \infty \ \text{ and } \lim_{r \searrow 0} \fint_{B(x,r)} \big(|u - u_{x,r}|^2 + |Du - (Du)_{x,r}|^2 \big) = 0 \Big\}$$

Then $\Omega \setminus \Sigma_u$ is open and u (the precise representative) is $C^{1,\alpha/2}_{loc}(\Omega \setminus \Sigma_u, \mathbb{R}^N)$.

Giaquinta & Giusti : Differentiability of the minima of non-differentiable functionals. Inv. Math. 72 (1982), 285–298.

See also [8] pp. 319 — 342.

Remark. Assume (H1-4), $\alpha = 1$ and $F = F(x,y,z) \in C^{\infty}$.

Then an F-minimizer $u \in W^{1,2}_g(\Omega, \mathbb{R}^N)$ is also an F-extremal, ie it solves the Euler-Lagrange equation for $\mathcal{F}$

(EL) $$\operatorname{div} F_z(x,u,Du) = F_y(x,u,Du) \text{ in } \Omega$$

Here $|F_y(x,u,Du)| \leq L(1+|Du|^2)$ by (H4) with $\alpha = 1$, so (EL) is an elliptic system with critical growth

($Du \in L^2$ so RHS $= F_y(x,u,Du) \in L^1$)

and it's known that there's no partial regularity theory for weak solutions of such systems (exx harmonic map systems, see Rivière). It's therefore important that we're dealing with minimizers and not merely extremals.

Idea of proof 'linearization strategy'

Given:

- $F = F(x,y,z)$ is continuous and (H1-4) hold
- u F-minimizer

technical reasons

Fix ball $B_{4R} = B(x_0, 4R) \subset \Omega$

Put $y_0 := (u)_{x_0,R}$, $z_0 := (Du)_{x_0,R}$ and

$$P(z) := F(x_0,y_0,z_0) + F_z(x_0,y_0,z_0)[z-z_0]$$
$$+ \tfrac{1}{2} F_{zz}(x_0,y_0,z_0)[z-z_0, z-z_0]$$

(the 2^{nd} Taylor polynomial for the 'frozen' integrand $F(x_0,y_0,\cdot)$ about z_0)

By (H1) well-defined and as $P''(z) = F_{zz}(x_0,y_0,z_0)$ (H2) yields

$$\ell|\lambda|^2 \leq P''(z)[\lambda,\lambda] \leq L|\lambda|^2 \quad \forall z,\lambda$$

$\therefore$ $v \mapsto \int_{B_R} P(Dv)$ quadratic, convex functional with Euler-Lagrange equation

(+) $\operatorname{div} P'(Dv) = 0$ in B_R.

$\oplus$ is a linear strongly elliptic system with constant coefficients.

Hence by the generalized Weyl Lemma (recall Part I) P-minimizers are C^∞ and we have good estimates: if $h \in W^{1,2}_u(B_R, \mathbb{R}^N)$ is P-minimizing, then

$$\circledast \quad \fint_{B_r(x_0)} |Dh - (Dh)_{x_0,r}|^2 \leq c\left(\frac{r}{R}\right)^2 \fint_{B_R(x_0)} |Du - (Du)_{x_0,R}|^2$$

for all $0 < r \leq R$, where $c = c(n, N, \frac{L}{\ell})$ is a constant.

Recall Campanato's integral characterization of Hölder continuity (Part I):

$f \in C^\alpha$ near x_0 iff

$$\fint_{B(x,r)} |f - f_{x,r}|^2 \sim c r^{2\alpha} \quad \text{as } r \searrow 0$$

for x near x_0.

Can we transfer $\circledast$ to u?

It requires precise estimate of

$$\fint_{B_R(x_0)} |Du - Dh|^2$$

and to get that we need various Caccioppoli inequalities and Gehring's Lemma.

<u>Proof outline</u>

Step 1 : Preliminary higher integrability via Caccioppoli inequality of the 1^{st} kind + Gehring's Lemma.

Step 2 : <u>The essential key step</u>

Caccioppoli inequality of the 2^{nd} kind

(= a reverse Poincaré inequality on increasing supports with 'good' error terms for $u - a$, where a is any affine map)

Step 3: Implementation of 'Linearization strategy'

Step 4: Iteration.

Proof of TH5: For the sake of notational simplicity assume $F = F(y,z)$. We can also assume that $\ell = 1$.

Step 1 We have already done this.

In view of (H3) u is a Q-minimizer for $\int_\Omega (1 + |Du|^2)$, hence by TH1 (with $p = 2$) there exist $q_0 = q_0(n, Q) = q_0(n, \frac{L}{\ell}) > 2$ and $c = c(n, \frac{L}{\ell})$ s.t. $u \in W^{1,q_0}_{loc}(\Omega, \mathbb{R}^N)$ and

(5.1) $$\fint_{B_R} |Du|^q \leq c + c\left(\fint_{B_{2R}} |Du|^2\right)^{\frac{q}{2}}$$

for all $q \in [2, q_0]$ and $B_{2R} \subset \Omega$.

Step 2

Lemma (Caccioppoli's inequality of 2nd kind)

There exists a constant $c = c(n, N, \frac{L}{\ell})$ with the following property. For any ball $B_R = B(x_0, R) \subset \Omega$, any $y_0 \in \mathbb{R}^N$, and any affine map $a : \mathbb{R}^n \to \mathbb{R}^N$:

$$(5.2) \quad \int_{B_{\frac{R}{2}}} |Du - Da|^2 \leq \frac{c}{R^2} \int_{B_R} |u-a|^2$$

$$+ c \int_{B_R} \omega_\alpha \left(|u - y_0| + |u - a| \right) \left(1 + |Du|^2 + |Da|^2 \right).$$

Remark The 'error term'

$$c \int_{B_R} \omega_\alpha \left(|u - y_0| + |u - a| \right) \left(1 + |Du|^2 + |Da|^2 \right)$$

is under our control because of the higher integrability of Du afforded by Step 1.

$\boxed{\text{Pf of Lemma:}}$ Put $\tilde{F}(z) := F(y_0, Da+z) - F(y_0, Da) - F_z(y_0, Da)[z]$

and note $\tilde{F}(z) = \int_0^1 (1-t)\, \tilde{F}''(tz)[z,z]\, dt =$

$\int_0^1 (1-t)\, F_{zz}(y_0, Da+tz)[z,z]\, dt$, hence by (H2)

$$(5.3) \quad \frac{1}{2}|z|^2 \leq \tilde{F}(z) \leq \frac{L}{2}|z|^2 \qquad \forall z$$

Likewise $\tilde{F}'(z) = \int_0^1 \tilde{F}''(tz)[z,\cdot]\, dt$, so by (H2) again

$$(5.4) \qquad |\tilde{F}'(z)| \leq L|z| \qquad \forall z$$

We'll use Widman's hole-filling trick.

Define $\tilde{u} := u - a$, let $\frac{R}{2} < r < s < R$

and $\rho \in W^{1,\infty}(B_R)$, $\mathbb{1}_{B_r} \leq \rho \leq \mathbb{1}_{B_s}$, $|D\rho| \leq \frac{1}{s-r}$

and $\varphi = \rho \tilde{u}$, $\psi = (1-\rho)\tilde{u}$.

Note: $\varphi \in W_0^{1,2}(B_s, \mathbb{R}^N)$, $\psi = \tilde{u}$ on ∂B_s, and $\varphi + \psi = \tilde{u}$.

By (5.3): $\frac{1}{2}\int_{B_r} |D\tilde{u}|^2 \leq \frac{1}{2}\int_{B_s} |D\varphi|^2 \leq \int_{B_s} \tilde{F}(D\varphi)$

$$= \int_{B_s} \tilde{F}(D\tilde{u} - D\varphi)$$

$$= \int_{B_s} \tilde{F}(D\tilde{u}) + \int_{B_s} \underbrace{\left(\tilde{F}(D\tilde{u} - D\varphi) - \tilde{F}(D\tilde{u})\right)}_{\substack{= 0 \text{ on } B_s \setminus B_r \\ = \int_0^1 \tilde{F}'(D\tilde{u} - tD\varphi)[D\varphi]\,dt}}$$

$$\overset{(5.4)}{\leq} \int_{B_s} \tilde{F}(D\tilde{u}) + cL \int_{B_s \setminus B_r} \left(|D\tilde{u}|^2 + \frac{|\tilde{u}|^2}{(s-r)^2} \right)$$

$$\leq \int_{B_s} \tilde{F}(D\tilde{u}) + cL \int_{B_s \setminus B_r} |D\tilde{u}|^2 + cL \int_{B_s} \left(\frac{|\tilde{u}|}{s-r} \right)^2 .$$

Here $\int_{B_s} \tilde{F}(D\tilde{u}) = \int_{B_s} \Big(F(y_0, Du) - F(y_0, Da)$

$- F_z(y_0, Da)[D\tilde{u}] \Big) =$

$$\int_{B_s} F(u, Du) + \int_{B_s} \left(F(y_0, Du) - F(u, Du) \right)$$

$$- \int_{B_s} \left(F(y_0, Da) + F_z(y_0, Da)[D\tilde{u}] \right) \underset{\& \tilde{u} = \varphi \text{ on } \partial B_s}{\overset{u\ F\text{-min}}{\leq}}$$

$$\int_{B_s} \left(F(u - \varphi, Du - D\varphi) - F(y_0, Da) - F_z(y_0, Da)[D\varphi] \right)$$

$$+ \int_{B_s} \left(F(y_0, Du) - F(u, Du) \right) \overset{(H4)}{\leq}$$

$$\leq \int_{B_s} \Big(F(\varphi + a, D\varphi + Da) - F(y_0, D\varphi + Da) \Big)$$

$$+ \int_{B_s} \underbrace{\Big(F(y_0, D\varphi + Da) - F(y_0, Da) - F_z(y_0, Da)[D\varphi] \Big)}_{= \tilde{F}(D\varphi)}$$

$$+ \int_{B_s} L\,\omega_\alpha(|u - y_0|)(|Du|^2 + 1)$$

$$\overset{(H4)\ \&\ (5.3)}{\leq} \int_{B_s} L\,\omega_\alpha(|\varphi + a - y_0|)(|D\varphi + Da|^2 + 1)$$

$$+ \frac{L}{2} \int_{B_s} |D\varphi|^2 + L \int_{B_s} \omega_\alpha(|y_0 - u|)(|Du|^2 + 1)$$

<u>Note</u>: $\quad |D\varphi| \leq |D\tilde{u}| + \dfrac{|\hat{u}|}{s - r}, \quad \varphi = 0$ on $B_s \setminus B_r$,

$$|\varphi + a - y_0| \leq |u - y_0| + |u - a|$$

Hence

$$\text{(5.5)} \quad \frac{1}{2} \int_{B_r} |D\tilde{u}|^2 \leq cL \int_{B_s \setminus B_r} |D\tilde{u}|^2 + cL \int_{B_s} \frac{|\tilde{u}|^2}{(s - r)^2}$$

$$+ cL \int_{B_s} \omega_\alpha(|u - y_0| + |u - a|)(|Du|^2 + |Da|^2 + 1).$$

We can rewrite (5.5) as

$$\int_{B_r} |D\tilde{u}|^2 \leq \theta \int_{B_s} |D\tilde{u}|^2 + A + \frac{B}{(s-r)^2}$$

for all $\frac{R}{2} < r < s < R$, where

$$\theta := \frac{2cL}{1+2cL} < 1,$$

$$A = 2cL \int_{B_R} \omega_\alpha (|u-y_0| + |u-a|)(1 + |Du|^2 + |Da|^2),$$

$$B = 2cL \int_{B_R} |u-a|^2 \quad \text{are constants.}$$

The conclusion now follows from the iteration Lemma quoted in L1&2. $\square$

Note that (5.2) can be rewritten as

new constant $= 2^n$ times old

$$\fint_{B_{\frac{R}{2}}} |Du - Da|^2 \leq \frac{c}{R^2} \fint_{B_R} |u-a|^2$$

$$+ c \fint_{B_R} \omega_\alpha (|u-y_0| + |u-a|)(1 + |Du|^2 + |Da|^2)$$

Invoking Poincaré–Sobolev's inequality

$$\frac{1}{R^2}\fint_{B_R}|u-a|^2 \leq c\Big[\fint_{B_R}|Du-Da|^{\frac{2n}{n+2}}\Big]^{\frac{n+2}{n}},$$

where we now assume that the affine map is chosen s.t. $(u-a)_{B_R}=0$ (the linear part is free, but the constant term in a is then fixed accordingly):

$$\fint_{B_{\frac{R}{2}}}|Du-Da|^2 \leq c\Big(\fint_{B_R}|Du-Da|^{\frac{2n}{n+2}}\Big)^{\frac{n+2}{n}}$$

$$+\,c\fint_{B_R}\omega_\alpha(|u-y_0|+|u-a|)\big(|Du|^2+|Da|^2+1\big).$$

Put $f:=|Du-Da|^{\frac{2n}{n+2}}$, $g:=\{c\,\omega_\alpha(|u-y_0|+|u-a|)\cdot(|Du|^2+|Da|^2+1)\}^{\frac{n}{n+2}}$ to get

$$\fint_{B_{\frac{R}{2}}} f^{\frac{n+2}{n}} \leq c\Big(\fint_{B_R} f\Big)^{\frac{n+2}{n}}+\fint_{B_R} g^{\frac{n+2}{n}}$$

for all balls $B_R\subset\Omega$. Consequently there exists $p_0 = p_0(n, c) > 2$ s.t.

$$\fint_{B_{\frac{R}{2}}} |Du - Da|^p \leqq c\left(\fint_{B_R} |Du - Da|^2\right)^{\frac{p}{2}} \qquad (5.6)$$

$$+ c \fint_{B_R} \omega_\alpha\big(|u - y_0| + |u - a|\big)\big(1 + |Du|^p + |Da|^p\big)$$

for all balls $B_R \subset \Omega$ (Gehring's Lemma, TH 3).