Partial regularity of minimizers, II

Recall from last lecture:

TH 5: (in the special case $F = F(y,z)$, $l=1$)

Assume $F : \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$ is continuous, and

(H1) $z \mapsto F(y,z)$ is C^2, and $(y,z) \mapsto F_{zz}(y,z)$ is continuous

(H2) $|\lambda|^2 \leq F_{zz}(y,z) \leq L|\lambda|^2$ for all $\lambda, z \in \mathbb{R}^{N\times n}$, $y \in \mathbb{R}^N$

(H3) $|z|^2 - L \leq F(y,z) \leq L(|z|^2+1)$

(H4) $|F(y_1,z) - F(y_2,z)| \leq L\,\omega_\alpha(|y_1-y_2|)(|z|^2+1)$,

where $\omega_\alpha(t) := \min\{1, t^\alpha\}$, $t \geq 0$, $0<\alpha\leq 1$.

If $u \in W^{1,2}_g(\Omega, \mathbb{R}^N)$ is an F-minimizer, then

$$\Sigma_u := \left\{ x \in \Omega : \ \overline{\lim_{r\searrow 0}} \left(|u_{x,r}| + |Du_{x,r}|\right) = \infty, \text{ or } \ \underline{\lim_{r\searrow 0}} \fint_{B(x,r)} \left(|u-u_{x,r}|^2 + |Du - Du_{x,r}|^2\right) > 0 \right\}$$

is relatively closed in Ω, and u is $C^{1,\frac{\alpha}{2}}_{loc}$ on $\Omega \setminus \Sigma_u$.

Step 1 Preliminary higher integrability.

$\exists\ q_0 = q_0(n,L) > 2$, $c = c(n,L) > 0$ s.t.

$u \in W^{1,q_0}_{loc}(\Omega, \mathbb{R}^N)$ and

(5.1) $$\fint_{B_R} |Du|^q \leq c + c\left(\fint_{B_{2R}} |Du|^2\right)^{\frac{q}{2}}$$

for all $q \in [2, q_0]$ and $B_{2R} \subset \Omega$.

(Done in L7&8. Pf used only (H3) & F-minimality)

Step 2 Caccioppoli's inequality of 2nd kind.

$\exists\ c_0 = c_0(n,N,L) > 0$ s.t.

(5.2) $$\fint_{B_{\frac{R}{2}}} |Du - Da|^2 \leq \frac{c_0}{R^2} \fint_{B_R} |u-a|^2 + c_0 \fint_{B_R} \omega_\alpha\left(|u-y_0|+|u-a|\right)\left(1+|Du|^2+|Da|^2\right)$$

for all $y_0 \in \mathbb{R}^N$, affine $a: \mathbb{R}^n \to \mathbb{R}^N$ and $B_R \subset \Omega$.

(5.2) + Poincaré–Sobolev & Gehring's Lemma (TH 3) yields

$\exists\ p_0 = p_0(n, c_0) > 2$, $c = c(n, c_0) > 0$ s.t.

$$(5.6) \quad \fint_{B_{\frac{R}{2}}} |Du - Da|^p \le c\Big(\fint_{B_R} |Du - Da|^2\Big)^{\frac{p}{2}}$$

$$+ c\fint_{B_R} \omega_\alpha\big(|u - y_0| + |u - a|\big)\big(1 + |Du|^p + |Da|^p\big)$$

for <u>all</u> $y_0 \in \mathbb{R}^N$, affine $a : \mathbb{R}^n \to \mathbb{R}^N$ s.t. $(u-a)_{B_R} = 0$, and $B_R \subset \Omega$.

(Done in L7 & 8. Pf used essentially all assumptions.)

<u>Step 3</u> Implementation of linearization strategy.

Let $m > 1$, and $0 < \varepsilon \le 1$.

We'll impose additional smallness conditions on ε during the proof below.

Fix $B_{4R} = B(x_0, 4R) \subset \Omega$ with $R \in (0,1]$.

Put $y_0 := u_{x_0,R}$, $z_0 := (Du)_{x_0,R}$.

Consider 'frozen' integrand

$$F^0(z) := F(y_0, z)$$

and its 2^{nd} Taylor polynomial about z_0

$$P(z) := F^0(z_0) + F^0_z(z_0)[z - z_0] + \frac{1}{2} F^0_{zz}(z_0)[z-z_0, z-z_0]$$

Observe:

$$|F^0(z) - P(z)| \leq \int_0^1 (1-t)\, |F^0_{zz}(z_0 + t(z-z_0)) - F^0_{zz}(z_0)|\, dt\, |z-z_0|^2$$

No uniform continuity assumption imposed on F_{zz} so we need bounds on y_0, z_0.
This causes some technical difficulties.

Assume

$$(6.1) \quad \begin{cases} |u_{x_0,4R}| < m, \quad |(Du)_{x_0,4R}| < m \\ \displaystyle\fint_{B(x_0,4R)} |Du - (Du)_{x_0,4R}|^2 < \varepsilon. \end{cases}$$

Note that $|(Du)_{x_0,R}| \leq |(Du)_{x_0,R} - (Du)_{x_0,4R}| + |(Du)_{x_0,4R}|$

$$< \fint_{B(x_0,R)} |Du - (Du)_{x_0,4R}| + m \overset{\text{Hölder}}{\leq}$$

$$\left(\fint_{B(x_0,R)} |Du-(Du)_{x_0,4R}|^2\right)^{\frac{1}{2}} + m$$

$$\leq \left(4^n \fint_{B(x_0,4R)} |Du-(Du)_{x_0,4R}|^2\right)^{\frac{1}{2}} + m$$

$$< 2^n \varepsilon^{\frac{1}{2}} + m < 1+m \quad \text{provided } \varepsilon < 4^{-n}.$$

Likewise $|u_{x_0,R}| < 1+m$ provided $\varepsilon < c_n$ where $c_n \in (0, 4^{-n}]$.

Consequently

$$(6.2) \qquad |u_{x_0,R}| < 1+m, \quad |(Du)_{x_0,R}| < 1+m$$

when $0 < \varepsilon < c_n$.

In view of (H1), $(y,z) \mapsto F_{zz}(y,z)$ is uniformly continuous on compact subsets of $\mathbb{R}^N \times \mathbb{R}^{N\times n}$, hence in particular on $\{(y,z) : |y| \leq 2+m, |z| \leq 2+m\}$.

We can therefore find a modulus of continuity $\omega = \omega_m : [0,\infty) \to [0,1]$ that is increasing, concave, $\omega(1)=1$ and so

$$|F_{zz}(y_1, z_1) - F_{zz}(y_2, z_2)| \leq c\,\omega(|y_1 - y_2| + |z_1 - z_2|)$$

for all $|y_i| \leq m+2$, $|z_i| \leq m+2$. (see lecture notes for Part I, L15&16, for the construction of such ω.)

Thus $|F^0(z) - P(z)| \leq c\,\omega(|z - z_0|)|z - z_0|^2$ when $|z| \leq m+2$ (note $|z_0| \leq m+1$).

In view of (H2) we also have

$$|F^0(z) - P(z)| \leq L|z - z_0|^2 \quad \forall z,$$

and since $\omega(|z - z_0|) = 1$ for $|z - z_0| \geq 1$ we conclude that

$$(6.3) \quad |F^0(z) - P(z)| \leq c\,\omega(|z - z_0|)|z - z_0|^2$$

for all $z \in \mathbb{R}^{N \times n}$.

Regarding P we also record

$$(6.4) \quad \begin{cases} P(z) - P(z_1) - P_z(z_1)[z - z_1] \geq \frac{1}{2}|z - z_1|^2 \\ |\lambda|^2 \leq P_{zz}(z)[\lambda, \lambda] \leq L|\lambda|^2 \end{cases}$$

for all $z, z_1, \lambda \in \mathbb{R}^{N\times n}$.

Let $h \in W^{1,2}_u(B_R, \mathbb{R}^N)$ be P-minimizing. Then by the generalized Weyl lemma (see Part I) h is C^∞ and

$$(6.5) \quad \fint_{B_r} |Dh - (Dh)_{x_0,r}|^2 \leq c\left(\frac{r}{R}\right)^2 \fint_{B_R} |Du - (Du)_{x_0,R}|^2$$

for all $0 < r \leq R$, where $c = c(n, N, L) > 0$.

Define $E(x,r) := \fint_{B(x,r)} |Du - (Du)_{x,r}|^2$

(usually called the 'quadratic excess').

Write $E(r) := E(x_0, r)$.

We seek a power-like decay of $E(r)$ as $r \searrow 0$, say $E(r) \lesssim r^\sigma$ for $r > 0$ small, and intend to then conclude by use of Campanato's integral characterization of Hölder continuity. The idea is of course to transfer some of the good decay in (6.5) to Du.

For that we must have an estimate of the 'approximation rate' $\fint_{B_R} |Du - Dh|^2$:

$$\frac{1}{2}\fint_{B_R} |Du-Dh|^2 \overset{(6.4)}{\leq} \fint_{B_R}\Big(P(Du) - P(Dh) - P_z(Dh)[Du-Dh]\Big)$$

h P-min

$$= \fint_{B_R}\Big(\big[P(Du) - F^0(Du)\big] + \big[F^0(Du) - F^0(Dh)\big]$$

$$+ \big[F^0(Dh) - P(Dh)\big]\Big) =: \mathrm{I} + \mathrm{II} + \mathrm{III}.$$

We use (6.3) to estimate

$$\mathrm{I} + \mathrm{III} \leq c\fint_{B_R}\Big(\omega(|Du-z_0|)|Du-z_0|^2$$

$$+ \omega(|Dh-z_0|)|Dh-z_0|^2\Big).$$

For $2 < p < \min\{p_0, q_0\}$ (where p_0, q_0 are defined in steps 2, 1, respectively) we get by use of Hölder with $\frac{p}{2}, \frac{p}{p-2}$:

$$\fint_{B_R} \omega(|Dh-z_0|)|Dh-z_0|^2 \leq \Big(\fint_{B_R}\omega^{\frac{p}{p-2}}\Big)^{\frac{p-2}{p}}\Big(\fint_{B_R}|Dh-z_0|^p\Big)^{\frac{2}{p}}$$

$$
\begin{aligned}
&\overset{\substack{\omega \leq 1, \\ \omega \text{ concave}}}{\leq} \omega\left(\fint_{B_R} |Dh - z_0| \right)^{1-\frac{2}{p}} \left(\fint_{B_R} |Dh - z_0|^p \right)^{\frac{2}{p}} \\
&\overset{\substack{\text{Hölder}, \\ \omega \text{ increasing}}}{\leq} \omega\left(\left(\fint_{B_R} |Dh - z_0|^2 \right)^{\frac{1}{2}} \right)^{1-\frac{2}{p}} \left(\fint_{B_R} |Dh - z_0|^p \right)^{\frac{2}{p}}.
\end{aligned}
$$

Here $\fint_{B_R} |Dh - z_0|^2 \leq c \fint_{B_R} |Du - z_0|^2$ and the p-integral is estimated by use of

<u>Lemma</u>

Assume $a : \mathbb{R}^{N\times n} \to \mathbb{R}^{N \times n}$ and

$$\langle a(z_1) - a(z_2), z_1 - z_2 \rangle \geq l |z_1 - z_2|^2,$$

$$|a(z_1) - a(z_2)| \leq L |z_1 - z_2|$$

for all $z_1, z_2 \in \mathbb{R}^{N\times n}$, where $0 < l < L$.

Let $g \in W^{1,p}(B_R, \mathbb{R}^N)$, $p \geq 2$.

If $v \in W^{1,2}_g(B_R, \mathbb{R}^N)$ and $\int_{B_R} a(Dv) . D\phi = 0$ for all $\phi \in W^{1,2}_0(B_R, \mathbb{R}^N)$, then $v \in W^{1,q}(B_R, \mathbb{R}^N)$ for $q \leq \min\{p, \frac{2n}{n-2}\}$ and for $c = c(n, p, \frac{L}{l})$,

$$\int_{B_R} |Dv|^q \leq c \int_{B_R} |Dg|^q.$$

(The pf is omitted here. It's a special case of Theorem 7.7 in [12]) ~~will be given in connection with the variational Calderón–Zygmund estimates in another lecture.~~

Hereby: $$I + III \le c\,\omega\left(E(R)^{\frac{1}{2}}\right)^{1-\frac{2}{p}} \left(\fint_{B_R} |Du - z_0|^p\right)^{\frac{2}{p}}.$$

We now invoke (5.6) from step 2.

Take $y_0 = u_{x_0,4R}$, $a(x) = (Du)_{x_0,4R}(x-x_0) + u_{x_0,4R}$

in (5.6) to get

$$\left(\fint_{B_R} |Du - z_0|^p\right)^{\frac{2}{p}} \le c\,E(2R) + c\left\{\fint_{B_{2R}} \omega_\alpha\left(|u-y_0| + |u-a|\right)\left(1 + |Du|^p + |Da|^p\right)\right\}^{\frac{2}{p}},$$

For $p < \tilde{p} < \min\{p_0, q_0\}$ we get by use of Hölder with $\frac{\tilde{p}}{p}, \frac{\tilde{p}}{\tilde{p}-p}$:

$$\fint_{B_{2R}} \omega_\alpha\left(|u-y_0| + |u-a|\right) |Du|^p$$

$$\le \left(\fint_{B_{2R}} \omega_\alpha\left(|u-y_0| + |u-a|\right)^{\frac{\tilde{p}}{\tilde{p}-p}}\right)^{1-\frac{p}{\tilde{p}}} \left(\fint_{B_{2R}} |Du|^{\tilde{p}}\right)^{\frac{p}{\tilde{p}}}$$

Using next (5.1) from step 1 and that $\omega_\alpha \leq 1$ is concave we get

$$\fint_{B_{2R}} \omega_\alpha(|u-y_0|+|u-a|)\,|Du|^p$$

$$\leq c\,\omega_\alpha\Big(\fint_{B_{2R}}(|u-y_0|+|u-a|)\Big)^{1-\frac{p}{\tilde p}}\Big(1+\Big(\fint_{B_{4R}}|Du|^2\Big)^{\frac{p}{2}}\Big)$$

ω_α increasing, Hölder & Poincaré

$$\leq c\,\omega_\alpha\Big(c\Big(\fint_{B_{4R}}|Du|^2\Big)^{\frac12}R+c\Big(\fint_{B_{4R}}|Du-Du|^2\Big)^{\frac12}R\Big)^{1-\frac{p}{\tilde p}}$$

$$\times\Big(1+|Da|^p+E(4R)^{\frac{p}{2}}\Big)$$

$$\overset{(6.1)}{\leq} c\,\omega_\alpha(cR)^{1-\frac{p}{\tilde p}} \leq c\,R^\sigma$$

where $c=c(m)$ and $\sigma := \alpha\left(1-\frac{p}{\tilde p}\right)\in(0,1)$.

$$\therefore \Big(\fint_{B_R}|Du-z_0|^p\Big)^{\frac{2}{p}} \leq c\,E(2R)+c\,R^\sigma$$

$$\leq c\,(4R)^\sigma + c\,E(4R).$$

Consequently: $\mathrm{I}+\mathrm{III} \leq$

$$c\,\omega\big(c\,E(4R)^{\frac12}\big)^{1-\frac{2}{p}}\big((4R)^\sigma+E(4R)\big)$$

and thus with

$$\tilde{\omega}(t) := \omega(ct^{\frac{1}{2}})^{1-\frac{2}{p}} \quad , \quad t \geqq 0,$$

we get

$$(6.6) \quad \mathrm{I} + \mathrm{III} \leqq c\left((4R)^{\sigma} + \tilde{\omega}(E(4R))E(4R) \right),$$

where $c = c(m)$.

$$\mathrm{II} = \fint_{B_R} \left(F^0(Du) - F^0(Dh) \right) =$$

$$\fint_{B_R} \Big(F(y_0, Du) - F(u, Du) + F(u, Du) - F(h, Dh)$$

$$+ F(h, Dh) - F(y_0, Dh) \Big) \overset{u \text{ F-min } \& \text{ (H4)}}{\leqq}$$

$$L \fint_{B_R} \Big(\omega_\alpha(|u - y_0|)(|Du|^2 + 1) + \omega_\alpha(|h - y_0|)(|Dh|^2 + 1) \Big).$$

Note: When $\delta > 1$

$$\fint_{B_R} \omega_\alpha(|h - y_0|)^\delta \overset{\omega_\alpha \leq 1, \text{ increasing}}{\leqq} \fint_{B_R} \omega_\alpha(|h - u| + |u - y_0|) \overset{\omega_\alpha \text{ concave}}{\leqq}$$

$$\omega_\alpha\left(\fint_{B_R} |h - u| + \fint_{B_R} |u - y_0| \right) \overset{\omega_\alpha \text{ increasing, Poincaré}}{\leqq}$$

$$\omega_\alpha\Big(c\big(⨍_{B_R} |Dh - Du|^2\big)^{\frac12} R + c\big(⨍_{B_R} |Du|^2\big)^{\frac12} R \Big)$$

$$\leq \omega_\alpha\Big(c\big(⨍_{B_R} |Dh - (Dh)_{x_0,R}|^2\big)^{\frac12} R + c\big(⨍_{B_R} |Du - (Du)_{x_0,R}|^2\big)^{\frac12} R$$

$$+ c\big(⨍_{B_{4R}} |Du - z_0|^2\big)^{\frac12} R + c|z_0| R \Big)$$

$$\overset{(6.1)}{\leq} \omega_\alpha\big(c\,E(4R)\,R + c|z_0| R\big) \leq \omega_\alpha(cR),$$

where $c = c(m)$. The estimate can therefore proceed as for I + III whereby we arrive at

$$(6.7)\quad ⨍_{B_R} |Du - Dh|^2 \leq c\Big((4R)^\sigma + \tilde{\omega}(E(4R))\,E(4R)\Big)$$

for $B_{4R} \subset \Omega$, $R \leq 1$, s.t. (6.1) holds with an $0 < \varepsilon < 2^{-n}$, and where $c = c(m)$.

Combining (6.5) and (6.7) we get ~~for~~ for $0 < r \leq R$:

$$E(r) = ⨍_{B(x_0,r)} |Du - (Du)_{x_0,r}|^2 \leq$$

$$⨍_{B_r} |Du - (Dh)_{x_0,r}|^2 \leq 2 ⨍_{B_r} |Du - Dh|^2 + 2 ⨍_{B_r} |Dh - (Dh)_{x_0,r}|^2$$

$$\leq 2\left(\frac{R}{r}\right)^n ⨍_{B_R} |Du - Dh|^2 + c\left(\frac{r}{R}\right)^2 E(R)$$

$E(R) \leq 4^n E(4R)$

$$\leq c\left((4R)^\sigma + \tilde{\omega}(E(4R))E(4R)\right)\left(\frac{R}{r}\right)^n + c\left(\frac{r}{R}\right)^2 E(4R)$$

$$= c(4R)^\sigma\left(\frac{R}{r}\right)^n + c\left[\left(\frac{r}{R}\right)^2 + \tilde{\omega}(E(4R))\left(\frac{R}{r}\right)^n\right]E(4R)$$

(6.1)
$$\leq c(4R)^\sigma\left(\frac{R}{r}\right)^n + c\left[\left(\frac{r}{R}\right)^2 + \tilde{\omega}(\varepsilon)\left(\frac{R}{r}\right)^n\right]E(4R),$$

where $c = c(m)$, $0 < \varepsilon < c_n$, where $0 < c_n \leq 4^{-n}$.

Let us change the notation a bit to express more clearly what we have proved (rescale $4R$ to R and adjust the constants $c = c(m)$):

Let $m > 1$, $0 < \varepsilon < \varepsilon_n$.

If $B_R \subset \Omega$ and

$|u_{x_0,R}| < m$, $|(Du)_{x_0,R}| < m$, $E(x_0,R) < \varepsilon$,

then

$$E(x_0,r) \leq c\left(\frac{R}{r}\right)^n R^\sigma + c\left[\left(\frac{r}{R}\right)^2 + \tilde{\omega}(\varepsilon)\left(\frac{R}{r}\right)^n\right] E(x_0,R)$$

for all $0 < r \leq \frac{R}{4}$.

Let $m > 1$, $0 < \varepsilon < \varepsilon_n$.

If $B_R \subset \Omega$ and

$|u_{x_0,R}| < m$, $|(Du)_{x_0,R}| < m$, $E(x_0,R) < \varepsilon$,

then

$$E(x_0,r) \leqq c\left(\frac{R}{r}\right)^n R^\sigma + c\left[\left(\frac{r}{R}\right)^2 + \tilde{\omega}(\varepsilon)\left(\frac{R}{r}\right)^n\right] E(x_0,R)$$

for all $0 < r \leqq \frac{R}{4}$.