## Outcome of Step 3:

Let $m > 1$, $0 < \varepsilon < 4^{-n}$.

There exist a constant $c = c(m, n, N, \frac{L}{\ell}) > 0$ and a bounded increasing function $\tilde{\omega} = \tilde{\omega}_m : [0, \infty) \to [0, \infty)$ s.t. $\tilde{\omega}(0+) = 0$ with the following property:

If $B(x_0, R) \subset \Omega$ and

(7.1) $|u_{x_0,R}| < m$, $|(Du)_{x_0,R}| < m$, $E(x_0, R) < \varepsilon$,

then

(7.2) $$E(x_0, r) \leq c\left(\frac{R}{r}\right)^n R^\sigma + c\left[\left(\frac{r}{R}\right)^2 + \tilde{\omega}(\varepsilon)\left(\frac{R}{r}\right)^n\right] E(x_0, R)$$

for all $0 < r < \frac{R}{4}$, where $\sigma \in (0, \alpha)$ is a constant (independent of $m, \varepsilon$).

We rewrite (7.2) as follows: put $r = \tau R$, where $0 < \tau < \frac{1}{4}$, and note

(7.2') $$E(x_0, \tau R) \leq c\tau^{-n} R^\sigma + c\left[\tau^2 + \tilde{\omega}(\varepsilon)\tau^{-n}\right] E(x_0, R)$$

## Step 4 Iteration.

We intend to iterate (7.2'), or rather, a more streamlined form of it.

### Selection of $\tau$:

Put $\beta := \frac{\sigma}{2}$ and take $\tau$ s.t.

$$c\tau^2 = \frac{1}{2}\tau^\beta, \quad \text{i.e.}$$

$$(7.3) \qquad \tau = (2c)^{-\frac{1}{2-\beta}}$$

We note that since $c = c(m, n, N, \frac{L}{\ell})$ and we can assume $c > 8$ we have that $\tau = \tau(m, n, N, \frac{L}{\ell}) \in (0, \frac{1}{4})$.

### Further smallness condition on $\varepsilon$:

Require that $c\,\tilde{\omega}(\varepsilon)\tau^{-n} \leq \frac{1}{2}\tau^\beta$.

Since $\tilde{\omega} = \tilde{\omega}_m$ is increasing and $\tilde{\omega}(0+)=0$ this corresponds to $\varepsilon \in (0, \varepsilon_1]$, where we can assume that $\varepsilon_1 < 4^{-n}$. Note that $\varepsilon_1 = \varepsilon_1(m, n, N, \frac{L}{\lambda}, F'')$.

(7.4) $\qquad 0 < \varepsilon < \varepsilon_1 \;, \quad c\,\tilde{\omega}(\varepsilon,)\tau^{-n} \leq \frac{1}{2}\tau^{\beta}$ 

Put $c_0 := c\tau^{-n}$ so that under assumptions (7.3), (7.4) the inequality (7.2') becomes

(7.5) $\qquad E(x_0, \tau R) \leq c_0 R^{2\beta} + \tau^{\beta} E(x_0, R)$.

We record that (7.5) is valid provided

$B(x_0, R) \subset \Omega$, $|u_{x_0,R}| < m$, $|(Du)_{x_0,R}| < m$, $E(x_0,R) < \varepsilon$

with the choices (7.3), (7.4).

In order to iterate (7.5) we require the following

Lemma For $B(x_0, r) \subset \Omega$ and $0 < t < 1$ we have

(i) $\quad |(Du)_{x_0,r} - (Du)_{x_0,tr}| \leq t^{-\frac{n}{2}} E(x_0,r)^{\frac{1}{2}}$

(ii) $\quad |u_{x_0,r} - u_{x_0,tr}| \leq c\, t^{-\frac{n}{2}} r\, E(x_0,r)^{\frac{1}{2}}$,

where $c = c(n)$.

$\boxed{\text{Pf:}}$ For $w \in L^2(\Omega, \mathbb{R}^d)$

$$|w_{x_0,r} - w_{x_0,tr}| \leq \fint_{B(x_0,tr)} |w_{x_0,r} - w|$$

$$\leq \left( \fint_{B(x_0,tr)} |w - w_{x_0,r}|^2 \right)^{\frac{1}{2}} \leq t^{-\frac{n}{2}} \left( \fint_{B(x_0,r)} |w - w_{x_0,r}|^2 \right)^{\frac{1}{2}}.$$

Taking $w = Du$ gives (i).

Taking $w = w(x) := u(x) - (Du)_{x_0,R}(x - x_0)$ gives

$$|u_{x_0,r} - u_{x_0,tr}| = |w_{x_0,r} - w_{x_0,tr}| \leq$$

$$t^{-\frac{n}{2}} \left( \fint_{B(x_0,r)} |w - w_{x_0,r}|^2 \right)^{\frac{1}{2}} = t^{-\frac{n}{2}} \left( \fint_{B(x_0,r)} |u(x) - (Du)_{x_0,r}(x - x_0) - u_{x_0,r}|^2 dx \right)^{\frac{1}{2}}$$

$$\overset{\text{Poincaré}}{\leq} c\, t^{-\frac{n}{2}}\, r\, E(x_0,r)^{\frac{1}{2}}. \quad \square$$

Fix $m > 1$ and choose $\tau, \varepsilon_1$ as in (7.3), (7.4). Let $B_R = B(x_0,R) \subset \Omega$ with $R \in (0, R_1]$, $R_1 > 0$ to be specified, and $\varepsilon \in (0, \varepsilon_1)$ to be specified. We assume that

(7.6) $|u_{x_0,R}| < m-1$, $|(Du)_{x_0,R}| < m-1$, $E(x_0,R) < \varepsilon$.

Write $E(r) := E(x_0,r)$ and recall (7.5):

$$E(\tau R) \leq c_0 R^{2\beta} + \tau^{\beta} E(R).$$

<u>Assertion</u> For suitable choices of $R_1, \varepsilon$ we have

$$\circledast_j \quad E(\tau^j R) \leq \tau^{j\beta} E(R) + c_0 (\tau^{j-1} R)^\beta \sum_{i=0}^{j-1} \tau^{i\beta}$$

for all $j \in \mathbb{N}$ provided (7.6) holds.

By induction on $j \in \mathbb{N}$.

$\circledast_1$ is just (7.5), or in fact a weaker version of it: $E(\tau R) \overset{(7.5)}{\leq} c_0 R^{2\beta} + \tau^\beta E(R)$

$< c_0 R^\beta + \tau^\beta E(R)$ <u>provided</u> $R \leq R_1 < 1$.

Assume $\circledast_j$ holds for $j \leq k \in \mathbb{N}$.

We estimate $E(\tau^k R)$, $|u_{x_0, \tau^k R}|$, $|(Du)_{x_0, \tau^k R}|$.

By $\circledast_k$ we have

$$\begin{aligned} E(\tau^k R) &\leq \tau^{k\beta} E(R) + c_0 (\tau^{k-1} R)^\beta \sum_{j=0}^{k-1} \tau^{j\beta} \\ &\overset{(7.6)}{<} \tau^{k\beta} \varepsilon + c_0 (\tau^{k-1} R)^\beta \frac{1}{1-\tau^\beta} \\ &\leq 2\varepsilon \tau^{k\beta} \end{aligned}$$

<u>provided</u> $c_0 (\tau' R)^\beta \frac{1}{1-\tau^\beta} \leq \varepsilon$, that is,

$$(7.7) \qquad R \leqq R_1 := \left( \frac{\tau^{\beta}(1-\tau^{\beta})}{c_0} \right)^{\frac{1}{\beta}}$$

With this assumption on $R$ we get

$E(\tau^k R) \leqq 2\varepsilon \tau^{k\beta} < \varepsilon_1$ <u>provided</u>

$$(7.8) \qquad \varepsilon < \frac{\varepsilon_1}{2\tau^{\beta}} .$$

Next, $|(Du)_{x_0,\tau^k R}| \leqq |(Du)_{x_0,R}| + \sum_{j=0}^{k-1} |(Du)_{x_0,\tau^j R} - (Du)_{x_0,\tau^{j+1} R}|$

$$\overset{(7.6)}{<} m-1 + \sum_{j=0}^{k-1} \tau^{-\frac{n}{2}} E(\tau^j R)^{\frac{1}{2}} \leqq m-1 + \tau^{-\frac{n}{2}} \sum_{j=0}^{k-1} \sqrt{2\varepsilon}\, \tau^{\frac{j\beta}{2}}$$

$$< m-1 + \sqrt{2\varepsilon}\, \tau^{-\frac{n}{2}} \frac{1}{1-\tau^{\beta/2}} < m$$

<u>provided</u> $\dfrac{\sqrt{2\varepsilon}\,\tau^{-\frac{n}{2}}}{1-\tau^{\beta/2}} < 1$, i.e.

$$(7.9) \qquad \varepsilon < \frac{1}{2} \tau^{n} (1-\tau^{\beta/2})^2 .$$

Likewise, $|u_{x_0,\tau^k R}| \leqq |u_{x_0,R}| + \sum_{j=0}^{k-1} |u_{x_0,\tau^j R} - u_{x_0,\tau^{j+1} R}|$

$$\overset{(7.6)}{<} m-1 + c\tau^{-\frac{n}{2}} R \sum_{j=0}^{k-1} E(\tau^j R)^{\frac{1}{2}} \tau^j$$

$$< m-1 + c\tau^{-\frac{n}{2}} R \sum_{j=0}^{k-1} \sqrt{2\varepsilon}\, \tau^{j\frac{\beta}{2}+j}$$

$$< m-1 + c\tau^{-\frac{n}{2}} R \sqrt{2\varepsilon} \frac{1}{1-\tau^{\frac{\beta}{2}+1}}$$

$$< m-1 + \frac{4}{3} c \tau^{-\frac{n}{2}} R_1 \sqrt{2\varepsilon} < m$$

provided $\frac{4}{3} c \tau^{-\frac{n}{2}} R_1 \sqrt{2\varepsilon} < 1$, i.e.

$$\varepsilon < \frac{\tau^n}{4c^2 R_1^2} . \tag{7.10}$$

Consequently, when (7.6) holds and we assume (7.7), (7.8), (7.9), (7.10), then

$E(\tau^k R) < \varepsilon_1$, $|u_{x_0, \tau^k R}| < m$, $|(Du)_{x_0, \tau^k R}| < m$

so that by (7.5):

$$E(\tau^{k+1} R) \leq c_0 (\tau^k R)^{2\beta} + \tau^\beta E(\tau^k R) \overset{\circledast_k}{\leq}$$

$$c_0 (\tau^k R)^{2\beta} + \tau^\beta \Big( \tau^{k\beta} E(R) + c_0 (\tau^{k-1} R)^\beta \sum_{j=0}^{k-1} \tau^{j\beta} \Big)$$

$$= \tau^{(k+1)\beta} E(R) + c_0 (\tau^k R)^{2\beta} + c_0 (\tau^k R)^\beta \sum_{j=0}^{k-1} \tau^{j\beta}$$

$$< \tau^{(k+1)\beta} E(R) + c_0 (\tau^k R)^\beta \sum_{j=0}^{k} \tau^{j\beta}, \text{ which}$$

is $\circledast_{k+1}$. This proves the assertion.

With the choices (7.7), (7.8), (7.9), (7.10) and when (7.6) holds, then $\oplus_k$ holds for all $k \in \mathbb{N}$:

$$E(\tau^k R) \leq \tau^{k\beta} E(R) + c_0 (\tau^{k-1} R)^\beta \sum_{j=0}^{k-1} \tau^{j\beta}$$

$$< \tau^{k\beta} E(R) + c_0 \frac{(\tau^{k-1} R)^\beta}{1 - \tau^\beta} = \tau^{k\beta}\left[E(R) + c_0 \frac{R^\beta}{\tau^\beta (1-\tau^\beta)}\right].$$

If $r \in (0, \tau R)$ we find $k \in \mathbb{N}$ s.t. $r \in [\tau^{k+1} R, \tau^k R)$ and so

$$E(r) \leq \tau^{-n} E(\tau^k R) < \tau^{-n} \tau^{k\beta}\left[E(R) + \frac{c_0 R^\beta}{\tau^\beta (1-\tau^\beta)}\right]$$

$$\leq \tau^{-n-\beta} \left(\frac{r}{R}\right)^\beta \left[E(R) + \frac{c_0 R^\beta}{\tau^\beta (1-\tau^\beta)}\right],$$ i.e.

$$E(r) \leq \left(\frac{r}{R}\right)^\beta \left[\tau^{-n-\beta} E(R) + \frac{c_0 R^\beta}{\tau^{n+2\beta}(1-\tau^\beta)}\right].$$

When $r \in [\tau R, R]$, $E(r) \leq \tau^{-n} E(R)$, and hence we have established that

(7.11) $$E(x_0, r) \leq \left(\frac{r}{R}\right)^\beta \left[\tau^{-n-\beta} E(x_0, R) + \frac{c_0 R^\beta}{\tau^{n+2\beta}(1-\tau^\beta)}\right]$$

holds for all $r \in (0, R)$ provided $B(x_0, R) \subset \Omega$ and

$$|u_{x_0,R}| < m+1, \quad |(Du)_{x_0,R}| < m+1, \quad E(x_0, R) < \varepsilon,$$

where we assume (7.7) — (7.10).

For a fixed $R \leqq R_1$ (see (7.7)) the functions $x \longmapsto |u_{x,R}|, |(Du)_{x,R}|, E(x,R)$ are continuous on the set $\{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > R\}$, and therefore the set

$$G_R = G_R^{m,\varepsilon} := \{x \in \Omega : B(x,R) \subset \Omega, \ |u_{x,R}| < m-1,$$

$$|(Du)_{x,R}| < m-1 \text{ and } E(x,R) < \varepsilon\}$$

is open, and we have for each $x \in G_R$ that

$$E(x,r) \leqq \cancel{\tilde{C}\left(\frac{r}{R}\right)} \ \tilde{C} \cdot r^\beta$$

for all $r \in (0,R)$, where

$$\tilde{C} = R^{-\beta}\left(2\tau^{-n-\beta} R^{-n} \int_\Omega |Du|^2 + \frac{C_0 R^\beta}{\tau^{n+2\beta}(1-\tau^\beta)}\right).$$

Consequently, $Du$ is $C^{0,\frac{\beta}{2}}$ on $G_R$ by Campanato's integral characterization of Hölder continuity.

Now if $x_0 \in \Omega \setminus \Sigma_u$, then

$$\overline{\lim_{r \searrow 0}} \left( |u_{x_0,r}| + |(Du)_{x_0,r}| \right) < \infty \quad \text{and}$$

$$\underline{\lim_{r \searrow 0}} \; E(x_0,r) = 0,$$ hence we can

find $m > 1$ and $\tilde{R} \in (0, \operatorname{dist}(x_0, \partial\Omega))$

s.t.

$$|u_{x_0,r}| < m-1, \quad |(Du)_{x_0,r}| < m-1$$

for all $r \in (0, \tilde{R})$. Let $\varepsilon = \varepsilon(m) > 0$ be ~~ti~~ corresponding number satisfying (7.8)–(7.10) and find $R \in \min\{\tilde{R}, R_1(m)\}$, where $R_1(m)$ is as in (7.7), s.t. $E(x_0, R) < \varepsilon$. It follows that $x_0 \in G_R^{m,\varepsilon}$, i.e. $\Omega \setminus \Sigma_u$ is open and $Du$ is locally $\beta/2$ - Hölder continuous on $\Omega \setminus \Sigma_u$.

In order to conclude the proof we need to revise parts of the above proof using that $Du$ is Hölder continuous on $\Omega \setminus \Sigma_u$

instead of merely higher integrable.

More precisely we should revise (6.7) using that $Du$ is bounded on $B(x_0,R)$ and hence $u$ is Lipschitz:

$$(7.12) \quad |u(x_1) - u(x_2)| \leq \|Du\|_{L^\infty(B(x_0,R))} |x_1 - x_2|$$

for all $x_1, x_2 \in B(x_0,R)$. Put $M := \|Du\|_{L^\infty(B(x_0,R))}$ and let $h \in W^{1,2}_u(B(x_0,R), \mathbb{R}^N)$ be $P$-harmonic, where $P$ is the 2nd Taylor polynomial for $F^0(z) := F(u_{x_0,R}, z)$ about $(Du)_{x_0,R}$. Then (6.5) still holds:

$$\fint_{B(x_0,r)} |Dh - (Dh)_{x_0,r}|^2 \leq c\left(\frac{r}{R}\right)^2 E(x_0,R).$$

Now instead of (6.7) we have using $\|Du\|_{L^\infty(B(x_0,R))} = M$ and (7.12):

$$\frac{1}{2}\fint_{B(x_0,R)} |Du - Dh|^2 \overset{(6.4)}{\leq} \fint_{B(x_0,R)} \big(P(Du) - P(Dh)\big) =$$

$$\fint_{B(x_0,R)} \Big\{ \big(P(Du) - F^0(Du)\big) + \big(F^0(Du) - F^0(Dh)\big) + \big(F^0(Dh) - P(Dh)\big)\Big\} =: I + II + III .$$

Using (6.3) we find as before:

$$I + III \leq c \fint_{B(x_0,R)} \Big( \omega(|Du - Du_{x_0,R}|)\,|Du - Du_{x_0,R}|^2 + \omega(|Dh - Dh_{x_0,R}|)\,|Dh - Dh_{x_0,R}|^2 \Big).$$

We now use that $Du \in C^{0,\frac{\beta}{2}}(B(x_0,R))$ and the global Schauder estimate (see Part I) whereby $Dh \in C^{0,\frac{\beta}{2}}(B(x_0,R))$ too with

$$[Dh]_{0,\frac{\beta}{2};B(x_0,R)} \leq c[Du]_{0,\frac{\beta}{2};B(x_0,R)} \leq cM$$

Consequently, $I + III \leq c\,\omega(cMR^{\frac{\beta}{2}})\,E(x_0,R)$.

Likewise, $II \leq \cancel{cM^\alpha R^\alpha E(x_0,R)}$ $\cancel{cM^{2+\alpha}}\; cR^\alpha$, $c = c(M)$, hence

$$(6.7)^{\text{revised}} \quad \fint_{B(x_0,R)} |Du - Dh|^2 \leq c\,\omega(cMR^{\frac{\beta}{2}})\,E(x_0,R) + c\cancel{M^{2+\alpha}}R^\alpha$$

We can proceed as before to get

$$E(x_0,r) \leq 2\left(\frac{R}{r}\right)^n \fint_{B(x_0,R)} |Du-Dh|^2 + 2c\left(\frac{r}{R}\right)^2 E(x_0,R)$$

$$\leq c\left(\frac{R}{r}\right)^n R^\alpha + \left[c\omega(cR^{\frac{\beta}{2}})\left(\frac{R}{r}\right)^n + c\left(\frac{r}{R}\right)^2\right]E(x_0,R)$$

where $c = c(m)$.

Write $r = \tau R$, $0<\tau<1$, so that

$$E(x_0,\tau R) \leq c\tau^{-n} R^\alpha + \left[c\omega(cR^{\frac{\beta}{2}})\tau^{-n} + c\tau^2\right]E(x_0,R).$$

$$= c_0 R^\alpha + \tau^{2\alpha}\left[c\omega(cR^{\beta/2})\tau^{-n-2\alpha} + c\tau^{2(1-\alpha)}\right]E(x_0,R)$$

Take $\tau \in (0,1)$ so $c\tau^{2(1-\alpha)} = \frac{1}{2}$ and $R_1 > 0$ so $c\omega(cR_1^{\beta/2})\tau^{-n-2\alpha} \leq \frac{1}{2}$. Then for $R \in (0, R_1]$,

$$E(x_0,\tau R) \leq c_0 R^\alpha + \tau^{2\alpha} E(x_0,R).$$

Consequently for $k \in \mathbb{N}$, $E(x_0,\tau^k R) \leq$

$$\tau^{2\alpha k} E(x_0,R) + c_0(\tau^{k-1}R)^\alpha \sum_{j=0}^{k-1} \tau^{j\alpha}$$

$$< \tau^{2\alpha k} E(x_0,R) + c_0(\tau^{k-1}R)^\alpha \frac{1}{1-\tau^\alpha}.$$

For $0<r<R$ we take $k\in\mathbb{N}_0$ so $\tau^{k+1}R<r\leq\tau^k R$, and get

$$
\begin{aligned}
E(x_0,r) &\leq \tau^{-n}E(x_0,\tau^k R)\\
&< \tau^{2\alpha k}E(x_0,R)\tau^{-n}+(\tau^{k-1}R)^\alpha\frac{c_0\tau^{-n}}{1-\tau^\alpha}\\
&< \left(\frac{r}{R}\right)^{2\alpha}E(x_0,R)\tau^{-n-2\alpha}+\left(\frac{r}{R}\right)^\alpha\frac{c_0\tau^{-n-2\alpha}R^\alpha}{1-\tau^\alpha}\\
&\leq \left\{E(x_0,R)\tau^{-n-2\alpha}+\frac{c_0\tau^{-n-2\alpha}}{1-\tau^\alpha}R^\alpha\right\}\left(\frac{r}{R}\right)^\alpha. \quad\square
\end{aligned}
$$