L13 & 14 The variational difference-quotient method, I

$\boxed{\text{Set-up}}$ $F : \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfies

(as in the simplified version of TH 5):

(H1) $z \mapsto F(y,z)$ is C^2, and

$(y,z) \mapsto F_{zz}(y,z)$ is continuous.

(H2) $|\lambda|^2 \leq F_{zz}(y,z)[\lambda,\lambda] \leq L|\lambda|^2$

(H3) $|z|^2 - L \leq F(y,z) \leq L(|z|^2+1)$

(H4) $|F(y_1,z) - F(y_2,z)| \leq L\omega_\alpha(|y_1-y_2|)(|z|^2+1)$

$\omega_\alpha(t) = \min\{1, t^\alpha\}$, $\quad 0<\alpha\leq 1$.

$u \in W^{1,2}(\Omega, \mathbb{R}^N)$ is F-minimizing.

From TH 5:

$$\Omega \setminus \Sigma_u = \left\{ x \in \Omega : \begin{array}{c} \overline{\lim}_{r \searrow 0} \left(|u_{x,r}| + |(Du)_{x,r}| \right) < \infty \\ \text{and} \\ \lim_{r\searrow 0} E(x,r) = 0 \end{array} \right\}$$

is open, and u is $C^{1,\frac{\alpha}{2}}_{\mathrm{loc}}$ on $\Omega\setminus\Sigma_u$.

From Lebesgue's differentiation theorem:

$$\mathcal{L}^n(\Sigma_u) = 0.$$

Better size estimates for the singular set Σ_n? Hausdorff dimension

Background results on Hausdorff measures

(We omit proofs here; the proofs can be found in, for instance, [4].)

Let $s \in [0,\infty)$, $\delta \in (0,\infty]$.

For any $A \subseteq \mathbb{R}^n$ put

$$\mathcal{H}^s_\delta(A) := \inf\left\{ \sum_{j=1}^{\infty} \omega_s \left(\frac{\operatorname{diam} C_j}{2}\right)^s : \begin{array}{l} A \subset \bigcup_{j=1}^{\infty} C_j \\ \operatorname{diam} C_j \leq \delta \end{array} \right\},$$

where $\omega_s := \dfrac{\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2}+1)}$ $\quad \left(\omega_n = \mathcal{L}^n(B(0,1)) \right)$

$$\mathcal{H}^s(A) := \lim_{\delta \searrow 0} \mathcal{H}^s_\delta(A) \quad \left(= \sup_{\delta > 0} \mathcal{H}^s_\delta(A)\right)$$

$\mathcal{H}^s$ is the s-dimensional Hausdorff measure on $\mathbb{R}^n$.

<u>Remark</u> $\mathcal{H}^s_\delta$ isn't an outer measure on $\mathbb{R}^n$ for $\infty > \delta > 0$, $0 < s < n$.

Facts

① $\mathcal{H}^s$ is a Borel regular outer measure on $\mathbb{R}^n$

② $\begin{cases} \mathcal{H}^0 \text{ is counting measure} \\ \mathcal{H}^s = 0 \quad \text{for} \quad s > n \\ \mathcal{H}^n = \mathcal{L}^n \end{cases}$

③ $\mathcal{H}^s \geqslant \mathcal{H}^s_{\delta'} \geqslant \mathcal{H}^s_{\delta''} \geqslant \mathcal{H}^s_\infty$ for $0 < \delta' < \delta'' < \infty$

but

$$\mathcal{H}^s_\infty(A) = 0 \Rightarrow \mathcal{H}^s(A) = 0.$$

$\mathcal{H}^s_\infty$ is called the s-dimensional Hausdorff capacity (or content) on $\mathbb{R}^n$.

④ For $0 \leq s < t < \infty$ and $A \subseteq \mathbb{R}^n$:

$$\mathcal{H}^s(A) < \infty \Rightarrow \mathcal{H}^t(A) = 0$$

$$\mathcal{H}^t(A) > 0 \Rightarrow \mathcal{H}^s(A) = \infty.$$

Def. The Hausdorff dimension of a set $A \subseteq \mathbb{R}^n$ is

$$\dim_H A := \inf\{s \geqslant 0 : \mathcal{H}^s(A) = 0\}$$

$$\left(\overset{③}{=} \inf\{s \geqslant 0 : \mathcal{H}^s_\infty(A) = 0\}. \right)$$

TH6: Assume (H1) — (H4) and that $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ is F-minimizing. If $u \in W^{1,p}_{loc}(\Omega, \mathbb{R}^N)$, $p > 2$, then

$$\dim_H(\Sigma_u) \leq n - \min\{\alpha, p-2\}.$$

Remark It follows from Gehring's Lemma (see Step 1 in proof of TH 5) that $u \in W^{1,\tilde{p}}_{loc}(\Omega, \mathbb{R}^N)$ for some $\tilde{p} > 2$.

(TH 6 is a special case of Th. 1.1 in [12].)

The theorem is a consequence of a higher differentiability result for u that we'll prove using a variational version of the difference-quotient method. Before stating it we need two definitions.

Def. (Nikolskii space)
Let $f \in L^2(\Omega)$ and $0 < \alpha \leq 1$. Then f satisfies a Nikolskii condition of order α on $\Omega' \Subset \Omega$ provided

$$(*) \quad \int_{\Omega'} |\Delta_{s,h} f|^2 \leq c|h|^{2\alpha}$$

for all increments $|h| < \operatorname{dist}(\Omega', \partial\Omega)$ and directions $1 \leq s \leq n$.

<u>Notation</u> $(\Delta_{s,h} f)(x) := f(x + h e_s) - f(x)$

where $e_1, \dots, e_n$ are standard basis vectors of $\mathbb{R}^n$.

<u>Remark</u> Recall from Part I that when $\alpha = 1$, then $(*)$ ~~is equivalent~~ implies that $f \in W^{1,2}(\Omega')$.

If $f \in W^{1,2}_{loc}(\Omega)$, then $(*)$ holds with $\alpha = 1$.

<u>Def.</u> (Sobolev-Slobodetskii space)

Let $0 < \theta < 1$. Then $f \in W^{\theta,2}(\Omega)$ iff

$$\|f\|_{W^{\theta,2}} := \|f\|_{L^2} + \underbrace{\left(\int_\Omega \int_\Omega \frac{|f(x) - f(y)|^2}{|x-y|^{n+2\theta}} \, dx \, dy \right)^{\frac{1}{2}}}_{=\,[f]_{\theta,2}} < \infty$$

Gagliardo seminorm $[f]_{\theta,2}$

Lemma Let $f \in L^2(2B)$, $0<\theta<1$.

Then

$$\int_B \int_B \frac{|f(x)-f(y)|^2}{|x-y|^{n+2\theta}}\,dx\,dy \leq c \sum_{s=1}^{n} \int_{-1}^{1} \int_B \frac{|\Delta_{s,h} f(x)|^2}{|h|^{1+2\theta}}\,dx\,dh$$

$\boxed{\text{Pf :}}$ Omitted. □ (See [1].)

Remarks ① If $|\Delta_{s,h} f| \leq c_\alpha |h|^\alpha$ on B for all $|h|<1$, $1 \leq s \leq n$, then for $0<\theta<\alpha$ we have

$$\sum_{s=1}^{n} \int_{-1}^{1} \int_B \frac{|\Delta_{s,h} f(x)|^2}{|h|^{1+2\theta}}\,dx\,dh \leq n c_\alpha^2 \int_{-1}^{1} \frac{dh}{|h|^{1+2(\theta-\alpha)}}$$

$$= \frac{n c_\alpha^2}{\alpha-\theta},$$ and so $\displaystyle\int_B \int_B \frac{|f(x)-f(y)|^2}{|x-y|^{n+2\theta}}\,dx\,dy \leq c\, c_\alpha^2$.

② All within scale of Besov spaces $B^{p,q}_\beta$:

$$W^{\theta,2} = B^{2,2}_\theta, \quad N^{\alpha,2} = B^{2,\infty}_\alpha,$$

$0<\theta, \alpha<1$.

<u>TH 7:</u> Assume (H1)–(H4) and that $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ is F-minimizing. If $u \in W^{1,p}_{loc}(\Omega, \mathbb{R}^N)$, $p > 2$, then $Du \in W^{\theta,2}_{loc}(\Omega, \mathbb{R}^{N\times n})$ for all $\theta < \frac{1}{2}\min\{\alpha, p-2\}$.

(Special case of Th 4.2 in [12].)

To see that TH 7 implies TH6 we require a Poincaré inequality for $W^{\theta,2}$-maps and a measure theoretic lemma.

<u>Lemma</u> (Poincaré's inequality for $W^{\theta,2}$)

If $f \in W^{\theta,2}(B_R)$, then

$$\int_{B_R} |f - f_{B_R}|^2 \leq c\, R^{2\theta} \int_{B_R}\int_{B_R} \frac{|f(x)-f(y)|^2}{|x-y|^{n+2\theta}}\,dx\,dy,$$

where $c = \dfrac{2^{n+2\theta}}{\omega_n}$

Pf: For $x \in B_R$:

$$|f(x) - f_{B_R}|^2 \le ⨍_{B_R} |f(x) - f(y)|^2 \, dy$$

and since $\frac{2R}{|x-y|} > 1$ for $x, y \in B_R$

we get

$$|f(x) - f_{B_R}|^2 \le ⨍_{B_R} |f(x) - f(y)|^2 \left(\frac{2R}{|x-y|}\right)^{n+2\theta} dy$$

and so

$$\int_{B_R} |f - f_{B_R}|^2 \le 2^{n+2\theta} \int_{B_R} ⨍_{B_R} \frac{|f(x)-f(y)|^2}{|x-y|^{n+2\theta}} \, dy \, dx \, R^{n+2\theta}$$

$$= \frac{2^{n+2\theta}}{\omega_n} R^{2\theta} \int_{B_R} \int_{B_R} \frac{|f(x)-f(y)|^2}{|x-y|^{n+2\theta}} \, dx \, dy. \square$$

Measure theoretic Lemma

Let $\lambda : \mathbb{B}(\Omega) \longrightarrow [0, \infty)$ be a monotone σ-super-additive finite set function, $0 \le s \le n$, and

$$\Lambda_s := \{ x \in \Omega ; \overline{\lim_{r \to 0}} \, r^{-s} \lambda(B(x,r)) > 0 \}.$$

Then $\dim_H(\Lambda_s) \le s$.

$\boxed{\text{Pf:}}$ For $\varepsilon>0$ let

$$\Lambda_{s,\varepsilon} := \{x\in\Omega : \overline{\lim_{r\to 0}}\ r^{-s}\lambda(B(x,r)) > \varepsilon\}.$$

Let $\delta>0$. For each $x\in\Lambda_{s,\varepsilon}$ take $r_x\in(0,\frac{\delta}{2})$ s.t. $B(x,5r_x)\subset\Omega$ and $r_x^{-s}\lambda(B(x,r_x))>\varepsilon$. Then $\{B(x,r_x) : x\in\Lambda_{s,\varepsilon}\}$ covers $\Lambda_{s,\varepsilon}$ and $\sup_{x\in\Lambda_{s,\varepsilon}} r_x < \operatorname{diam}\Omega<\infty$ so by the Vitali-Wiener covering lemma there exists an at most countable sub-family $\{B(x,r_x) : x\in J\subset\Lambda_{s,\varepsilon}\}$ consisting of disjoint balls and so

$$\Lambda_{s,\varepsilon} \subseteq \bigcup_{x\in J} B(x,5r_x).$$

Now $\displaystyle\sum_{x\in J}\omega_s(5r_x)^s = 5^s\omega_s\sum_{x\in J} r_x^s <$

$$\frac{5^s\omega_s}{\varepsilon}\sum_{x\in J}\lambda(B(x,r_x)) \leq \frac{5^s\omega_s}{\varepsilon}\lambda\Big(\bigcup_{x\in J}B(x,r_x)\Big)$$

$\leq \dfrac{5^s\omega_s}{\varepsilon}\lambda(\Omega)$, hence $\mathcal{H}^s_\delta(\Lambda_{s,\varepsilon}) \leq \dfrac{5^s\omega_s}{\varepsilon}\lambda(\Omega)$

Consequently, $\mathcal{H}^t(\Lambda_{s,\varepsilon}) = 0 \quad \forall t > s$, and so by σ-subadditivity of $\mathcal{H}^t$, $\mathcal{H}^t(\Lambda_s) = 0$. But then $\dim_H(\Lambda_s) \leq s$. $\square$

<u>Corollary</u> Assume $Du \in W^{\theta,2}_{loc}(\Omega, \mathbb{R}^{N\times n})$, where $0 < \theta < 1$. Then $\dim_H(\Sigma_u) \leq n - 2\theta$.

<u>Recall</u> $$\Sigma_u = \left\{ x \in \Omega : \begin{array}{c} \overline{\lim\limits_{r \searrow 0}} \left(|u_{x,r}| + |(Du)_{x,r}| \right) = \infty \\ \text{or} \\ \underline{\lim\limits_{r\searrow 0}}\, E(x,r) > 0 \end{array} \right\}$$

$\boxed{\text{Pf:}}$ Put $$\lambda(A) := \int_A \int_A \frac{|Du(x) - Du(y)|^2}{|x-y|^{n+2\theta}} \, dx\, dy.$$

By Poincaré: $E(x,r) \leq c\, r^{2\theta - n} \lambda(B(x,r))$.

Put $$\Sigma^1 = \{ x \in \Omega : \underline{\lim_{r \searrow 0}}\, E(x,r) > 0 \},$$
$$\Sigma^2 = \{ x \in \Omega : \overline{\lim_{r\searrow 0}} \left(|u_{x,r}| + |(Du)_{x,r}| \right) = \infty \}.$$

For $\Omega' \Subset \Omega$,
$$\Sigma^1 \cap \Omega' \subset \{ x \in \Omega' : \underline{\lim_{r\searrow 0}}\, r^{2\theta - n} \lambda(B(x,r)) \}$$

and so by the measure theoretic lemma

$$\dim_H(\Sigma^1 \cap \Omega') \leq n - 2\theta,$$

and thus $\dim_H(\Sigma^1) \leq n - 2\theta$.

Note $\Sigma^2 \subseteq \{x \in \Omega : \overline{\lim_{r \searrow 0}} \fint_{B(x,r)} (|u|^{\cancel{2}} + |Du|^{\cancel{2}}) = \infty\}$
$=: \tilde{\Sigma}$.

Let $\varepsilon > 0$.

Suppose $x_0 \in \Omega$ and $\overline{\lim_{r \searrow 0}}\, r^{2\theta - n - \varepsilon} \lambda(B(x_0, r)) = 0$,

ie $x_0 \notin \Lambda_{-2\theta + n + \varepsilon}$

<u>Claim</u> $x_0 \notin \tilde{\Sigma}^{\cancel{2}}$ (and hence $\tilde{\Sigma}^{\cancel{2}} \subseteq \Lambda_{n+\varepsilon-2\theta}$,

so $\dim_H(\Sigma^2) \leq n + \varepsilon - 2\theta \quad \forall \varepsilon > 0$)

Fix $B(x_0, R) \subset \Omega$. Then for $k \in \mathbb{N}$,

$$\left| \fint_{B(x_0, 2^{-k-1}R)} |Du|^{\cancel{2}} - \fint_{B(x_0, 2^{-k}R)} |Du|^{\cancel{2}} \right|^2 \leq$$

$$\fint_{B(x_0, 2^{-k-1}R)} \left| |Du| - |Du|_{x_0, 2^{-k}R} \right|^2 \leq$$

$$2^n \fint_{B(x_0, 2^{-k}R)} \left| |Du| - |Du|_{x_0, 2^{-k}R} \right|^2 \leq$$

$$2^n \fint_{B(x_0, 2^{-k}R)} \left| |Du| - |(Du)_{x_0, 2^{-k}R}| \right|^2 \leq$$

$$2^n E(x_0, 2^{-k}R) \leq c\,(2^{-k}R)^{2\theta - n} \lambda(B(x_0, 2^{-k}R))$$

$$= c\,(2^{-k}R)^{\varepsilon} \cdot \left[(2^{-k}R)^{2\theta - n - \varepsilon} \lambda(B(x_0, 2^{-k}R)) \right]$$

$$\leq \tilde{C} R^{\varepsilon} (2^{-\varepsilon})^k, \quad \text{and so}$$

$$|Du|_{x_0, 2^{-j}R} \leq |Du|_{x_0,R} + \sum_{k=0}^{j-1} \left| |Du|_{x_0, 2^{-k-1}R} - |Du|_{x_0, 2^{-k}R} \right|$$

$$\leq |Du|_{x_0,R} + \sum_{k=0}^{j-1} \sqrt{\tilde{C}}\, R^{\frac{\varepsilon}{2}} (2^{-\frac{\varepsilon}{2}})^k$$

$$< |Du|_{x_0,R} + \sqrt{\tilde{C}}\, R^{\frac{\varepsilon}{2}} \frac{1}{1 - 2^{-\frac{\varepsilon}{2}}} < \infty.$$

Now if $r_j \searrow 0$, then take $k_j \in \mathbb{N}$ so $r_j \in (2^{-k_j}R, 2^{-k_j+1}R]$. Note

$$|Du|_{x_0, r_j} \leq 2^n |Du|_{x_0, 2^{-k_j+1}R}$$

so $\overline{\lim_{r \searrow 0}} |Du|_{x_0, r} < \infty$.

Similarly, $\overline{\lim_{r \searrow 0}} |u|_{x_0, r} < \infty$ proving claim. $\square$

Review of difference-quotient method

Assume $f : \mathbb{R}^{N\times n} \to \mathbb{R}$ is C^2 and

$$(*) \quad \ell |\lambda|^2 \leq f_{zz}(z)[\lambda,\lambda] \leq L|\lambda|^2 \quad \forall z, \lambda,$$

where $0 < \ell \leq L < \infty$ are constants.

If $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ is f-minimizing, then $u \in W^{2,2}_{loc}(\Omega, \mathbb{R}^N)$, and for any cube $3Q \subset \Omega$ we have

$$(7.1) \quad \int_Q |\Delta_{s,h} Du|^2 \leq \left(\frac{4L}{\ell}\right)^2 \frac{h^2}{|Q|^{\frac{2}{n}}} \int_{3Q} |Du - Da|^2$$

for all affine maps $a : \mathbb{R}^n \to \mathbb{R}^N$, all directions $1 \leq s \leq n$ and all increments $|h| < \frac{1}{2}|Q|^{\frac{1}{n}}$.

<u>Notation</u> Let $e_1, \dots, e_n$ be the canonical basis vectors of $\mathbb{R}^n$ and define

$$\Delta_{s,h} Du\,(x) := Du(x + he_s) - Du(x)$$

for $x \in \Omega$, $h \in \mathbb{R}$ s.t. $x + he_s \in \Omega$.

$\boxed{\text{Pf:}}$ As u is f-minimizing we have

E-L : $\int_\Omega f_z(Du)\cdot D\varphi = 0 \quad \forall \varphi \in W_0^{1,2}(\Omega,\mathbb{R}^N)$

Fix $3Q \subset \Omega$. Take a Lipschitz cut-off $\rho : \Omega \to [0,1]$ verifying $\mathbb{1}_Q \leq \rho \leq \mathbb{1}_{2Q}$, $|D\rho| \leq \frac{2}{|Q|^{\frac{1}{n}}}$. Put $\varphi = \Delta_{s,-h}(\rho^2 \Delta_{s,h}(u-a))$ for $|h| < \frac{1}{2}|Q|^{\frac{1}{n}}$, $1 \leq s \leq n$. Then $\varphi \in W_0^{1,2}(\Omega,\mathbb{R}^N)$, hence

$$0 = \int_\Omega f_z(Du)\cdot D\varphi = \int_\Omega f_z(Du)\cdot \Delta_{s,-h} D(\rho^2 \Delta_{s,h}(u-a))$$

$$= \int_\Omega \Delta_{s,h} f_z(Du)\cdot\left(\rho^2 \Delta_{s,h}(Du - Da) + \Delta_{s,h}(u-a)\otimes D(\rho^2)\right)$$

<u>Note</u> $\quad \Delta_{s,h} f_z(Du)\cdot \Delta_{s,h} Du =$

$$\int_0^1 f_{zz}(Du + t\Delta_{s,h}Du)[\Delta_{s,h}Du, \Delta_{s,h}Du]\,dt \begin{cases} \geq \ell |\Delta_{s,h}Du|^2 \\ \leq L|\Delta_{s,h}Du|^2 \end{cases}$$

and consequently

$$0 \geqslant \ell \int_\Omega \rho^2 |\Delta_{s,h}Du|^2 - L\int_\Omega |\Delta_{s,h}Du|\,|\Delta_{s,h}(u-a)|\,|D(\rho^2)|$$

So

$$\int_\Omega \rho^2 |\Delta_{s,h} Du|^2 \leq 2\frac{L}{\ell}\int_\Omega \rho |\Delta_{s,h} Du| \cdot |\Delta_{s,h}(u-a)| |D\rho|$$

Cauchy-Schwarz

$$\leq 2\frac{L}{\ell}\left(\int_\Omega \rho^2 |\Delta_{s,h} Du|^2\right)^{\frac{1}{2}}\left(\int_\Omega |\Delta_{s,h}(u-a)|^2 |D\rho|^2\right)^{\frac{1}{2}},$$

whereby: $$\int_\Omega \rho^2 |\Delta_{s,h} Du|^2 \leq \left(\frac{2L}{\ell}\right)^2 \int_\Omega |\Delta_{s,h}(u-a)|^2 |D\rho|^2.$$

and invoking properties of ρ:

~~Note~~ $$\int_Q |\Delta_{s,h} Du|^2 \leq \left(\frac{4L}{\ell}\right)^2 \frac{1}{|Q|^{\frac{2}{n}}}\int_{2Q} |\Delta_{s,h}(u-a)|^2.$$

<u>Note</u> $$\int_{2Q} |\Delta_{s,h}(u-a)|^2 = \int_{2Q}\left|\int_0^1 D_s(u-a)(x+the_s)\,dt\right|^2 h^2$$

$$\leq \int_{2Q}\int_0^1 |D_s(u-a)(x+the_s)|^2\,dt\,dx\; h^2$$

$$= \int_0^1 \int_{2Q} |D_s(u-a)(x+the_s)|^2\,dx\,dt\; h^2$$

$$= \int_0^1 \int_{\underbrace{2Q+the_s}_{\subset 3Q \text{ for } |h|<\frac{1}{2}|Q|^{\frac{1}{n}}}} |D_s(u-a)|^2\,dx\,dt\; h^2 \leq \int_{3Q} |D_s(u-a)|^2 h^2.$$

$\square$

(See for instance Part I for difference-quotient characterization of Sobolev spaces.)