L15 & 16 The variational difference-quotient method, II

It remains to prove:

TH 7 Assume $F : \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfies

(H1) $z \mapsto F(y,z)$ is C^2, and $(y,z) \mapsto F_{zz}(y,z)$ is continuous

(H2) $|\lambda|^2 \leq F_{zz}(y,z)[\lambda,\lambda] \leq L|\lambda|^2$

(H3) $|z|^2 - L \leq F(y,z) \leq L(|z|^2+1)$

(H4) $|F(y_1,z) - F(y_2,z)| \leq L\,\omega_\alpha(|y_1-y_2|)(|z|^2+1)$

$\omega_\alpha(t) := \min\{t^\alpha, 1\}$, $0 < \alpha \leq 1$.

If $u \in W^{1,p}_{loc}(\Omega, \mathbb{R}^N)$ is F-minimizing and $p > 2$, then $Du \in W^{\theta,2}_{loc}(\Omega, \mathbb{R}^{N\times n})$ for all $\theta < \frac{1}{2}\min\{\alpha, p-2\}$.

Fix a ball $B = B(x_0, R) \Subset \Omega$.

Put $\tilde{F}(z) := F(u_B, z)$. Then by (H1), (H2) $\tilde{F}$ is C^2, and

(8.1) $|\lambda|^2 \leq \tilde{F}_{zz}(z)[\lambda, \lambda] \leq L|\lambda|^2 \quad \forall z, \lambda.$

Let $v \in W^{1,2}_u(B, \mathbb{R}^N)$ be $\tilde{F}$-minimizing.

For a cube $Q = Q(x_0, \frac{R}{3\sqrt{n}})$ we have by the difference-quotient method

$$\int_Q |\Delta_{s,h} Dv|^2 \leq c \frac{h^2}{|Q|^{\frac{2}{n}}} \int_{3Q} |Dv - Da|^2$$

for all affine $a: \mathbb{R}^n \to \mathbb{R}^N$, directions $1 \leq s \leq n$ and increments $4|h| \leq \frac{1}{2}|Q|^{\frac{1}{n}}$.

Clearly, $\int_{3Q} |Dv - Da|^2 \leq \int_B |Dv - Da|^2$.

Since $u = v$ on ∂B, $(Du)_B = (Dv)_B$. Take a s.t. $Da = (Du)_B$.

<u>Observe</u> For $z, z_0 \in \mathbb{R}^{N\times n}$ we get by (8.1):

(8.2) $\tilde{F}(z) - \tilde{F}(z_0) - \tilde{F}_z(z_0)\cdot(z - z_0) = \int_0^1 (1-t)\tilde{F}_{zz}(z_0 + t(z-z_0))dt$

$[z - z_0, z - z_0] \begin{cases} \geq \frac{1}{2}|z - z_0|^2 \\ \leq \frac{L}{2}|z - z_0|^2 \end{cases}$

Consequently, $\frac{1}{2}\int_B |Dv-(Du)_B|^2 \leq$

$$\int_B \tilde{F}(Dv) - \tilde{F}((Du)_B) - \tilde{F}_z((Du)_B)\cdot(Dv-(Du)_B) \leq$$

$$\int_B \tilde{F}(Du) - \tilde{F}((Du)_B) - \tilde{F}_z((Du)_B)\cdot(Du-(Du)_B) \leq$$

$\frac{L}{2}\int_B |Du-(Du)_B|^2$, i.e.

$$(8.3) \qquad \int_Q |\Delta_{s,h} Dv|^2 \leq c\,\frac{h^2}{|Q|^{\frac{2}{n}}}\int_B |Du-(Du)_B|^2 ,$$

where $Q = Q(x_0, \frac{R}{3\sqrt{n}}) \subset 3Q = Q(x_0, \frac{R}{\sqrt{n}}) \subset B = B(x_0, R)$.

[By F-minimality and (H3) + Gehring's Lemma $\exists\, \tilde{p} > 2$ s.t. $u \in W^{1,\tilde{p}}_{loc}(\Omega, \mathbb{R}^N)$, so we can always take $p = \tilde{p}$.]

From Lemma on Variational Calderón–Zygmund estimates we have

$$(8.4) \qquad \int_B |Dv|^q \leq c_q \int_B (|Du|^q + 1)$$

for $q < \frac{2n}{n-2}$. [Special case of Theorem 7.7 in [12].]

<u>Lemma</u> Put $\sigma := \min\{\alpha, p-2\}$.
Then in the above notation we have

$$(8.5) \quad \int_B |Du - Dv|^2 \leq c \int_B \left(|Du|^{2+\sigma} + 1 \right) \cdot R^\sigma$$

<u>Dimension check</u>: LHS $= n-2$, RHS $= n-2-\sigma+\sigma$
$= n-2$.

<u>Note</u>: $2+\sigma \leq p$ so $|Du| \in L^{2+\sigma}(B)$.

$\boxed{\text{Pf:}}$ From (8.2): $\frac{1}{2}\int_B |Du-Dv|^2 \leq$

$$\int_B \Big(\tilde{F}(Du) - \tilde{F}(Dv) - \underbrace{\cancel{\tilde{F}_z(Dv)\cdot(Du-Dv)}}_{0} \Big) =$$

$$\int_B \Big(F(u_B, Du) - F(u, Du) + F(u, Du) - F(v, Dv)$$

$$+ F(v, Dv) - F(u_B, Dv) \Big) =: I + II + III.$$

u F-min: $II \leq 0$.

By (H4): $III \leq \int_B L\omega_\alpha(|v - u_B|)(|Dv|^2 + 1)$

Young with $\frac{2+\sigma}{\sigma}$, $\left(\frac{2+\sigma}{\sigma}\right)' = \frac{2+\sigma}{2}$

$$\leq L \int_B \left[R^{-2-\sigma} \omega_\alpha(|v-u_B|)^{\frac{2+\sigma}{\sigma}} + (|Dv|^2+1)^{\frac{2+\sigma}{2}} \right] R^\sigma$$

$\omega_\alpha(t) \leq t^\sigma$

$$\leq c \int_B \left[R^{-2-\sigma} \left(|u-u_B|^{2+\sigma} + |u-v|^{2+\sigma} \right) + |Dv|^{2+\sigma} + 1 \right] R^\sigma$$

Poincaré

$$\leq c \int_B \left[|Du|^{2+\sigma} + |Du-Dv|^{2+\sigma} + |Dv|^{2+\sigma} + 1 \right] R^\sigma$$

$$\leq c \int_B \left[|Du|^{2+\sigma} + |Dv|^{2+\sigma} + 1 \right] R^\sigma$$

(8.4)

$$\leq c \int_B \left(|Du|^{2+\sigma} + 1 \right) R^\sigma .$$

Similarly for I. □

$\boxed{\text{Pf of Th. 7:}}$ In 3 steps.

<u>Step 1</u>: For $\theta < \sigma/(\sigma+2)$ $Du \in W^{\theta,2}_{loc}(\Omega, \mathbb{R}^{N \times n})$ and for

$\Omega' \Subset \Omega'' \Subset \Omega$ $\exists\, c = c(n, N, \sigma, \theta, L, dist(\Omega', \partial\Omega''))$ s.t.

$$(8.6) \qquad \|Du\|^2_{\theta,2;\Omega'} \leq c \int_{\Omega''} \left(1 + |Du|^{2+\sigma}\right).$$

Fix $\Omega' \Subset \Omega'' \Subset \Omega$.

For $\beta \in (0,1)$ (to be specified)

consider increments $h \in \mathbb{R}$ satisfying

$$(8.7) \qquad |h| < \min\left\{ \left(\mathrm{dist}(\Omega', \partial\Omega'')\right)^{\frac{1}{\beta}}, \left(\frac{1}{3\sqrt{n}}\right)^{\frac{1}{1-\beta}} \right\}$$

For $x_0 \in \Omega'$ put $B = B(x_0, |h|^\beta)$.

Clearly, $B \subset \Omega''$, and recall the notation

$$Q = Q\left(x_0, \frac{|h|^\beta}{3\sqrt{n}}\right) \subset 3Q = Q\left(x_0, \frac{|h|^\beta}{\sqrt{n}}\right) \subset B.$$

Let $\tilde{F}(z) := F(u_B, z)$, and $v \in W^{1,2}_{u}(B, \mathbb{R}^N)$ be $\tilde{F}$-minimizing. From (8.3)

$$\int_Q |\Delta_{s,h} Dv|^2 \leq c \frac{h^2}{|Q|^{\frac{2}{n}}} \int_B |Du - (Du)_B|^2$$

$$\leq c\, |h|^{2-2\beta} \int_B |Du|^2 .$$

Valid for all h satisfying (8.7) (since then in particular $|h| \leq \frac{1}{2}|Q|^{\frac{1}{n}} = \frac{|h|^\beta}{3\sqrt{n}}$).

From (8.5): $\int_B |Du - Dv|^2 \leq c \int_B (|Du|^{2+\sigma} + 1)|h|^{\sigma\beta}$

and so $\int_Q |\Delta_{s,h} Du|^2 \leq 2 \int_Q (|\Delta_{s,h}(Du - Dv)|^2$

$$+ |\Delta_{s,h} Dv|^2) \underset{\substack{Q \subset B,\\ Q + he_s \subset B}}{\leq} 8 \int_B |Du - Dv|^2 + 2 \int_B |\Delta_{s,h} Dv|^2$$

$$\leq c \int_B (|Du|^{2+\sigma} + 1) \cdot (|h|^{\sigma\beta} + |h|^{2-2\beta})$$

Take $\beta \in (0,1)$ so $\sigma\beta = 2 - 2\beta$, i.e. $\beta = \frac{2}{2+\sigma}$. Hereby

$$\int_Q |\Delta_{s,h} Du|^2 \leq c \int_B (|Du|^{2+\sigma} + 1) |h|^{\frac{2\sigma}{2+\sigma}},$$

for all $1 \leq s \leq n$ and $h \in \mathbb{R}$ satisfying (8.7).

Fix $1 \leq s \leq n$, $h \in \mathbb{R}$ satisfying (8.7).
Cover Ω' by <u>non-overlapping</u> closed cubes $\overline{Q} = Q(x_0, \frac{|h|^\beta}{3\sqrt{n}})$ as above.
Observe that the corresponding balls $B = B(x_0, |h|^\beta)$ have bounded overlap

property independent of h:
B intersects at most $(12\sqrt{n})^n$ other such balls. Consequently,

$$\int_{\Omega'} |\Delta_{s,h} Du|^2 \leq \sum_{j\in J} \int_{Q_j} |\Delta_{s,h} Du|^2$$

$$\leq c \sum_{j \in J} \int_{B_j} \left(|Du|^{2+\sigma} + 1 \right) |h|^{\frac{2\sigma}{2+\sigma}}$$

$$\overset{*)}{\leq} c \int_{\Omega''} \left(|Du|^{2+\sigma} + 1 \right) |h|^{\frac{2\sigma}{2+\sigma}} .$$

*) From (8.7) follows $B_j \subset \Omega''$.

Hence Du satisfies a Nikolskii condition of order $\frac{\sigma}{2+\sigma}$ on Ω', and so the desired conclusion follows from the remark following the lemma on p. 6 in L13 & 14.

<u>Step 2:</u> Let $0 < \theta < \frac{\sigma}{2}$. If $Du \in W^{\theta,2}_{loc}(\Omega, \mathbb{R}^{N\times n})$

and for all $\Omega' \Subset \Omega'' \Subset \Omega$ $\exists\, c = c(n, N, \sigma, \theta, L, dist(\Omega', \partial\Omega''))$ s.t.

$$(8.8) \quad \|Du\|^2_{\theta,2;\Omega'} \le c\int_{\Omega''} (|Du|^{2+\sigma}+1),$$

then $\quad Du \in W^{t,2}_{loc}(\Omega,\mathbb{R}^{N\times n})$ for $t<\dfrac{\sigma}{2(1-\theta)+\sigma}$

and (8.8) holds with $\theta = t$ too.

<u>Remark:</u> $\dfrac{\sigma}{2(1-\theta)+\sigma} > \theta$ iff $\theta < \frac{\sigma}{2}$.

We proceed as in Step 1, but revise the estimate for the comparison map v.

With the notation from Step 1:

$$\int_Q |\Delta_{s,h} Dv|^2 \le c\frac{h^2}{|Q|^{\frac{2}{n}}}\int_B |Du-(Du)_B|^2$$

$$\overset{\text{Poincaré}}{\le} c\frac{h^2}{|Q|^{\frac{2}{n}}}|B|^{\frac{2\theta}{n}}\int_B\int_B \frac{|Du(x)-Du(y)|^2}{|x-y|^{n+2\theta}}\,dx\,dy$$

$$= c\,|h|^{2-2\beta+2\theta\beta}\,[Du]^2_{\theta,2;B}$$

$$= c\,[Du]^2_{\theta,2;B}\cdot |h|^{2(1-(1-\theta)\beta)}.$$

Otherwise we proceed as in Step 1:

$$\int_Q |\Delta_{s,h} Du|^2 \leq c\,[Du]^2_{\theta,2;B}\, |h|^{2(1-(1-\theta)\beta)} + c\int_B (|Du|^{2+\sigma}+1)\,|h|^{\beta\sigma}$$

$$\leq c\Big([Du]^2_{\theta,2;B} + \int_B (|Du|^{2+\sigma}+1)\Big)\Big(|h|^{2(1-(1-\theta)\beta)} + |h|^{\sigma\beta}\Big)$$

Take $\beta \in (0,1)$ s.t. $2(1-(1-\theta)\beta) = \sigma\beta$, i.e.

$$\beta = \frac{2}{\sigma + 2(1-\theta)}.$$

Hereby

$$\int_Q |\Delta_{s,h} Du|^2 \leq c\Big([Du]^2_{\theta,2;B} + \int_B (|Du|^{2+\sigma}+1)\Big)|h|^{\frac{2\sigma}{\sigma+2(1-\theta)}}$$

and we conclude as before with a covering argument:

$$\int_{\Omega'} |\Delta_{s,h} Du|^2 \leq c\Big([Du]^2_{\theta,2;\Omega''} + \int_{\Omega''} (|Du|^{2+\sigma}+1)\Big)|h|^{\frac{2\sigma}{\sigma+2(1-\theta)}},$$

and thus $Du \in W^{t,2}_{loc}(\Omega, \mathbb{R}^{N\times n})$ and (8.8) holds with $\theta = t$, $t < \frac{\sigma}{\sigma+2(1-\theta)}$.

Step 3: Conclusion.

Put $H(t) := \frac{\sigma}{2(1-t)+\sigma}$, $t \in [0,1]$.

Then H is increasing, $t < H(t) < \frac{\sigma}{2}$ for $0 < t < \frac{\sigma}{2}$ and $H(\frac{\sigma}{2}) = \frac{\sigma}{2}$.

Note Step 2 yields: $Du \in W^{\theta,2}_{loc}$, $0<\theta<\frac{\sigma}{2}$ implies $Du \in W^{t,2}_{loc}$, $\forall\, t < H(\theta)$.

Define sequences $(\theta_j)_{j=0}^{\infty}$, $(\gamma_j)_{j=0}^{\infty}$ recursively as follows:

Choose $\theta_0 \in (0, \frac{\sigma}{2+\sigma})$.

Put $\gamma_0 := \frac{\theta_0}{2} + \frac{\sigma}{2(2+\sigma)} < \frac{\sigma}{2+\sigma}$.

Define $\theta_j := H(\theta_{j-1})$, $\gamma_j := \frac{\theta_j + H(\gamma_{j-1})}{2}$.

Then $0 < \theta_j < \frac{\sigma}{2}$, $\theta_j \nearrow \frac{\sigma}{2}$,

$\theta_j < \gamma_j < \frac{\sigma}{2}$, $\gamma_j \nearrow \frac{\sigma}{2}$.

<u>Claim</u> $Du \in W^{\gamma_j, 2}_{loc}(\Omega, \mathbb{R}^{N\times n})$

and $\| Du \|^2_{\gamma_j, 2; \Omega'} \leq c_j \int_{\Omega''} (|Du|^{2+\sigma} + 1) \quad \forall j$

By induction.

$j = 0$ follows from Step 1.

Assume it's true for $j = k \in \mathbb{N}_0$.

Then by Step 2, $Du \in W^{t,2}_{loc} \quad \forall t < H(\gamma_k)$.

Now $H(\gamma_k) \geq \dfrac{H(\gamma_k) + H(\theta_k)}{2} = \dfrac{H(\gamma_k) + \theta_{k+1}}{2} = \gamma_{k+1}$

and so the conclusion follows for $j = k+1$. □