A NECESSARY AND SUFFICIENT CONDITION FOR LOWER SEMICONTINUITY

Jan Kristensen

Abstract

Results on relaxation and semicontinuity are obtained for variational integrals $\int_\Omega F(\nabla u)$, when the integrand F is extended real-valued and when it satisfies a p, q growth condition, respectively.

Key words: *Quasiconvexity, Compensated Compactness, Young Measures*

1 Introduction

When the integrand F satisfies the growth conditions $|\xi|^p \leq F(\xi) \leq c(|\xi|^p + 1)$ and $p > 1$ it has been shown by Dacorogna (see [15]) that the lower semicontinuous envelope of the integral $\int_\Omega F(\nabla u)$ in the weak topology of $W^{1,p}$ equals $\int_\Omega \tilde{F}(\nabla u)$, where $\tilde{F}$ denotes the $W^{1,p}$ quasiconvex envelope of F. We refer to Section 2 for notation and terminology. The main results of this paper establish similar relaxation formulas in situations where F is extended real-valued or where it satisfies a p, q growth condition. We proceed to describe and state the main results.

Throughout the paper $\Omega \subset \mathbb{R}^n$ denotes a fixed bounded and open set with $\mathcal{L}^n(\partial\Omega) = 0$ and $p \in (1, \infty)$ an exponent.

Let $F \colon \mathbb{R}^{N\times n} \to [0, \infty]$ be an extended real-valued Borel function. For a matrix field $V \in L^p(\Omega, \mathbb{R}^{N\times n})$ define

$$\bar{\mathrm{I}}\,[V] = \inf_{\{V_j\}} \left\{ \liminf_{j\to\infty} \int_\Omega F(V_j) \right\}, \tag{1.1}$$

where the infimum is taken over all sequences $\{V_j\}$ of L^p matrix fields on Ω satisfying

$$\begin{aligned} &V_j \rightharpoonup V \text{ weakly in } L^p, \\ &\operatorname{curl} V_j \to \operatorname{curl} V \text{ strongly in } W^{-1,p}. \end{aligned} \tag{1.2}$$

Here curl on an $N \times n$ matrix field on Ω is understood row-wise and in the distributional sense, $W^{-1,p}$ is the dual space of $W_0^{1,p'}$ under the usual distributional duality pairing and $p' = p/(p-1)$ is the Hölder conjugate exponent. This is a special case of the compensated compactness setting considered in e.g. [16], [27], [28], [46] and [55]. The first main result of the paper is the following relaxation formula.

Theorem 1.1. *Let* $F\colon \mathbb{R}^{N\times n} \to [0,\infty]$ *be a Borel function, and denote by* S *the lower semicontinuous envelope of* $\bar{\mathrm{I}}$ *in the strong topology of* L^p. *Then for any* $V \in L^p(\Omega, \mathbb{R}^{N\times n})$,

$$\mathrm{S}[V] = \int_\Omega \bar{F}(V) \tag{1.3}$$

where $\bar{F}$ *denotes the closed* $W^{1,p}$ *quasiconvex envelope of* F.

The function F is closed $W^{1,p}$ quasiconvex provided it is lower semicontinuous and Jensen's inequality holds for F and every homogeneous $W^{1,p}$ gradient Young measure (see [49]). The closed $W^{1,p}$ quasiconvex envelope is the largest closed $W^{1,p}$ quasiconvex function below F.

It is not clear to the author whether in the general situation considered in Theorem 1.1 the functionals $\bar{\mathrm{I}}$ and S coincide. However, it is easy to show that they do under mild conditions on F.

Corollary 1.2. *Suppose that the Borel function* $F\colon \mathbb{R}^{N\times n} \to [0,\infty]$ *satisfies that* $F(\xi) \geq \varepsilon|\xi|^p$ *for some* $\varepsilon > 0$ *when* $|\xi|$ *is sufficiently large. Then* $\mathrm{S}[V] = \bar{\mathrm{I}}\,[V]$ *for all matrix fields* $V \in L^p$.

We record another easy consequence of Theorem 1.1 in the following corollary.

Corollary 1.3. *Let* $F\colon \mathbb{R}^{N\times n} \to [0,\infty]$ *be Borel,* $1 < p < \infty$ *and* $\Omega \subseteq \mathbb{R}^n$ *be open. Then*

$$\liminf_{j\to\infty} \int_\Omega F(V_j) \geq \int_\Omega F(V) \tag{1.4}$$

holds for all sequences $\{V_j\}$ *and* V *satisfying* (1.2) *if and only if* F *is closed* $W^{1,p}$ *quasiconvex.*

For a Sobolev map $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ define

$$\tilde{\mathrm{I}}\,[\nabla u] = \inf_{\{u_j\}} \left\{ \liminf_{j\to\infty} \int_\Omega F(\nabla u_j) \right\}, \tag{1.5}$$

where the infimum is taken over all sequences $\{u_j\} \subset W^{1,p}(\Omega, \mathbb{R}^N)$ satisfying $u_j \rightharpoonup u$ weakly in $W^{1,p}$, i.e.

$$\begin{aligned} &u_j \rightharpoonup u \text{ weakly in } L^p, \\ &\nabla u_j \rightharpoonup \nabla u \text{ weakly in } L^p. \end{aligned} \tag{1.6}$$

It is clear that $\tilde{\mathrm{I}}\,[\nabla u] \geq \bar{\mathrm{I}}\,[\nabla u]$ for all $W^{1,p}$ maps u. The following elementary example is based on the T_4 configuration of four matrices (see [50] and [56]). It shows that the inequality can be strict for lower semicontinuous extended real-valued integrands F.

Example 1.4. *Let*

$$K = \left\{ \pm\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \pm\begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} \right\}.$$

The following three facts are well-known (see [47] *and the references therein):*

(i) There exists a laminate ν *which is supported on* K *and has centre of mass* $\bar{\nu} = 0$.

(ii) Laminates with compact support are homogeneous $W^{1,p}$ gradient Young measures for any p.

(iii) If $\Omega \subset \mathbb{R}^2$ is open and connected, $u \in W^{1,1}(\Omega, \mathbb{R}^2)$ and $\nabla u \in K$ a.e., then ∇u is constant a.e.
(See also [13] *for far reaching generalizations.)*

If therefore we define

$$F(\xi) = \begin{cases} 0 & \text{if } \xi \in K \\ \infty & \text{else,} \end{cases}$$

then as a consequence of (iii) the integral $\int_\Omega F(\nabla u)$ is weakly lower semicontinuous on $W^{1,1}(\Omega, \mathbb{R}^2)$, so that $\tilde{\mathrm{I}}\,[\nabla u] = \int_\Omega F(\nabla u)$. However, in view of (i) and (ii), F cannot be closed $W^{1,p}$ quasiconvex for any p, since Jensen's inequality fails for F and ν. In fact, it is not difficult to show that

$$\bar{F}(\xi) = \begin{cases} 0 & \text{if } \xi \in K^{rc} \\ \infty & \text{else,} \end{cases}$$

where K^{rc} denotes the rank-1 convex hull of K.

It is not clear if a similar example in which the integrand is continuous could be constructed. It is therefore still possible that for continuous integrands $F\colon \mathbb{R}^{N\times n} \to [0, \infty]$, closed $W^{1,p}$ quasiconvexity is also a necessary condition for the weak lower semicontinuity of $\int_\Omega F(\nabla u)$ on $W^{1,p}$. However, in this connection notice that even when F also satisfies the growth condition

$$0 \le F(\xi) \le c(|\xi|^q + 1) \tag{1.7}$$

where $c > 0$ with $1 < p < q < \infty$ and $q/p > 1$ is arbitrarily close to 1 the necessity of closed $W^{1,p}$ quasiconvexity for weak lower semicontinuity on $W^{1,p}$ is an open problem. At present the necessity of closed $W^{1,p}$ quasiconvexity has only been established for integrands satisfying (1.7) with $p = q$, or under special structure assumptions on F. (See [1], [6], [27], [37] or [40].) In these cases closed $W^{1,p}$ quasiconvexity can be shown to be equivalent to $W^{1,p}$ quasiconvexity as introduced in [5]. There it was shown that $W^{1,p}$ quasiconvexity is a necessary condition for sequential weak lower semicontinuity on $W^{1,p}$. Corollary 1.7 below states that, with an appropriate growth condition, $W^{1,p}$ quasiconvexity is also sufficient for sequential weak lower semicontinuity on $W^{1,p}$. However, for general continuous integrands the question of whether $W^{1,p}$ quasiconvexity implies sequential weak lower semicontinuity on $W^{1,p}$ remains open.

The second main result of the paper concerns the functional $\tilde{\mathrm{I}}$.

Theorem 1.5. *Let $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ be a Borel function satisfying the growth condition* (1.7). *Assume that $1 < p \le q < \infty$ satisfy*

$$q < \frac{np}{n-1}. \tag{1.8}$$

For maps $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ the functional at (1.5) *can be expressed as*

$$\tilde{\mathrm{I}}\,[\nabla u] = \lim_{\delta \searrow 0} \int_\Omega \tilde{F}_\delta(\nabla u)$$

where $\tilde{F}_\delta$ denotes the $W^{1,p}$ quasiconvex envelope of $F_\delta = F + \delta|\cdot|^p$.

When the $W^{1,p}$ map u satisfies the condition $\tilde{\mathrm{I}}\,[\nabla u] < \infty$, then the above result simplifies to

$$\tilde{\mathrm{I}}\,[\nabla u] = \int_\Omega \tilde{F}(\nabla u).$$

It is unclear to the author whether in the general case considered in Theorem 1.5 there can exist a $W^{1,p}$ map u such that

$$\tilde{\mathrm{I}}\,[\nabla u] = \infty > \int_\Omega \tilde{F}(\nabla u). \tag{1.9}$$

This is the reason for the slightly elaborate statement involving F_δ. However, under additional mild conditions on F (1.9) is easy to exclude. This is the content of the next corollary.

Corollary 1.6. *In addition to the hypotheses of Theorem* 1.5 *suppose that one of the following conditions holds:*

(i) $p = q$

(ii) There exists $\varepsilon > 0$ *such that* $F(\xi) \geq \varepsilon|\xi|^p$ *when* $|\xi|$ *is sufficiently large.*

Then we have for the functional at (1.5) *and for each* $W^{1,p}$ *map* u *that*

$$\tilde{\mathrm{I}}\,[\nabla u] = \int_\Omega \tilde{F}(\nabla u)$$

where $\tilde{F}$ *denotes the* $W^{1,p}$ *quasiconvex envelope of* F*.*

Finally, we also record the following consequence of Theorem 1.5, which has also previously been recorded in [51][Theorem 4.4].

Corollary 1.7. *Assume that* F *satisfies the conditions of Theorem* 1.5*. Then*

$$\liminf_{j\to\infty} \int_\Omega F(\nabla u_j) \geq \int_\Omega F(\nabla u) \tag{1.10}$$

holds for all sequences $\{u_j\}$ *and maps* u *satisfying* (1.6) *if and only if* F *is* $W^{1,p}$ *quasiconvex.*

The main difficulty in proving Theorem 1.5 is to obtain the inequality

$$\tilde{\mathrm{I}}\,[\nabla u] \leq \lim_{\delta\searrow 0} \int_\Omega \tilde{F}_\delta(\nabla u).$$

The opposite inequality is established by use of standard methods. The proof of the above inequality is based on an approximation result, which loosely stated says that a $W^{1,p}$ map can be approximated in energy by countably piecewise affine $W^{1,p}$ maps. We refer to Proposition 4.7 in Section 4 for a precise statement and note that it is the main new technical point of the proof. Condition (1.8) on p and q is essential for the method of proof, and means that there exists a compact, linear extension operator from $W^{1,p}$ on the $(n-1)$-sphere $\partial\mathrm{B}$ to $W^{1,q}$ on $\mathbb{R}^n$.

We do not know whether (1.8) is really necessary for the result to be true. (However, see [5] Ex. 3.5.) The extension operator is used to glue $W^{1,p}$ maps, and this idea can be traced back to Meyers [42], though the way we do it follows the works of Malý [39] and Fonseca & Malý [24]. The idea to use approximation by piecewise affine maps for proving semicontinuity is natural in connection with quasiconvexity, and it goes back to Morrey [45]. The need to use countably piecewise affine maps instead of merely piecewise affine maps arises when $p < q$, and is connected to Lavrentiev's phenomenon (see [4], [11], [43], [44] and the references therein).

Finally, we point out that the approach to relaxation and semicontinuity adopted here differs from the Lebesgue-Serrin extension approach, though of course some of the proofs have technical points in common (viz. the use of the extension operator to fix boundary values, and the blow-up method). In the context of quasiconvex integrals the Lebesgue-Serrin extension was first investigated by Marcellini in [41], and subsequently many works have followed, see e.g. [8], [24], [25], [28], [35, 36], [39] and [43] and the references therein.

The organization of the paper is as follows. In Section 2 we recall the main definitions and state some basic results on quasiconvex envelopes and Young measures that are used in the following sections. The proofs of Theorem 1.1 and its corollaries 1.2 and 1.3 are presented in Section 3. Section 4 contains the proofs of Theorem 1.5, Corollaries 1.6 and 1.7.

2 Notation and preliminary results

The purpose of this section is to recall definitions and some auxiliary results.

We use standard notation for maps and function spaces as can be found in for instance [22]. When $f\colon S \subset \mathbb{R}^n \to H$ is a (Lebesgue-) integrable map defined on a measurable set S and with values in a finite-dimensional inner-product space H and $T \subseteq S$ is a measurable set of positive and finite Lebesgue measure the average of f over T is denoted by $⨍_T f$, i.e.

$$⨍_T f = \frac{1}{\mathcal{L}^n(T)} \int_T f(x)\, dx.$$

In connection with a sequence of maps, the symbols $\to$, $\rightharpoonup$ and $\overset{*}{\rightharpoonup}$ denote strong, weak and weak* convergence, respectively. We frequently use the Greek letters ξ, ζ and η to denote matrices, and we consider $\mathbb{R}^{N\times n}$ with the Hilbert-Schmidt norm: $|\xi|^2 = \mathrm{trace}(\xi^T\xi)$.

The notation and terminology for Young measures follow essentially that of [32, 33], [47] and [37]. Hence a $W^{1,p}$ gradient Young measure on Ω is a measure ν on $\Omega \times \mathbb{R}^{N\times n}$ for which there exists a weakly convergent sequence $\{u_j\}$ in $W^{1,p}(\Omega, \mathbb{R}^N)$, such that

$$\int_\Omega \Phi(x, \nabla u_j(x))\, dx \to \langle \nu, \Phi \rangle \text{ as } j \to \infty \tag{2.1}$$

for each $\Phi\colon \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}$ which is continuous and vanishes outside of some compact set. It is not difficult to show from this definition that the measure ν must have the property that $\nu(O \times \mathbb{R}^{N\times n}) = \mathcal{L}^n(O)$ for each open subset O of Ω. A general fact of measure theory, then implies that ν can be disintegrated as $\nu = \int_\Omega \delta_x \otimes \nu_x\, dx$, where ν_x are probability measures on

$\mathbb{R}^{N\times n}$. The formula means that for any non-negative Borel function $F\colon \Omega\times\mathbb{R}^{N\times n}\to[0,\infty]$ the function $x\mapsto\langle\nu_x,F(x,\cdot)\rangle$ is measurable and

$$\int_{\Omega\times\mathbb{R}^{N\times n}} F\,d\nu=\int_{\Omega}\int_{\mathbb{R}^{N\times n}} F(x,\cdot)\,d\nu_x\,dx.$$

We refer to the above references for the basic properties of Young measures.

The following definition by Ball and Murat in [5] extends the original notion of quasiconvexity introduced by Morrey in [45].

Definition 1. *A function $F\colon\mathbb{R}^{N\times n}\to[-\infty,\infty]$ is $W^{1,p}$ quasiconvex at $\xi\in\mathbb{R}^{N\times n}$ if the inequality*

$$\int_{(0,1)^n} F(\xi+\nabla\varphi(x))\,dx\geq F(\xi)$$

holds for all $\varphi\in W_0^{1,p}((0,1)^n,\mathbb{R}^N)$ for which the left-hand side makes sense as a Lebesgue integral (possibly with values $\pm\infty$). F is $W^{1,p}$ quasiconvex if it is quasiconvex at every ξ.

It is well-known that the condition of $W^{1,p}$ quasiconvexity depends on $p\in[1,\infty]$, where $W^{1,1}$ quasiconvexity is the strongest condition and $W^{1,\infty}$ the weakest (see [5]).

The $W^{1,p}$ quasiconvex envelope of $F\colon\mathbb{R}^{N\times n}\to[-\infty,\infty]$ is the largest $W^{1,p}$ quasiconvex function that is below F. We denote it by $\tilde{F}$, and do not specify p in the notation, the value being understood from the context. Thus,

$$\tilde{F}(\xi)=\sup\left\{G(\xi):\ G\text{ is }W^{1,p}\text{ quasiconvex},\ G\leq F\right\},$$

and it is not excluded that $\tilde{F}$ is identically $\pm\infty$.

The following result is crucial for the proofs of Theorems 1.1 and 1.5. It is due to Dacorogna [15], though the form in which it is stated here is slightly more general. The proof below is the same as those of [23] and [32], and is given here for the convenience of the reader.

Lemma 2.1. *Let $F\colon\mathbb{R}^{N\times n}\to\mathbb{R}$ be Borel, and $\Omega\neq\emptyset$ a bounded, open subset of $\mathbb{R}^n$ with $\mathcal{L}^n(\partial\Omega)=0$. Then the $W^{1,\infty}$ quasiconvex envelope is*

$$\tilde{F}(\xi)=\inf\left\{\fint_{\Omega}F(\xi+\nabla\varphi(x))\,dx:\ \varphi\in W_0^{1,\infty}(\Omega,\mathbb{R}^N)\right\}. \tag{2.2}$$

For the proof of Lemma 2.1 we need some auxiliary results. The following existence result from [31] is the key.

Lemma 2.2. *Let ξ_0, $\xi_1\in\mathbb{R}^{N\times n}$ satisfy $\xi_1-\xi_0=a\otimes b$ for vectors $a\in\mathbb{R}^N$ and $b\in\mathbb{R}^n$. Put $\xi=t\xi_1+(1-t)\xi_0$ for $t\in(0,1)$. Let $b_3,\dots,b_n\in\mathbb{R}^n$ be $n-2$ vectors with the property that $0\in\mathbb{R}^n$ is an interior point of the convex hull of $\{b,-b,b_3,\dots,b_n\}$. Let Ω be an open and bounded (non-empty) subset of $\mathbb{R}^n$ and let $\varepsilon>0$. There exists a Lipschitz map $u\colon\Omega\to\mathbb{R}^N$, such that*

(i) $u(x)=\xi x$ *for* $x\in\partial\Omega$, *and* $\sup_{x\in\Omega}|u(x)-\xi x|<\varepsilon$,

(ii) $\nabla u(x) \in \{\xi_0, \xi_1, \xi + a \otimes b_3, \dots, \xi + a \otimes b_n\}$ *for a.e.* $x \in \Omega$,

(iii) $\mathcal{L}^n(\{x \in \Omega : \nabla u(x) = \xi_0\}) > (1-\varepsilon)t\mathcal{L}^n(\Omega)$ *and* $\mathcal{L}^n(\{x \in \Omega : \nabla u(x) = \xi_1\}) > (1-\varepsilon)(1-t)\mathcal{L}^n(\Omega)$.

Proof. See [31], Lemma 2.2, p. 10. □

Corollary 2.3. *Let* $\Omega \neq \emptyset$ *be an open and bounded subset of* $\mathbb{R}^n$*. Suppose that* $F\colon \mathbb{R}^{N\times n} \to [-\infty, \infty)$ *(*∞ *excluded) satisfies*

$$⨍_{\Omega} F(\xi + \nabla\varphi(x))\,dx \geq F(\xi)$$

for all $\xi \in \mathbb{R}^{N\times n}$ *and all piecewise affine* $\varphi \in W_0^{1,\infty}(\Omega, \mathbb{R}^N)$*. Then either* $F \equiv -\infty$ *or* $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ *is locally Lipschitz and* $W^{1,\infty}$ *quasiconvex.*

Proof. It follows by use of Lemma 2.2 that F is rank-1 convex, i.e. that $\mathbb{R} \ni t \mapsto F(\xi + ta \otimes b)$ is convex for all $\xi \in \mathbb{R}^{N\times n}$ and $a \in \mathbb{R}^N$, $b \in \mathbb{R}^n$. (See also the proof of Th. 2.4 in [23].) Now a rank-1 convex function $F\colon \mathbb{R}^{N\times n} \to [-\infty, \infty)$ is either $F \equiv -\infty$ or $F(\xi) > -\infty$ for all ξ (see e.g. [23]). If therefore F is not identically $-\infty$, then F is real-valued, and because of the rank-1 convexity it is then locally Lipschitz (see [45] p. 112 or [17]). The $W^{1,\infty}$ quasiconvexity follows because we can show that

$$⨍_{\Omega} F(\xi + \nabla\varphi(x))\,dx \geq F(\xi)$$

holds for all $\xi \in \mathbb{R}^{N\times n}$ and $\varphi \in W_0^{1,\infty}(\Omega, \mathbb{R}^N)$ by approximating φ with piecewise affine maps (e.g., first apply [21], Prop. 2.1, p. 309, to smooth compactly supported φ and then use the definition of $W_0^{1,\infty}$). The proof is finished with a standard covering argument (see e.g. [31], Construction 2.1, p. 9). □

Proof of Lemma 2.1. Let $H(\xi)$ denote the right-hand side of (2.2). Clearly, $\tilde{F}(\xi) \leq H(\xi) \leq F(\xi)$ holds for all ξ. It therefore suffices to show that H is $W^{1,\infty}$ quasiconvex, and we do this by use of Corollary 2.3. By use of a covering argument it is not hard to show that $H(\xi)$ is independent of Ω, in the sense that whenever $O \neq \emptyset$ is a bounded, open subset of $\mathbb{R}^n$ with $\mathcal{L}^n(\partial O) = 0$, then

$$H(\xi) = \inf\left\{ ⨍_{O} F(\xi + \nabla\varphi) : \varphi \in W_0^{1,\infty}(O, \mathbb{R}^N) \right\}.$$

Fix a $\xi \in \mathbb{R}^{N\times n}$ for which $H(\xi) > -\infty$. (If no such ξ exists, then we are done.) Let $\varphi \in W_0^{1,\infty}(\Omega, \mathbb{R}^N)$ be piecewise affine. Suppose that $\nabla\varphi$ is constant on each of the open sets

$T_1, \dots, T_I$, where $\mathcal{L}^n(\Omega \setminus \bigcup T_i) = 0$. Now for an $\varepsilon > 0$ and each i we select $\phi_i \in W_0^{1,\infty}(T_i, \mathbb{R}^N)$, such that (for a.e. $x \in T_i$)

$$\fint_{T_i} F(\xi + \nabla\varphi(x) + \nabla\phi_i(y))\,\mathrm{d}y \le H(\xi + \nabla\varphi(x)) + \varepsilon.$$

Define $\varphi_\varepsilon = \phi_i$ on T_i, $i \in \{1, \dots, I\}$ and $\varphi_\varepsilon = 0$ otherwise. Then $\varphi + \varphi_\varepsilon \in W_0^{1,\infty}(\Omega, \mathbb{R}^N)$ and consequently,

$$\int_\Omega H(\xi + \nabla\varphi(x))\,\mathrm{d}x \ge \int_\Omega F(\xi + \nabla\varphi(x) + \nabla\varphi_\varepsilon(x))\,\mathrm{d}x - \varepsilon\mathcal{L}^n(\Omega),$$

and the latter is by definition of H not less than $(H(\xi) - \varepsilon)\mathcal{L}^n(\Omega)$. □

Let $p \in [1, \infty]$ and denote by $\mathcal{M}^p$ the set of probability measures μ on $\mathbb{R}^{N\times n}$ with a finite p-th moment (bounded support when $p = \infty$), such that

$$\int_{\mathbb{R}^{N\times n}} F(\xi)\,\mathrm{d}\mu \ge F(\bar{\mu})$$

holds for all $W^{1,\infty}$ quasiconvex functions $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ for which there exists a constant $c = c(F)$, such that $|F(\xi)| \le c(1 + |\xi|^p)$. When $p = \infty$ we interpret this growth condition as vacuous. By [32, 33] the set $\mathcal{M}^p$ coincides with the homogeneous $W^{1,p}$ gradient Young measures. We require the following result about these measures.

Lemma 2.4. *For each $p \in (1, \infty)$,*

$$\mathcal{M}^1 \cap \left\{ \nu\colon \int_{\mathbb{R}^{N\times n}} |\xi|^p\,\mathrm{d}\nu(\xi) < \infty \right\} = \mathcal{M}^p$$

Proof. See [37] Cor. 1.8. □

The $W^{1,p}$ quasiconvexity condition is closely related to the following condition that is due to Pedregal [49].

Definition 2. *A lower semicontinuous function $F\colon \mathbb{R}^{N\times n} \to \mathbb{R} \cup \{\infty\}$ that is bounded from below, is closed $W^{1,p}$ quasiconvex provided that*

$$\int_{\mathbb{R}^{N\times n}} F(\xi)\,\mathrm{d}\mu \ge F(\bar{\mu})$$

holds for all $\mu \in \mathcal{M}^p$. Here $\bar{\mu} = \langle \mu, id \rangle$ denotes the center of mass of μ and $p \in [1, \infty]$.

The closed $W^{1,p}$ quasiconvex envelope of $F\colon \mathbb{R}^{N\times n} \to \mathbb{R} \cup \{\infty\}$, assumed bounded from below, is the largest closed $W^{1,p}$ quasiconvex function that is below F. We denote it by $\bar{F}$, where, as with the other envelopes, p is understood from the context and not specified in the notation. Hence,

$$\bar{F}(\xi) = \sup\{G(\xi)\colon G \text{ is closed } W^{1,p} \text{ quasiconvex}, G \le F\}.$$

We end the section with some general remarks. Recall that the bounded weak topology on a Banach space X by definition is the finest topology that coincides with the weak topology on all closed balls of X. The uniform boundedness principle and the fact that the weak topology on a reflexive, separable Banach space is metrizable on closed balls yield the following result.

Lemma 2.5. *Let X be a reflexive, separable Banach space and $T\colon X \to \mathbb{R} \cup \{\infty\}$ a functional. Then T is lower semicontinuous in the bounded weak topology on X if and only if it is sequentially lower semicontinuous in the weak topology of X.*

In light of this remark and Theorem 1.5 we can interpret the functional $\tilde{\mathrm{I}}$ defined at (1.5) as follows. Let for $W^{1,p}$ maps u, $T[u] = \int_\Omega F(\nabla u)$ and let $\tilde{T}$ be the lower semicontinuous envelope of T on $W^{1,p}$ with bounded weak topology. Assume that F satisfies (1.7) with $q < np/(n-1)$. Then $\tilde{T}[u] = \tilde{\mathrm{I}}\,[\nabla u]$ for all $W^{1,p}$ maps u for which $\tilde{\mathrm{I}}\,[\nabla u] < \infty$. It is unclear if the situation $\tilde{\mathrm{I}}\,[\nabla u] = \infty > \tilde{T}[u]$ can occur. By virtue of Corollary 1.6 it can be excluded when either $p = q$, or $p < q < np/(n-1)$ and $F(\xi) \geq \varepsilon|\xi|^p$ holds for some $\varepsilon > 0$ when $|\xi|$ is sufficiently large.

Finally, observe that, regardless of the growth conditions on F, when $F(\xi) \geq \varepsilon|\xi|^p$ holds for some $\varepsilon > 0$ when $|\xi|$ is sufficiently large, or more generally when F is p-mean coercive, then $u \mapsto \tilde{\mathrm{I}}\,[\nabla u]$ is the lower semicontinuous envelope of the integral $\int_\Omega F(\nabla u)$ in the weak topology of $W^{1,p}$. It appears that this ceases to be true without a coercivity assumption, even in the case where (1.7) is imposed with $p = q$.

3 Proofs of Theorem 1.1 and its corollaries

The proof of Theorem 1.1 is performed in two steps. The first step consists in proving the 'if' part of Corollary 1.3, i.e. the sufficiency of closed $W^{1,p}$ quasiconvexity for lower semicontinuity. We state it in a separate lemma.

Lemma 3.1. *Let $F\colon \mathbb{R}^{N\times n} \to [0,\infty]$ be lower semicontinuous and closed $W^{1,p}$ quasiconvex, where $p \in (1,\infty)$. Then it holds for any open set $\Omega \subseteq \mathbb{R}^n$ that*

$$\liminf_{j\to\infty} \int_\Omega F(V_j) \geq \int_\Omega F(V)$$

whenever the matrix fields V_j, $V\colon \Omega \to \mathbb{R}^{N\times n}$ satisfy $V_j \rightharpoonup V$ in L^p, $\operatorname{curl} V_j \to \operatorname{curl} V$ in $W^{-1,p}$ on Ω.

The proof of this result relies on standard properties of Young measures and the following well-known result about the Helmholtz decomposition on $\mathbb{R}^n$ (see e.g. [27], [37] or [46]).

Lemma 3.2. *Let P denote orthogonal projection of $L^2 = L^2(\mathbb{R}^n, \mathbb{R}^n)$ onto the closed subspace $\{V \in L^2 : \operatorname{div} V = 0\}$. Then P admits an extension as a continuous linear operator $\mathrm{P}\colon L^p \to L^p$ for each $p \in (1,\infty)$. This extension has the additional property that $\operatorname{curl}(V - \mathrm{P}[V]) = 0$, and for each $p \in (1,\infty)$ there exists $c_p \in \mathbb{R}$, such that*

$$\|\mathrm{P}[V]\|_{L^p} \leq c_p \|\operatorname{curl} V\|_{W^{-1,p}}$$

for all $V \in L^p$.

If $\mathcal{F}$ denotes the Fourier transformation, then it can be shown that P admits the representation

$$\mathrm{P}[V] = \mathcal{F}^{-1}(m\mathcal{F}V),$$

where the Fourier multiplier $m\colon \mathbb{R}^n \to \mathbb{R}^{n\times n}$ has components $m_{r,s}(y) = \delta_{r,s} - y_r y_s/|y|^2$ ($\delta_{r,s}$ is the Kronecker symbol). The proof of the lemma is then finished by use of standard results on Fourier multipliers. See [27] where it is carried out in a periodic setting. We omit the details here.

Proof of Lemma 3.1. Let $\{V_j\}$, V be as in the statement of the lemma. By considering a subsequence if necessary we can assume that

$$\liminf_{j\to\infty}\int_\Omega F(V_j) = \lim_{j\to\infty}\int_\Omega F(V_j)$$

and that $\{V_j\}$ generates a Young measure $\nu = \int_\Omega \delta_x \otimes \nu_x \,\mathrm{d}x$. We can clearly also assume that this limit is finite (though this is not important for the argument). Because F is nonnegative and lower semicontinuous we have in this situation that

$$\lim_{j\to\infty}\int_\Omega F(V_j) \geq \int_\Omega \int_{\mathbb{R}^{N\times n}} F(\xi)\,\mathrm{d}\nu_x(\xi)\,\mathrm{d}x.$$

To establish the desired lower semicontinuity we show that for almost all x,

$$\int_{\mathbb{R}^{N\times n}} F(\xi)\,\mathrm{d}\nu_x(\xi) \geq F(V(x)). \tag{3.1}$$

It is clear that for almost all x, $V(x)$ equals the centre of mass $\overline{\nu}_x$ of ν_x, so that (3.1) is Jensen's inequality for F and ν_x. We therefore conclude the proof by showing that $\nu_x \in \mathcal{M}^p$ for almost all x. To this end notice first that the sequence $\{V_j - V\}$ generates the Young measure $\tilde{\nu}$, where $\tilde{\nu}_x = \nu_x \star \delta_{-V(x)}$ (convolution of measures). Since $\nu_x \in \mathcal{M}^p$ if $\tilde{\nu}_x \in \mathcal{M}^p$ we may therefore assume that $V = 0$, that is, $\operatorname{curl} V_j \to 0$ in $W^{-1,p}$ on Ω.

Because $\operatorname{curl}(\phi V_j) = \phi \operatorname{curl} V_j + \nabla\phi \wedge V_j$ for smooth functions $\phi\colon \Omega \to \mathbb{R}$, it follows that we may assume that each V_j has support in some bounded set, and hence that $V_j \rightharpoonup 0$ in L^p and $\operatorname{curl} V_j \to 0$ in $W^{-1,p}$ on $\mathbb{R}^n$. Then by applying Helmholtz decomposition to each row of V_j it follows (see Lemma 3.2) that $V_j = \nabla u_j + E_j$, where $\nabla u_j \rightharpoonup 0$ in L^p and $E_j \to 0$ in L^p on $\mathbb{R}^n$. The Young measure ν is therefore generated by the sequence $\{\nabla u_j\}$, and so is a $W^{1,p}$ gradient Young measure. It then follows from the localization principle for gradient Young measures (see [32, 33] or [37] Lemma 8.2) that $\nu_x \in \mathcal{M}^p$ for almost all x. ☐

Proof of Theorem 1.1. By virtue of Lemma 3.1 the inequality

$$\mathrm{S}[V] \geq \int_\Omega \bar{F}(V)$$

holds for all L^p matrix fields V. To prove the opposite inequality we fix an L^p matrix field V such that $\int_\Omega \bar{F}(V) < \infty$. For integers j, $k \in \mathbb{N}$ define the sets

$$E_{k,l} = \{x \in \Omega :\ \bar{F}(V(x)) \in [(l-1)2^{-k}, l2^{-k})\},$$

and select $x_{k,l} \in E_{k,l}$ (when $E_{k,l} \neq \emptyset$) such that

$$\int_{E_{k,l}} |\bar{F}(V) - \bar{F}(\xi_{k,l})| \leq 2^{-k}\mathcal{L}^n(E_{k,l}).$$

If we localize in Ω (i.e., replace the sets $E_{k,l}$ by $\mathrm{B} \cap E_{k,l}$ where $\mathrm{B} \subset \Omega$ ranges over a family of sufficiently small disjoint balls that exhaust Ω) we can obtain points $x_{k,l} = x_{k,l}(\mathrm{B}) \in E_{k,l} \cap \mathrm{B}$ such that additionally $\int_{E_{k,l}\cap\mathrm{B}} |V - V(x_{k,l})|^p < 2^{-k}\mathcal{L}^n(E_{k,l} \cap \mathrm{B})$. In order not to complicate the notation we assume in the following that this is true already for the points $x_{k,l} \in E_{k,l}$.

Put $\xi_{k,l} = V(x_{k,l})$ and $V_k = \sum_{l=1}^{l(k)} \xi_{k,l} 1_{E_{k,l}}$, where $l(k) \in \mathbb{N}$ is so large that

$$\int_\Omega \Big(|\bar{F}(V) - \bar{F}(V_k)| + |V - V_k|^p \Big) \leq 2^{1-k}\mathcal{L}^n(\Omega)$$

for all $k \in \mathbb{N}$. Define $F_\delta(\xi) = F(\xi) + \delta|\xi|^p$ for $\delta > 0$, and observe that $\bar{F}_\delta(\xi) \searrow \bar{F}(\xi)$ as $\delta \searrow 0$ pointwise in ξ. Consequently, for each $k \in \mathbb{N}$ we may find $\delta = \delta(k) > 0$ such that $\bar{F}_\delta(V_k) \leq \bar{F}(V_k) + 2^{-k}$ a.e. on Ω.

Fix $k \in \mathbb{N}$ and define $G_j(\xi) = \min\{F_\delta(\xi), j(1 + |\xi|^p)\}$. Then G_j is Borel and

$$\delta|\xi|^p \leq \tilde{G}_j(\xi) \leq G_j(\xi) \leq j(1 + |\xi|^p) \tag{3.2}$$

where $\tilde{G}_j$ denotes the $W^{1,\infty}$ quasiconvex envelope of G_j. It is obvious that $\int_\Omega \tilde{G}_j(V_k) \leq \int_\Omega \bar{F}_\delta(V_k)$ for all j. We use $\tilde{G}_j$, V_k to construct matrix fields W_j satisfying (1.2) (with V_k instead of V) and such that $\lim_{j\to\infty} \int_\Omega F_\delta(W_j) \leq \int_\Omega \bar{F}_\delta(V_k)$. In view of the definitions of V_k, $\bar{\mathrm{I}}$ and S this will finish the proof.

Now $W \mapsto \int_\Omega \tilde{G}_j(W)$ is continuous on L^p for each j, and we may therefore take $M_j \in L^p$, such that M_j is constant on each open dyadic cube T that intersects Ω and corresponds to some lattice $(2^{-m_j}\mathbb{Z})^n$, where $m_j \in \mathbb{Z}^+$, $\|V_k - M_j\|_{L^p} \leq 1/j$ and

$$\int_\Omega \tilde{G}_j(M_j) < \int_\Omega \bar{F}_\delta(V_k) + \varepsilon_j \quad (\varepsilon_j \searrow 0 \text{ as } j \nearrow \infty).$$

Let $\mathcal{T}_j$ denote this finite family of open dyadic cubes. By virtue of Lemma 2.1 we may for each $T \in \mathcal{T}_j$ select $\varphi_T \in W_0^{1,\infty}(T, \mathbb{R}^N)$, such that $\|\varphi_T\|_{L^\infty} \leq 1/j$ and

$$\varepsilon_j \frac{\mathcal{L}^n(T \cap \Omega)}{\mathcal{L}^n(\Omega)} + \int_{T\cap\Omega} \tilde{G}_j(M_j) > \int_{T\cap\Omega} G_j(M_j + \nabla\varphi_T(y))\,\mathrm{d}y.$$

Define

$$\varphi_j(x) = \begin{cases} \varphi_T(x) & \text{if } x \in T \cap \Omega,\ T \in \mathcal{T}_j, \\ 0 & \text{else.} \end{cases}$$

Then $\varphi_j \in W_0^{1,\infty}(\Omega, \mathbb{R}^N)$, $\|\varphi_j\|_{L^\infty} \leq 1/j$ and

$$\int_\Omega G_j(M_j + \nabla\varphi_j) < \int_\Omega \bar{F}_\delta(V_k) + 2\varepsilon_j.$$

Because $F_\delta(\xi) \geq \delta|\xi|^p$ it follows that $\{\varphi_j\}$ is bounded in $W_0^{1,p}(\Omega, \mathbb{R}^N)$, and hence that $\varphi_j \rightharpoonup 0$ in $W_0^{1,p}$. Put $E_j = \{x \in \Omega : \, F_\delta(M_j(x) + \nabla\varphi_j(x)) > j(1 + |M_j(x) + \nabla\varphi_j(x)|^p)\}$. Then

$$\int_{\Omega \setminus E_j} F_\delta(M_j + \nabla\varphi_j) < \int_\Omega \bar{F}_\delta(V) + 2\varepsilon_j$$

and $\int_{E_j}(1 + |M_j + \nabla\varphi_j|^p) \to 0$. Put $V_j = (M_j + \nabla\varphi_j)1_{\Omega\setminus E_j} + \xi_0 1_{E_j}$, where ξ_0 is chosen so $F(\xi_0) < \infty$. Then $V_j \rightharpoonup V$ in L^p, $\operatorname{curl} V_j \to \operatorname{curl} V$ in $W^{-1,p}$ on Ω and

$$\int_\Omega \bar{F}_\delta(V_k) \geq \liminf_{j\to\infty} \int_{\Omega\setminus E_j} F_\delta(M_j + \nabla\varphi_j) =$$

$$\liminf_{j\to\infty} \left(\int_\Omega F_\delta(V_j) - F_\delta(\xi_0)\mathcal{L}^n(E_j) \right) \geq \bar{\mathrm{I}}\,[V_k].$$

□

The proof of Corollary 1.2 is an easy variation of the above proof, and we leave it to the interested reader to check it. The same applies to the ‘only if’ part of Corollary 1.3. We record two further corollaries that follow by minor adaptations of the above proof.

Corollary 3.3. *Suppose that F satisifes the hypotheses of Corollary 1.2. Then F is closed $W^{1,p}$ quasiconvex if and only if there exists a sequence $\{F_j\}$ of $W^{1,\infty}$ quasiconvex functions $F_j \colon \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfying $F_j(\xi) \leq j(1 + |\xi|^p)$, such that $F_j(\xi) \nearrow F(\xi)$ as $j \nearrow \infty$ pointwise in ξ.*

Note that we can take F_j to be the $W^{1,\infty}$ quasiconvex envelope of the function $\xi \mapsto \min\{F(\xi), j(1 + |\xi|^p)\}$.

Corollary 3.4. *Let $F \colon \mathbb{R}^{N\times n} \to [0, \infty]$ be a Borel function. Then F is closed $W^{1,p}$ quasiconvex if and only if*

$$\int_B F(\nabla u)$$

is sequentially weakly lower semicontinuous on $W^{1,p}(\Omega, \mathbb{R}^N)$ for all measurable subsets $B \subseteq \Omega$. (It suffices for the ‘if’ part of the statement to take compact sets B.)

4 Proofs of Theorem 1.5 and its Corollaries

We start by recalling some elementary facts about precise representatives. For a real-valued $f \in L^1_{\mathrm{loc}}(\Omega)$ its precise representative is defined for every $x \in \Omega$ as $f^*(x) = \limsup_{r\searrow 0} \fint_{\mathrm{B}(x,r)} f(y)\,\mathrm{d}y$. Hereby f^* is an extended real-valued Borel function, and if we adopt the convention $(+\infty) - (+\infty) = \infty - \infty = \infty$, then $(f + g)^*(x) \leq f^*(x) + g^*(x)$ holds for all $x \in \Omega$ and for all f, $g \in L^1_{\mathrm{loc}}(\Omega)$. Of course, the two sides are equal for almost all x by virtue of Lebesgue’s differentiation theorem. The precise representative of a map $u \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ or a matrix field $V \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^{N\times n})$ is defined component-wise.

Let $\mathrm{BB}(0,1)$ be the open unit ball in $\mathbb{R}^n$. The Sobolev space $W^{1,p}(\partial\mathrm{B})$ can be defined as the completion of $C^1(\partial\mathrm{B})$ in the norm

$$\|f\|_{W^{1,p}(\partial\mathrm{B})} = \left(\int_{\partial\mathrm{B}} (|f|^p + |\nabla_{\tan} f|^p)\right)^{\frac{1}{p}},$$

where $\nabla_{\tan} f$ is the gradient of $f \colon \partial\mathrm{B} \to \mathbb{R}$. Recall that if $g \colon \mathbb{R}^n \to \mathbb{R}$ is any C^1 extension of f, then $\nabla_{\tan} f(x) = \pi_x(\nabla g(x))$ for each $x \in \partial\mathrm{B}$, where π_x is orthogonal projection of $\mathbb{R}^n$ onto the tangent space of $\partial\mathrm{B}$ at x. In particular, $|\nabla_{\tan} f(x)| \le |\nabla g(x)|$ holds for all $x \in \partial\mathrm{B}$.

For notational convenience we write B_r for the open ball $\mathrm{B}(0,r)$ in the sequel.

Lemma 4.1. *Let* $p \in (1,\infty)$ *and* $f \in W^{1,p}(\mathrm{B})$. *Then for almost all* $r \in (0,1)$ *the pointwise restriction of the precise representative* $f^*|_{\partial\mathrm{B}_r}$ *coincides with the functional analytic two-sided trace* $\mathrm{Tr}[f]$ *on* $\partial\mathrm{B}_r$ *and belongs to* $W^{1,p}(\partial\mathrm{B}_r)$. *Furthermore, for such* r,

$$|\nabla_{\tan}(f^*|_{\partial\mathrm{B}_r})(x)| \le |(\nabla f)^*(x)| \quad \textit{for a.e. } x \in \partial\mathrm{B}_r,$$

where $\nabla_{\tan}(f^*|_{\partial\mathrm{B}_r})$ *denotes the weak gradient of* $f^*|_{\partial\mathrm{B}_r}$.

Proof. The proof is standard, but we outline it here for the convenience of the reader. Put $f_j(x) = \fint_{\mathrm{B}(x,1/j)} f$ for $|x| < 1 - 1/j$. Then f_j is C^1 on $\mathrm{B}_{1-1/j}$, $f_j(x) \to f^*(x), \nabla f_j(x) \to (\nabla f)^*(x)$ pointwise in a.e. $x \in \mathrm{B}$ as $j \to \infty$ and $f_j \to f$ in $W^{1,p}_{\mathrm{loc}}(\mathrm{B})$ as $j \to \infty$. It follows by use of Fubini's theorem that for a.e. $r \in (0,1)$

$$f_j(x) \to f^*(x), \quad \nabla f_j(x) \to (\nabla f)^*(x) \text{ pointwise in a.e. } x \in \partial\mathrm{B}_r \tag{4.1}$$

as $j \to \infty$. Let $h_j = |f^* - f_j| + |(\nabla f)^* - \nabla f_j|$ on $\mathrm{B}_{1-1/j}$. Then $h_j(x) \le |f^*(x)| + |(\nabla f)^*(x)| + \sup_{0<t<1-|x|} \fint_{\mathrm{B}(x,t)}(|f| + |\nabla f|)$, and the latter is of class L^p on B by the local version of the Hardy–Littlewood-Wiener maximal inequality (see e.g. [2]). Hence we can use the dominated convergence theorem and Fubini's theorem to show that for a.e. $r \in (0,1)$

$$f_j \to f^*, \quad \nabla f_j \to (\nabla f)^* \quad \text{in } L^p(\partial\mathrm{B}_r) \tag{4.2}$$

as $j \to \infty$. (A simpler argument yields (4.2) for a subsequence.) Note that (4.1), (4.2) hold for a.e. $r \in (0,1)$. Fix such an r, and infer from (4.2) that $\{f_j|_{\partial\mathrm{B}_r}\}$ is a Cauchy sequence in $W^{1,p}(\partial\mathrm{B}_r)$. Hence $f^*|_{\partial\mathrm{B}_r}$ belongs to $W^{1,p}(\partial\mathrm{B}_r)$. By use of (4.1) it follows that $|\nabla_{\tan}(f^*|_{\partial\mathrm{B}_r})| \le |(\nabla f)^*|$ a.e. on $\partial\mathrm{B}_r$. To finish the proof we notice that according to the trace theorem (see [22] pp. 133-135, and note that the proof also works for two-sided traces), $f_j|_{\partial\mathrm{B}_r} = \mathrm{Tr}[f_j] \to \mathrm{Tr}[f]$ in $L^p(\partial\mathrm{B}_r)$, so that $f^*|_{\partial\mathrm{B}_r} = \mathrm{Tr}[f]$ in $L^p(\partial\mathrm{B}_r)$, where Tr denotes the two-sided trace operator $\mathrm{Tr} \colon W^{1,p}(\mathrm{B}) \to L^p(\partial\mathrm{B}_r)$. □

In the sequel we consider all maps and matrix fields in terms of their precise representatives, and we omit the asterisk from the notation. We proceed with two auxiliary results about extension and gluing of Sobolev maps, respectively.

The extension procedure goes as follows. Let $f \in L^1(\partial \mathrm{B})$ and

$$\mathrm{PI}[f](x) = c_n \int_{\partial \mathrm{B}} \frac{1-|x|^2}{|y-x|^n} f(y)\, \mathrm{d}\mathcal{H}^{n-1}(y)$$

be the Poisson integral of f, i.e., the harmonic extension of f to $\mathbb{R}^n \backslash \partial \mathrm{B}$. When $u = (u_1, \dots, u_N) \in L^1(\partial \mathrm{B}, \mathbb{R}^N)$ we put $\mathrm{E}[u] = (\mathrm{PI}[u_1], \dots, \mathrm{PI}[u_N])\rho$, where $\rho \colon \mathrm{B}_2 \to [0,1]$ is a C^1 cut-off function, such that $\rho = 1$ on B and $\rho = 0$ near $\partial \mathrm{B}_2$.

Lemma 4.2. *Let $p \in (1, \infty)$. Then $\mathrm{E} \colon W^{1,p}(\partial \mathrm{B}, \mathbb{R}^N) \to W_0^{1,\frac{np}{n-1}}(\mathrm{B}_2, \mathbb{R}^N)$ is a bounded linear operator. The operator is compact when considered as an operator into $W_0^{1,q}(\mathrm{B}_2, \mathbb{R}^N)$ with $q < np/(n-1)$.*

We omit the proof of this well–known result, and only remark that it makes use of standard properties of the Laplacian and the general embedding and trace theorems for fractional order Sobolev spaces.

Lemma 4.3. *Let $r \in [1, \infty]$ and suppose that $u \in W^{1,r}(\mathrm{B}, \mathbb{R}^N)$ and $v \in W^{1,r}(\mathrm{B}_2 \setminus \mathrm{B}, \mathbb{R}^N)$ have the same trace on $\partial \mathrm{B}$. Then the map*

$$w = \begin{cases} u & \text{on } \mathrm{B} \\ v & \text{on } \mathrm{B}_2 \setminus \overline{\mathrm{B}}, \end{cases}$$

belongs to $W^{1,r}(\mathrm{B}_2, \mathbb{R}^N)$.

This is well–known (see [22] for the ingredients of a proof). We turn to the proof of Theorem 1.5, and start by recording the following result, which was also observed in [51][Theorem 4.4]. Notice that it is the ‘if’ part of Corollary 1.7. (The ‘only if’ part of Corollary 1.7 is a consequence of [5], Th. 3.1. and holds without any growth condition on the integrand.)

Lemma 4.4. *Let $F \colon \mathbb{R}^{N \times n} \to \mathbb{R}$ be a $W^{1,p}$ quasiconvex function satisfying the growth condition* (1.7) *with exponents p, q satisfying* (1.8)*. Then*

$$\liminf_{j \to \infty} \int_\Omega F(\nabla u_j) \geq \int_\Omega F(\nabla u)$$

holds for all sequences $\{u_j\}$ such that $u_j \rightharpoonup u$ weakly in $W^{1,p}$.

Corollary 4.5. *Let $F \colon \mathbb{R}^{N \times n} \to \mathbb{R}$ be a Borel function satisfying the growth condition* (1.7) *with exponents p, q satisfying* (1.8)*. If for $\delta > 0$ we define $F_\delta(\xi) = F(\xi) + \delta|\xi|^p$, then*

$$\tilde{\mathrm{I}}\,[\nabla u] \geq \lim_{\delta \searrow 0} \int_\Omega \tilde{F}_\delta(\nabla u)$$

holds for all $W^{1,p}$ maps u, where $\tilde{\mathrm{I}}\,[\nabla u]$ is defined at (1.5) *and $\tilde{F}_\delta$ is the $W^{1,p}$ quasiconvex envelope of F_δ.*

We omit the straightforward proof, and only remark that it relies on the boundedness of weakly convergent sequences.

The rest of this section is devoted to proving the opposite inequality. We prove a little more, namely that the inequality holds also under trace constraints and a slightly less restrictive growth condition. First, for a Borel function $F\colon \mathbb{R}^{N\times n} \to [0,\infty]$ and a map $u \in W^{1,p}(\Omega,\mathbb{R}^N)$ define

$$\tilde{\mathrm{I}}_{\mathrm{tr}}[u] = \inf_{\{u_j\}} \left\{ \liminf_{j\to\infty} \int_\Omega F(\nabla u_j) \right\} \tag{4.3}$$

where the infimum is taken over all sequences $\{u_j\}$ in $u + W^{1,p}_0(\Omega,\mathbb{R}^N)$ such that $u_j \rightharpoonup u$ weakly in $W^{1,p}$.

Proposition 4.6. *Let $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ be a Borel function satisfying the growth condition* (1.7) *with $q = np/(n-1)$. Then with F_δ as defined above*

$$\tilde{\mathrm{I}}_{\mathrm{tr}}[u] \leq \lim_{\delta\searrow 0} \int_\Omega \tilde{F}_\delta(\nabla u)$$

holds for $W^{1,p}$ maps u, where $\tilde{F}_\delta$ denotes the $W^{1,p}$ quasiconvex envelope of F_δ.

The key ingredient in the proof of this result is Proposition 4.7 below about approximation in energy of $W^{1,p}$ maps by countably piecewise affine maps. As remarked in the Introduction this is also the main novelty in the proof of Theorem 1.5. It is stated in a slightly more general form than is strictly needed for the proof of Proposition 4.6. First recall that a map $v \in W^{1,p}(\Omega,\mathbb{R}^N)$ is called countably piecewise affine if there exists disjoint open sets $O_j \subset \Omega$, such that $\mathcal{L}^n(\Omega \setminus \bigcup_{j=1}^\infty O_j) = 0$ and the restrictions $v|_{O_j}$ are affine.

Proposition 4.7. *Let $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ be a Borel function satisfying the growth condition $|F(\xi)| \leq c(|\xi|^{\frac{np}{n-1}} + 1)$ for all $\xi \in \mathbb{R}^{N\times n}$, where $p \in (1,\infty)$ and $c \geq 1$. Assume that $u \in W^{1,p}(\Omega,\mathbb{R}^N)$ is such that $\int_\Omega |F(\nabla u)| < \infty$, and let $\varepsilon > 0$. Then there exists a countably piecewise affine map $v \in u + W^{1,p}_0(\Omega,\mathbb{R}^N)$ such that $\int_\Omega \bigl(|F(\nabla u) - F(\nabla v)| + |\nabla u - \nabla v|^p\bigr) < \varepsilon$.*

We postpone the proof of this result till after the proof of Proposition 4.6. As a first step towards the latter we establish the following formula for $W^{1,p}$ quasiconvex envelopes.

Lemma 4.8. *Let $G\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ be a Borel function satisfying for some δ, $c > 0$ and $p > 1$ the growth condition $\delta|\xi|^p \leq G(\xi) \leq c(|\xi|^{\frac{np}{n-1}} + 1)$. Then the $W^{1,p}$ quasiconvex envelope is given by the formula*

$$\tilde{G}(\xi) = \inf\left\{ \fint_\Omega G(\xi + \nabla\varphi(x))\,\mathrm{d}x : \varphi \in W^{1,p}_0(\Omega,\mathbb{R}^N) \right\}.$$

Proof. The argument is standard once we take Proposition 4.7 for granted. We outline it here for the convenience of the reader. Let

$$R_\Omega(\xi) = \inf\left\{ \fint_\Omega G(\xi + \nabla\varphi) : \varphi \in W^{1,p}_0(\Omega,\mathbb{R}^N) \right\},$$

and note that $\tilde{G}(\xi) \leq R_\Omega(\xi)$ for all ξ. To prove the opposite inequality it suffices to show that R_Ω is $W^{1,p}$ quasiconvex. First, observe that according to [23] Lemma 2.16 the definition of $R_\Omega(\xi)$ is independent of Ω, i.e., if Ω' is another bounded, open nonempty set, then $R_{\Omega'}(\xi) = R_\Omega(\xi)$. Henceforth we suppress Ω from the notation and write simply $R(\xi)$ instead of $R_\Omega(\xi)$. Next, R satisfies the same growth condition as G does, and is therefore especially finite everywhere, hence according to [23] Th. 2.17 it is rank-1 convex. In particular, R is Borel measurable and $\delta|\xi|^p \leq R(\xi) \leq c(|\xi|^{\frac{np}{n-1}} + 1)$ for all ξ. Let $\varphi \in W_0^{1,p}(\Omega, \mathbb{R}^N)$ and suppose that $\int_\Omega R(\xi + \nabla\varphi) < \infty$. For $\varepsilon > 0$ we refer to Proposition 4.7 and find a countably piecewise affine map $\psi \in W_0^{1,p}(\Omega, \mathbb{R}^N)$ such that $\int_\Omega R(\xi + \nabla\varphi) \geq -\varepsilon + \int_\Omega R(\xi + \nabla\psi)$. Denote by $\{O_j\}_{j\in J}$ the partition of Ω into open sets where ψ is affine. Then using the definition of R (with the sets O_j instead of Ω) we find for each $j \in J$ a map $\psi_j \in W_0^{1,p}(O_j, \mathbb{R}^N)$ such that on O_j: $\mathcal{L}^n(O_j)R(\xi + \nabla\psi) + \varepsilon\mathcal{L}^n(O_j)/\mathcal{L}^n(\Omega) > \int_{O_j} G(\xi + \nabla\psi + \nabla\psi_j)$. Extend each ψ_j by 0 outside O_j and define $\phi = \psi + \sum_{j\in J} \psi_j$. Clearly, ϕ is a well-defined map pointwise a.e., and by an induction argument using Lemma 4.3 we infer that $\sum_{j\in J: j\leq k} \psi_j$ is of class $W_0^{1,1}$ on Ω for each integer $k \in \mathbb{N}$. By the coercivity of G

$$\delta \int_{O_j} |\nabla\psi_j|^p \leq \Big(R(\xi + \psi) + \varepsilon/\mathcal{L}^n(\Omega)\Big)\mathcal{L}^n(O_j) \quad \text{on } O_j, \tag{4.4}$$

hence $\delta \int_{O_j} |\nabla\psi_j| \leq (R(\xi + \psi) + \delta + \varepsilon/\mathcal{L}^n(\Omega))\mathcal{L}^n(O_j)$ on O_j, and therefore

$$\delta \sum_{j\in J} \|\nabla\psi_j\|_{L^1} \leq 2\varepsilon + \delta\mathcal{L}^n(\Omega) + \int_\Omega R(\xi + \nabla\varphi) < \infty.$$

Since $W_0^{1,1}$ is complete it follows that $\phi = \psi + \sum_{j\in J} \psi_j$ is of class $W_0^{1,1}$ on Ω. Next, use (4.4) to infer that ϕ is of class $W_0^{1,p}$ on Ω, and consequently that $\int_\Omega R(\xi + \nabla\varphi) \geq -2\varepsilon + \int_\Omega G(\xi + \nabla\phi) \geq -2\varepsilon + R(\xi)\mathcal{L}^n(\Omega)$, where the last inequality follows from the definition of $R(\xi)$. Hence R is $W^{1,p}$ quasiconvex at ξ. □

Remark 4.9. *Let* $F\colon \mathbb{R}^{N\times n} \to [0, \infty]$ *be a Borel function and*

$$R(\xi) = \inf \left\{ ⨍_\Omega F(\xi + \nabla\varphi(x))\, dx : \varphi \in W_0^{1,p}(\Omega, \mathbb{R}^N) \right\}.$$

Then for each matrix ξ *there exists a sequence* $\{\varphi_k\}$ *in* $W_0^{1,p}(\Omega, \mathbb{R}^N)$ *such that* $\varphi_k \to 0$ *strongly in* L^p *and* $⨍_\Omega F(\xi + \nabla\varphi_k) \to R(\xi)$.

Proof of Remark 4.9. First, observe that the definition of R is independent of the specific set Ω (see [23] Lemma 2.16). Next, for positive integers k take $\psi_k \in W_0^{1,p}(\mathrm{B}, \mathbb{R}^N)$ such that $⨍_\mathrm{B} F(\xi + \nabla\psi_k) < R(\xi) + 1/k$, where $\mathrm{B} = \mathrm{B}(0, 1)$. Fix $R_k > 0$ so $R_k\|\psi_k\|_{L^p}\mathcal{L}^n(\Omega)/\mathcal{L}^n(\mathrm{B}) < 1/k$ and use an exhaustion argument to find disjoint balls $\mathrm{B}(x_j, r_j) \subset \Omega$ of radii $r_j = r_j(k) < R_k$ ($j \in J_k$) such that $\mathcal{L}^n(\Omega \setminus \bigcup_{j\in J_k} \mathrm{B}(x_j, r_j)) = 0$. If we define

$$\varphi_k(x) = \begin{cases} r_j\psi_k(\frac{x-x_j}{r_j}) & \text{if } x \in \mathrm{B}(x_j, r_j),\ j \in J_k, \\ 0 & \text{else,} \end{cases}$$

then $\varphi_k \in W_0^{1,p}(\Omega,\mathbb{R}^N)$, $\|\varphi_k\|_{L^p} < 1/k$ and $⨍_\Omega F(\xi+\nabla\varphi_k) = ⨍_{\mathrm{B}} F(\xi+\nabla\varphi_k) < R(\xi)+1/k$. This concludes the proof of the remark. $\square$

Proof of Proposition 4.6. Fix $\delta > 0$. The goal is to prove that

$$\tilde{\mathrm{I}}_{\mathrm{tr}}[u] \le \int_\Omega \tilde{F}_\delta(\nabla u) \tag{4.5}$$

holds for all maps $u \in W^{1,p}$. Clearly, it suffices to consider $W^{1,p}$ maps u with $\int_\Omega \tilde{F}_\delta(\nabla u) < \infty$. By virtue of Proposition 4.7 we can then find a sequence $\{v_j\}$ of countably piecewise affine maps verifying $v_j \in u + W_0^{1,p}(\Omega,\mathbb{R}^N)$, $v_j \to u$ strongly in $W^{1,p}$ and $\tilde{F}_\delta(\nabla v_j) \to \tilde{F}_\delta(\nabla u)$ strongly in L^1. Since F_δ satisfies the assumptions of Lemma 4.8 we can for each of the affine pieces of v_j, say $O_k = O_k(j) \subset \Omega$, $k \in K = K(j) \subseteq \mathbb{N}$, find maps $\varphi_k \in W_0^{1,p}(O_k,\mathbb{R}^N)$ such that $\int_{O_k} |\varphi_k|^p < \mathcal{L}^n(O_k)/j$ and $⨍_{O_k} F_\delta(\nabla v_j + \nabla\varphi_k) < \tilde{F}_\delta(\nabla v_j) + 1/j$. Extend φ_k by 0 outside O_k and define $u_j = v_j + \sum_{k\in K}\varphi_k$. Then using the coercivity of F_δ as in the proof of Lemma 4.8 we infer that $u_j \in u + W_0^{1,p}$ and $u_j \rightharpoonup u$ weakly in $W^{1,p}$. By construction,

$$\limsup_{j\to\infty} \int_\Omega F_\delta(\nabla u_j) \le \int_\Omega \tilde{F}_\delta(\nabla u),$$

which in view of the definition (4.3) of $\tilde{\mathrm{I}}_{\mathrm{tr}}[u]$ establishes (4.5). $\square$

It is not difficult to deduce Corollary 1.6 from Theorem 1.5. We leave it to the interested reader and turn instead to the proof of Proposition 4.7. Two auxiliary results will be needed. The first is a result about approximation in $W^{1,q}$ by countably piecewise affine maps with fixed trace. It is stated in a slightly more general form than is needed for the proof of Proposition 4.7. (We only need the case where Ω is the open set between two concentric balls.)

Lemma 4.10. *Let $u \in W^{1,q}(\Omega,\mathbb{R}^N)$ and $\varepsilon > 0$. There exists a countably piecewise affine map $v \in u + W_0^{1,q}(\Omega,\mathbb{R}^N)$, such that $\int_\Omega (|u-v|^q + |\nabla u - \nabla v|^q) < \varepsilon$.*

The result is well-known to experts and we merely outline the proof.

Proof. According to [10], Ch. 2, Th. 1, we can assume that u is of class $W^{1,q} \cap C^\infty$ on Ω. To define the map v we consider a net $\mathcal{N}$ of n-simplexes S contained in Ω refining towards the boundary of Ω, say with diameters $\operatorname{diam} S \le \theta(\operatorname{dist}(S,\partial\Omega))$, where $\theta\colon (0,\infty) \to (0,\infty)$ is an increasing function that will be specified below. Let $v\colon \Omega \to \mathbb{R}^N$ be the countably piecewise affine map which coincides with u at the vertices of the net. Put $\omega(t) = \sup\{|\nabla^2 u(x)| : x \in \Omega,\ \operatorname{dist}(x,\partial\Omega) \ge t\}$ (with the convention $\sup\emptyset = 0$) and for $\bar\varepsilon > 0$

$$\theta(t) = \bar\varepsilon \frac{\min\{1,t\}}{1+\omega(t)}.$$

For each n-simplex $S \in \mathcal{N}$ we estimate by Taylor's formula and the definitions, $\operatorname{osc}(\nabla u, S) \le \max_{\overline{S}} |\nabla^2 u| \operatorname{diam} S \le \omega(\operatorname{dist}(S,\partial\Omega))\theta(\operatorname{dist}(S,\partial\Omega))$, so that $|\nabla u - \nabla v| \le \bar\varepsilon \operatorname{dist}(S,\partial\Omega)$ on S. It follows in particular that $\int_\Omega |\nabla u - \nabla v|^q \le \bar\varepsilon^q \mathcal{L}^n(\Omega)$. Since also $\max_{\overline{S}} |u-v| \le \bar\varepsilon \operatorname{dist}(S,\partial\Omega)$ it is not hard to show that $u - v \in W_0^{1,q}$ on Ω. The thesis follows by an appropriate choice of $\bar\varepsilon$. $\square$

The second result builds on Lemma 4.10, and forms together with a covering argument the ingredients of the proof of Proposition 4.7.

Lemma 4.11. *There exists a constant $C = C(n, N, p)$ (depending on n, N and p only) with the following property. For $u \in W^{1,p}(\mathrm{B}_2, \mathbb{R}^N)$ and $\delta \in (1, 2)$ there exist $v \in W^{1,p}(\mathrm{B}_2, \mathbb{R}^N)$ and $r \in (\delta, 2)$ such that $v = 0$ in B, v is generalized piecewise affine in $\mathrm{B}_r \setminus \overline{\mathrm{B}}$, $v = u$ in $\mathrm{B}_2 \setminus \overline{\mathrm{B}}_r$ and*

$$\int_{\mathrm{B}_r \setminus \mathrm{B}} |\nabla v|^{\frac{np}{n-1}} \leq C \left(\frac{1}{(\delta-1)^p (2-\delta)} \int_{\mathrm{B}_2} (|u|^p + |\nabla u|^p) \right)^{\frac{n}{n-1}} .$$

Proof. In view of Lemma 4.1, $u|_{\partial \mathrm{B}_r} \in W^{1,p}(\partial \mathrm{B}_r, \mathbb{R}^N)$ for a.e. r, and since

$$\int_\delta^2 \int_{\partial \mathrm{B}_r} (|u|^p + |\nabla u|^p) \leq \int_{\mathrm{B}_2} (|u|^p + |\nabla u|^p),$$

we can find such an $r \in [\delta, 2]$ with the additional property that

$$\int_{\partial \mathrm{B}_r} (|u|^p + |\nabla u|^p) \leq \frac{1}{2-\delta} \int_{\mathrm{B}_2} (|u|^p + |\nabla u|^p).$$

Define $u_r(y) = \frac{1}{r} u(ry)$ and $w(y) = \mathrm{E}(u_r|_{\partial \mathrm{B}})(y)$, where E is the extension operator from Lemma 4.2. Then $w \in W_0^{1,\frac{np}{n-1}}(\mathrm{B}_2, \mathbb{R}^N)$ and

$$\int_{\mathrm{B}_2} |\nabla w|^{\frac{np}{n-1}} \leq c_1 \left(\int_{\partial \mathrm{B}} (|u_r|^p + |\nabla u_r|^p) \right)^{\frac{n}{n-1}} \leq$$

$$c_1 \left(\int_{\partial \mathrm{B}_r} (|u|^p + |\nabla u|^p) \right)^{\frac{np}{n-1}} \leq c_1 \left(\frac{1}{2-\delta} \int_{\mathrm{B}_2} (|u|^p + |\nabla u|^p) \right)^{\frac{n}{n-1}},$$

where the third and fourth inequalities rely on that $r^{-1} \leq \delta^{-1} < 1$. Let ρ be a Lipschitz function satisfying $1_{\mathrm{B}_1} \leq \rho \leq 1_{\mathrm{B}_r}$ and $\|\nabla \rho\|_{L^\infty} \leq 1/(\delta - 1)$. Define

$$v_0(x) = \begin{cases} u(x) & \text{in } \mathrm{B}_2 \setminus \overline{\mathrm{B}}_r, \\ (1-\rho(x)) r w(\frac{x}{r}) & \text{in } \mathrm{B}_r. \end{cases}$$

Then $v_0 \in W^{1,p}(\mathrm{B}_2, \mathbb{R}^N)$ by Lemma 4.3, $v_0 = 0$ in B and

$$\int_{\mathrm{B}_r} |\nabla v_0|^{\frac{np}{n-1}} \leq c_2 \int_{\mathrm{B}_r \setminus \mathrm{B}} \left(|\nabla w|^{\frac{np}{n-1}} + \left(\frac{|rw(\frac{x}{r}) - b|}{\delta - 1} \right)^{\frac{np}{n-1}} \right) \mathrm{d}x,$$

where $b \in \mathbb{R}^N$ is any vector. If we take $b = \fint_{\mathrm{B}_r \setminus \mathrm{B}} rw(x/r)\, \mathrm{d}x$, use Poincaré's inequality and change coordinates we arrive at

$$\int_{\mathrm{B}_r \setminus \mathrm{B}} |\nabla v_0|^{\frac{np}{n-1}} \leq C \left(\frac{1}{(\delta-1)^p (2-\delta)} \int_{\mathrm{B}_2} (|u|^p + |\nabla u|^p) \right)^{\frac{n}{n-1}},$$

where $C = C(n, N, p)$. To finish the proof note that v_0 is of class $W^{1,\frac{np}{n-1}}$ on $\mathrm{B}_r \setminus \overline{\mathrm{B}}$. By virtue of Lemma 4.10 we can find a countably piecewise affine map $\varphi \in v_0 + W_0^{1,q}(\mathrm{B}_r \setminus \overline{\mathrm{B}})$, such that

$$\int_{\mathrm{B}_r \setminus \mathrm{B}} |\nabla\varphi|^{\frac{np}{n-1}} \leq 2 \int_{\mathrm{B}_r \setminus \mathrm{B}} |\nabla v_0|^{\frac{np}{n-1}}.$$

To finish the proof refer to Lemma 4.3 and define $v = \varphi$ on $\mathrm{B}_r \setminus \overline{\mathrm{B}}$ and $v = v_0$ elsewhere in B_2. □

Proof of Proposition 4.7. We use Lemma 4.11 (rescaled and translated) in connection with Vitali's covering theorem. Let $\bar{\varepsilon} > 0$ and $\delta \in (1, 2)$. We will make a specific choice later. Denote by $\mathcal{V}$ the family of open balls $\mathrm{B}(x_0, r_0)$ with the following properties: x_0 is a Lebesgue point for u and for ∇u, $\mathrm{B}(x_0, 2r_0) \subset \Omega$,

$$\fint_{\mathrm{B}(x_0,2r_0)} |\nabla u(x) - \nabla u(x_0)|^p \,\mathrm{d}x < \bar{\varepsilon},$$

$$\frac{1}{(2r_0)^p} \fint_{\mathrm{B}(x_0,2r_0)} |u(x) - u(x_0) - \nabla u(x_0)(x - x_0)|^p \,\mathrm{d}x < \bar{\varepsilon},$$

$$\fint_{\mathrm{B}(x_0,2r_0)} |F(\nabla u(x)) - F(\nabla u(x_0))| \,\mathrm{d}x < \bar{\varepsilon},$$

and there exist $v_0 \in u + W_0^{1,p}(\mathrm{B}(x_0, r_0), \mathbb{R}^N)$ and $r \in [r_0/\delta, r_0]$ such that $v_0(x) = u(x_0) + \nabla u(x_0)(x - x_0)$ a.e. on $\mathrm{B}(x_0, r)$, v_0 is countably piecewise affine on $\mathrm{B}(x_0, r_0) \setminus \mathrm{B}[x_0, r]$ and

$$r_0^{-n} \int_{\mathrm{B}(x_0,r_0) \setminus \mathrm{B}(x_0,r)} |\nabla v_0|^{\frac{np}{n-1}} \leq C \left(\frac{\bar{\varepsilon}}{(\delta - 1)^p (2 - \delta)} \right)^{\frac{n}{n-1}},$$

where $C = C(n, N, p)$ is the constant from Lemma 4.11. By standard properties of Sobolev maps and Lemma 4.11 the family $\mathcal{V}$ is fine at almost all points of Ω. Hence by Vitali's covering theorem we can find disjoint balls $\mathrm{B}_j \in \mathcal{V}$, $j \in J$, such that $\mathcal{L}^n(\Omega \setminus \bigcup_{j \in J} \mathrm{B}_j) = 0$. For each $j \in J$ we denote by v_j the map corresponding to $\mathrm{B}_j = \mathrm{B}(x_j, R_j)$, i.e., $v_j \in u + W_0^{1,p}(\mathrm{B}_j, \mathbb{R}^N)$, $v_j(x) = u(x_j) + \nabla u(x_j)(x - x_j)$ a.e. on $r_j \mathrm{B}_j = \mathrm{B}(x_j, r_j R_j)$ for some $r_j \in [1/\delta, 1]$, v_j is countably piecewise affine on $\mathrm{B}_j \setminus (r_j \overline{\mathrm{B}}_j)$ and

$$\int_{\mathrm{B}_j \setminus (r_j \mathrm{B}_j)} |\nabla v_j|^{\frac{np}{n-1}} \leq C \left(\frac{\bar{\varepsilon}}{(\delta - 1)^p (2 - \delta)} \right)^{\frac{n}{n-1}} R_j^n.$$

Extend each map $v_j - u$ by 0 outside B_j and define $v = u + \sum_{j \in J} (v_j - u)$. As in the proof of Lemma 4.8 it is not hard to show that $v - u$ is of class $W_0^{1,p}$ on Ω. By construction v is countably piecewise affine, and using the integral bounds and properties of v_j we find for some constant $c_3 = c_3(n, N, p)$:

$$\int_{\Omega} \Bigl(|F(\nabla u) - F(\nabla v)| + |\nabla u - \nabla v|^p \Bigr) =$$

$$\sum_{j\in J}\int_{\mathrm{B}_j}\Big(|F(\nabla u)-F(\nabla v_j)|+|\nabla u-\nabla v_j|^p\Big)\leq$$

$$c_3\Big(\bar{\varepsilon}+\Big(\frac{\bar{\varepsilon}}{(\delta-1)^p(2-\delta)}\Big)^{\frac{n}{n-1}}\Big)\mathcal{L}^n(\Omega)+\int_{\bigcup_{j\in J}(\mathrm{B}_j\setminus r_j\mathrm{B}_j)}\Big(F(\nabla u)+c_3(1+|\nabla u|^p)\Big).$$

Now $\mathcal{L}^n(\bigcup_{j\in J}(\mathrm{B}_j\setminus r_j\mathrm{B}_j))\leq(1-\delta^{-n})\mathcal{L}^n(\Omega)$ and if therefore for a given $\varepsilon>0$ we first choose a $\delta\in(1,2)$ close to 1, and next a sufficiently small $\bar{\varepsilon}>0$ we can obtain that $\int_\Omega\big(|F(\nabla u)-F(\nabla v)|+|\nabla u-\nabla v|^p\big)<\varepsilon$. □

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Mathematical Institute, University of Oxford, 24–29 St. Giles’, Oxford OX1 4AU, England
e-mail: *kristens@maths.ox.ac.uk*