Scattering Amplitudes and Holomorphic Linking

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based on:

1009.2225, L. Mason & D.S.
1101.1329, M. Bullimore & D.S.

Introduction

Given a (trivial) G-bundle over an oriented 3-manifold M we have the **Chern-Simons functional**

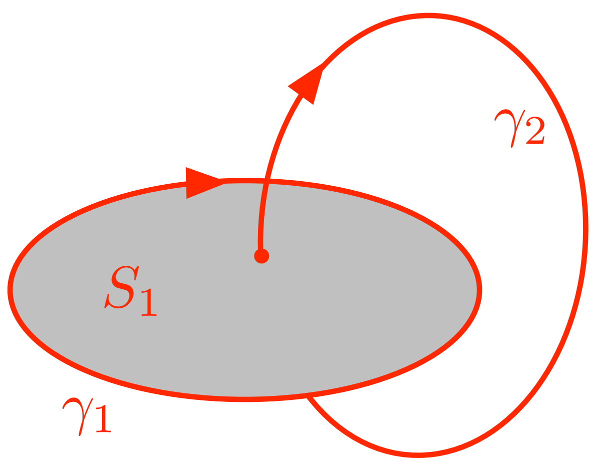
$$S = \frac{k}{4\pi}\int_M \mathrm{Tr}\left(A\mathrm{d}A + \frac{2}{3}A^3\right)$$

If $\gamma_i \subset M$ are closed oriented curves and R_i are representations of G, then the correlation functions

$$Z(M;\gamma_i,R_i) \equiv \int \mathrm{D}A\, \prod_i W_{R_i}(\gamma_i)\,\mathrm{e}^{\mathrm{i}S} \qquad W_R[\gamma] = \mathrm{Tr}_R\,\mathrm{P}\exp\left(-\oint_\gamma A\right)$$

famously[Witten] compute link invariants such as the Jones and HOMFLY polynomials.

e.g.



$$\mathrm{lk}(\gamma_1,\gamma_2) = \int_{\gamma_1\times\gamma_2} \langle A(x)A(y)\rangle$$

$$= \int_{S_1\times\gamma_2} \mathrm{d}\langle A(x)A(y)\rangle = \sum_{x\in S_1\cap\gamma_2} \pm 1$$

It has long been the twistor theorist's ambition[Penrose; Atiyah] that by trading topological invariance for holomorphic invariance in twistor space, one would be able to use linking ideas to encode the dynamics of interacting QFTs purely in terms of twistor geometry. **This is what we will investigate today.**

Holomorphic Chern-Simons & holomorphic linking

The natural complex analogue of an oriented 3-manifold is a Calabi-Yau threefold X, where we can define the **holomorphic Chern-Simons functional**[Witten; Donaldson, Thomas; ...]

$$S[\mathcal{A}] = \int_X \Omega \wedge \mathrm{Tr}\left(\mathcal{A}\,\bar{\partial}\mathcal{A} + \frac{2}{3}\mathcal{A}^3\right)$$

where $\mathcal{A}$ is a connection (0,1)-form on a complex bundle $E \to X$ (assumed trivial).

- The eom say $\mathcal{F}^{(0,2)} = 0$, so that E becomes holomorphic on-shell.
- In place of the real Wilson Loop, in the ***Abelian*** case we pick an algebraic curve $C \subset X$ with $g \geq 1$, together with $\omega \in H^0(C, K_C)$ and define

$$W[C;\omega] \equiv \exp \int_C \omega \wedge \mathcal{A} \qquad \text{(gauge invariant since } \bar{\partial}\omega = 0\text{)}.$$

Expectation values of Abelian holomorphic Wilson Loops are then given in terms of the **holomorphic linking**[Frenkel, Todorov; Thomas; Khesin, Rosly]

$$\mathrm{hlk}(\{C_i, \omega_i\}) \equiv \int_{C_1 \times C_2} \omega_1 \wedge \omega_2 \wedge \langle \mathcal{A}_1 \wedge \mathcal{A}_2 \rangle = \sum_{z \in S_1 \cap C_2} \left.\frac{\mu_1 \wedge \omega_2}{\Omega}\right|_z$$

The holomorphic linking depends holomorphically on the curves, with poles where they intersect. Frenkel & Todorov interpret this homologically in terms of Serre classes[Atiyah] and analytically in terms of a complexified version of the Gauss integral[Penrose]

Twistors and $\mathcal{N} = 4$ SYM

While twistor space is not CY, with $\mathcal{N} = 4$ supersymmetry it becomes a CY supermanifold

$$\mathbb{CP}^{3|4} := \frac{\mathbb{C}^{4|4} - \mathbb{C}^{0|4}}{\mathbb{C}^*}$$

- Homogeneous coordinates (Z^a, ψ^A)
- $\Omega \equiv Z\mathrm{d}^3 Z\, \mathrm{d}^4\psi$ a nowhere vanishing holomorphic section of Berezinian

We promote $\mathcal{A}$ to a twistor superfield[Ferber; Pilato]

$$\mathcal{A}(Z,\psi) = a(Z) + \psi^A \Gamma_A(Z) + \frac{1}{2}\psi^A\psi^B\phi_{AB}(Z) + \frac{\epsilon_{ABCD}}{3!}\psi^A\psi^B\psi^c\,\tilde{\Gamma}^D(Z) + \frac{\epsilon_{ABCD}}{4!}\psi^A\psi^B\psi^C\psi^D\, g(Z)$$

and consider same holomorphic Chern-Simons action as before[Witten]

$$\boxed{\begin{aligned} S[\mathcal{A}] &= \int_{\mathbb{CP}^{3|4}} \Omega \wedge \mathrm{Tr}\left(\mathcal{A}\bar{\partial}\mathcal{A} + \frac{2}{3}\mathcal{A}^3\right) \\ &= \int_{\mathbb{CP}^3} Z\mathrm{d}^3 Z \wedge \mathrm{Tr}\left(g \wedge \mathcal{F}_a^{(0,2)} + \tilde{\Gamma}^A \wedge \bar{D}\Gamma_A + \frac{\epsilon^{ABCD}}{4}\phi_{AB}\wedge\bar{D}\phi_{CD} + \frac{\epsilon^{ABCD}}{2}\Gamma_A\Gamma_B\phi_{CD}\right) \end{aligned}}$$

- Linearised eom $\bar{\partial}\mathcal{A} = 0$ mod gauge give $\mathcal{N} = 4$ multiplet with helicities $\left(-1^1, -\tfrac{1}{2}^4, 0^6, \tfrac{1}{2}^4, 1^1\right)$
- Non-linearly, have asd background together with susy completion (1/2 BPS background).

Which curve?

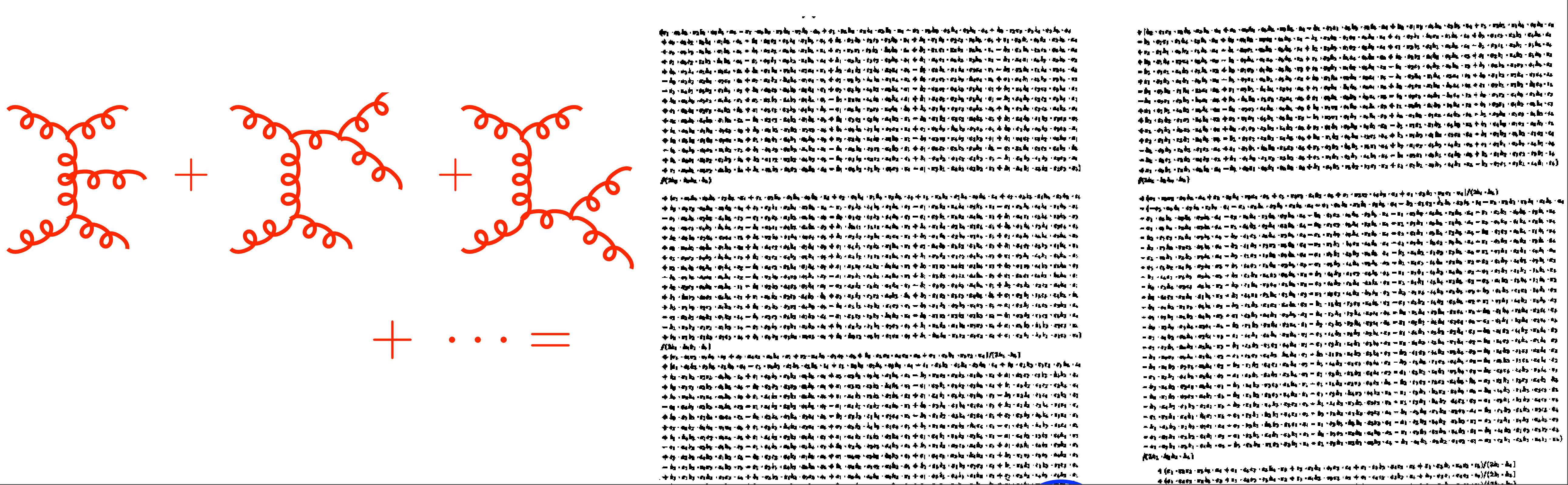
We will actually make a rather degenerate choice, motivated by a conjecture of Alday & Maldacena involving **scattering amplitudes**.

- Scattering amplitudes are among the most important objects in a QFT, both encoding its dynamics and being closely related to objects that experimentalists actually measure.

- In the idealized case that the scattering process involves some definite number $n = n_{\mathrm{in}} + n_{\mathrm{out}}$ of particles, each with definite momentum p_i, the amplitudes are subject to the constraints

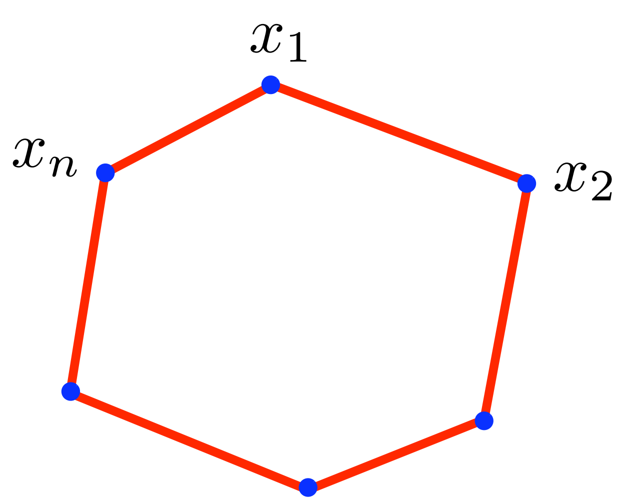
$$\boxed{p_i^2 = m_i^2 \quad \text{and} \quad \sum_i p_i = 0}$$

- Even for small numbers of particles, Feynman diagrams rapidly become horribly complicated



Which curve?

Given an ordering of the external states, the momentum conservation constraint provides us with a closed polygon in momentum space. If the external states are massless, the polygon is piecewise null:



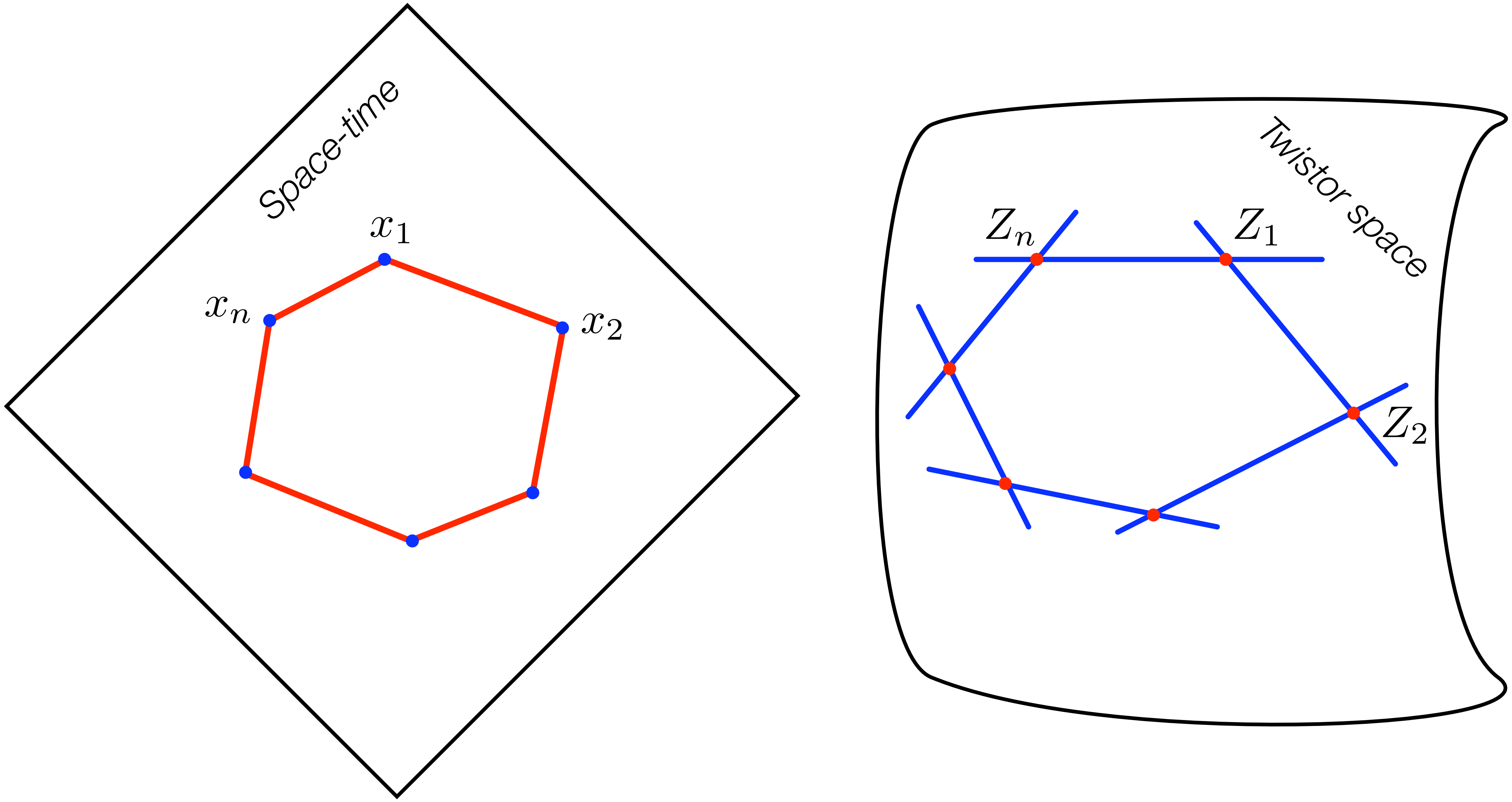
x_i are coordinates on affine momentum space, with $x_i - x_{i+1} = p_i$

According to the conjecture, certain (**MHV**) scattering amplitudes in planar $\mathcal{N}=4$ SYM are determined by the expectation value of a fundamental Wilson Loop around this polygon, **treated as a curve in space-time**

- Much supporting evidence has now been found, both at strong[Alday, Gaiotto, Maldacena, Sever, Vieira] and weak[Drummond, Henn, Korchemsky, Sokatchev; Anastasiou, Brandhuber, Heslop, Khoze, Spence, Travaglini; Del Duca, Duhr, Smirnov] coupling.
- Was not known how to extend the Wilson Loop / amplitude correspondence beyond the MHV sector.

Which curve?

In twistor space, this piecewise null polygon becomes a set of intersecting twistor lines, or an elliptic curve that has been pinched n times.



- The twistor data is **unconstrained**: given *arbitrary* Z_i , the twistor lines intersect by construction, so the corresponding space-time vertices are inevitably null separated.
- **We take this to be our curve.** Since nodal, ω is allowed to have simple poles at the nodes.

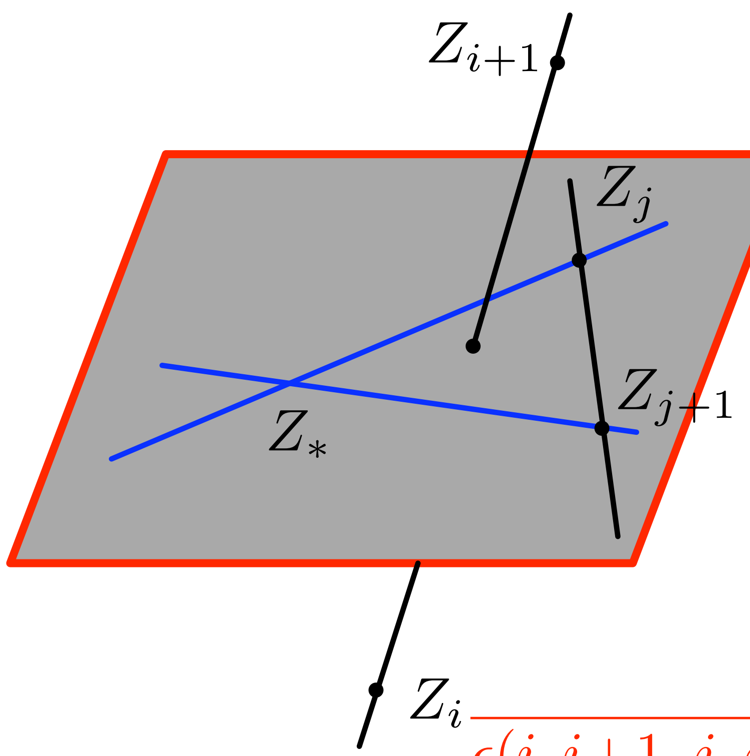
The Abelian case

To define the self holomorphic linking, pick a framing - a nowhere vanishing holomorphic section of the normal bundle to C. Then, as before $\langle W[C;\omega]\rangle = \exp(\mathrm{hlk}(C,C'))$

Easily computable: introduce currents for the line (Z_i, Z_{i+1}) and plane (Z_j, Z_{j+1}, Z_*)

$$\int_{(Z_i,Z_{i+1})} \omega_i \wedge \mathcal{A}_i = \int_{\mathbb{CP}^{3|4}} \Omega \wedge \bar{\delta}^{2|4}(Z, Z_i, Z_{i+1}) \wedge \mathcal{A}_i$$

$$\int_{(Z_j,Z_{j+1},Z_*)} \mu_j \wedge \mathcal{F}_j = \int_{\mathbb{CP}^{3|4}} \Omega \wedge \bar{\delta}^{1|4}(Z, Z_j, Z_{j+1}, Z_*) \wedge \mathcal{F}_j$$



$$\mathrm{hlk}(C,C') = \frac{1}{2}\sum_{i,j}\int \Omega \wedge \bar{\delta}^{1|4}(Z, Z_j, Z_{j+1}, Z_*) \wedge \bar{\delta}^{2|4}(Z, Z_i, Z_{i+1})$$

$$=: \frac{1}{2}\sum_{i,j}[*, i, i{+}1, j, j{+}1]$$

$$\frac{\delta^{0|4}(\psi_* \epsilon(i, i{+}1, j, j{+}1) + \text{cyclic})}{\epsilon(i, i{+}1, j, j{+}1)\, \epsilon(i{+}1, j, j{+}1, *)\, \epsilon(j, j{+}1, *, i)\, \epsilon(j{+}1, *, i, i{+}1)(*, i, i{+}1, j)}$$

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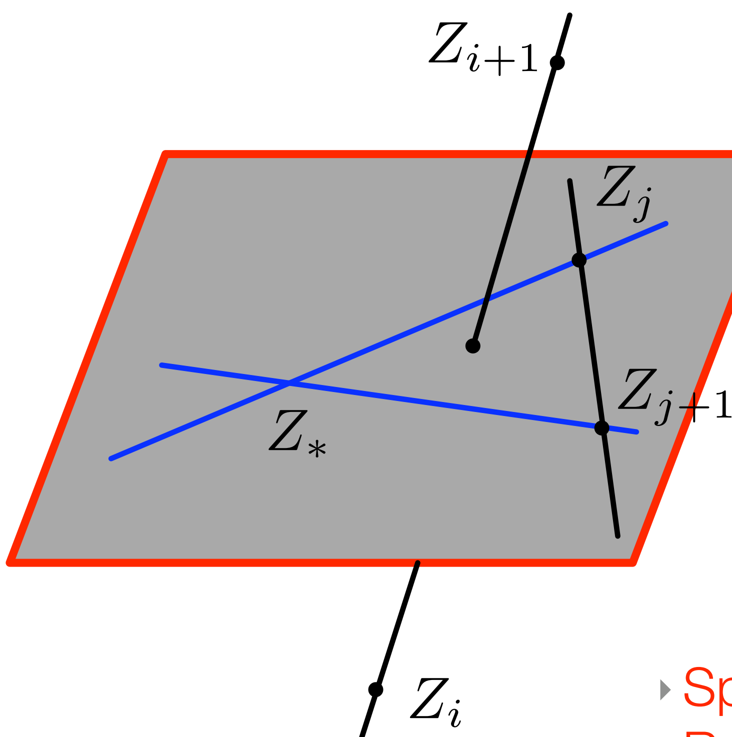

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$$=: \frac{1}{2}\sum_{i,j}[*, i, i{+}1, j, j{+}1]$$

- Spurious poles cancel in sum, as shared by previous / subsequent plane.
- Remarkably, can take framing to zero; consequence of supersymmetry.

The resulting linking gives exactly the **NMHV tree super-amplitude**! (The corresponding space-time calculation is considerably more difficult[Caron-Huot; Belitsky, Korchemsky, Sokatchev].)

Non-Abelian holomorphic Wilson lines

For general curves, difficult to consider non-Abelian case (non-trivial moduli space of semi-stable bundles). However, for our degenerate curves, it is straightforward.

- On each rational component, we have a unique holomorphic frame $U(Z, Z_i)$ obeying

$$(\bar{\partial} + \mathcal{A})\big|_{\mathrm{X}_i} U(Z, Z_i) = 0 \qquad\qquad U(Z_i, Z_i) = \mathrm{id}$$

- In Abelian case $U(Z, Z_i) = \exp(-\bar{\partial}^{-1}\mathcal{A})$ so

$$\prod_i U(Z_{i+1}, Z_i) = \exp\left(-\int_C \omega_i \wedge \mathcal{A}_i\right)$$ is the holomorphic Wilson Loop

- For non-Abelian case, solve perturbatively by Born series

$$U(Z, Z_i) = \mathrm{id} - \bar{\partial}^{-1}\mathcal{A} + \bar{\partial}^{-1}(\mathcal{A}\bar{\partial}^{-1}\mathcal{A}) + \cdots$$

$$= \mathrm{P}\,\exp\left(-\int \omega \wedge \mathcal{A}\right)$$ with the standard concatenation and inversion properties $U(Z_2, Z_1)U(Z_1, Z_0) = U(Z_2, Z_0)$ and $U(Z_2, Z_1)^{-1} = U(Z_1, Z_2)$

For the $U(N)$ theory, we define the fundamental Wilson Loop as

$$W[C] = \frac{1}{N}\mathrm{Tr}_{\Box}\mathrm{P}\exp\left(-\int_C \omega \wedge \mathcal{A}\right)$$

an ordered product of holomorphic frames around the curve.

Varying the Wilson Loop

To compute the non-Abelian correlator, we can either work perturbatively[Mason, DS] or else consider how the correlator changes as we vary the curve[Bullimore, DS].

Recall that under a smooth change in the curve, a real Wilson Loop obeys

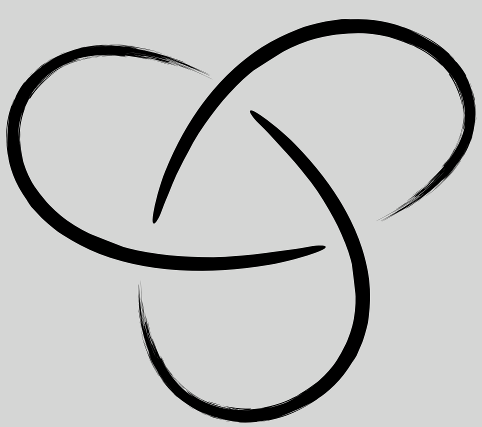
$$\delta W[\gamma] = -\int_\gamma \mathrm{d}x^\mu \wedge \delta x^\nu \,\mathrm{Tr}\left[F_{\mu\nu}(x)\,\mathrm{P}\exp\left(-\oint_x A\right)\right]$$

saying (*e.g.*) that the Wilson Loop is unchanged if γ varies only in a region where the field-strength vanishes.

Similarly, the holomorphic Wilson Loop obeys

$$\bar\delta \mathrm{W}[C] = -\int_C \omega\wedge \mathrm{d}\bar Z^{\bar a}\wedge\bar\delta\bar Z^{\bar b}\,\mathrm{Tr}\left[\mathcal{F}_{\bar a\bar b}(Z)\,\mathrm{P}\exp\left(-\int\omega\wedge\mathcal{A}\right)\right]$$

under a holomorphic change in C. This says that, if $\mathcal{F}^{(0,2)}=0$, then $\mathrm{W}[C]$ varies *holomorphically* over this holomorphic family of curves.



- The Loop Equations[Migdal, Makeenko; Polyakov] tell us how the correlation function, rather than the trace of the classical holonomy, behaves.
- In real Chern-Simons theory, the loop equations give (poor man's) derivation of the **skein relations** - *i.e.* recursion relations for the knot polynomial[Cotta-Rasmusino, Guadagnini, Martellini, Mintchev].

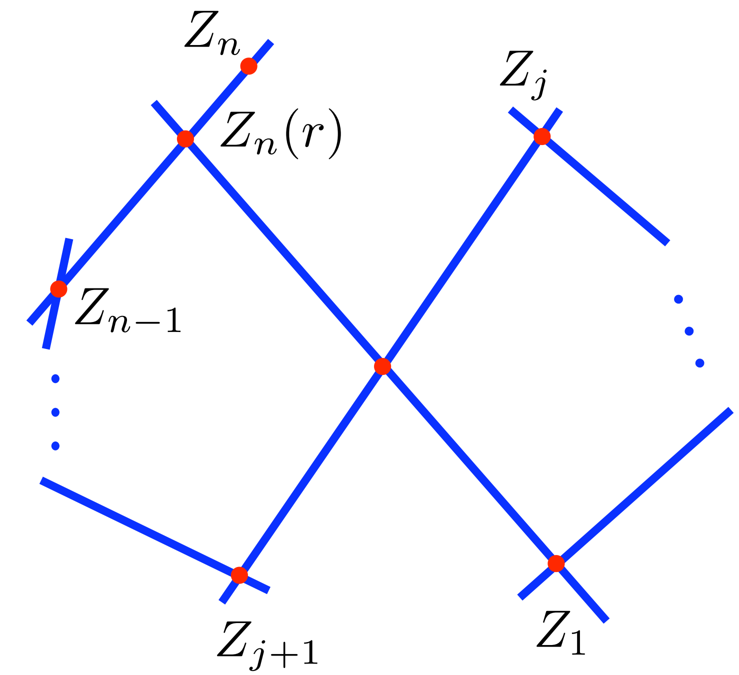
Loop equations & BCFW recursion

Beginning in the pure holomorphic Chern-Simons theory, integration by parts in the path integral gives

$$
\begin{aligned}
\langle \bar{\delta}\mathrm{W}[C(r)]\rangle &= -\int_C \omega\wedge\left\langle \mathrm{Tr}\,\mathcal{F}^{(0,2)}(Z)\,\mathrm{P}\exp\left(-\int\omega\wedge\mathcal{A}\right)\right\rangle_{\mathrm{hCS}} \\
&= -\int_C \omega\wedge\left\langle \mathrm{Tr}\,\frac{\delta S_{\mathrm{hCS}}}{\delta\mathcal{A}(Z)}\,\mathrm{P}\exp\left(-\int\omega\wedge\mathcal{A}\right)\right\rangle_{\mathrm{hCS}} \\
&= \int_{C\times C}\omega\wedge\omega'\wedge\bar{\delta}^{3|4}(Z,Z')\,\langle \mathrm{W}[C']\,\mathrm{W}[C'']\rangle_{\mathrm{hCS}}
\end{aligned}
$$

for $U(N)$ theory

ensures only get contribution if $C(r)$ self-intersects as we deform



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&= - \int_C \omega \wedge \left\langle \mathrm{Tr}\, \frac{\delta S_{\mathrm{hCS}}}{\delta \mathcal{A}(Z)}\, \mathrm{P} \exp \left(- \int \omega \wedge \mathcal{A} \right) \right\rangle_{\mathrm{hCS}} \\
&= \int_{C \times C} \omega \wedge \omega' \wedge \bar{\delta}^{3|4}(Z, Z')\, \langle \mathrm{W}[C']\, \mathrm{W}[C''] \rangle_{\mathrm{hCS}} \\
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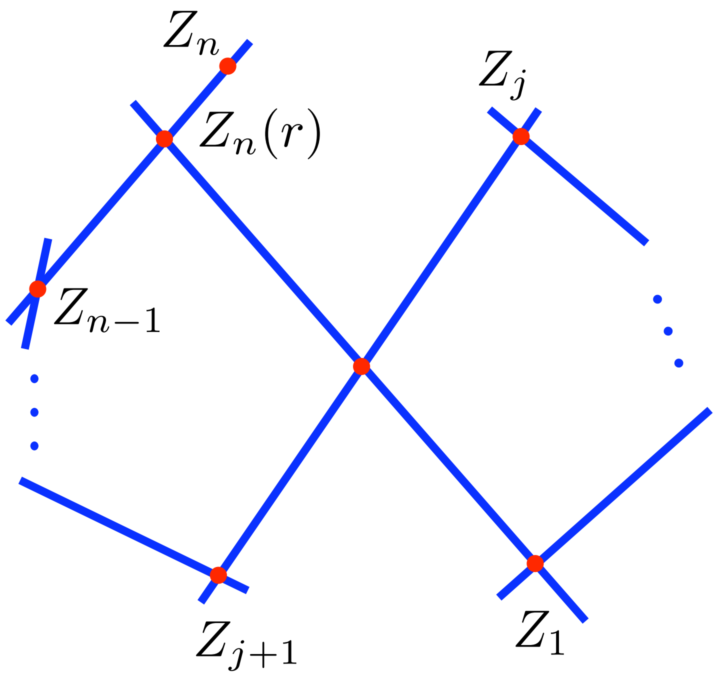

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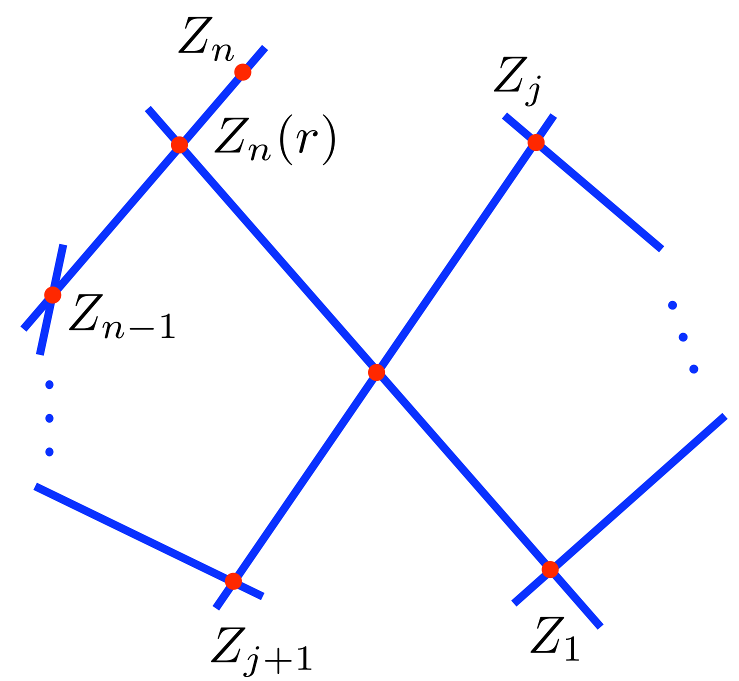
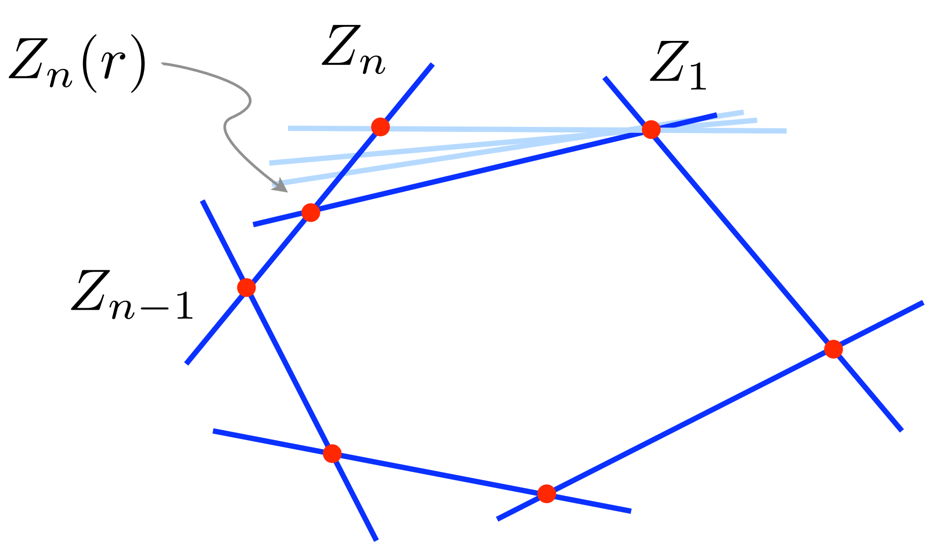
\end{aligned}
$$



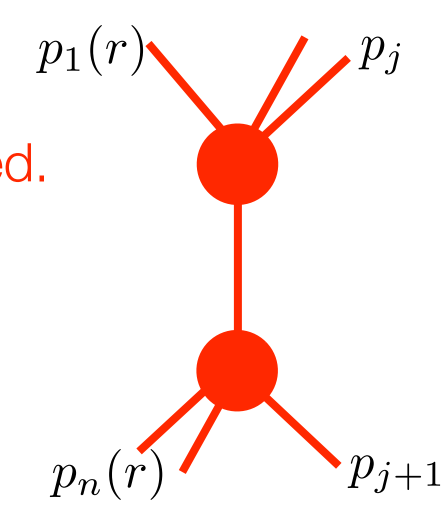
An interesting holomorphic family of curves corresponds to BCFW deforming the scattering amplitude



- When the line $(n_r 1)$ intersects $(j\, j{+}1)$, the corresponding space-time points are null separated. For the amplitude this implies

$$(p_1(r) + p_2 + \cdots + p_j)^2 = 0$$

which is a BCFW factorization channel.



Loop equations & BCFW recursion

Integrating both sides with respect to the BCFW measure gives

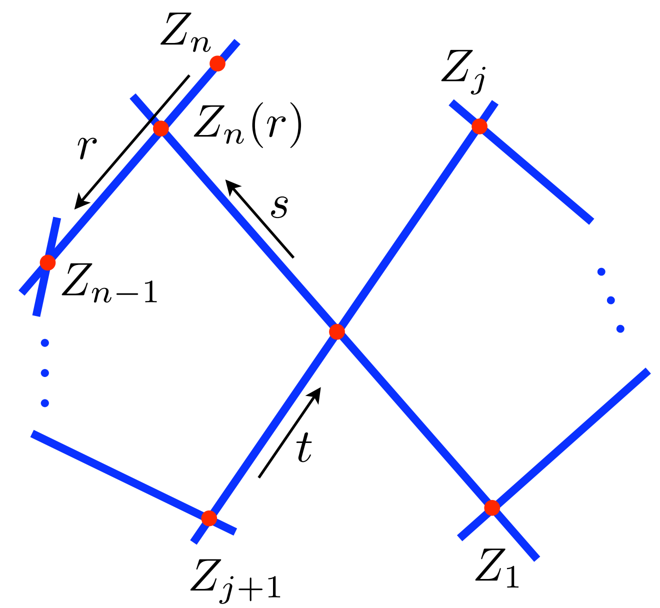
$$\int \frac{\mathrm{d}r}{r} \wedge \mathrm{d}\bar{r} \frac{\partial}{\partial \bar{r}} \langle \mathrm{W}[C(r)] \rangle = \int \frac{\mathrm{d}r}{r} \int_{C\times C} \omega \wedge \omega' \wedge \bar{\delta}^{3|4}(Z, Z') \langle \mathrm{W}[C'] \rangle \langle \mathrm{W}[C''] \rangle$$

Carrying out the three integrals is just a matter of finding a Jacobian.

$$Z(s,r) = Z_1 + sZ_n(r)$$

$$Z'(t) = Z_j + tZ_{j+1}$$

$$\omega \wedge \omega' = \frac{\mathrm{d}s}{s} \frac{\mathrm{d}t}{t}$$



$$\langle \mathrm{W}[1,\ldots,n] \rangle = \langle \mathrm{W}[1,\ldots,n-1] \rangle + \sum_{j=2}^{n-2} [n-1, n, 1, j, j+1] \; \langle \mathrm{W}[1,\ldots,j,Z_I] \rangle \langle \mathrm{W}[Z_I, j+1,\ldots,n_r] \rangle$$

This is the **tree-level BCFW recursion relation**, summed over MHV degree. Arises here as an analogue of the skein relations for holomorphic invariants.

- **Complete classical S-matrix arises from correlator in theory with no amplitudes!**
- Provides rationale for amplitude / Wilson Loop correspondence; both meromorphic with same poles & residues.

Quantum Corrections

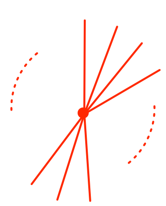
To also obtain quantum corrections, we compute the expectation value of the same holomorphic Wilson Loop, but now in the twistor QFT for complete, not just asd, $\mathcal{N}=4$ SYM. This can be described by the twistor space action[Nair; Witten; Boels, Mason, DS]

$$S=\int \mathrm{D}^{3|4}Z\wedge \mathrm{Tr}\left(\mathcal{A}\bar{\partial}\mathcal{A}+\frac{2}{3}\mathcal{A}^3\right)+\mathrm{g}^2\int \mathrm{d}^{4|8}x\ \log\det\left(\bar{\partial}+\mathcal{A}\right)_X$$

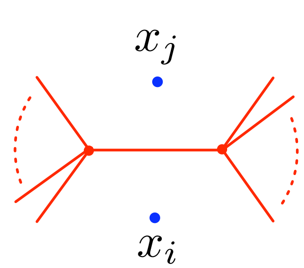
- Expanding in powers of the field, the terms proportional to the coupling are

$$\int \mathrm{d}^{4|8}x\ \log\det\left(\bar{\partial}+\mathcal{A}\right)_{\mathrm{X}}=\sum_{n=2}^{\infty}\frac{1}{n}\int \mathrm{d}^{4|8}x\,\mathrm{Tr}\,\underbrace{\left(\bar{\partial}^{-1}\mathcal{A}\,\bar{\partial}^{-1}\mathcal{A}\cdots\bar{\partial}^{-1}\mathcal{A}\right)}_{n\ \text{terms}}$$

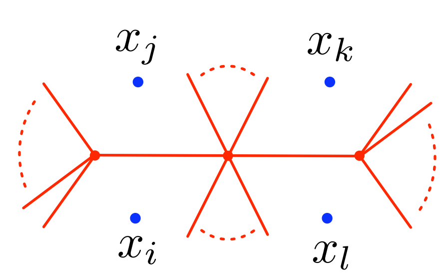
giving an infinite sum of MHV vertices.



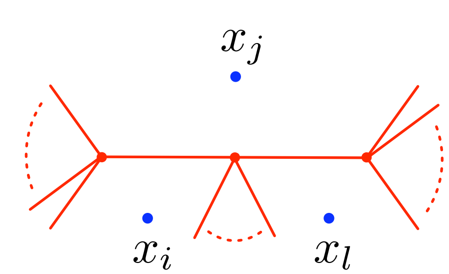
$+\sum_{i<j-1}$



$+\sum_{i<j-1<k<l-1}$



$+\sum_{i<j-1<l-2}$



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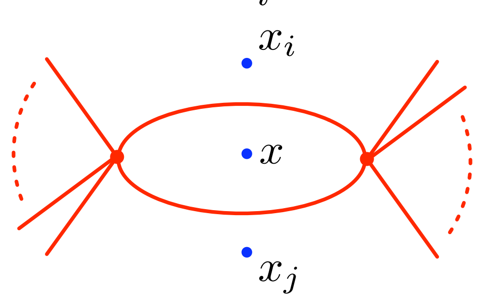


$+\cdots$ (explored by many groups[Cachazo, Svrcek, Witten; Brandhuber, Spence, Travaglini; Boels, Mason, DS; Bianchi, Elvang, Freedman, Kiermaier; Adamo, Bullimore, ...])

Repeating the Loop Equations calculation theory defined by above action leads to the all-loop extension[Arkani-Hamed, Bourjaily, Caron-Huot, Cachazo, Trnka] of the BCFW recursion relations for the integrand of the planar amplitude.

Space-time formulation

What does this supersymmetric twistor space Wilson Loop correspond to on space-time?[c.f. Caron-Huot]

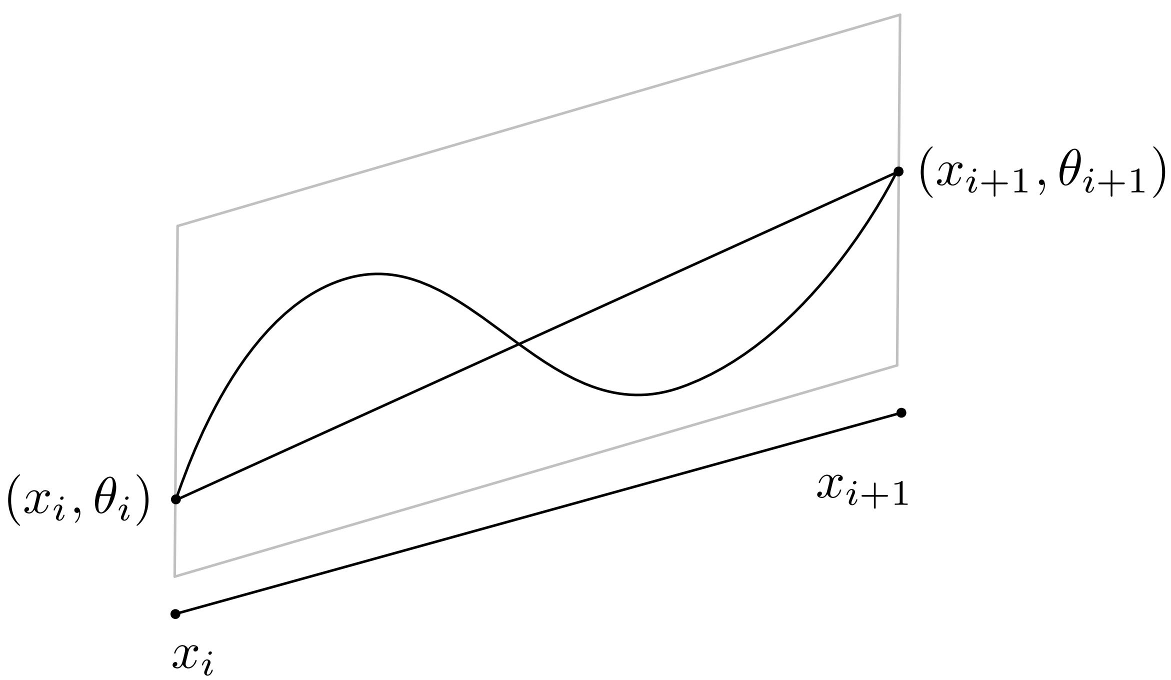
- A natural generalisation of the space-time connection is a (chiral) superconnection

$$\mathrm{d}x^\mu\left(\frac{\partial}{\partial x^\mu}+A_\mu(x,\theta)\right)+\mathrm{d}\theta^{A\dot\alpha}\left(\frac{\partial}{\partial\theta^{A\dot\alpha}}+\Gamma_{A\dot\alpha}(x,\theta)\right)$$

- However, the supersymmetric incidence relations

$$\omega^\alpha=\mathrm{i}x^{\alpha\dot\alpha}\pi_{\dot\alpha}\qquad \psi^A=\theta^{A\dot\alpha}\pi_{\dot\alpha}$$

mean that a (null) twistor no longer corresponds to a null ray, but to a (chiral) **super null ray**[Witten; Harnad, Hurtubise, Shnider]. These are best thought of as 1|4 dimensional.



To reflect the twistor geometry, the space-time superconnection must be integrable over super null rays

$$\left[D^{\mathrm{bos}}_{\dot\alpha(\alpha}\,,\,D^{\mathrm{ferm}}_{\beta)B}\right]=0\qquad\left\{D^{\mathrm{ferm}}_{A(\alpha}\,,\,D^{\mathrm{ferm}}_{\beta)B}\right\}=0$$

This gives correct space-time version[Mason, DS; Caron-Huot]

- These integrablity constraints make the space-time version considerably more difficult to work with.
- In addition, a good regularisation scheme is currently lacking[Belitsky, Korchemsky, Sokatchev].

Conclusions & related directions

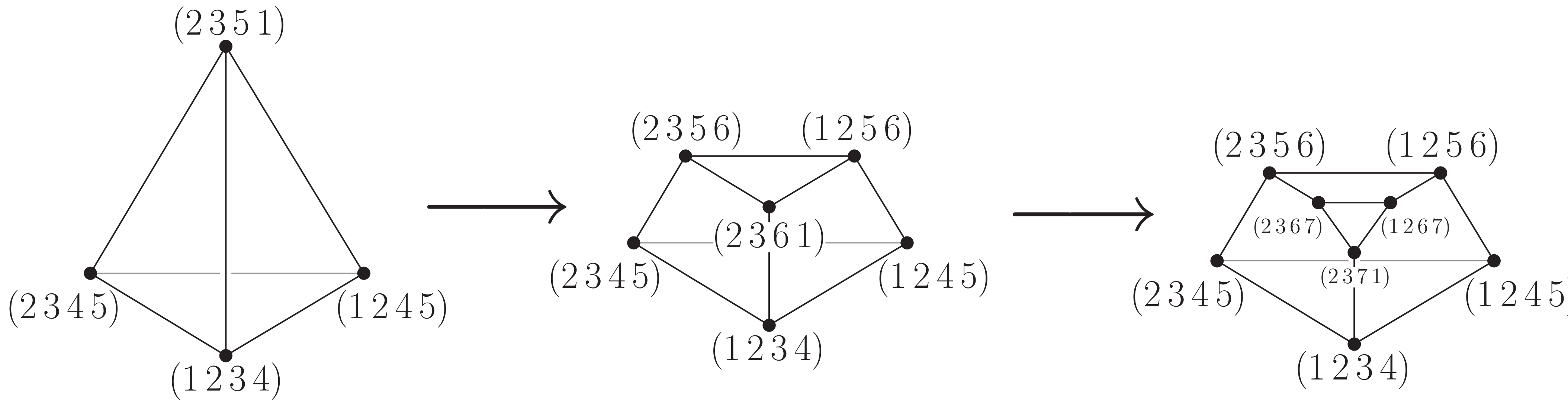
Holomorphic linking in twistor space, and more sophisticated non-Abelian invariants, are intimately related to dynamical processes in space-time QFT.

- Through this and other means, we are beginning to understand how to encode fully non-linear (but perturbative) space-time QFT in terms of twistor geometry.

- The resulting picture provides one of the most powerful & efficient techniques physicists currently posses for computing scattering amplitudes, at least in supersymmetric gauge theories. In particular, provides a **vast** improvement over textbook Feynman diagrams *(some QFT textbooks now use twistor methods...)* and also improves on `modern unitarity methods' of *circa* 2005.

There are a number of closely related topics:

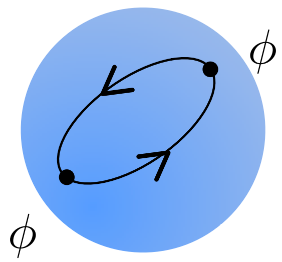
1) Scattering amplitudes as volumes of polytopes[Hodges; Arkani-Hamed, Bourjaily, Cachazo, Trnka]



Conclusions & related directions

2) How about other representations?

Choosing *e.g.* the adjoint representation makes contact with a conjecture[Alday, Eden, Heslop, Korchemsky, Maldacena, Sokatchev] relating Wilson Loops to correlation functions of gauge-invariant local operators, again in a null-separated limit.



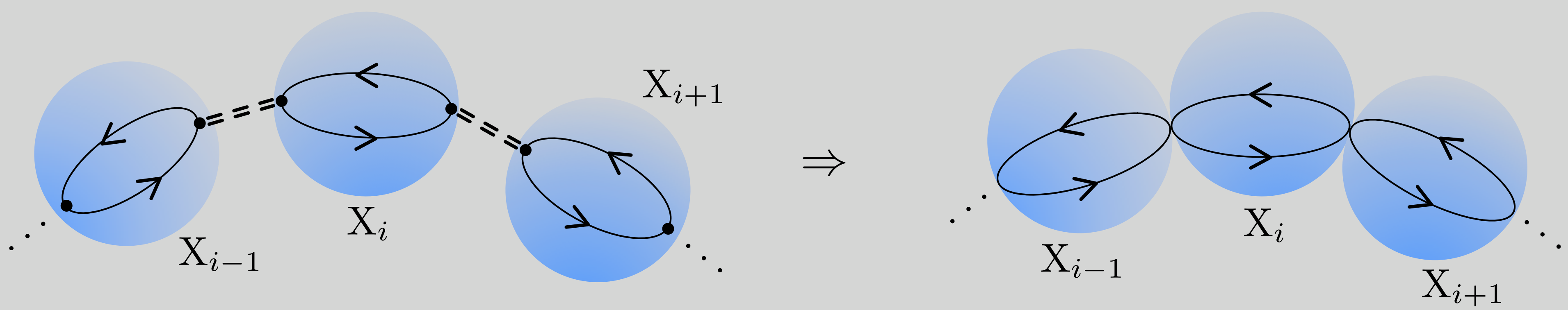
This is straightforward to see on twistor space[Adamo, Bullimore, Mason, DS]:

Local operators on space-time are non-local on twistor space.

e.g. $$\mathrm{Tr}\,\Phi^2(x) = \int_{\mathrm{X}\times\mathrm{X}} \mathrm{d}\sigma\,\mathrm{d}\sigma'\,\mathrm{Tr}\left(\phi(\sigma)\mathrm{U}(\sigma,\sigma')\phi(\sigma')\mathrm{U}(\sigma',\sigma)\right)$$

and easy to generalize to complete (chiral) operator supermultiplets.

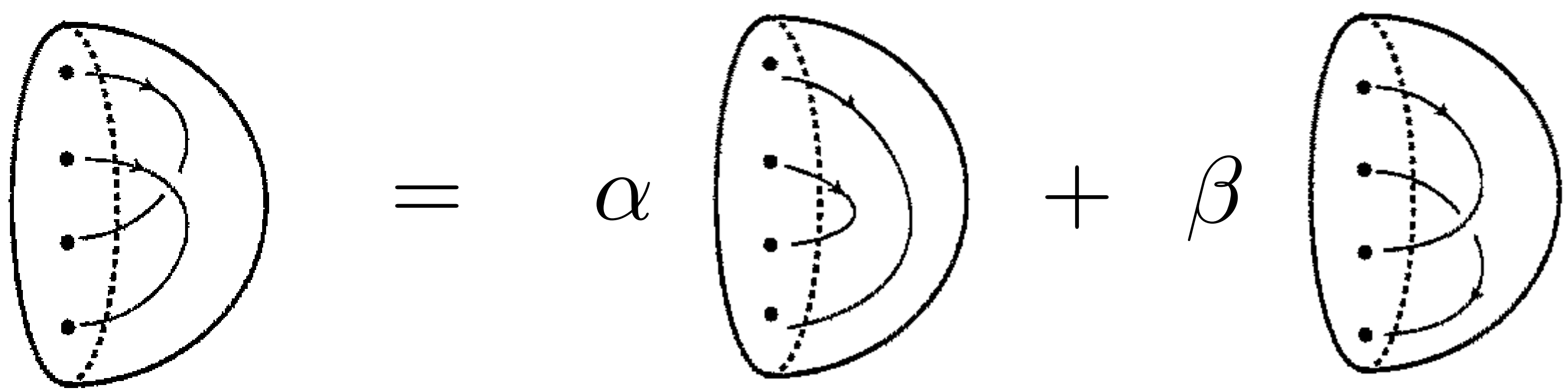
In the limit that the twistor lines intersect, factoring out a singular piece from the integrand leaves us with a super Wilson Loop in the adjoint, and in the planar limit this equals (scattering amplitude)^2



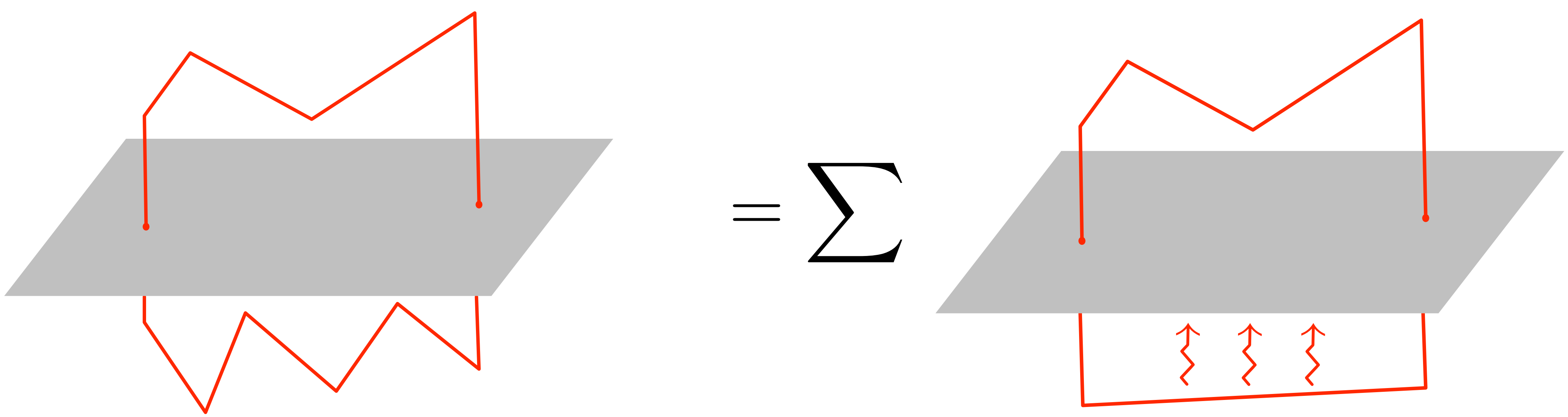
Conclusions & related directions

3) Hamiltonian framework?

Witten's derivation of the skein relations for the Jones polynomial did not use the loop equations, but rather cutting open the 3-manifold on a Riemann surface and identifying the associated Hilbert space.



- In SYM, a related idea is the Wilson Loop OPE[Alday, Gaiotto, Maldacena, Sever, Vieira] for space-time version



There should be a generalization of the Wilson Loop OPE that handles the supersymmetric extension / non-MHV sector. It will be interesting to see what this looks like in twistor space.