Polytopes and amplitudes

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1. Pure N=4 gauge theory amplitudes

2. Linearized gravity amplitudes

Basics:

Helicity representation, $p_a = \pi_{A'}\tilde{\pi}_A$ makes Lorentz transformations into Möbius on CP^1

Basic notation: $\pi_{1A'}\pi_2^{A'}$ written as $\langle 12\rangle$, $\pi_{1A}\pi_2^{A}$ written as $[12]$

Twistor representation makes conformal transformations into linear maps on CP^3.
(RP, 1964 onwards.)

Amplitudes become higher-dimensional contour integrals in twistor spaces (RP, 1970 onwards)

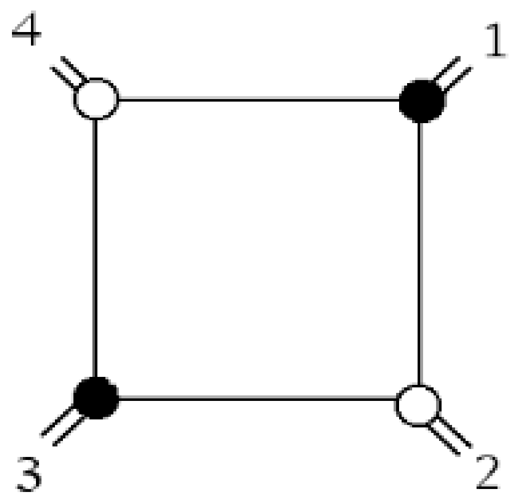
Gauge theory amplitudes

$$A(1^-2^+3^-4^+) = \frac{\langle 13\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\,\delta(\sum_1^4 p_i)$$

$$= \frac{[24]^4}{[12][23]34][41]}\,\delta(\sum_1^4 p_i)$$

$$= \frac{\langle 13\rangle^2[24]^2}{s_{12}s_{13}}\,\delta(\sum_1^4 p_i)$$

All represented by a twistor diagram, defining an integral:



Lines denote boundaries of the region of integration, manifestly conformally invariant.

The diagram embodies the momentum conservation.
Identities become simple steps of integration by parts.

BCFW rule for higher amplitudes is a simple joining operation on diagrams
(AH 2005, extending RP's original work for QED, 1972)

Diagrams naturally extend to N=4 supersymmetric gauge theory by using N=4 super-twistors

Super-amplitudes from super-volumes with super-conformally invariant super-boundaries.

BUT the diagrams are not very convenient — far too much information, with all the infra-red detail.

Grassmannian structure (Nima Arkani-Hamed, Freddy Cachazo + group) identifies the most important combinatorial information, and extends analysis into leading singularities....

For MHV amplitudes very simple.

For NMHV and beyond, much more complicated.

The amplitudes are given by sums of terms from BCFW recursion, equivalent to twistor diagrams.

Terms individually have spurious pole singularities. Hard to see the amplitude as a single object.

Spurious poles

The simplest case: two expressions for A($1^-2^-3^-4^+5^+6^+$):

$$\left(\frac{[4|5+6|1\rangle^3}{[34][23]\langle 56\rangle\langle 61\rangle[2|3+4|5\rangle S_{234}} + \frac{[6|1+2|3\rangle^3}{[61][12]\langle 34\rangle\langle 45\rangle[2|3+4|5\rangle S_{612}} \right) \delta(\sum_{i=1}^{6} p_i)$$

$$\left(\frac{(S_{123})^3}{[12][23]\langle 45\rangle\langle 56\rangle[1|2+3|4\rangle[3|4+5|6\rangle} + \frac{\langle 12\rangle^3[45]^3}{\langle 16\rangle[34][3|4+5|6\rangle[5|6+1|2\rangle S_{612}} + \frac{\langle 23\rangle^3[56]^3}{\langle 34\rangle[16][1|2+3|4\rangle[5|6+1|2\rangle S_{234}} \right) \delta(\sum_{i=1}^{6} p_i)\,.$$

Dual conformal invariance and momentum-twistors

In a colour sector with order {1234....n}, the momenta p_i only appear in ‘consecutive’ combinations like $(p_2 + p_3 + p_4)$

This is equivalent to having only functions of $(x_i - x_k)$ where the x_i are in ‘region space’,
$x_i = x_0 + p_1 + p_2 + p_3 \ldots + p_i$

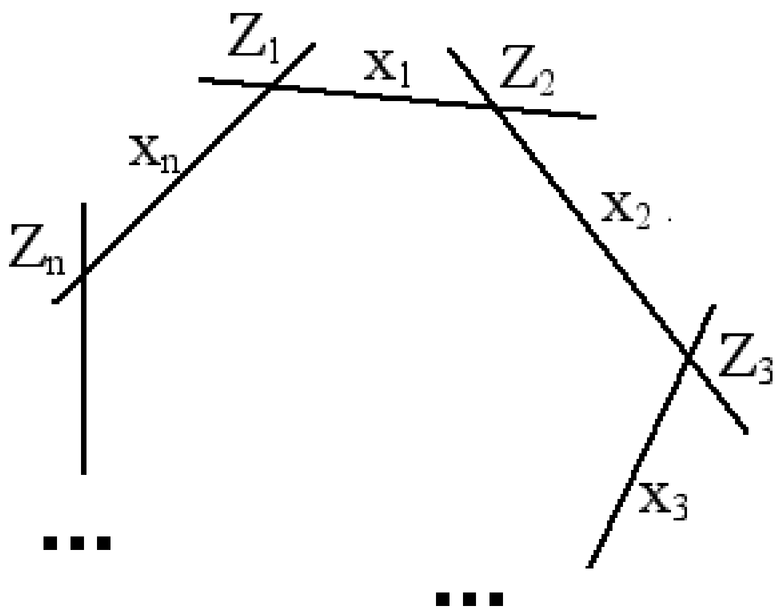
More than this: dual conformal symmetry.

Define momentum-twistor coordinates to express dual conformal symmetry.

This means mapping 2n spinors into n twistors; a non-linear transformation.

Momentum conservation is simple in the resulting algebra.

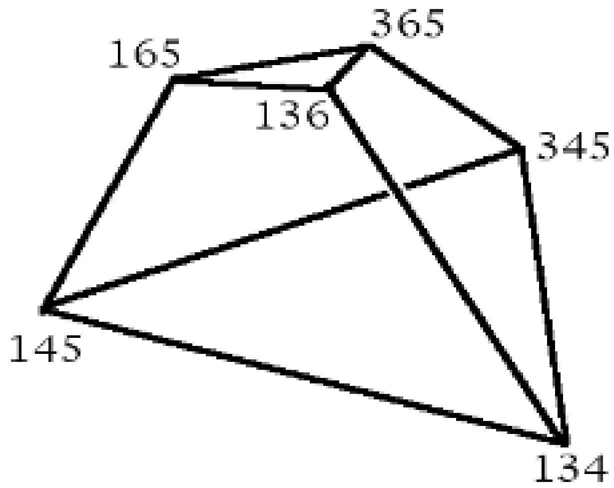
Natural emergent geometric concept, an ordered loop of momentum twistors
and a dual system of planes defined by three consecutive twistors:



$$p_2^2 = 0 \Leftrightarrow (x_1 - x_2)^2 = 0 \Leftrightarrow Z_2^\alpha \text{ exists.}$$

$$W_{2\alpha} = \frac{\epsilon_{\alpha\beta\gamma\delta} Z_1^\beta Z_2^\gamma Z_3^\delta}{I_{\alpha\beta} Z_1^\alpha Z_2^\beta \; I_{\gamma\delta} Z_2^\gamma Z_3^\delta}$$

The amplitude $A(1^-2^-3^-4^+5^+6^+)$ is easily translated into these coordinates.
It is equivelent to the volume of a polyhedron in dual momentum twistor space:



The two-term expression is the difference of two tetrahedra, with faces 1,3,5,4 and 1,3,5,6
The spurious pole corresponds to the cancelled vertex 135.
Surviving vertices correspond to physical poles.

But this polyhedron can also be represented as sum of three tetrahedra (3456, 4561, 1346), with cancelled vertices 146, 456, 346, and this gives the three-term expression.

The 5-term identity simply corresponds to the sum of 5 tetrahedra vanishing in 3-dimensional space.

Other divisions of the polyhedron into tetrahedra give rise to explicit expressions for the amplitude without spurious poles.

Agreement with Berends-Giele formulas (1988).

This extends to all NMHV amplitudes.

In general need 4-dimensional polytopes, with an essential six-term identity. The extra dimension absorbs the supersymmetric extension which removes the restriction to $A(1^-2^-3^-4^+5^+6^+\ldots)$.

The 4-polytopes are simple objects with dihedral symmetry, defined by multiplying loops.

Multiplying (Wilson) Loops

Given two ordered loops of points in CP^4, $\{x_1, x_2, \ldots x_m\}$, $\{y_1, y_2, \ldots y_n\}$, (with $x_{m+1} = x_1$ etc.)

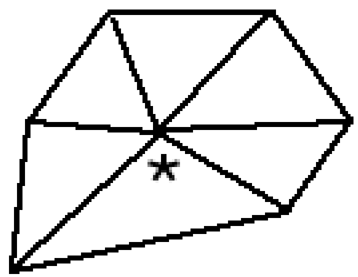
form the oriented tetrahedra of form $\{x_i, x_{i+1}, y_j, y_{j+1}\}$, and sum them over all i, j.

The boundary of the sum vanishes. Hence the sum is the boundary of a unique 4-polytope

$\{x_1, x_2, \ldots x_m\} \times \{y_1, y_2, \ldots y_n\}$

which can be defined by sum of the 4-simplexes $\{*, x_i, x_{i+1}, y_j, y_{j+1}\}$, where * is arbitrary.

This is the analogue of triangulating a polygon by an arbitrary *.



Dihedral polytopes for all NMHV amplitudes

For NMHV amplitudes, we define a 4-polytope which is 1/2 the square $\{z_1, z_2, \ldots z_n\} x \{z_1, z_2, \ldots z_n\}$.
The amplitude is the volume (with respect to a natural measure) of the dual of this polytope.

Shorter formulas arise from taking * = z_i but these lose manifest dihedral aymmetry.
Such formulas correspond exactly to the different expressions arising from BCFW recursion.

This polytope defines a single geometric entity for the amplitude,
possessing every symmetry of the theory.

Analogous constructions give 1-loop MHV amplitudes.

The momentum twistor variables are more generally useful for many-loop expressions.

Geometry of NNMHV $A(1^- 2^- 3^- 4^- 5^+ 6^+ 7^+)$ is much harder...

Gravitational amplitudes

The analogous structures are more complicated.
Even for 4 gravitons, a non-trivial multiplicity of representations.

Use reduced amplitude $\bar{M}_4(1234)$ which is totally symmetric in {1234}

(stripping off the delta-function and the helicity prefix <ij>^{8}).

$$\begin{aligned}
\bar{M}_4(1234) &= \frac{[13]}{\langle 12\rangle\langle 34\rangle\langle 41\rangle^2\langle 23\rangle^2\langle 24\rangle} - \frac{[13]}{\langle 13\rangle\langle 41\rangle^2\langle 23\rangle^2\langle 24\rangle^2} \\
&= \frac{[13]}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle\langle 13\rangle\langle 24\rangle^2} = \frac{[34]^4}{\langle 12\rangle^4 s_{12}s_{13}s_{24}} \\
&= s_{12}\frac{1}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\,\frac{1}{\langle 12\rangle\langle 24\rangle\langle 43\rangle\langle 31\rangle}\,.
\end{aligned}$$

Look for twistor representations to supply a unified geometric picture of these expressions.

Twistor representations will make the conformal symmetry-breaking explicit.

Conformal symmetry-breaking only through a numerator (RP 1972).

BCFW recursion is again valid. It is based on 3-amplitudes with twistor representations which were essentially also given in (RP 1972).

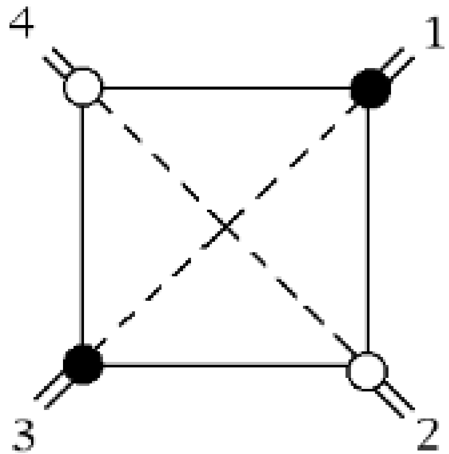
BUT... the obvious BCFW recursion methods bring in spurious double poles, all cancelling in a summation over terms.

Can these be eliminated?

New developments:

(1) Use a recursion in N=7 supersymmetry (not N=8), which eliminates spurious double poles.

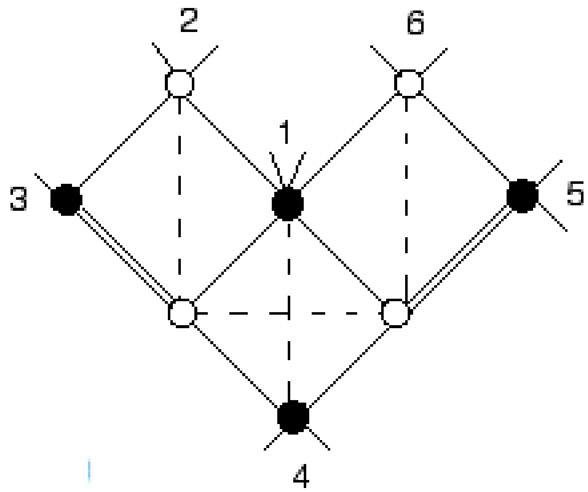
The four-graviton amplitude is then a single twistor diagram with numerator <12>[34]



Again, momentum conservation is embodied in the diagram,
and we have a unified geometrical object for the amplitude.
Not complete symmetry (because of N=7 representation) but free from spurious poles.

Amplitude for n=6 is sum of six diagrams:

$$\sum_{\sigma(345)}$$

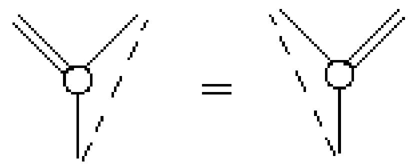


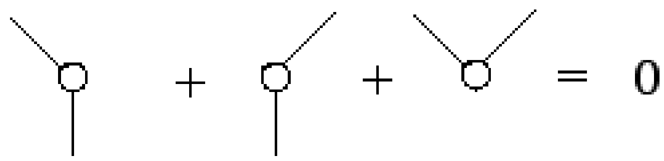
where the terms correspond directly to the terms in the BGK-Mason-Skinner formula

$$\sum_{\sigma(345)} \frac{\langle 26\rangle^7 [23][56][4|5+6|1\rangle}{\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 12\rangle\langle 13\rangle\langle 14\rangle\langle 15\rangle\langle 16\rangle}.$$

The boundary contours can be summed in (relative) homology.

Use integration by parts [addition of total derivatives] and Jacobi relation of contours, both purely graph-theoretic:





The twistor diagram then becomes a single object with S_4 symmetry in {1345}.

More generally, summation over (n–3)! diagrams, but symmetry in (n–2) in homology.

(2) Recursion relation derived from N=7 formalism:

$$\bar{M}_n(123\ldots n-1,n)=\sum_{p=3}^{n-1}\frac{[pn]}{\langle pn\rangle}\frac{\langle 1p\rangle\langle 2p\rangle}{\langle 1n\rangle\langle 2n\rangle}\bar{M}_{n-1}(\hat{1}_{(p)}23\ldots\hat{p}\ldots n-1)$$

and this has new solutions for n=6 and beyond....

(3) Define the phase factor

$$\psi^i_j = \frac{[ij]}{\langle ij\rangle} \text{ (for } i \neq j\text{)}, \quad \psi^i_i = 0$$

Then gravitational amplitudes are antisymmetrized phase factors:

$$\bar{M}_3(123) = \frac{1}{\langle 12\rangle\langle 23\rangle\langle 31\rangle\ \langle 12\rangle\langle 23\rangle\langle 31\rangle},$$

$$\bar{M}_4(1234) = \frac{\psi^1_4}{\langle 12\rangle\langle 23\rangle\langle 31\rangle\ \langle 23\rangle\langle 34\rangle\langle 42\rangle},$$

$$\bar{M}_5(12345) = \frac{\psi^1_{[4}\psi^2_{5]}}{\langle 12\rangle\langle 23\rangle\langle 31\rangle\ \langle 34\rangle\langle 45\rangle\langle 53\rangle},$$

and remarkably

$$\bar{M}_6(123456) = \frac{\psi^1_{[4}\psi^2_5\psi^3_{6]}}{\langle 12\rangle\langle 23\rangle\langle 31\rangle\ \langle 45\rangle\langle 56\rangle\langle 64\rangle}\,.$$

which is symmetric in 123456 by simple spinor identities alone.

Extension to 7, 8... but not yet in a general form.

(4) These expressions are connected with yet another representation of the amplitudes using momentum twistors (with respect to any order).

Conjecture:

$$\bar{M}_n(123\ldots n) = \frac{N_n(123\ldots n)}{\prod_i \langle i, i+1\rangle \ \prod_{i<j}\langle ij\rangle}$$

where N_n is a polynomial of degree $(n-3)$ in each of the n momentum twistors, and of degree $(n-3)(n-4)/2$ in I.

Example: for n=4, yet another expression:

$$\bar{M}_4(1234) = \frac{\langle 1234\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle \ \langle 12\rangle\langle 13\rangle\langle 14\rangle\langle 23\rangle\langle 24\rangle\langle 34\rangle}$$

That each N is a polynomial, rather than rational, is non-trivial.

(5) N...HMV amplitudes are also much simpler in this N=7 formalism, but not yet proved correct.

At this stage, emergent features of new geometric structures rather than complete theory.
A grand tradition:

R. Penrose and M. A. H. MacCallum, *Twistor theory: an approach to the quantisation of fields and space-time,* Physics Reports **4**, 241 (1972)
R. Britto, F. Cachazo, B. Feng, E. Witten hep-th/0501052 (2005) BCFW recursion
AH hep-th/0512336 (2005) twistor diagrams for N=4 gauge theory
N. Arkani-Hamed, F. Cachazo, C. Cheung, J. Kaplan 0903.2110 (2009) take up diagrams
AH 0905.1473 momentum twistors and polytopes (2009)
L. Mason, D. Skinner 0909.0250 (2009) SUSY momentum twistors
J. Bourjaily 1011.2447 (2010) computing with momentum twistors
N. Arkani-Hamed, J. Bourjaily, F. Cachazo, J. Trnka, AH 1012.6030 (2010) more on polytopes
N. Arkani-Hamed, J. Bourjaily F. Cachazo, J. Trnka 1012.6032 (2010) momentum twistors in loop amplitudes

L. Mason, D. Skinner 0808.3907 (2008), and
D. Nguyen, M. Spradlin, A. Volovich, C. Wen 0907.2276 (2009) on gravitational amplitudes