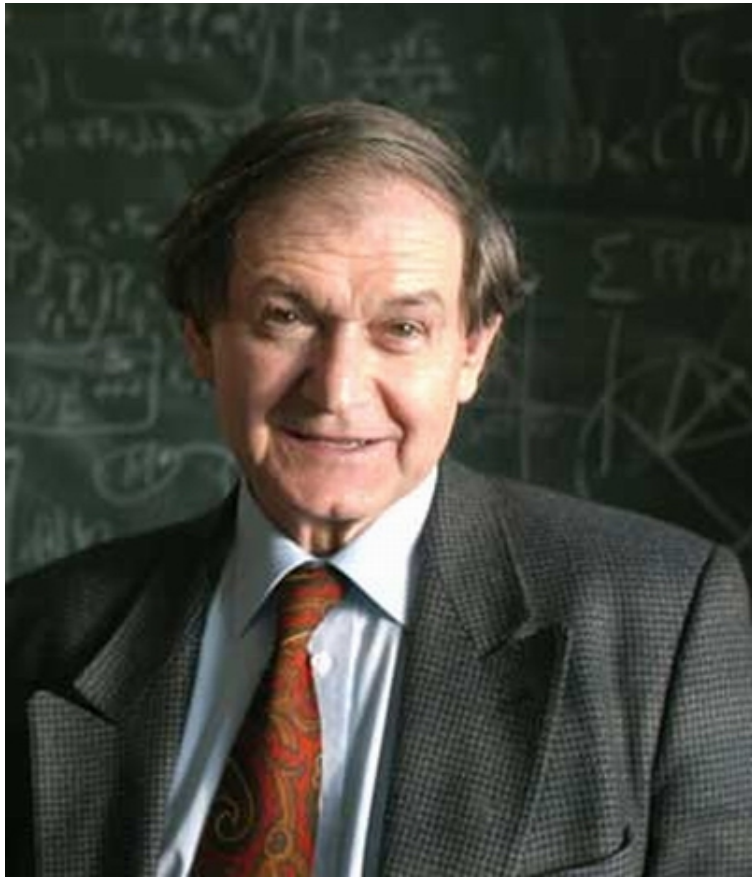
On

Hermitian, Einstein

4-Manifolds

Claude LeBrun
Stony Brook University

For Roger Penrose



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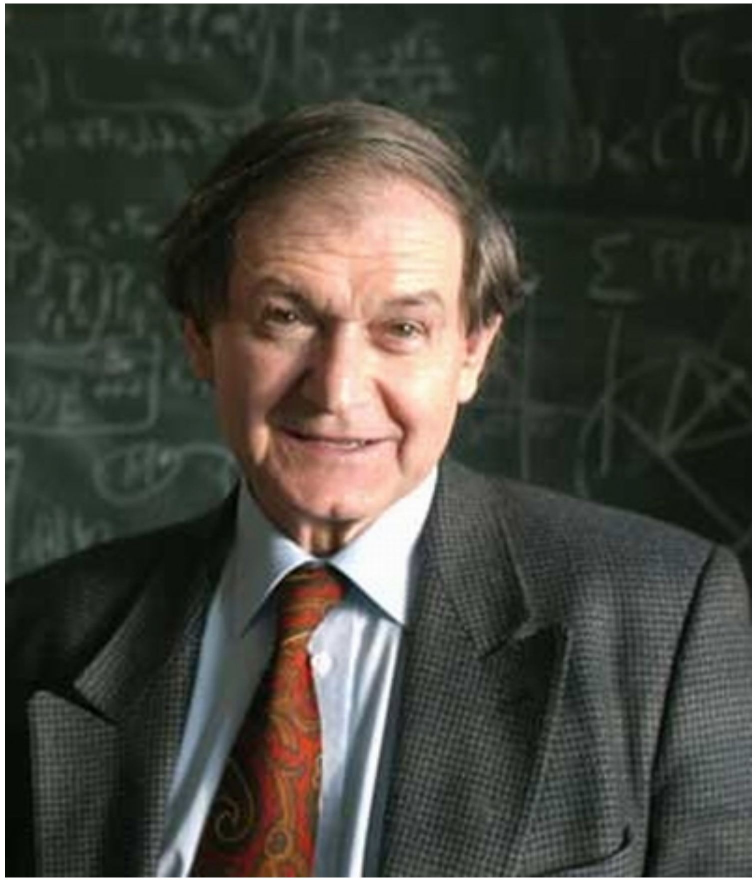

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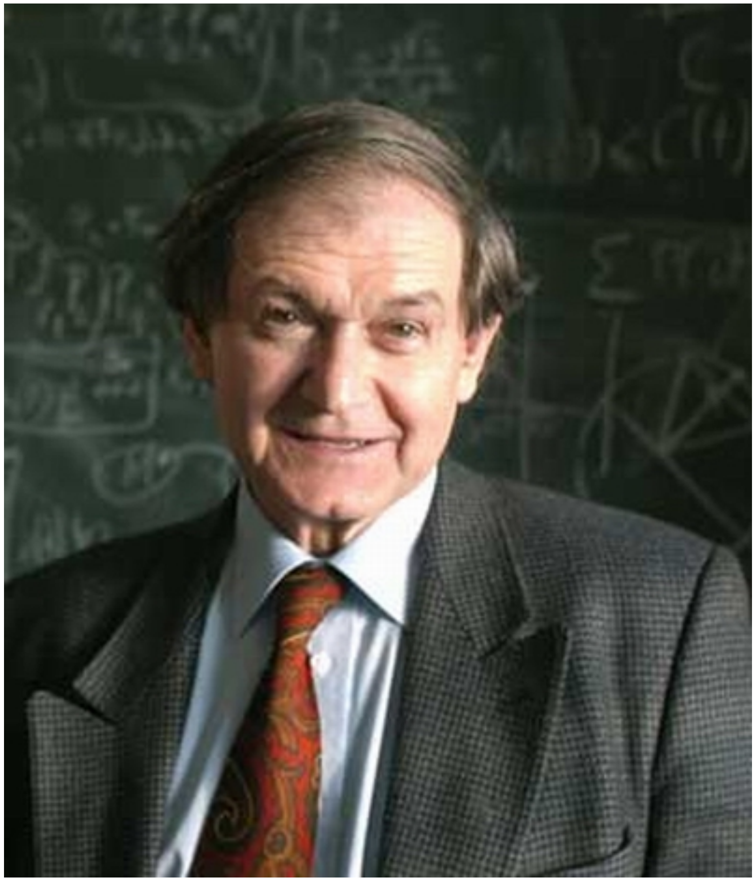

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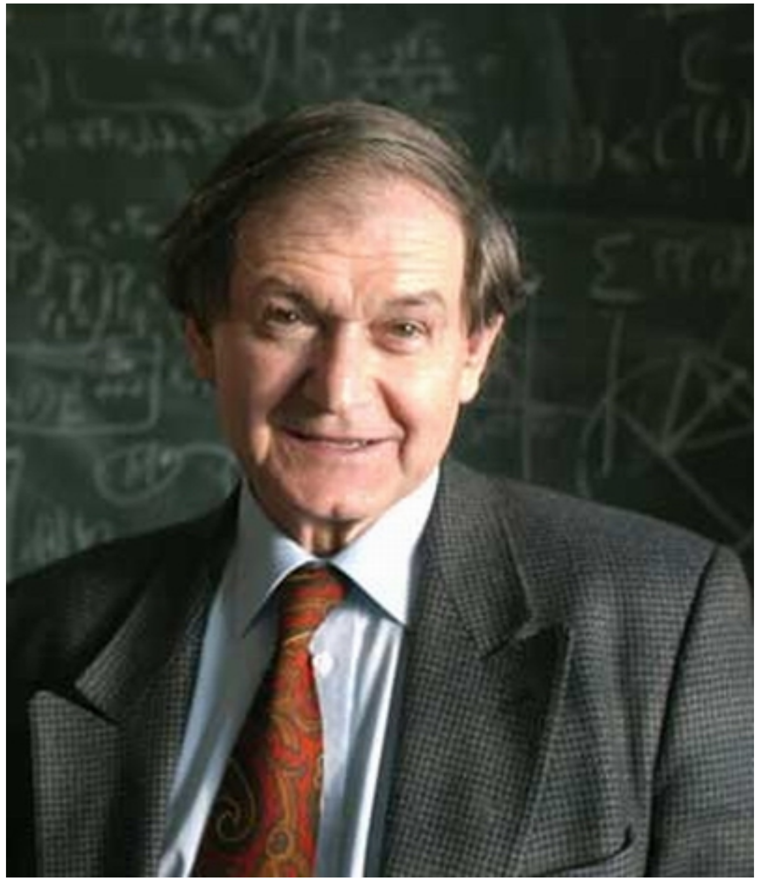

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Einstein's equations and complex geometry.

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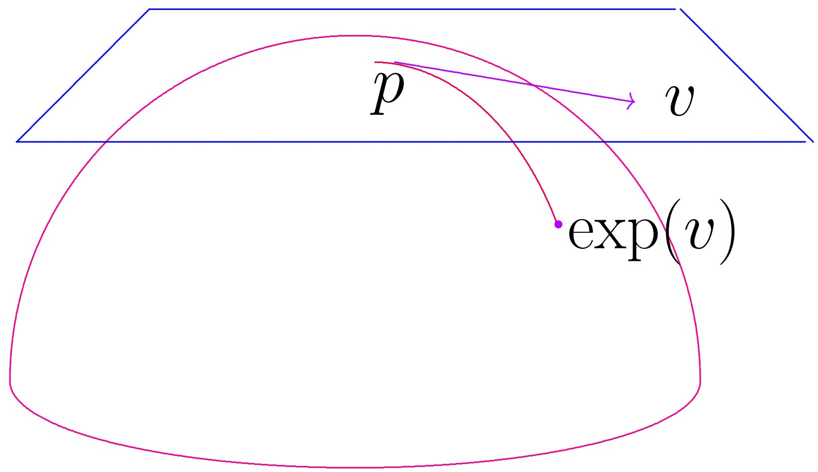
Manifestly special for Einstein metrics, too.

Ricci curvature measures
volume distortion by exponential map:



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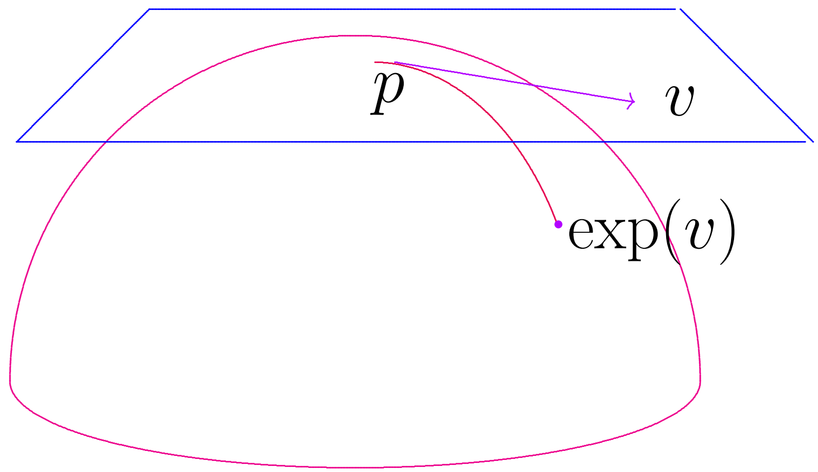

In "geodesic normal coordinates"
metric volume measure is

$$d\mu_g = \left[1 - \frac{1}{6}\, r_{jk}\, x^j x^k + O(|x|^3)\right] d\mu_{\mathrm{Euclidean}},$$

where r is the *Ricci tensor* $r_{jk} = \mathcal{R}^i{}_{jik}$.

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"...the greatest blunder of my life!"

— A. Einstein, to G. Gamow

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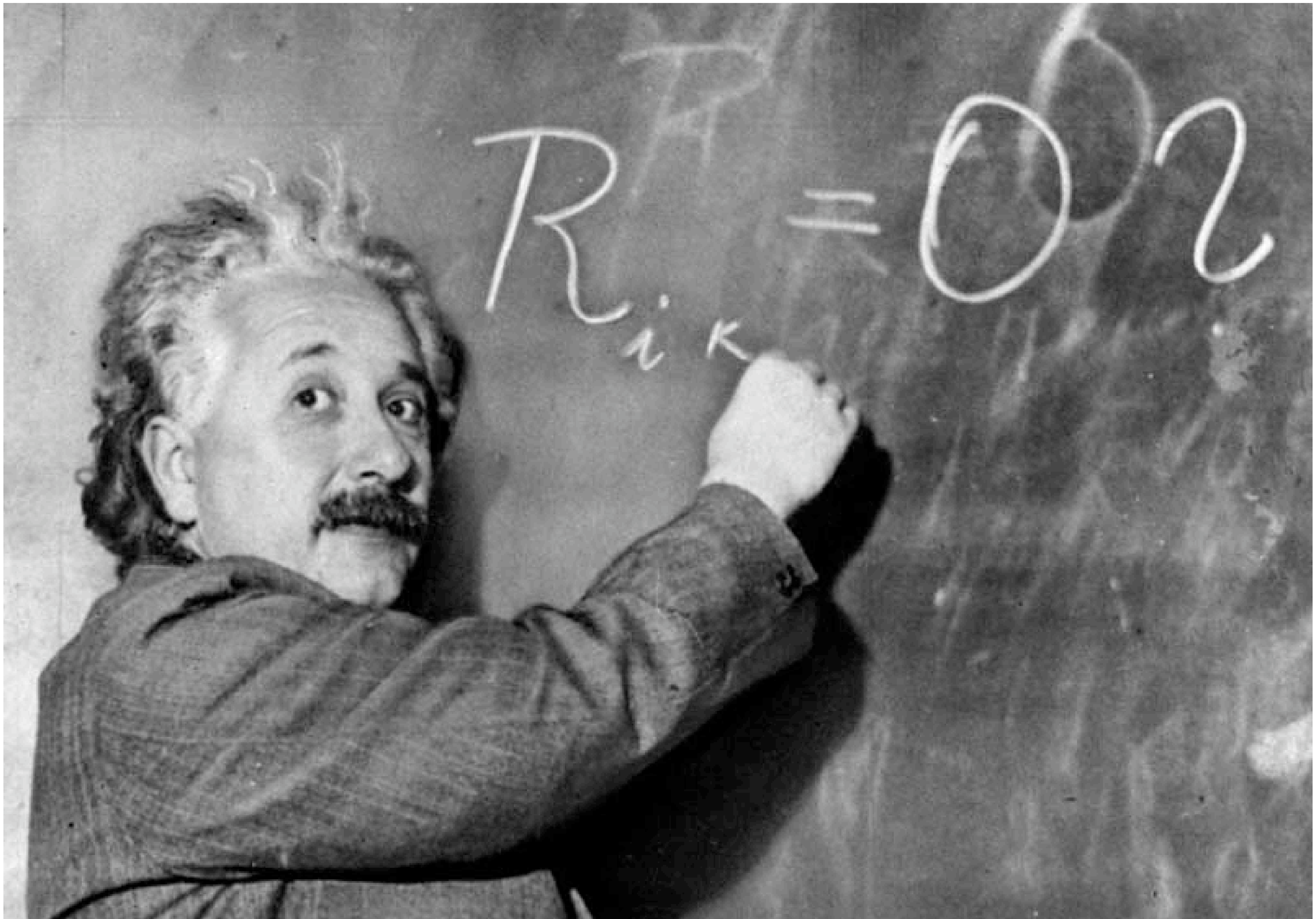

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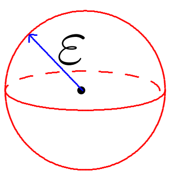
$$\frac{\mathrm{vol}_h(B_\varepsilon(p))}{c_n \varepsilon^n} = 1 - s\,\frac{\varepsilon^2}{6(n+2)} + O(\varepsilon^4)$$



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$[\omega] \in H^2(M, \mathbb{R})$ called the Kähler class.

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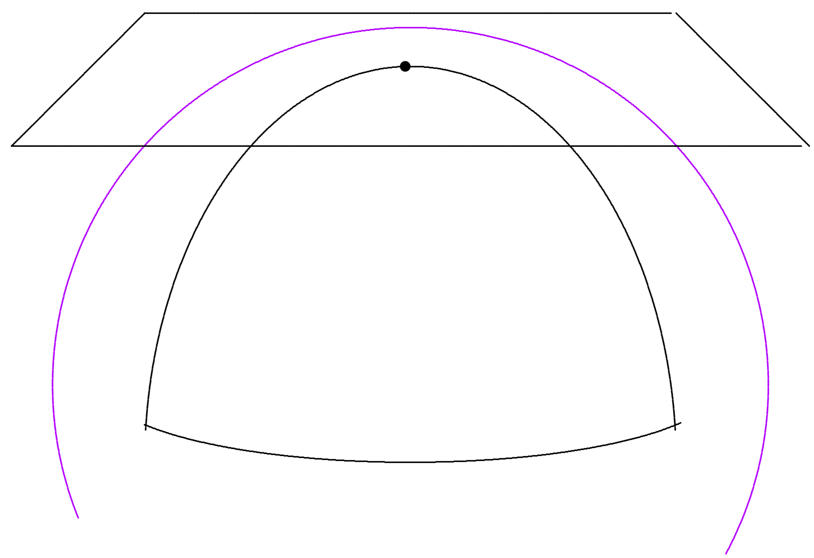

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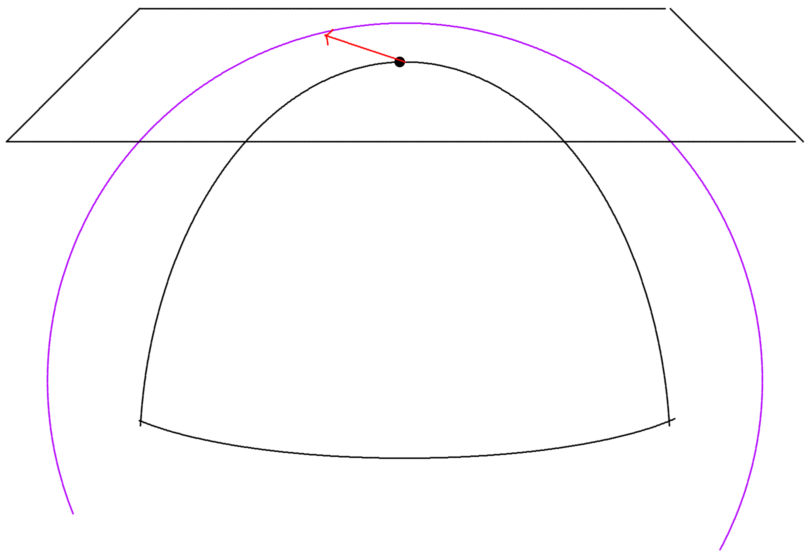

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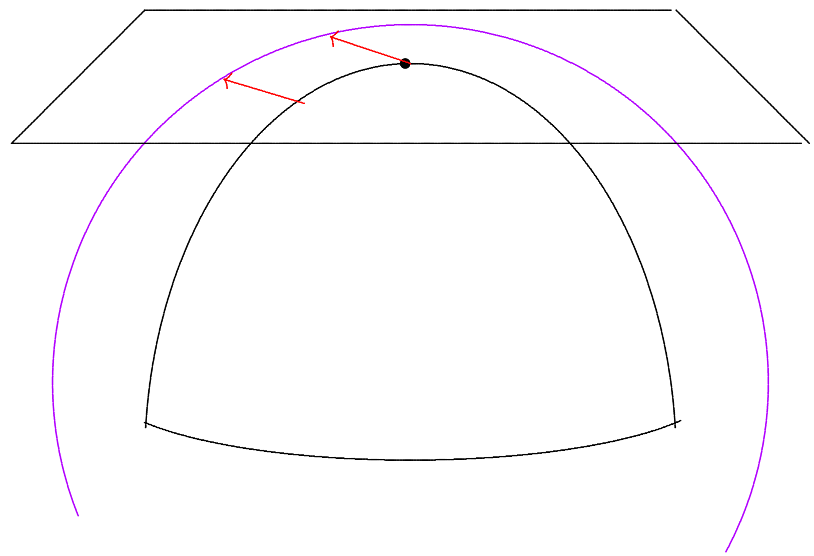

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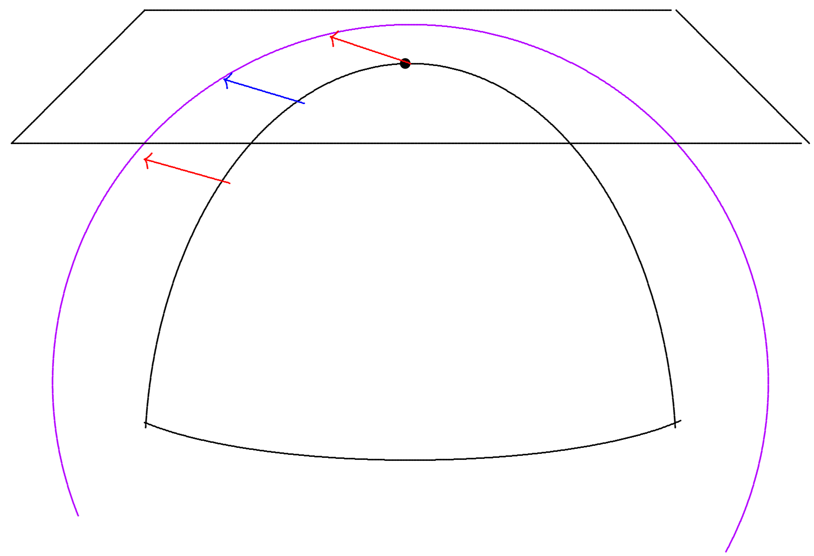

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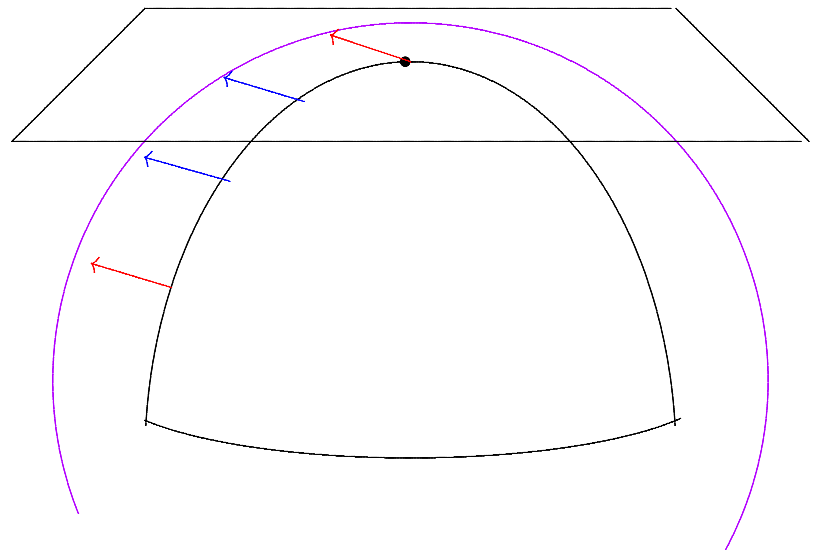

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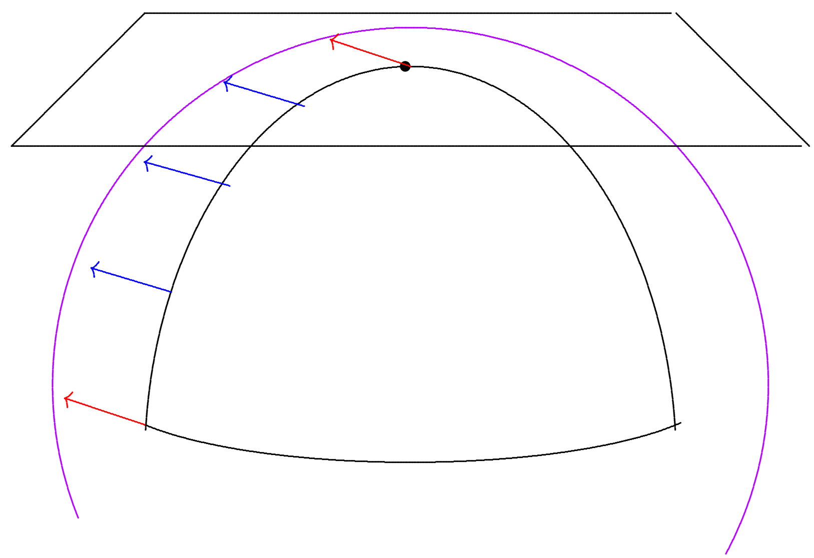

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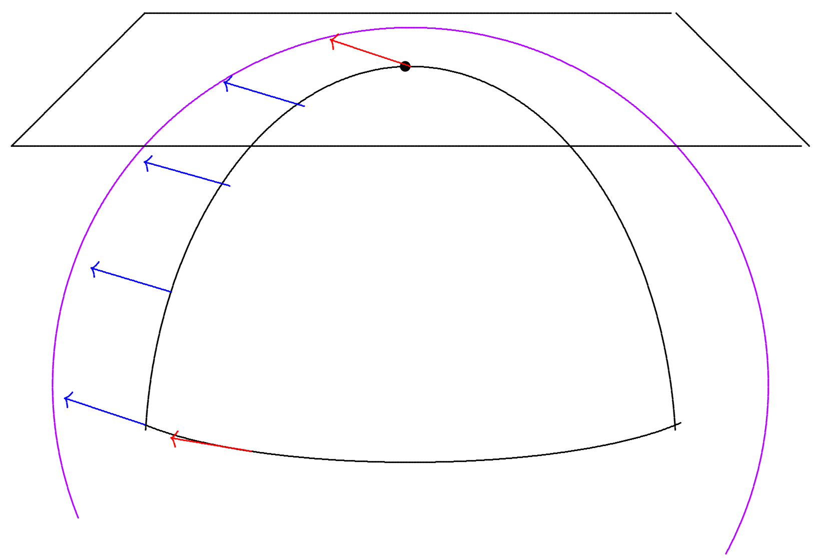

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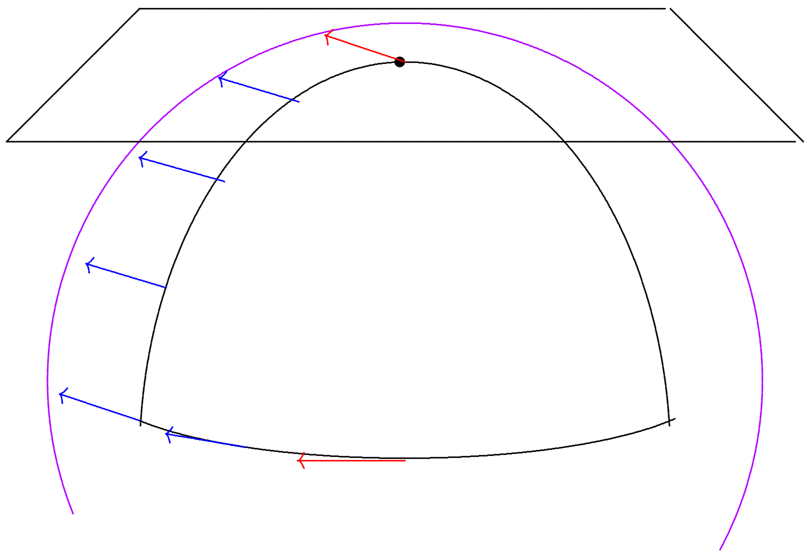

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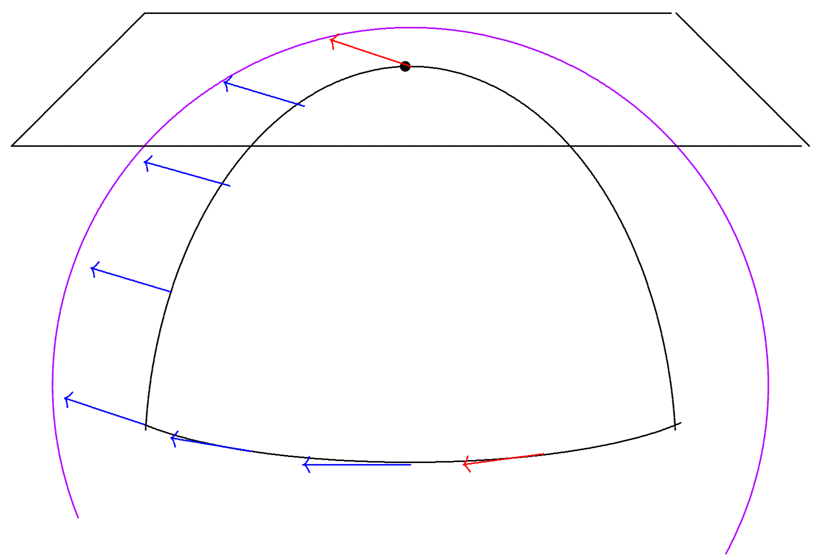

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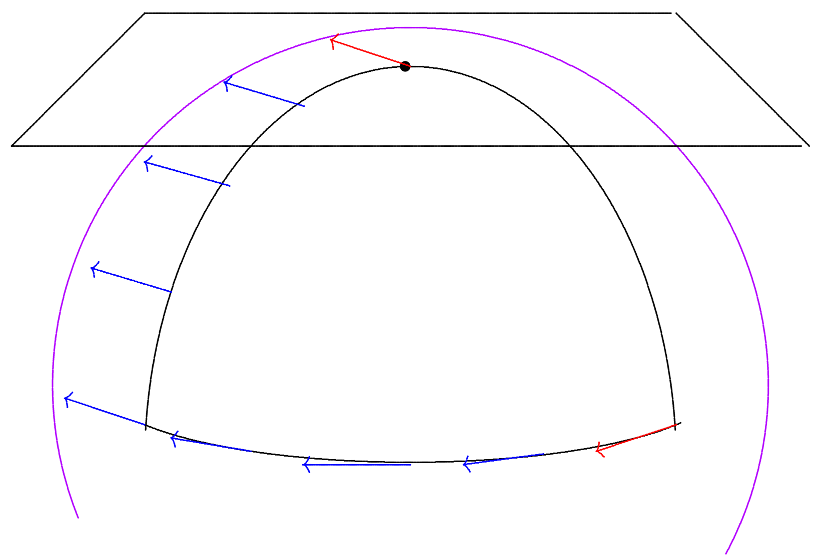

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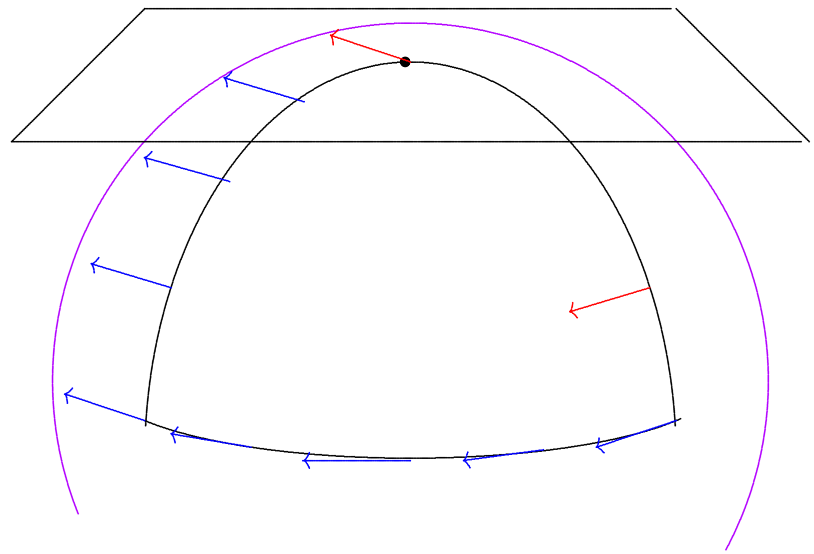

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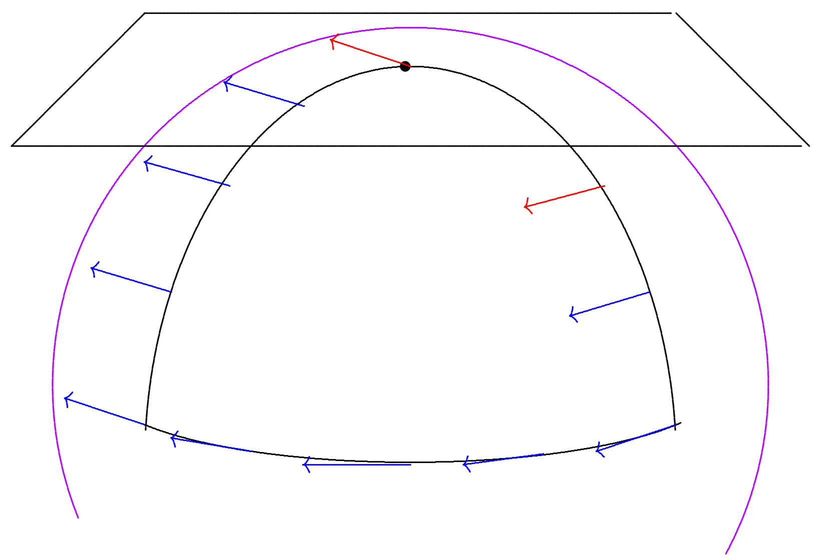

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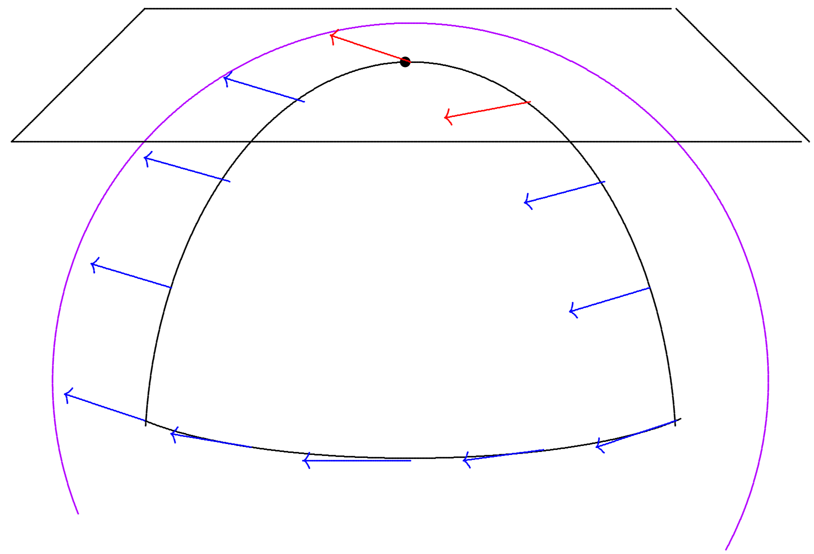

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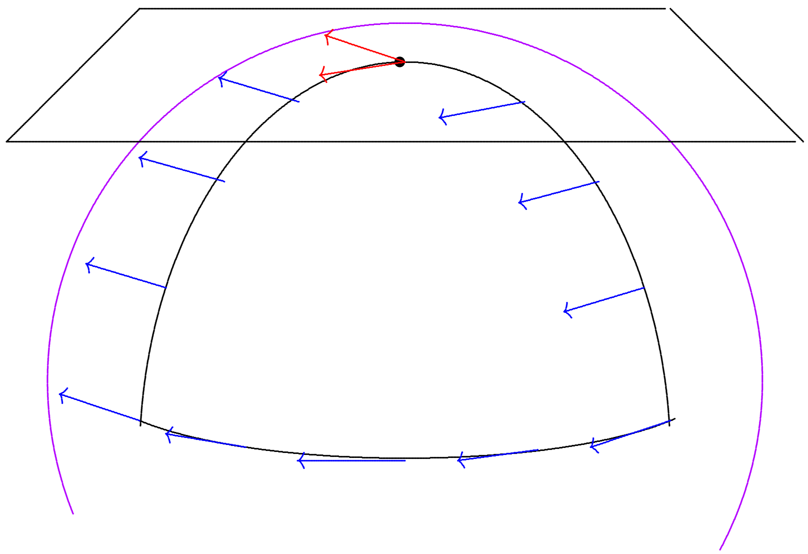

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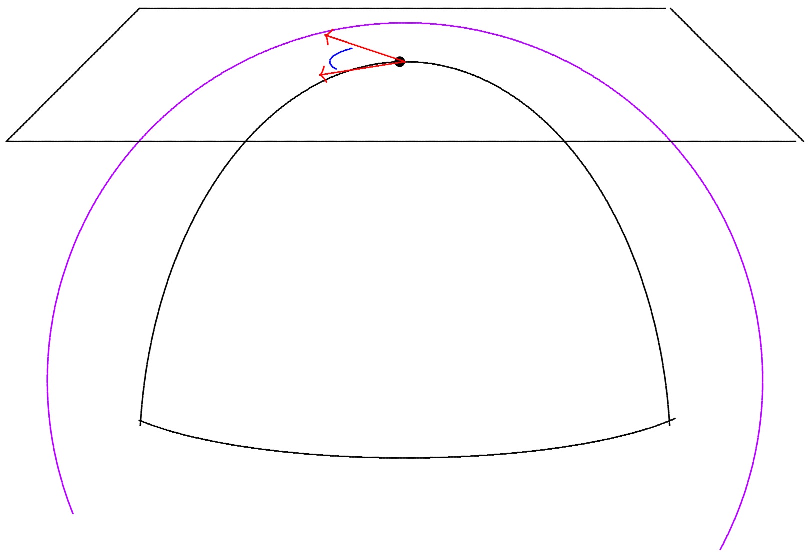
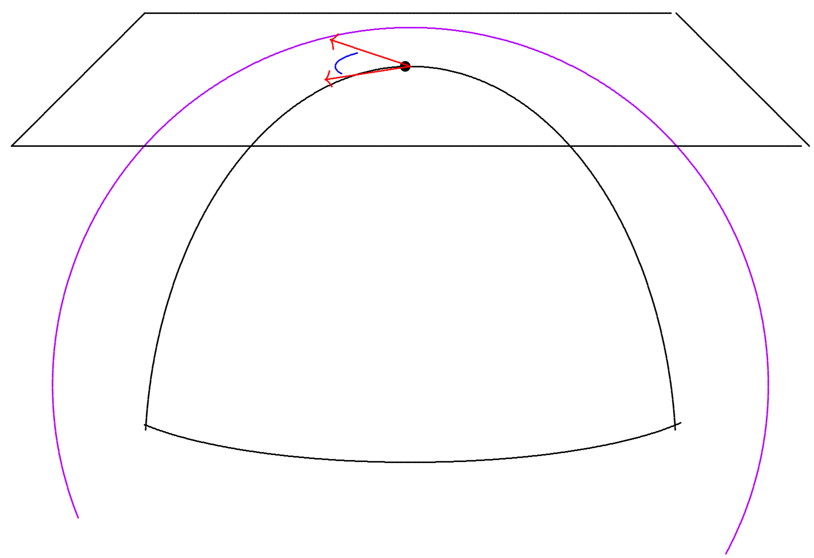


(M^n, g):   holonomy $\subset O(n)$



Kähler metrics:

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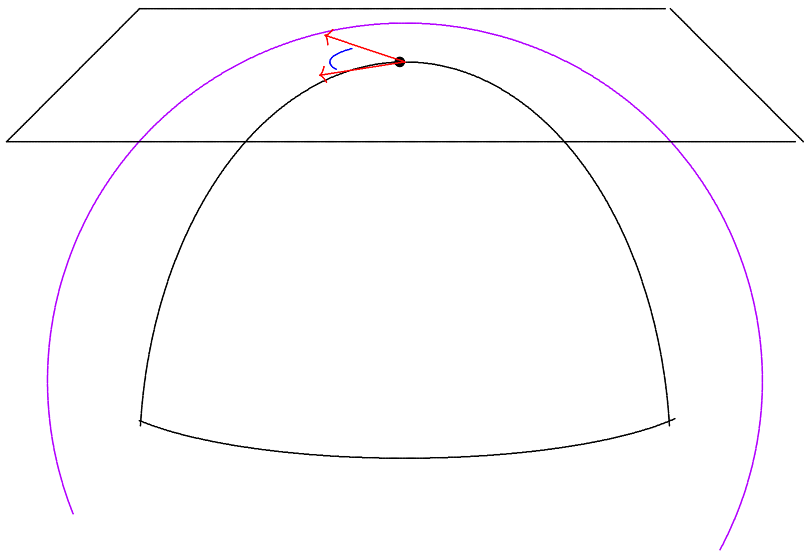

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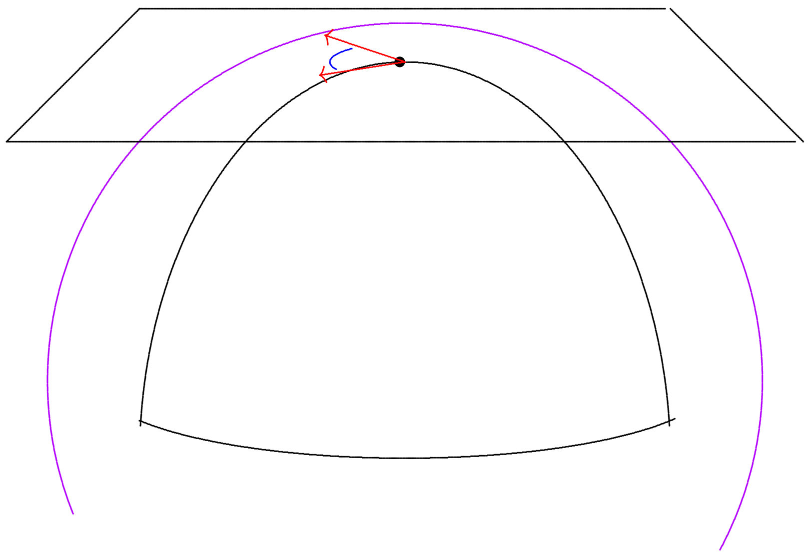

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is curvature of canonical line bundle $K = \Lambda^{m,0}$.

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In local complex coordinates

$$r_{j\bar{k}} = -\frac{\partial^2}{\partial z^j \partial \bar{z}^k} \log \det [h_{\ell\bar{m}}]$$

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Both are actually Hermitian.

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Recall:

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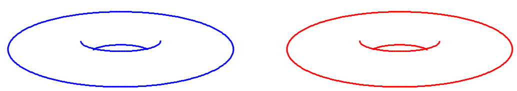

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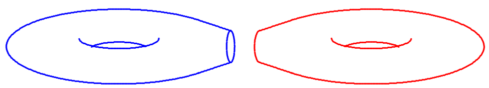

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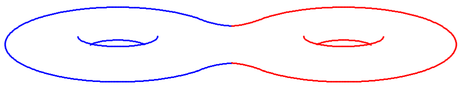

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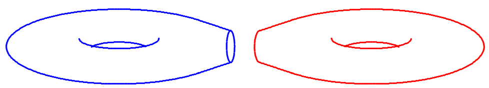

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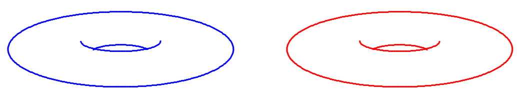

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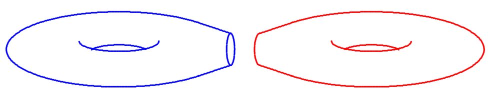

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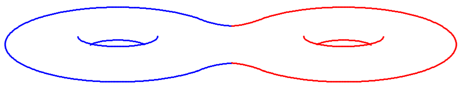

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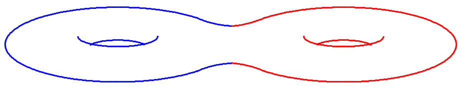

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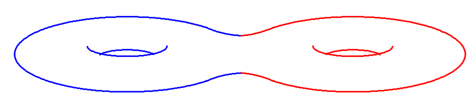

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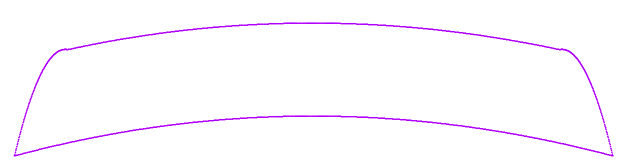

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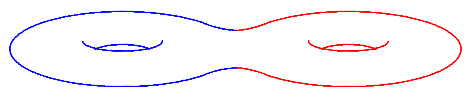

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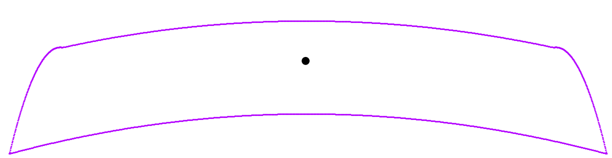

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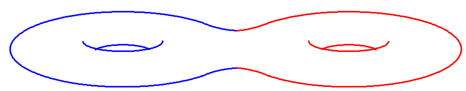

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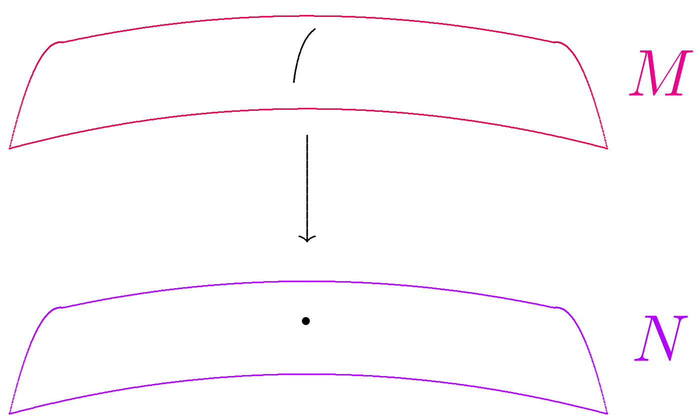

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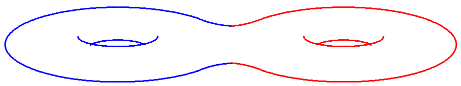

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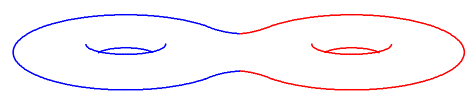

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Theorem A. *Let* (M^4, J) *be a compact complex surface, and suppose that* h *is an Einstein metric on* M *which is Hermitian with respect to* J*:*

$$h(J\cdot, J\cdot) = h.$$

Then either

- (M, J, h) *is Kähler-Einstein; or*
- $M \approx \mathbb{CP}_2 \# \overline{\mathbb{CP}}_2$*, and* h *is a constant times the Page metric; or*
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Warning: when h is non-Kähler, its relation to ω is surprisingly complicated!

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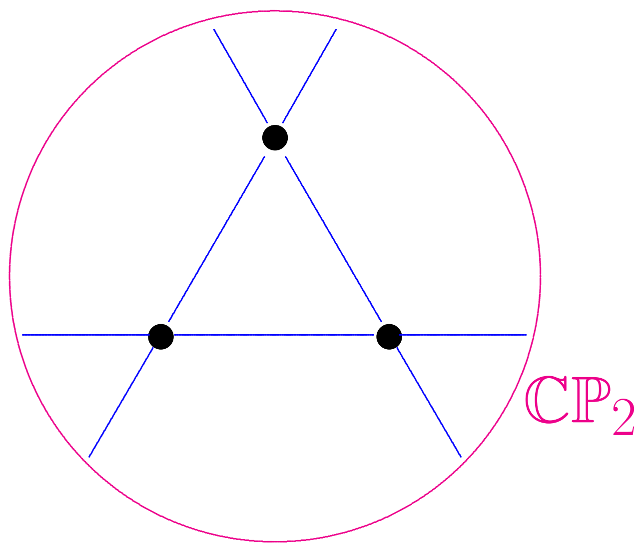

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In other words,

$$h = fg$$

$\exists$ Kähler metric g, smooth function $f : M \to \mathbb{R}^+$.

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Zeroes of $\xi^A \mapsto (W_+)_{ABCD}\xi^A\xi^B\xi^C\xi^D$ conj. inv.

$\Longrightarrow$ Petrov form $\{1111\}$, $\{22\}$, or $\{-\}$.

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$$W_{+} + \frac{s}{12} = \begin{pmatrix} 0 & & \\ & 0 & \\ & & \frac{s}{4} \end{pmatrix}$$

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Notice that W_{+} has a repeated eigenvalue:

$(W_{+})_{ABCD} = \frac{s}{8}\,\omega_{(AB}\,\omega_{CD)}$ type $\{22\}$ or $\{-\}$.

Lemma. *Let* (M^4, J) *be a compact complex surface, and suppose that* h *is an Einstein metric on* M *which is Hermitian with respect to* J*:*

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$\nabla \cdot W_+ = 0$, while $T^{1,0}M$ isotropic & involutive.

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conformally invariant with conformal weight.

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Derdziński. ASD only when hyper-Kähler: Boyer.

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Conformally related g must actually be be an extremal Kähler metric in sense of Calabi!

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X.X. Chen: always minimizers.

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Donaldson/Mabuchi/Chen-Tian:
unique in Kähler class, modulo bihomorphisms.

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In fact, for Kähler metrics,

$$B = \frac{1}{12}\left[2s\mathring{r} + \mathrm{Hess}_0(s) + 3J^*\mathrm{Hess}_0(s)\right]$$

where Hess_0 denotes trace-free part of $\nabla\nabla$.

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- *$g_t = g + tB$ is Kähler metric for small t.*

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So the critical metrics of restriction of $\mathcal{W}_+$ to {Kähler metrics} are Bach-flat Kähler metrics.

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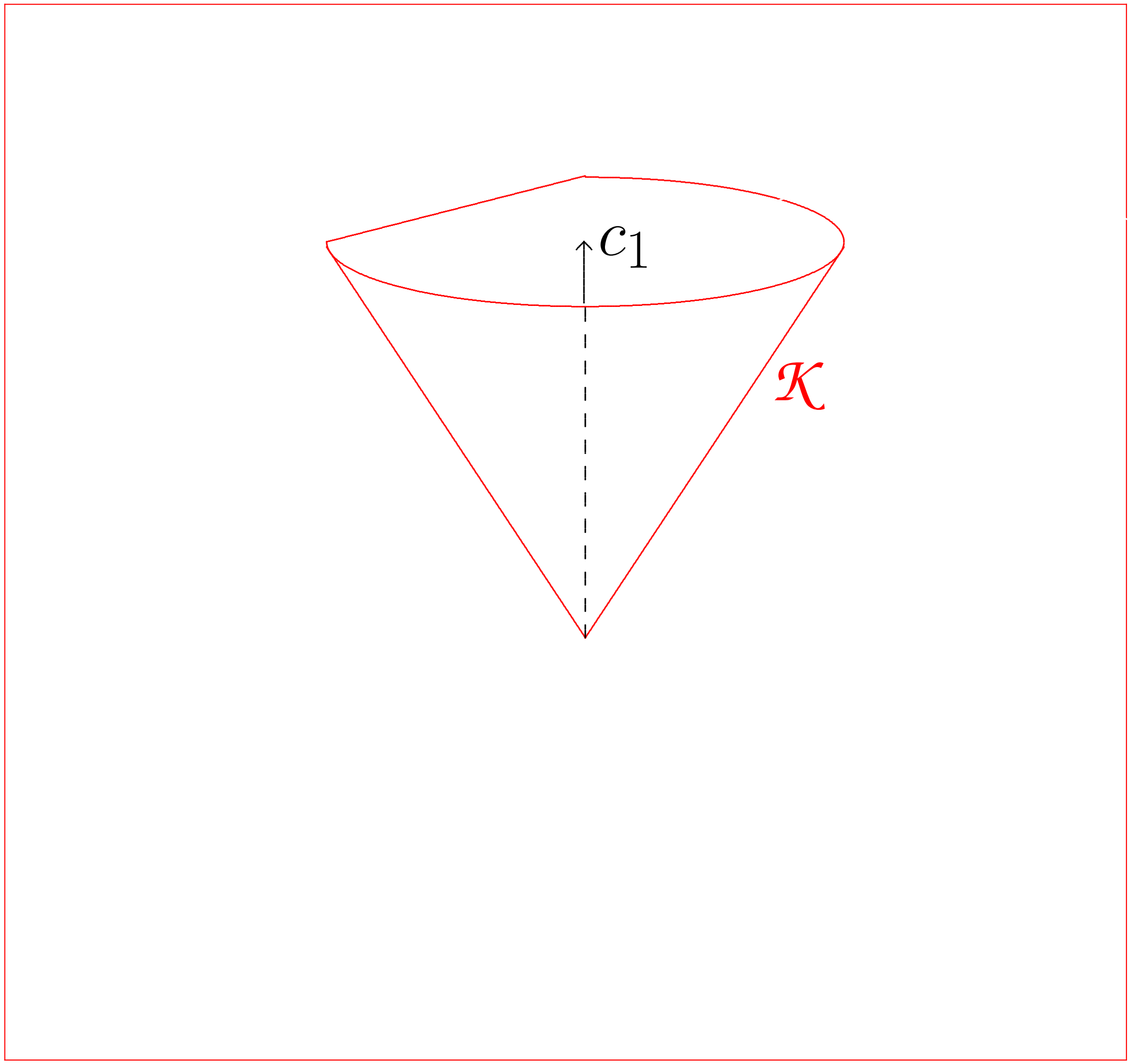

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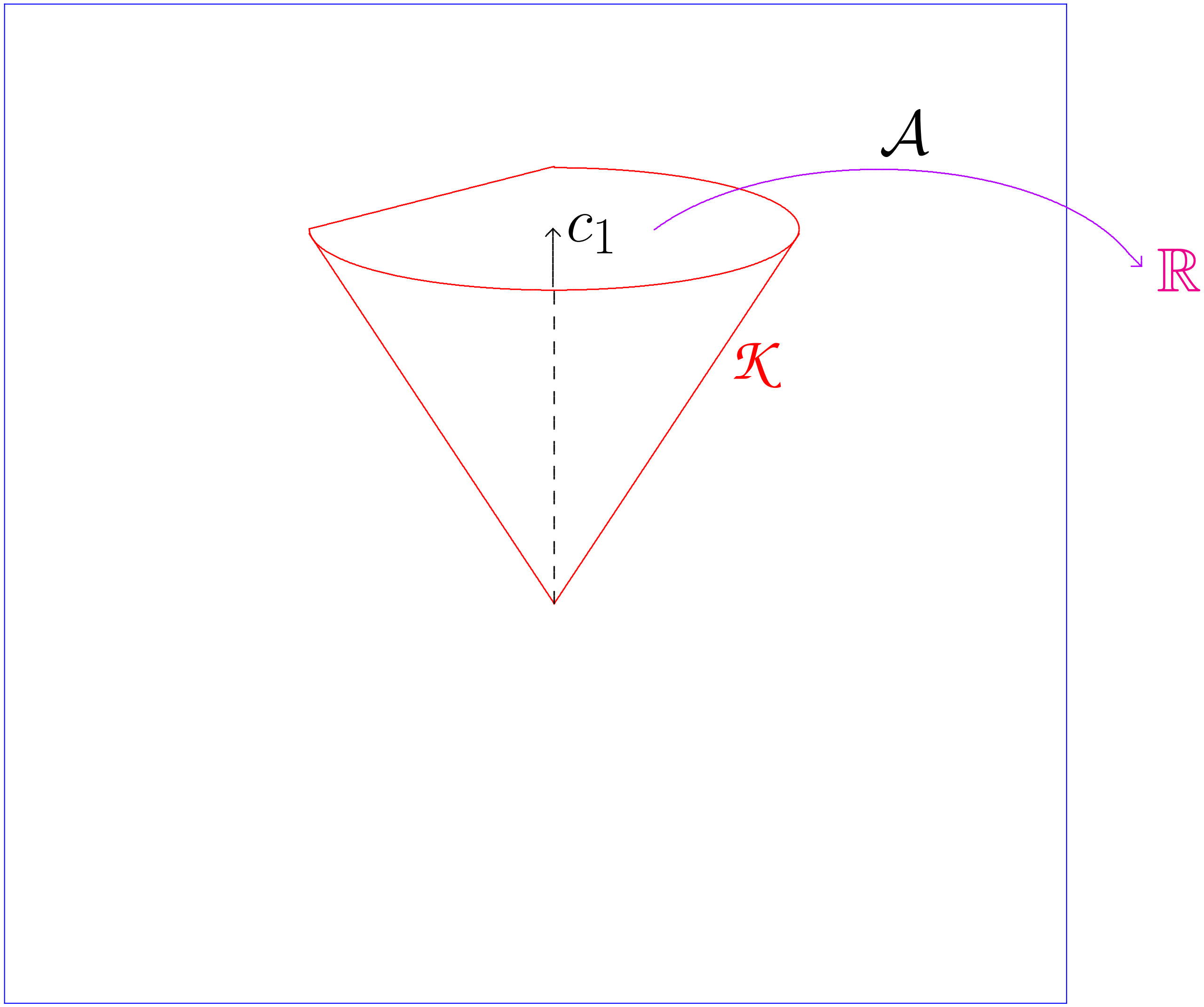

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$$0 = 6s^{-1}B = \mathring{r} + 2s^{-1}\mathrm{Hess}_0(s)$$

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Theorem. *Let* (M^4, J) *be a Del Pezzo surface. Then, up to automorphisms and rescaling, there is a unique Bach-flat Kähler metric* g *on* M. *This metric is characterized by the fact that it minimizes the Calabi functional*

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Hermitian, Einstein metric then given by

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and uniqueness Theorem A follows.

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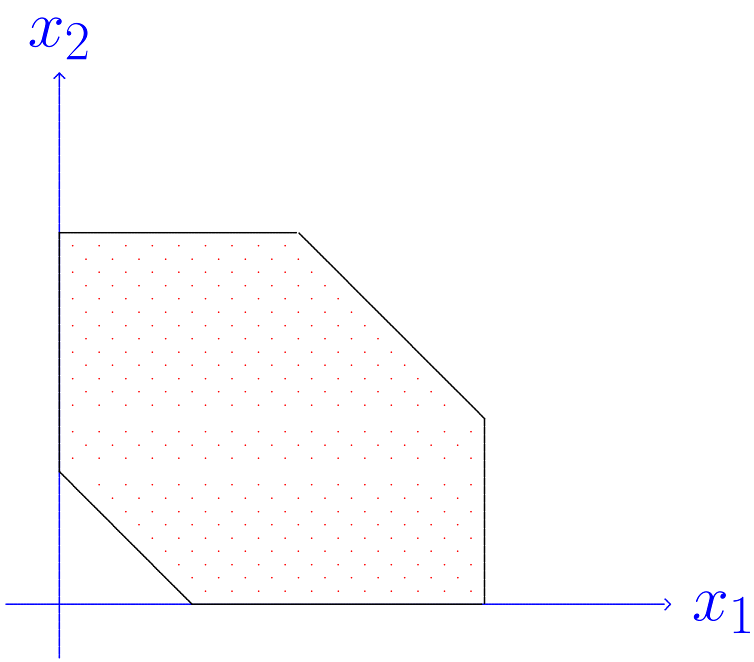
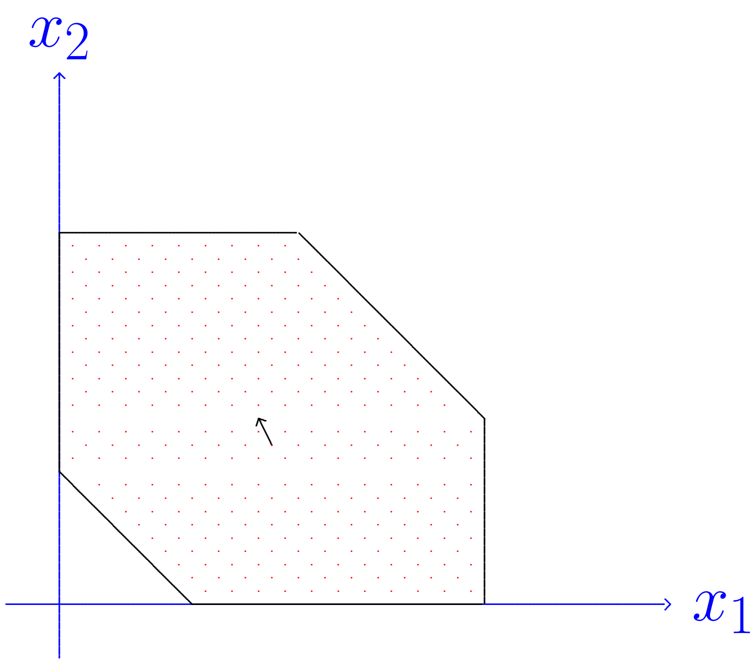
Only three cases are non-trivial:

$$\mathbb{CP}_2 \# k \overline{\mathbb{CP}}_2, \qquad k = 1, 2, 3.$$

The non-trivial cases are toric, and the action $\mathcal{A}$ can be directly computed from moment polygon.



The non-trivial cases are toric, and the action $\mathcal{A}$ can be directly computed from moment polygon. Formula involves barycenters, moments of inertia.



$$\mathcal{A}([\omega]) = \frac{|\partial P|^2}{2}\left(\frac{1}{|P|} + \vec{\mathfrak{D}} \cdot \Pi^{-1}\vec{\mathfrak{D}}\right)$$

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$$3\Big[3+28\gamma+96\gamma^2+168\gamma^3+164\gamma^4+80\gamma^5+16\gamma^6+16\beta^6(1+\gamma)^4+16\alpha^6(1+\beta+\gamma)^4+16\beta^5(5+24\gamma+43\gamma^2+37\gamma^3+15\gamma^4+2\gamma^5)+4\beta^4(41+228\gamma+478\gamma^2+496\gamma^3+263\gamma^4+$$

$$60\gamma^5+4\gamma^6)+8\beta^3(21+135\gamma+326\gamma^2+392\gamma^3+248\gamma^4+74\gamma^5+8\gamma^6)+4\beta(7+58\gamma+176\gamma^2+270\gamma^3+228\gamma^4+96\gamma^5+16\gamma^6)+4\beta^2(24+176\gamma+479\gamma^2+652\gamma^3+478\gamma^4+$$

$$172\gamma^5+24\gamma^6)+16\alpha^5(5+2\beta^5+24\gamma+43\gamma^2+37\gamma^3+15\gamma^4+2\gamma^5+\beta^4(15+14\gamma)+\beta^3(37+70\gamma+30\gamma^2)+\beta^2(43+123\gamma+108\gamma^2+30\gamma^3)+\beta(24+92\gamma+123\gamma^2+70\gamma^3+$$

$$14\gamma^4))+4\alpha^4(41+4\beta^6+228\gamma+478\gamma^2+496\gamma^3+263\gamma^4+60\gamma^5+4\gamma^6+\beta^5(60+56\gamma)+\beta^4(263+476\gamma+196\gamma^2)+8\beta^3(62+169\gamma+139\gamma^2+35\gamma^3)+2\beta^2(239+876\gamma+1089\gamma^2+$$

$$556\gamma^3+98\gamma^4)+4\beta(57+263\gamma+438\gamma^2+338\gamma^3+119\gamma^4+14\gamma^5))+8\alpha^3(21+135\gamma+326\gamma^2+392\gamma^3+248\gamma^4+74\gamma^5+8\gamma^6+8\beta^6(1+\gamma)+2\beta^5(37+70\gamma+30\gamma^2)+4\beta^4(62+$$

$$169\gamma+139\gamma^2+35\gamma^3)+4\beta^3(98+353\gamma+428\gamma^2+210\gamma^3+35\gamma^4)+2\beta^2(163+735\gamma+1179\gamma^2+856\gamma^3+278\gamma^4+30\gamma^5)+\beta(135+736\gamma+1470\gamma^2+1412\gamma^3+676\gamma^4+140\gamma^5+$$

$$8\gamma^6))+4\alpha(7+58\gamma+176\gamma^2+270\gamma^3+228\gamma^4+96\gamma^5+16\gamma^6+16\beta^6(1+\gamma)^3+4\beta^5(24+92\gamma+123\gamma^2+70\gamma^3+14\gamma^4)+4\beta^4(57+263\gamma+438\gamma^2+338\gamma^3+119\gamma^4+14\gamma^5)+$$

$$2\beta^3(135+736\gamma+1470\gamma^2+1412\gamma^3+676\gamma^4+140\gamma^5+8\gamma^6)+4\beta^2(44+278\gamma+645\gamma^2+735\gamma^3+438\gamma^4+123\gamma^5+12\gamma^6)+2\beta(29+210\gamma+556\gamma^2+736\gamma^3+526\gamma^4+184\gamma^5+$$

$$24\gamma^6))+4\alpha^2(24+176\gamma+479\gamma^2+652\gamma^3+478\gamma^4+172\gamma^5+24\gamma^6+24\beta^6(1+\gamma)^2+4\beta^5(43+123\gamma+108\gamma^2+30\gamma^3)+2\beta^4(239+876\gamma+1089\gamma^2+556\gamma^3+98\gamma^4)+4\beta^3(163+$$

$$735\gamma+1179\gamma^2+856\gamma^3+278\gamma^4+30\gamma^5)+4\beta(44+278\gamma+645\gamma^2+735\gamma^3+438\gamma^4+123\gamma^5+12\gamma^6)+\beta^2(479+2580\gamma+5058\gamma^2+4716\gamma^3+2178\gamma^4+432\gamma^5+24\gamma^6))\Big]\Big/$$

$$\Big[1+10\gamma+36\gamma^2+64\gamma^3+60\gamma^4+24\gamma^5+24\beta^5(1+\gamma)^5+24\alpha^5(1+\beta+\gamma)^5+12\beta^4(1+\gamma)^2(5+20\gamma+23\gamma^2+10\gamma^3)+16\beta^3(4+28\gamma+72\gamma^2+90\gamma^3+57\gamma^4+15\gamma^5)+$$

$$12\beta^2(3+24\gamma+69\gamma^2+96\gamma^3+68\gamma^4+20\gamma^5)+2\beta(5+45\gamma+144\gamma^2+224\gamma^3+180\gamma^4+60\gamma^5)+12\alpha^4(1+\beta+\gamma)^2(5+20\gamma+23\gamma^2+10\gamma^3+10\beta^3(1+\gamma)+\beta^2(23+46\gamma+$$

$$16\gamma^2)+2\beta(10+30\gamma+23\gamma^2+5\gamma^3))+16\alpha^3(4+28\gamma+72\gamma^2+90\gamma^3+57\gamma^4+15\gamma^5+15\beta^5(1+\gamma)^2+3\beta^4(19+57\gamma+50\gamma^2+13\gamma^3)+3\beta^3(30+120\gamma+155\gamma^2+78\gamma^3+$$

$$13\gamma^4)+3\beta^2(24+120\gamma+206\gamma^2+155\gamma^3+50\gamma^4+5\gamma^5)+\beta(28+168\gamma+360\gamma^2+360\gamma^3+171\gamma^4+30\gamma^5))+12\alpha^2(3+24\gamma+69\gamma^2+96\gamma^3+68\gamma^4+20\gamma^5+20\beta^5(1+\gamma)^3+$$

$$\beta^4(68+272\gamma+366\gamma^2+200\gamma^3+36\gamma^4)+4\beta^3(24+120\gamma+206\gamma^2+155\gamma^3+50\gamma^4+5\gamma^5)+2\beta(12+84\gamma+207\gamma^2+240\gamma^3+136\gamma^4+30\gamma^5)+\beta^2(69+414\gamma+864\gamma^2+$$

$$824\gamma^3+366\gamma^4+60\gamma^5))+2\alpha(5+45\gamma+144\gamma^2+224\gamma^3+180\gamma^4+60\gamma^5+60\beta^5(1+\gamma)^4+12\beta^4(15+75\gamma+136\gamma^2+114\gamma^3+43\gamma^4+5\gamma^5)+12\beta^2(12+84\gamma+207\gamma^2+$$

$$240\gamma^3+136\gamma^4+30\gamma^5)+8\beta^3(28+168\gamma+360\gamma^2+360\gamma^3+171\gamma^4+30\gamma^5)+3\beta(15+120\gamma+336\gamma^2+448\gamma^3+300\gamma^4+80\gamma^5))\Big]$$

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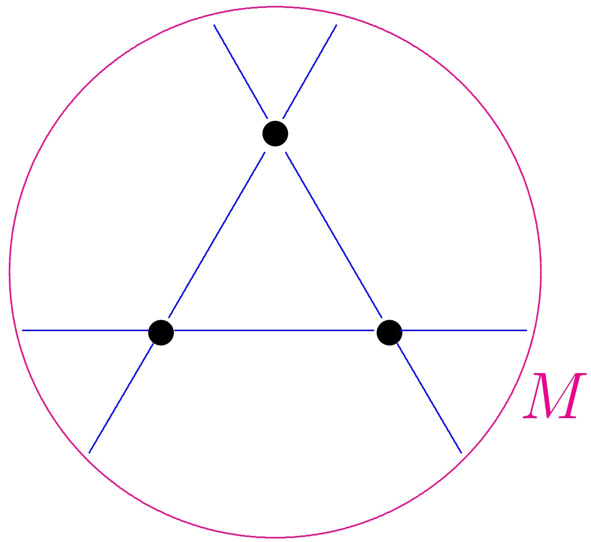
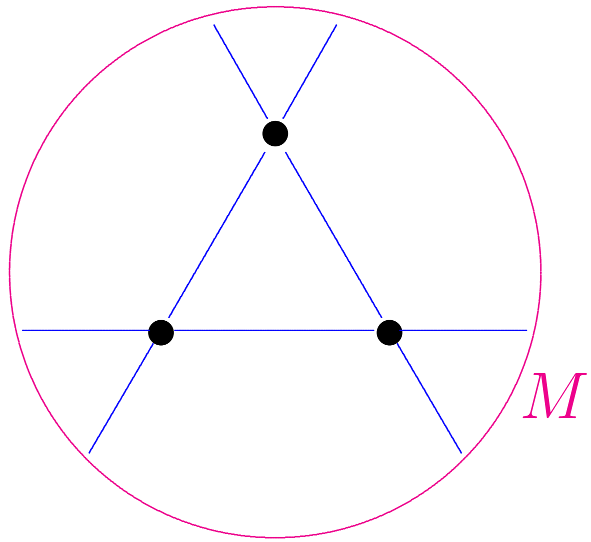

M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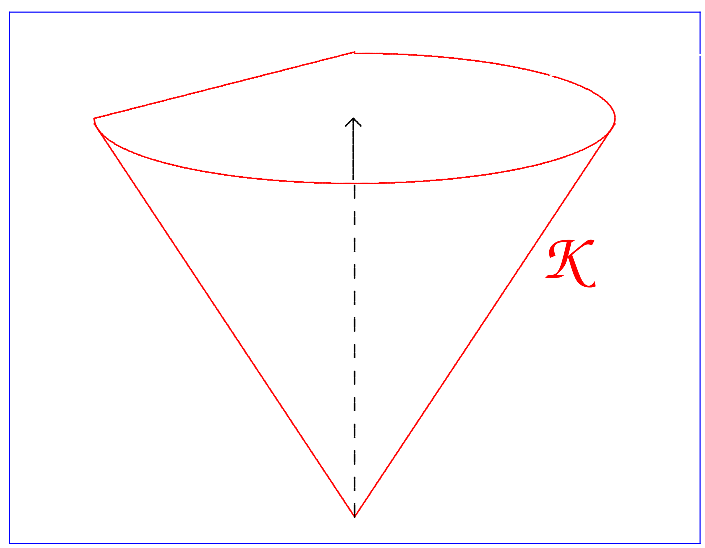
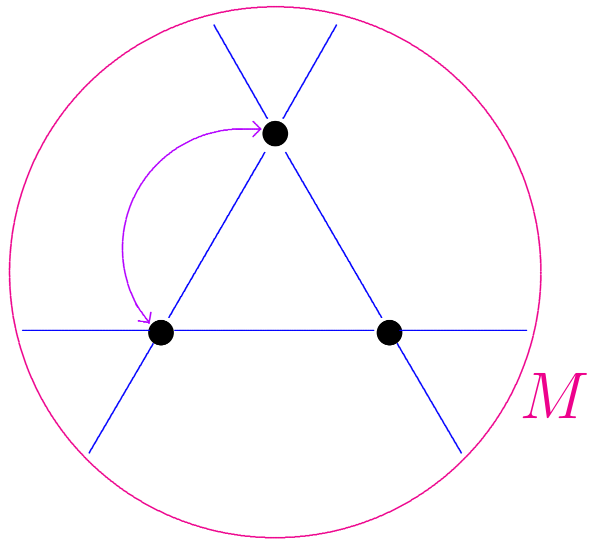

$\mathcal{K}$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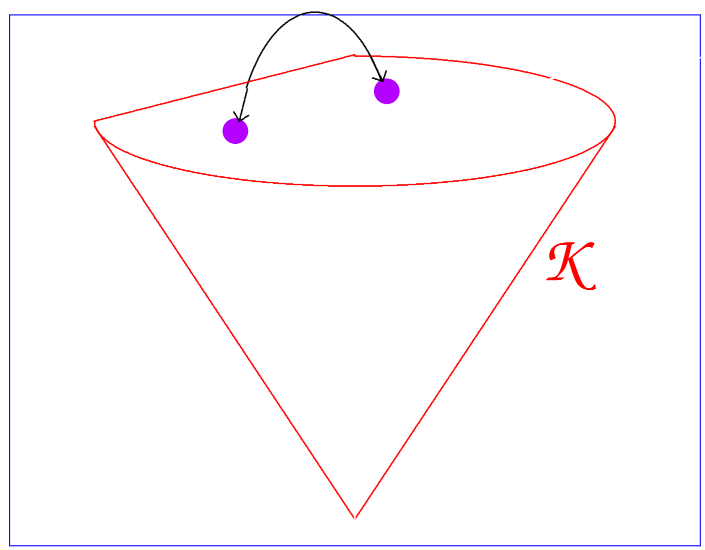
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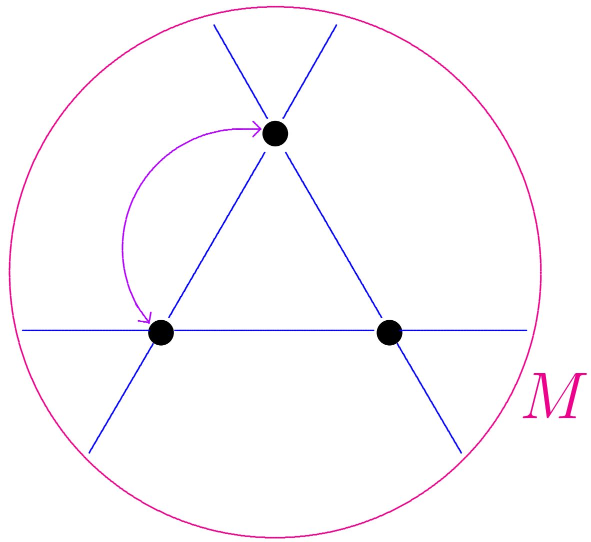
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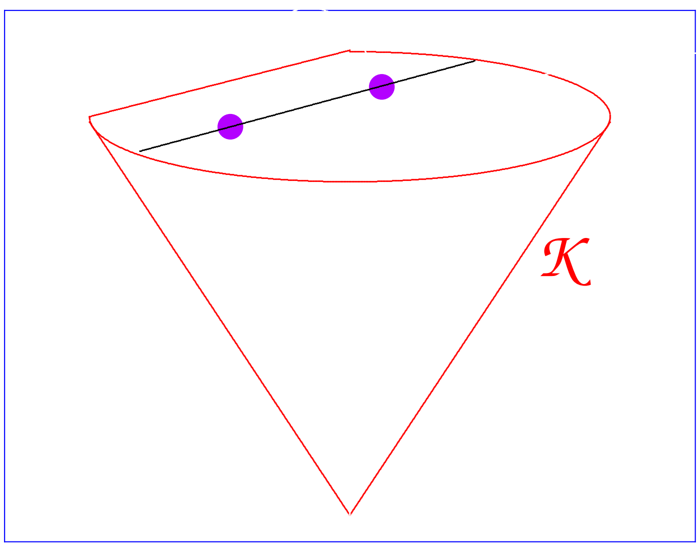
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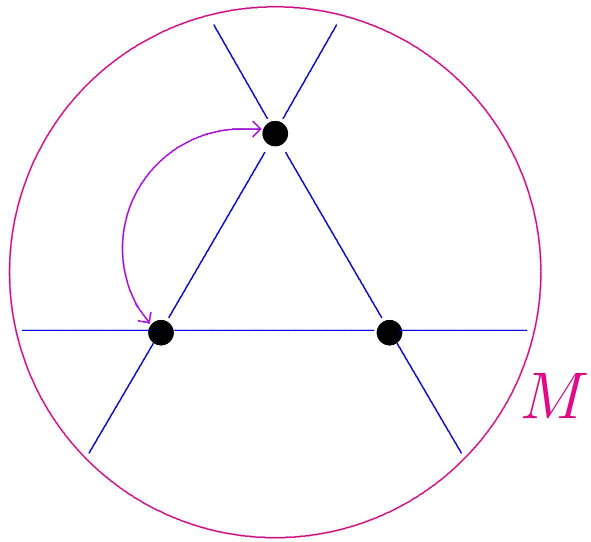
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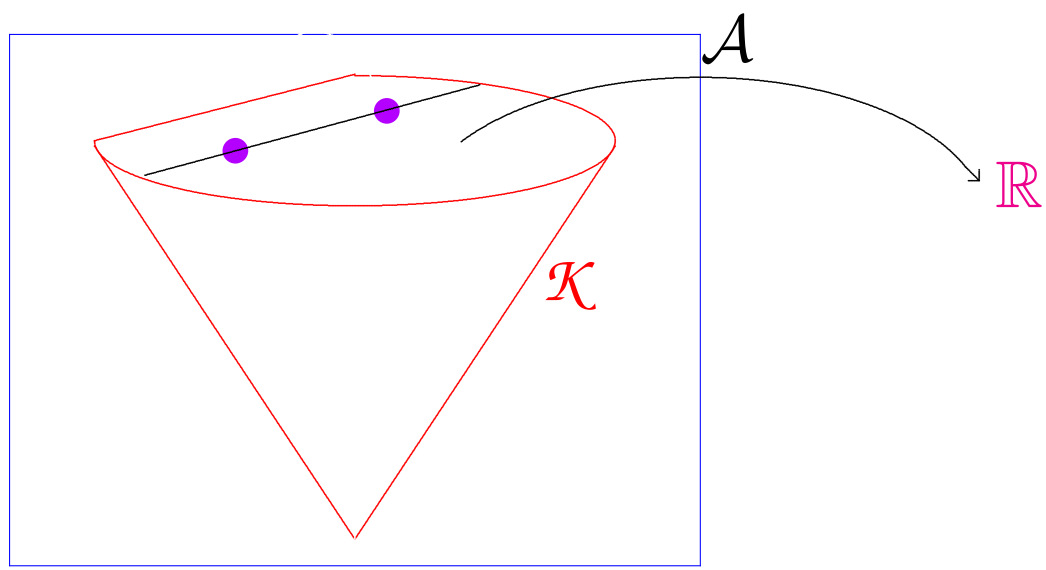
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Final step then just calculus in one variable. . .

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Similar calculations also led to new existence proof. . .

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Could also reconstruct Page metric this way. . .