Conformal foliations and CR geometry

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[joint work with Paul Baird]

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Disclaimers and references

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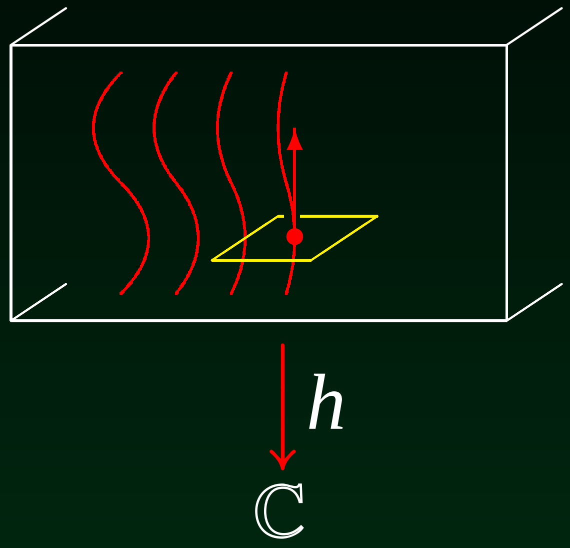
Conformal foliations

U = unit vector field on $\Omega^{\text{open}} \subseteq \mathbb{R}^3$.



U is (transversally) conformal $\Leftrightarrow \mathcal{L}_U$ preserves the conformal metric orthogonal to its leaves

isothermal coördinates

$\rightsquigarrow$ $h = f + ig \qquad \langle \nabla f, \nabla g \rangle = 0$

$\|\nabla f\| = \|\nabla g\|$

conjugate functions

Conjugate functions on $\mathbf{R}^2$

$$f = f(r,s) \quad g = g(r,s) \quad \text{s.t.} \begin{cases} \langle \nabla f, \nabla g \rangle = 0 \\ \|\nabla f\| = \|\nabla g\| \end{cases}$$

- $\boxed{f = r \quad g = s}$
- $\boxed{f = r^2 - s^2 \quad g = 2rs}$
- $\boxed{f = \dfrac{r}{r^2 + s^2} \quad g = \dfrac{s}{r^2 + s^2}}$
- $\boxed{f = e^r \cos s \quad g = e^r \sin s}$

$h \equiv f + ig$ is (anti-)holomorphic in $z \equiv r + is$

Conjugate functions on R^3

$$f = f(q,r,s) \quad g = g(q,r,s) \quad \text{s.t.} \begin{cases} \langle \nabla f, \nabla g \rangle = 0 \\ \|\nabla f\| = \|\nabla g\| \end{cases}$$

- $\boxed{f = r \quad g = s}$
- $\boxed{f = q^2 - r^2 - s^2 \quad g = 2q\sqrt{r^2 + s^2}}$
- $\boxed{f = r\dfrac{q^2 + r^2 + s^2}{r^2 + s^2} \quad g = s\dfrac{q^2 + r^2 + s^2}{r^2 + s^2}}$
- $$f = \frac{(1 - q^2 - r^2 - s^2)r + 2qs}{r^2 + s^2}$$ $$g = \frac{(1 - q^2 - r^2 - s^2)s - 2qr}{r^2 + s^2}$$

$$\begin{array}{ccc} \mathbb{R}^3 \hookrightarrow S^3 & & \\ \downarrow & \text{Hopf} & \\ \mathbb{R}^2 \leftarrow S^2 \setminus \{*\} & & \end{array}$$

Almost Hermitian structures

NB: $J(p,q,r,s) : \mathbb{R}^4 \to \mathbb{R}^4$ satisfies

- $J^2 = -\mathrm{Id}$
- $J \in SO(4)$

$$\Longleftrightarrow J = \begin{bmatrix} 0 & -u & -v & -w \\ u & 0 & -w & v \\ v & w & o & -u \\ w & -v & u & 0 \end{bmatrix}$$

$u^2 + v^2 + w^2 = 1$, two-sphere

Consider $\mathbb{R}^3 = \{(p,q,r,s) \in \mathbb{R}^4 \,|\, p = 0\} \subset \mathbb{R}^4$

NB: $U \equiv \left(J\frac{\partial}{\partial p}\right)\Big|_{\mathbb{R}^3} = \left(u\frac{\partial}{\partial q} + v\frac{\partial}{\partial r} + w\frac{\partial}{\partial s}\right)\Big|_{\mathbb{R}^3}$

unit vector field

also $\rightsquigarrow$ two-sphere

Sphere bundles

bundle of
unit vectors

bundle of almost
Hermitian structues

$$\begin{tikzcd} Q_\circ \arrow[d] \arrow[phantom, r, "\subset"] & Z_\circ \arrow[d, "\tau"'] \\ \mathbb{R}^3 \arrow[u, bend left] \arrow[phantom, r, "\subset"] & \mathbb{R}^4 \arrow[u, bend right] \end{tikzcd}$$

section
$\updownarrow$
unit vector field

section
$\updownarrow$
almost Hermitian structure

Hermitian structures

Lemma

J is integrable $\Longrightarrow U \equiv \left(J\frac{\partial}{\partial p}\right)\Big|_{\mathbb{R}^3}$ is conformal

Conversely??

NB: J integrable $\Longrightarrow J$ real-analytic

Question: U conformal $\Longrightarrow U$ real-analytic??

Answer: $\boxed{\boxed{\text{NO!}}}$

However: U real-analytic and conformal
$\Longrightarrow U$ extends uniquely to an integrable J.

WHY?

Twistor geometry

bundle of almost
Hermitian structues

$$
\begin{array}{ccccl}
Q_\circ & \subset & Z_\circ & = & \mathbb{CP}_3 \setminus \{z_3 = z_4 = 0\} \ni [z_1, z_2, z_3, z_4] \\
\downarrow & & \tau\downarrow & & \qquad \tau\downarrow \qquad\qquad\qquad\qquad \downarrow \\
\mathbb{R}^3 & \subset & \mathbb{R}^4 & = & \mathbb{C}^2 \ni \begin{pmatrix} p + iq \\ r + is \end{pmatrix} = \dfrac{1}{|z_3|^2 + |z_4|^2} \begin{pmatrix} z_2\bar{z}_3 + z_4\bar{z}_1 \\ z_1\bar{z}_3 - z_4\bar{z}_2 \end{pmatrix}
\end{array}
$$

compactify

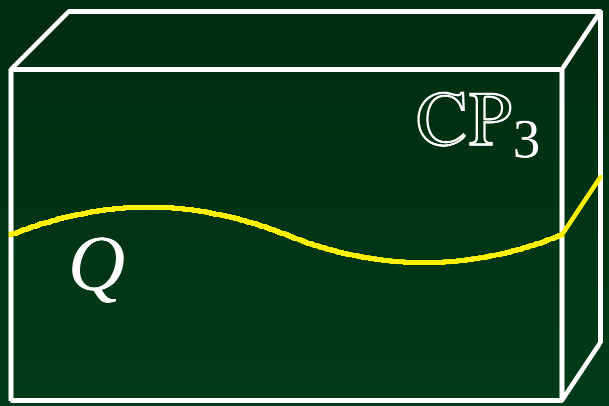
$$
\begin{array}{c}
\mathbb{CP}_3 \\
\tau\downarrow \\
S^4
\end{array}
$$

twistor fibration (cf. Hopf)

Twistor geometry cont'd

$$\begin{tikzcd} Q_\circ \arrow[d] & \subset & Z_\circ \arrow[d] \\ \mathbb{R}^3 & \subset & \mathbb{R}^4 \end{tikzcd} \quad \underset{\hookrightarrow}{\text{compactify}} \quad \begin{tikzcd} Q \arrow[d] & \subset & \mathbb{CP}_3 \arrow[d, "\tau"'] \\ S^3 & \subset & S^4 \end{tikzcd}$$

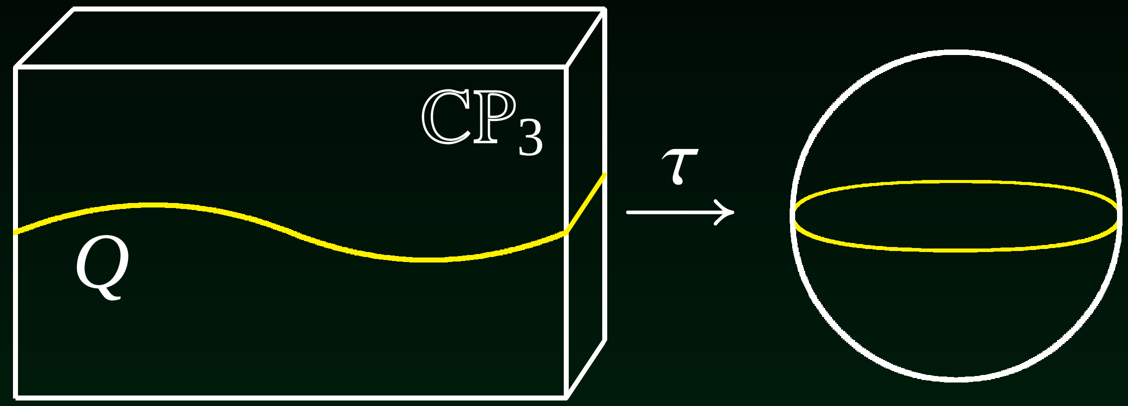
$$\begin{aligned} Q &= \{[z] \in \mathbb{CP}_3 \,|\, \Re(z_2\overline{z_3} + z_4\overline{z_1}) = 0\} \\ &\cong \{[Z] \in \mathbb{CP}_3 \,|\, |Z_1|^2 + |Z_2|^2 = |Z_3|^2 + |Z_4|^2\} \\ &\equiv \underline{\text{Levi-indefinite hyperquadric}} \end{aligned}$$



(cf. saddle)

Twistor results

$$\begin{tikzcd} Q \arrow[r, phantom, "\subset"] \arrow[d] & \mathbb{CP}_3 \arrow[d, "\tau"'] \\ S^3 \arrow[r, phantom, "\subset"] & S^4 \end{tikzcd}$$



Theorem A section $S^4 \supseteq {}^{\text{open}}\Omega \xrightarrow{J} \mathbb{CP}_3$ of τ defines an integrable Hermitian structure if and only if $\tilde{M} \equiv J(\Omega)$ is a complex submanifold.

Theorem A section $S^3 \supseteq {}^{\text{open}}\Omega \xrightarrow{U} Q$ of $\tau : Q \to S^3$ defines a conformal foliation if and only if $M \equiv U(\Omega)$ is a CR submanifold.

CR submanifolds and functions

$\boxed{M \subset Q \subset \mathbb{CP}_3 \text{ is a ‘CR submanifold’?}}$

It means: $TM \cap JTQ$ is preserved by J.

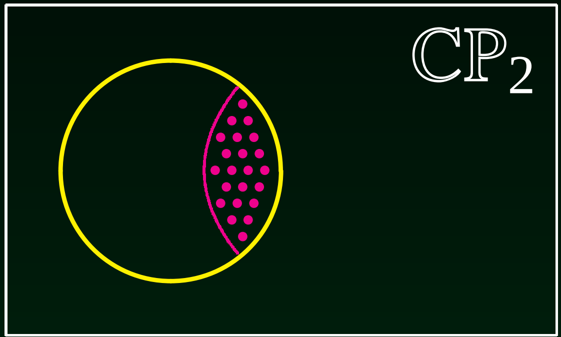
It does not mean: $M = \{f = 0\}$ where f is a CR function: $(X + iJX)f = 0 \ \ \forall X \in \Gamma(TQ \cap JTQ)$.

Implicit function theorem
is false in the CR category

- CR functions on Q are real-analytic.
- conformal foliations on S^3 need not be.

CR functions

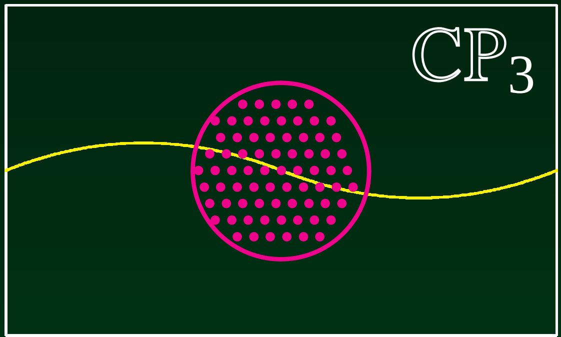
$\{[Z] \in \mathbb{CP}_2 \mid |Z_1|^2 + |Z_2|^2 = |Z_3|^2\} =$ three-sphere



Theorem (H. Lewy 1956)

CR $\Rightarrow$ holomorphic extension

$\{[Z] \in \mathbb{CP}_3 \mid |Z_1|^2 + |Z_2|^2 = |Z_3|^2 + |Z_4|^2\} = Q$



Corollary

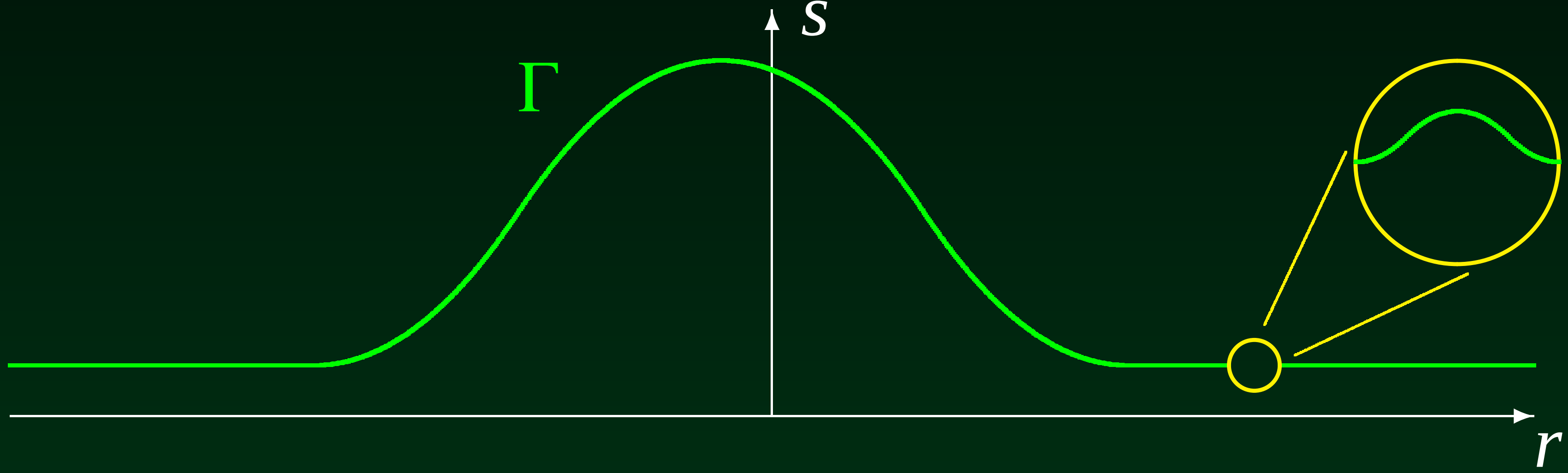
CR $\Rightarrow$ holomorphic extension

Hence, $\boxed{\text{a CR function on } Q \text{ is real-analytic!}}$

Smooth conjugate functions

Eikonal equation: $\left(\frac{\partial f}{\partial r}\right)^2 + \left(\frac{\partial f}{\partial s}\right)^2 = 1$

Plenty of non-analytic solutions:



$f =$ signed distance to Γ

$$\left.\begin{aligned} f(q,r,s) &= f(r,s) \\ g(q,r,s) &= q \end{aligned}\right\} \Rightarrow \boxed{\begin{aligned} \langle \nabla f, \nabla g \rangle &= 0 \\ \|\nabla f\| &= \|\nabla g\| \end{aligned}}$$

$\boxed{\text{QED}}$

Real-analytic refinements

U = unit vector field on $\Omega^{\text{open}} \subseteq \mathbb{R}^3$.
Choose ω a $\mathbb{C}$-valued null 1-form on Ω s.t. $U \lrcorner\, \omega = 0$.

Lemma U conformal $\Leftrightarrow \omega \wedge d\omega = 0$.

Consider $h = f + ig : \Omega \to \mathbb{C}$ and let $\omega = dh$.
Remark • $d\omega = 0$ • f and g conjugate $\Leftrightarrow \omega$ is null.

Theorem Suppose $\omega : \Omega \to \mathbb{C}$ is real-analytic null.

- $\omega \wedge d\omega = 0 \leftrightarrow \tilde{M} \hookrightarrow \mathbb{CP}_3$ (s.t. $M = \tilde{M} \cap Q$)
- $\omega^{\perp} \wedge d\omega = 0 \leftrightarrow \tilde{S} \hookrightarrow \mathbb{C}^4 \setminus \{0\}$ s.t. $\pi(\tilde{S}) = \tilde{M}$
- $d\omega = 0 \leftrightarrow \tilde{S} \hookrightarrow \mathbb{C}^4 \setminus \{0\}$ s.t. $\tilde{S}$ is Lagrangian

Holomorphic function of two complex variables

THANK YOU

HAPPY BIRTHDAY ROGER!