TWISTORS AND THE OCTONIONS

Penrose 80

Nigel Hitchin

Oxford July 21st 2011

- **8th August 1931**

- **8th August 1931**

- **1851 "... an oblong arrangement of terms consisting, suppose, of lines and columns. This will not in itself represent a determinant, but is, as it were, a Matrix out of which we may form various systems of determinants.."**

  JJ Sylvester, *An Essay on Canonical Forms, Supplement to a Sketch of a Memoir on Elimination, Transformation and Canonical Forms*

- **26th December 1843**

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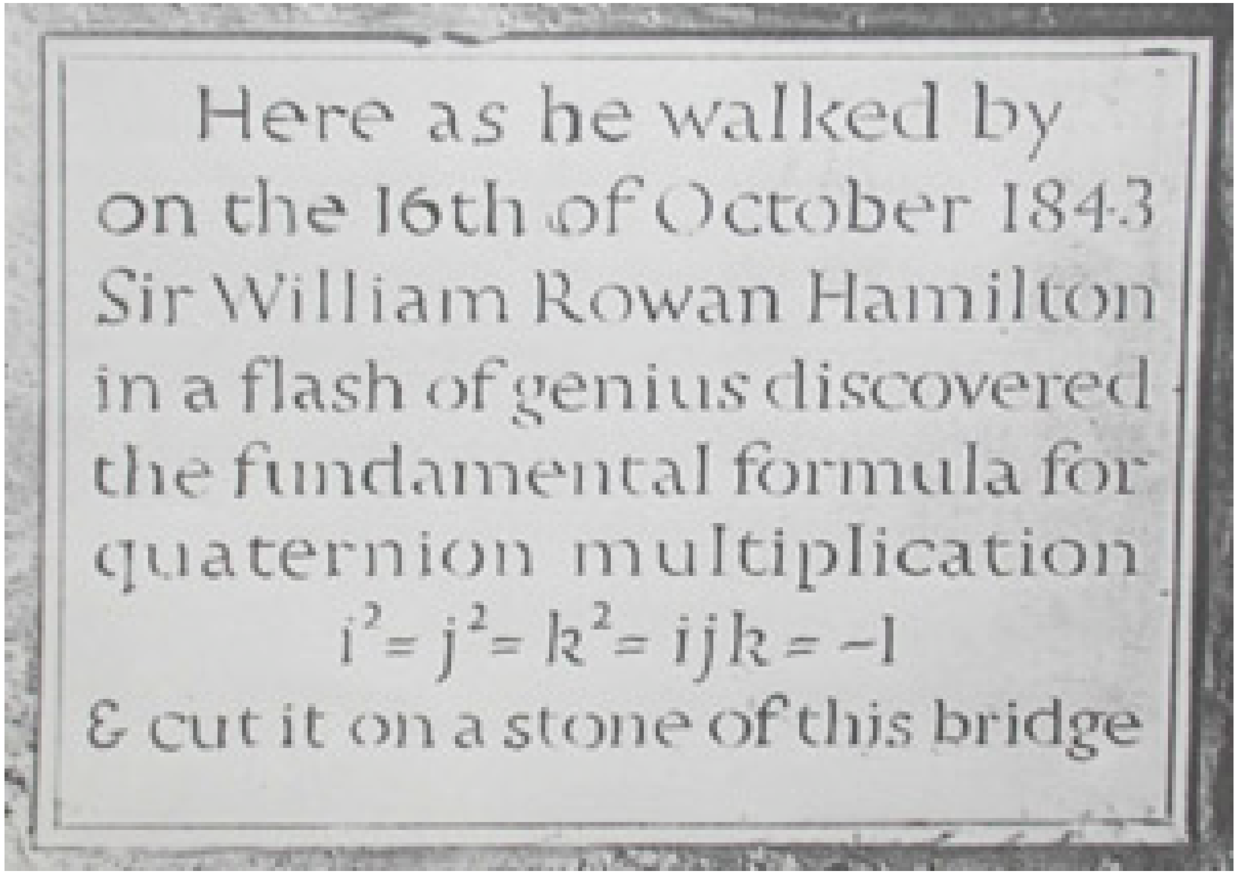
John T Graves

16th October 1843


Here as he walked by
on the 16th of October 1843
Sir William Rowan Hamilton
in a flash of genius discovered
the fundamental formula for
quaternion multiplication
$i^2 = j^2 = k^2 = ijk = -1$
& cut it on a stone of this bridge

Hamilton to Graves October 1843

(Copy of a) Letter from Sir William R. Hamilton to John T. Graves, Esq. on Quaternions

Observatory, October 17, 1843

My dear Graves,—A very curious train of mathematical speculation occurred to me yesterday, which I cannot but hope will prove of interest to you. You know that I have long wished, and I believe that you have felt the same desire, to possess a Theory of Triplets, analogous to my published Theory of Couplets, and also to Mr. Warren's geometrical representation of imaginary quantities. Now I think that I discovered[1] yesterday a *theory of quaternions* which includes such a theory of *triplets*.[2]

My train of thoughts was of this kind. Since $\sqrt{-1}$ is in a certain well-known sense, a line perpendicular to the line 1, it seemed natural that there should be some other imaginary to express a line perpendicular to the former; and because the rotation from this to this also being doubled conducts to -1, it ought also to be a square root of negative unity, though not to be confounded with the former. Calling the old root, as the Germans often do, i, and the new one j, I inquired what laws ought to be assumed for multiplying together $a + ib + jc$ and $x + iy + jz$. It was natural to assume that the product

Graves to Hamilton December 1843

$$(a^2+b^2+c^2+d^2+e^2+f^2+g^2+h^2)(m^2+n^2+o^2+p^2+q^2+r^2+s^2+t^2)$$

$$= v_1^2 + v_2^2 + v_3^2 + v_4^2 + v_5^2 + v_6^2 + v_7^2 + v_8^2$$

where,

$$v_1 = am - bn - co - dp - eq - fr - gs - ht$$

$$v_2 = bm + an + do - cp + fq - er - hs + gt$$

$$v_3 = cm - dn + ao + bp + gq + hr - es - ft$$

etc.

21.

ON JACOBI'S ELLIPTIC FUNCTIONS, IN REPLY TO THE REV. B. BRONWIN; AND ON QUATERNIONS.

[From the *Philosophical Magazine*, vol. XXVI. (1845), pp. 208, 211.]

The first part of this Paper is omitted, see [17]: only the Postscript on Quaternions, pp. 210, 211, is printed.

It is possible to form an analogous theory with seven imaginary roots of (-1) (? with $\nu = 2^n - 1$ roots when ν is a prime number). Thus if these be i_1, i_2, i_3, i_4, i_5, i_6, i_7, which group together according to the types

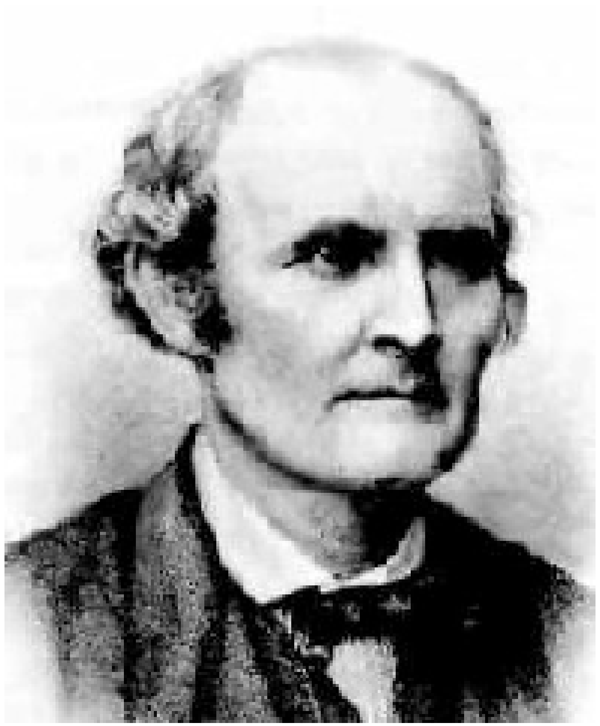
$$123, \quad 145, \quad 624, \quad 653, \quad 725, \quad 734, \quad 176,$$

i.e. the type 123 denotes the system of equations

$$\begin{aligned} i_1 i_2 &= i_3, \quad & i_2 i_3 &= i_1, \quad & i_3 i_1 &= i_2, \\ i_2 i_1 &= -i_3, \quad & i_3 i_2 &= -i_1, \quad & i_1 i_3 &= -i_2, \end{aligned}$$

&c. We have the following expression for the product of two factors:

CAYLEY NUMBERS

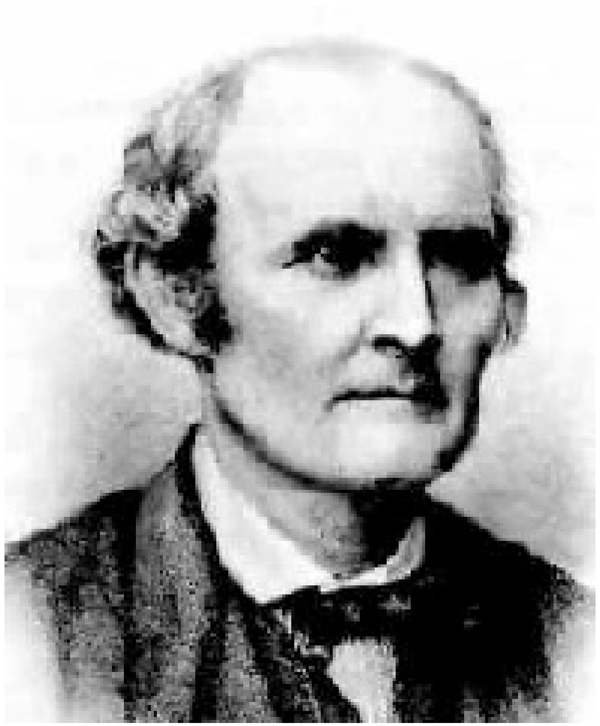


$$e_i^2 = e_j^2 = e_k^2 = e_i e_j e_k = -1$$

$$(ijk) = (123), (145), (624), (653), (725), (734), (176)$$

CAYLEY NUMBERS

= OCTONIONS



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AUTOMORPHISMS

- quaternions $x_0 + \underline{x_1 i + x_2 j + x_3 k}$

- automorphism group $SO(3)$

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- automorphism group $SO(3)$

- octonions $x_0 + x_1 i_1 + x_2 i_2 + x_3 i_3 + x_4 i_4 + x_5 i_5 + x_6 i_6 + x_7 i_7$

- automorphism group $G_2 \subset SO(7)$

- 14-dimensional compact simple Lie group “the crazy old uncle nobody lets out of the attic”

J C Baez, *The Octonions*, BAMS **39** 145–205 (2002)

- Matrices

- Octonions $\mathbf{O}$

- Matrices
- Octonions $\mathbf{O}$

What is $SL(2, \mathbf{O})$?

Lorentz group and the conformal group of Minkowski space to be discussed in terms of groups of complex matrices:

$$\widetilde{SO}(3,1) \cong SL(2,\mathbb{C}) , \tag{1}$$

$$\widetilde{SO}(4,2) \cong SU(2,2) \cong Sp^{\dagger}(4,\mathbb{C}) , \tag{2}$$

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In the critical dimension of the fermionic string, namely $(9+1)$-dimensional Minkowski space–time M^{10}, there are similar isomorphisms involving groups of octonionic matrices [1]:

$$\widetilde{SO}(9,1) \cong SL(2,\mathbb{O}) , \tag{5}$$

$$\widetilde{SO}(10,2) \cong Sp^{\dagger}(4,\mathbb{O}) . \tag{6}$$

KW Chung & A Sudbery, *Octonions and the Lorentz and conformal groups of ten-dimensional space-time,* **Phys Lett B 198 (1987)**

- $Spin(9,1)$

- spinors S, S^* 16-dimensional real spaces

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Claim: $GL(2,\mathbb{O})$ is an open set in $S \otimes \mathbb{R}^2$

TWISTORS

- conformal transformations of S^4: $SO(5,1)$
- spin representations S, S^* complex 4-dimensional
- 2-dimensional quaternionic spaces
- $Spin(5,1) \cong SL(2,\mathbf{H})$

- a complex vector space S is quaternionic if it has an antilinear automorphism J such that $J^2 = -1$

- $q = (a_0 + ia_1) + (a_2 + ia_3)J$

- $A : S \to S$ complex linear is *quaternionic* if $AJ = JA$

- (left action of a quaternionic matrix commutes with right multiplication by q)

- eigenvalues: $Av = \lambda v$

- $AJv = JAv = J(\lambda v) = \bar{\lambda} Jv$

- complex determinant of A is real and ≥ 0

- quaternionic $n \times n$ matrix $A \Rightarrow \det A$ is a real polynomial of degree $2n$

REALIZATION IN CONFORMAL GEOMETRY

- spinor bundles S^+, S^-

- Dirac operator

$$D\psi = \sum_i e_i \cdot \nabla_i \psi$$

- Twistor operator

$$\bar{D}\psi = \sum_i e_i \otimes \nabla_i \psi + \frac{1}{n} e_i \otimes e_i \cdot D\psi$$

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conformal weight $\frac{n-1}{2}$

- Twistor operator

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conformal weight $-\frac{1}{2}$

- on S^4, $\bar{D}\psi = 0$ with $\psi \in S^+$ has a 4-dimensional space of solutions – the *space of twistors* $\mathbf{T}$

- for $\psi \in S^-$ the solutions are the dual twistor space $\mathbf{T}^*$

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- conformal transformations act on $\mathbf{T}, \mathbf{T}^*$ as the representations S, S^* of $Spin(5,1)$.

- on $\mathbf{R}^4$

$$\bar{\Delta} f = (\nabla^2 f)_{ij} - \frac{1}{n}\delta_{ij}\Delta f$$ conformal weight -1

- $\bar{\Delta} f = 0$ has a 6-dimensional space of solutions

- $f = ar^2 + b_i x_i + c$: $f^{-2}\delta_{ij}$ is an Einstein metric

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- Lorentzian inner product $(f, f) = b_i b_i - 4ac =$ - scalar curvature of the metric

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- $\bar{\Delta} f = 0$

- $f = \langle \psi, \psi \rangle$ real $\Rightarrow$ Einstein metric

- $(f, f) = 0$ – flat metric on $S^4 \backslash pt.$

- $\psi \in \mathbf{T}$
- $f = \langle \psi, \psi \rangle$ vanishes at a point in S^4
- $f = x \cdot \varphi$ vanishes at $x = 0$:

  one-dimensional quaternionic subspace $\cong S_0^+$ of $\mathbf{T}$
- $S^4 = \mathbf{HP}^1 = \mathbf{P}(\mathbf{T})$

$$GL(2, \mathbb{H})$$

- $\mathbf{T} \otimes \mathbf{R}^2$
- left action of $\mathrm{End}_{\mathbf{H}}(\mathbf{T})$
- right action of $\mathbf{H}$ + right action of $\mathrm{End}(\mathbf{R}^2)$
- $\Rightarrow \mathbf{T} \otimes \mathbf{R}^2 = \mathrm{Hom}_{\mathbf{H}}(\mathbf{H}^2, \mathbf{T})$

DETERMINANT

- $\rho = (\psi_1, \psi_2) \in \mathbf{T} \otimes \mathbf{R}^2$
- $\langle \psi_a, \psi_b \rangle = f_{ab}$
- $(f_{11}, f_{22}) - (f_{12}, f_{21})$ (real) quartic function $\mu(\rho)$
- $\mu(\rho) = -3 \det A, \ A : \mathbf{H}^2 \to \mathbf{T}$

- $\{\rho \in \mathbf{T} \otimes \mathbf{R}^2 : \mu(\rho) \neq 0\} \cong \mathrm{Iso}_{\mathbf{H}}(\mathbf{H}^2, \mathbf{T})$

- $\sim$ quaternionic bases of twistor space

- any two differ by an action of $GL(2, \mathbf{H})$

$$GL(2, \mathbf{o})$$

- conformal transformations of S^8: $SO(9,1)$
- spin representations S, S^* real 16-dimensional
- twistor space $\mathbf{T} \cong S$
- $\psi = x \cdot \varphi^- + \varphi^+$ vanishes at a point: $S^8 \cong \mathbf{OP}^1$

- $\rho = (\psi_1, \psi_2) \in \mathbf{T} \otimes \mathbf{R}^2$

- $\langle \psi_a, \psi_b \rangle = f_{ab}$

- $(f_{11}, f_{22}) - (f_{12}, f_{21})$ real quartic $\mu(\rho)$

PROP: $\{\rho \in \mathbf{T} \otimes \mathbf{R}^2 : \mu(\rho) \neq 0\}$ is an open orbit of the group $Spin(9,1) \times GL(2,\mathbf{R})$ with stabilizer $G_2 \times SL(2,\mathbf{R})$.

T.Kimura, *Introduction to prehomogeneous vector spaces, AMS (2003)*

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- $G_2 \times SL(2,\mathbf{R}) \to G_2 \times SO(2,1) \subset SO(7) \times SO(2,1) \subset SO(9,1)$

- $\mathbf{T} \cong \mathbf{O} \oplus \mathbf{O}$

- $\mathbf{T} \otimes \mathbf{R}^2 \cong 2 \times 2$ octonionic matrices

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- $\mathbf{T} \cong \mathbf{O} \oplus \mathbf{O}$
- $\mathbf{T} \otimes \mathbf{R}^2 \cong 2 \times 2$ octonionic matrices
- $\{\rho \in \mathbf{T} \otimes \mathbf{R}^2 : \mu(\rho) \neq 0\} =$ "octonionic bases" in $\mathbf{T}$

IN TWISTOR TERMS

- twistors $\psi = x \cdot \varphi^- + \varphi^+$

- φ^-, φ^+ constant spinors in 8 dimensions

- e unit vector $\varphi^- \mapsto e \cdot \varphi^-$ isomorphism as representations of $Spin(7)$

IN TWISTOR TERMS

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- φ^{-}, φ^{+} constant spinors in 8 dimensions

- e unit vector $\varphi^{-} \mapsto e \cdot \varphi^{-}$ isomorphism as representations of $Spin(7)$

- $G_2 \subset Spin(7)$ stabilizer of a spinor φ

- $\rho = (\psi_1, \psi_2) = (x \cdot \varphi, e \cdot \varphi)$

- $Spin(9,1)$

- spinors S, S^* 16-dimensional real spaces

Claim: $GL(2,\mathrm{O})$ is an open set in $S \otimes \mathrm{R}^2$

- $\mathfrak{so}(9,1) + \mathfrak{g}l(2) = \mathfrak{g}_2 + \mathfrak{s}l(2) + \mathfrak{g}l(2) \otimes \mathbf{O}$

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.

- μ=const.

- $A \in \mathfrak{g}l(2) \otimes \mathbf{O}$ tangent space if tr A is imaginary

- Definition: $\{\rho \in \mathbf{T} \otimes \mathbf{R}^2 : \mu(\rho) = -3\} = SL(2, \mathbf{O})$

THE INVARIANT METRIC

- $SL(2,\mathbf{O})$ = hypersurface $\mu = -3$ in $\mathbf{R}^{32}$

- Hessian metric $g = \nabla^2\mu$ invariant under $Spin(9,1) \times SL(2,\mathbf{R})$

- tangent space $\mathfrak{sl}(2) + \mathfrak{gl}(2) \otimes \mathrm{im}\,\mathbf{O}$

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- Replace $\mathbf{O}$ by $\mathbf{H}$

- metric $=$ Killing form on $SL(2, \mathbf{H}) \cong Spin(5, 1)$

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maximal compact
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- Replace $\mathbf{O}$ by $\mathbf{H}$
- metric = Killing form on $SL(2, \mathbf{H}) \cong Spin(5, 1)$
- signature $(2 + 1 \times 3, 1 + 3 \times 3) = (5, 10)$ ← maximal compact $Spin(5)$

STIEFEL MANIFOLDS

- $V_2(\mathbf{F}^n)$ = orthonormal pairs of vectors

- $V_2(\mathbf{C}^2) = U(2)$

- $V_2(\mathbf{H}^2) = Sp(2)$

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- $V_2(\mathbf{O}^2) = Spin(9)/G_2$

- dim $= 36 - 14 = 22$

- $M = Spin(9)/G_2$

- $T_xM \cong 2 \times 2$ octonionic matrices A such that...

- $\bar{A}^T = -A$

- $H^*(U(2)) = H^*(S^1 \times S^3)$

- $H^*(Sp(2)) = H^*(S^3 \times S^7)$

- $H^*(Spin(9)/G_2) = H^*(S^7 \times S^{15})$

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- $H^*(Sp(2)) = H^*(S^3 \times S^7)$
- $H^*(Spin(9)/G_2) = H^*(S^7 \times S^{15})$

$$Spin(7)/G_2 \to Spin(9)/G_2 \to Spin(9)/Spin(7)$$

$Spin(7)/G_2 = S^7$, $Spin(9)/Spin(7) = S^{15}$

- $Spin(9)/G_2$ has trivial tangent bundle

  WA Sutherland, *A note on the parallelizability of sphere-bundles over spheres*, J. London Math. Soc **39** 55–62 (1964)

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- The product of two harmonic forms is harmonic

  D Kotschick, D & S Terzic, *Geometric formality of homogeneous spaces and of biquotients*, Pacific J. Math. **249** 157-176 (2011)

REAL FORMS

- $\mu : S \otimes \mathbf{R}^2 \to \mathbf{R}$

- $d\mu(\rho) = \widehat{\rho} \in S^* \otimes \mathbf{R}^2$

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- $d\mu(\rho) = \widehat{\rho} \in S^* \otimes \mathbf{R}^2$

- $Spin(5,1)$, $\rho \sim$ quaternionic matrix A

- $\widehat{A} = (\bar{A}^T)^{-1}$

- $v \in V, (v,v) \neq 0$

- $\psi \mapsto v \cdot \psi$ defines $S \cong S^*$

- real form $\widehat{\rho} = v \cdot \rho$

- $(v,v) < 0 \Rightarrow Sp(2)$, $(v,v) > 0 \Rightarrow Sp(1,1)$

- $\mu : S \otimes \mathbf{R}^2 \to \mathbf{R}$

- $Spin(9,1)$

- $(v,v) < 0 \Rightarrow Spin(9)/G_2 = SU(2;\mathbf{O})$

- $(v,v) > 0 \Rightarrow Spin(8,1)/G_2 = SU(1,1;\mathbf{O})$

WHAT NEXT?

8-DIMENSIONAL RIEMANNIAN GEOMETRY?

- M^8 Riemannian manifold

- principal $Spin(8)$-bundle P

- P/G_2 modelled on $SU(2, \mathbf{O})$

$D = 10, \mathcal{N} = (2,0)$ SUPERGRAVITY?

- $M^{9,1}$ space time
- supermanifold $S \otimes \mathbf{R}^2 \to M$
- "principal $SL(2, \mathbf{O})$ bundle" ?

".... Of course, mathematical beauty is a worthy end in itself, but it would be even more delightful if the octonions turned out to be built into the fabric of nature. As the story of the complex numbers and countless other mathematical developments demonstrates, it would hardly be the first time that purely mathematical inventions later provided precisely the tools that physicists need."

J C Baez & J Huerta, *"The Strangest Numbers in String Theory*, Scientific American, May (2011)