Scattering Without Spacetime

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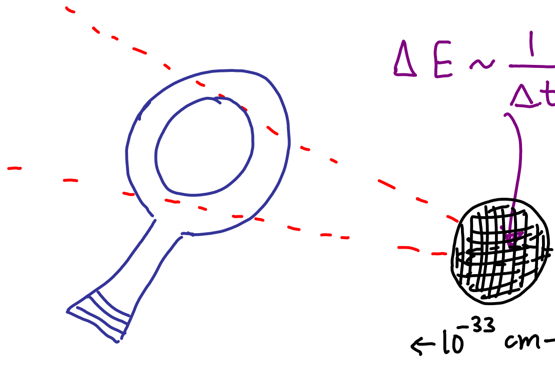
I.A.S

Motivations

Gravity + QM

$$\Downarrow$$

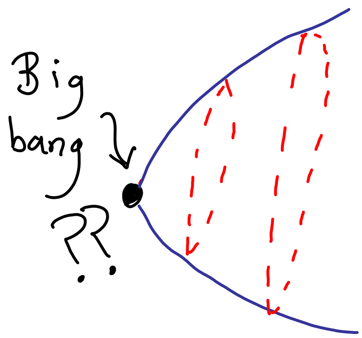
"Space-time is Doomed"

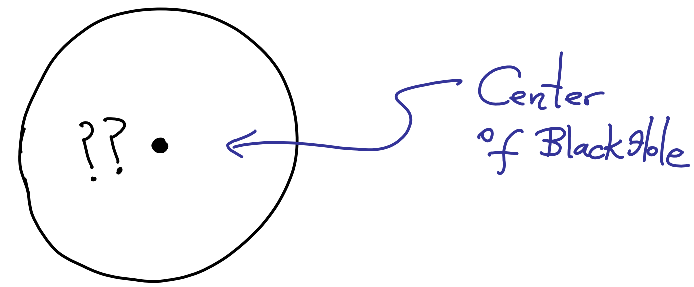


$\Delta E \sim \frac{1}{\Delta t} \longrightarrow$ eventually make Black Hole!

No Operational meaning to distance $< 10^{-33}$ cm, times $< 10^{-43}$ s, ...

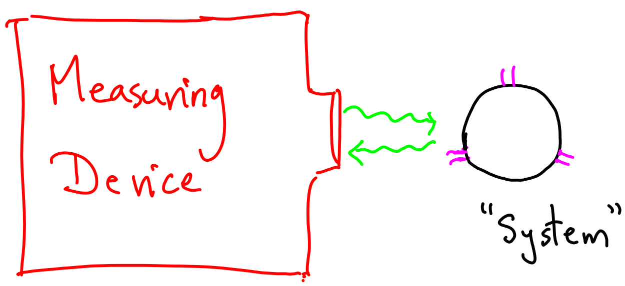
End of Space-Time





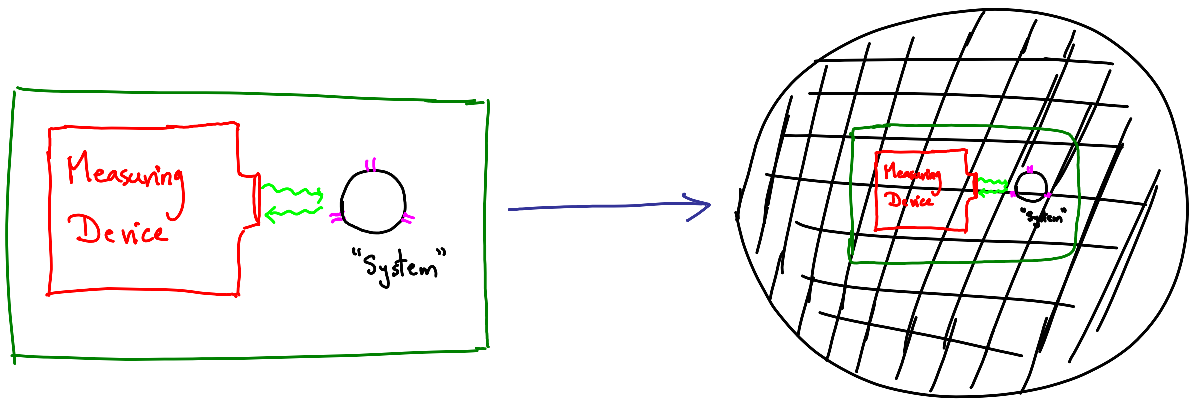
Our theories just <u>break</u> <u>down</u> when gravity is strong and quantum gravity effects are dominant.

Exact Quantum Observables

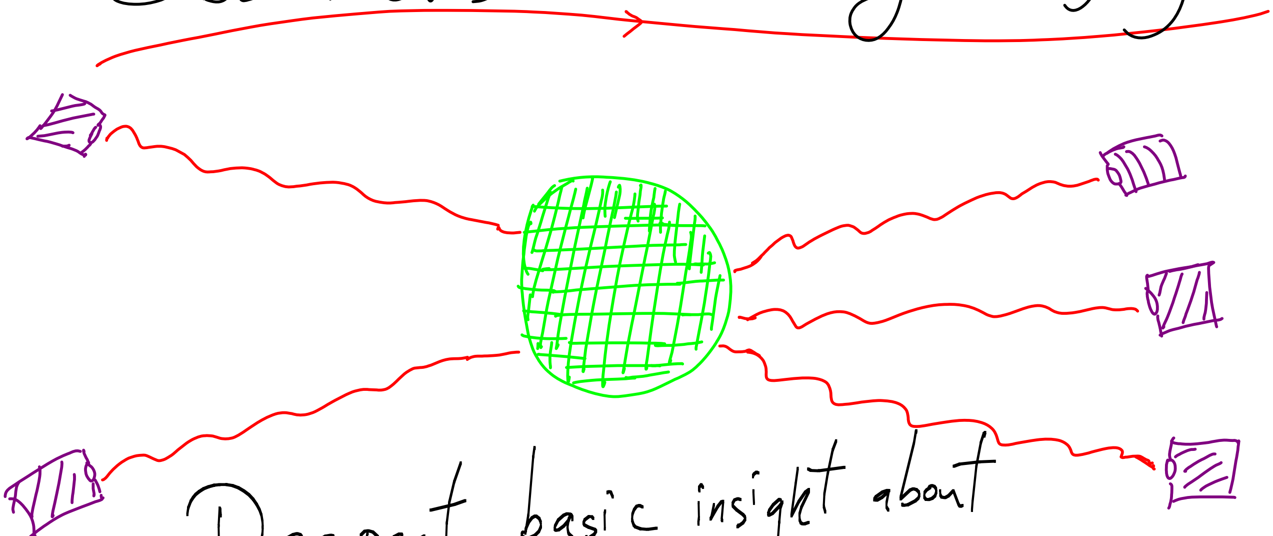


Infinitely many measurements with an Infinitely large measuring apparatus!

No Local Observables!



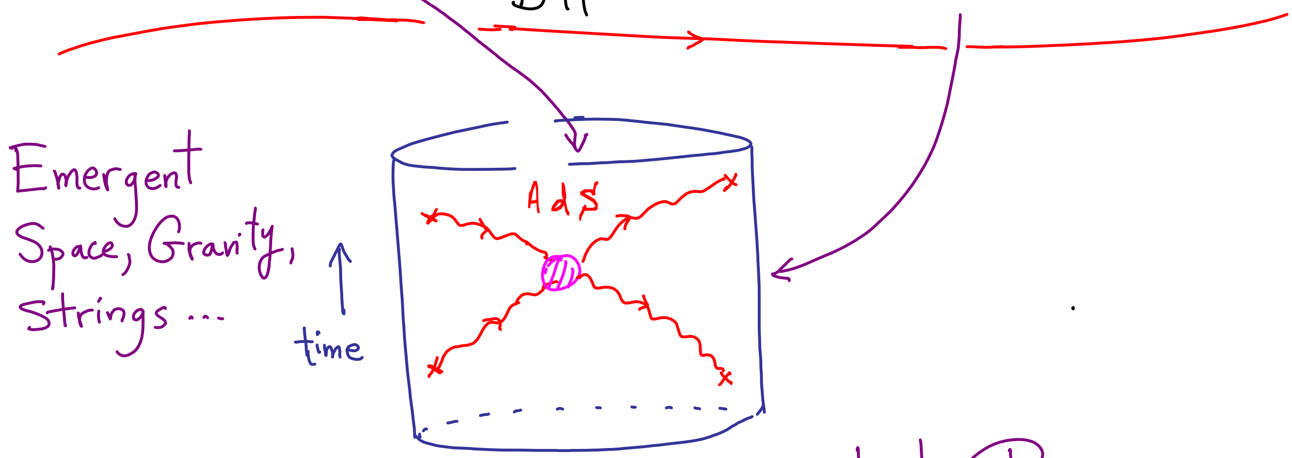
Observables on "Boundary at Infinity"



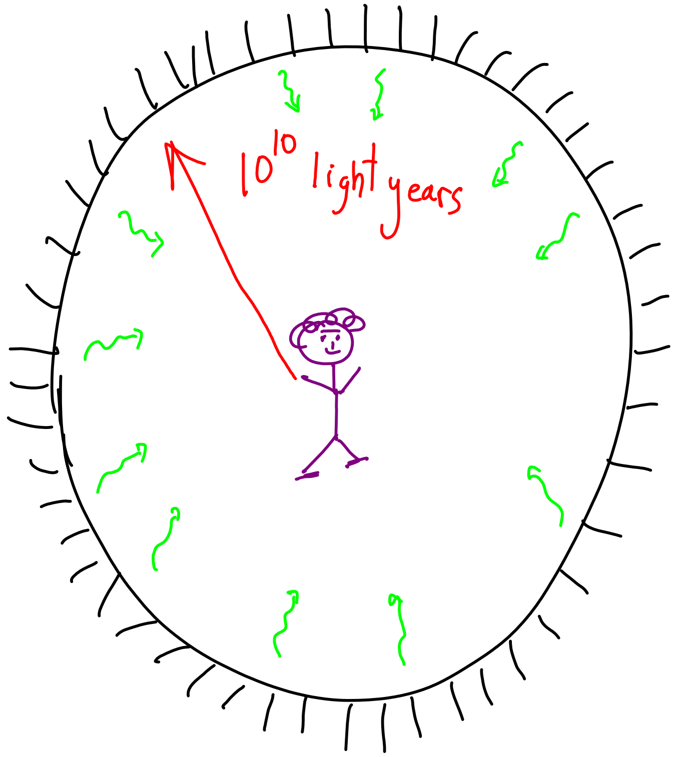
Deepest basic insight about quantum gravity $\longleftrightarrow$ Holography.

[Understood 20 yrs earlier than anyone by Roger + B. deWitt]

$$(\text{Quantum Gravity})_{D+1} = (\text{Quantum Field Theory})_D$$



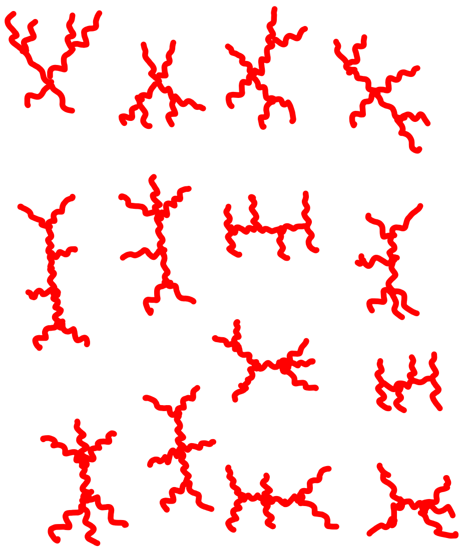
$$\text{String Theory} = \text{Particle Physics}$$



What are the correct observables ?!?!

Emergent Space-time ?

Feynman Explosion



$+ \cdots$

220 Diagrams

10's of thousands of terms ...

Result of a brute force calculation:

$k_1 \cdot k_4 \, \varepsilon_2 \cdot k_1 \, \varepsilon_1 \cdot \varepsilon_3 \, \varepsilon_4 \cdot \varepsilon_5$

+ 24 more pages ....

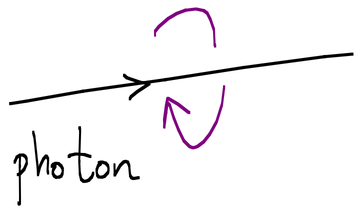
$$\mathrm{Amp}(1^+2^-3^+4^-5^+6^+) = \frac{\langle 24\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle} \; !$$

"MHV Amplitudes": i^-, j^-, rest plus

$$\frac{\langle i\,j\rangle^4}{\langle 12\rangle\langle 23\rangle \cdots \langle n1\rangle}$$

Q. What makes Feynman Diagrams so complicated, obscuring simplicity of answer?

A. Insistence on Manifest Locality + Unitarity!



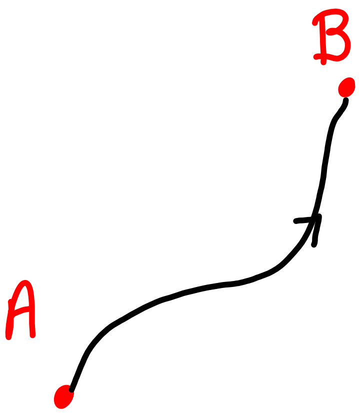
<u>2</u> helicities ± 1.

Locality $\Longrightarrow$ Field $\underbrace{A_\mu(x) = \epsilon_\mu}_{\text{4 components!}} e^{ipx}$

$$\epsilon \cdot p = 0 \quad , \quad \epsilon_\mu \sim \epsilon_\mu + \alpha p_\mu$$

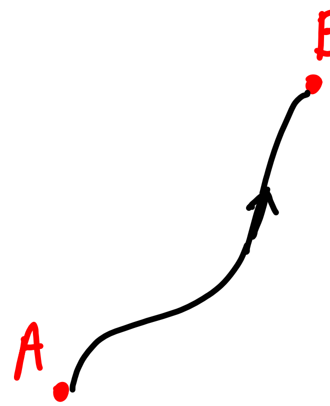
$$A_\mu \sim A_\mu + \partial_\mu \Lambda$$

Gauge Redundancy $\longrightarrow$ All the trouble!



$$m\ddot{x} = -\frac{\partial V}{\partial x}$$

Manifestly Determinstic

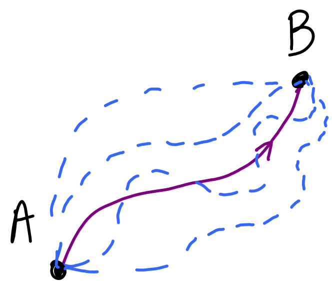


$x(t)$ minimizes action

$$S = \int dt \left[\tfrac{1}{2} m\dot{x}^2 - V(x)\right]$$

__Not__ manifestly determinstic

Quantum Mechanics



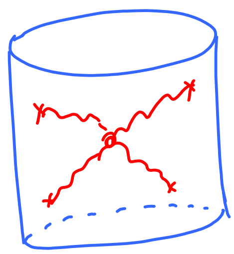
All paths are taken.

$$\text{Amp} = \sum_{\text{paths}} e^{iS/\hbar}$$

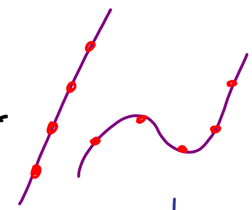
$\hbar \to 0$ limit of QM = Least action principle,

<u>not</u> $F = ma$!

Sitting Under our Noses for 60 yrs

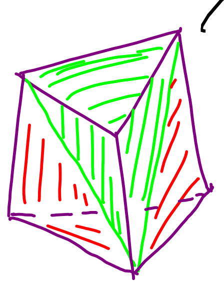


String Theory

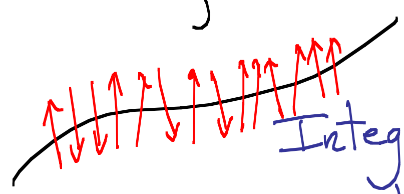


Twistor Theory

Scattering without
Spacetime – emergent
locality + unitarity



Algebraic Geometry



Integrable
systems

Roger - 1987

"On the origin of twistor theory"

2. SOME BACKGROUND IDEAS

Let me try to set in perspective my own state of mind some twenty years ago, and to explain some of the reasons why I felt that a different viewpoint with regard to space-time structure, of the kind provided by twistor theory, was needed. I had, for a good many years earlier, been of the opinion that the space-time continuum picture of reality would prove inadequate on some small scale. I do not propose to discuss all the reasons for this - and in any case it is a view that is hardly original with me. Indeed, that the quantum nature of reality should affect the very structure of space-time at some scale is now a more-or-less accepted viewpoint among those physicists who have examined this question in some depth (cf. Schrödinger 1952, Wheeler 1962). But I think that most physicists would believe that such effects should be relevant only at the absurdly small quantum gravity scale of 10-33 cm. (or smaller). My own attitude was somewhat different from this. While it might be that only at 10-33 cm is it necessary to invoke a description of space-time radically removed from that of a manifold, my view was (and still is) that even at the much larger levels of elementary particles, or perhaps atoms, where quantum behaviour holds sway, the standard space-time descriptions have ceased to be the most physically

Space-time descriptions of the normal kind can, of course, be used at the atomic or particle level provided that the quantum rules are correctly applied, and they have implications that are extraordinarily accurate. Thus, this new geometrical picture must, at that level, be mathematically equivalent to the normal space-time picture - in the sense that some kind of mathematical transformation must exist between the two pictures. However, the new description ought to incorporate quantum behaviour more readily and naturally than the old. Moreover, at the quantum gravity level of 10-33 cm, or at the level of space-time singularities, it ought to provide an essentially different and more accurate picture of things.

An idea - Hamiltonian theory, for example - may have immense utility and lead to new insights without, in this sense, having any new physical content. Thus in this sense it was, I suppose, the advent of quantum theory which saved the Hamiltonian viewpoint from the dustbin!

What was missing 40 years ago?

"Data"!

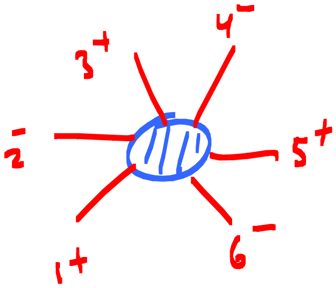
'04 – '05 : Witten's twistor string

$\longrightarrow$ BCFW recursion relations

for tree amplitudes, + big

insights from AdS/CFT...

BCFW 6pt



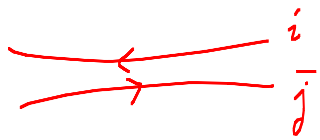
$$= \frac{\langle 46\rangle^4 [13]^4}{[12][23]\langle 45\rangle\langle 56\rangle} \frac{1}{(p_1+p_2+p_3)^2} \times \frac{1}{\langle 6|5+4|3]} \frac{1}{\langle 4|5+6|1]}$$

"Spurious" Poles: Don't occur in local theories!

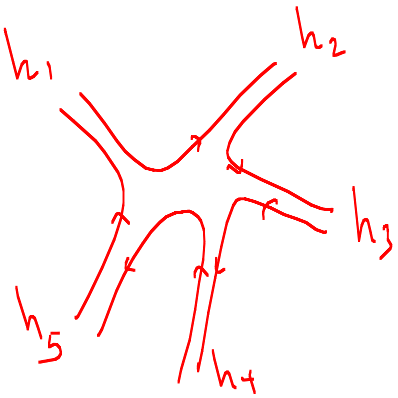
$$+ \{i \to i+2\} + \{i \to i+4\}$$

Cast of Characters

Gluon Scattering Amplitudes

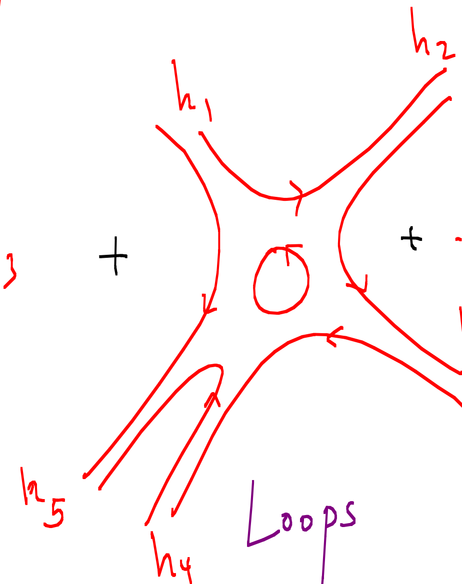


N_c colors.



trees

$+$

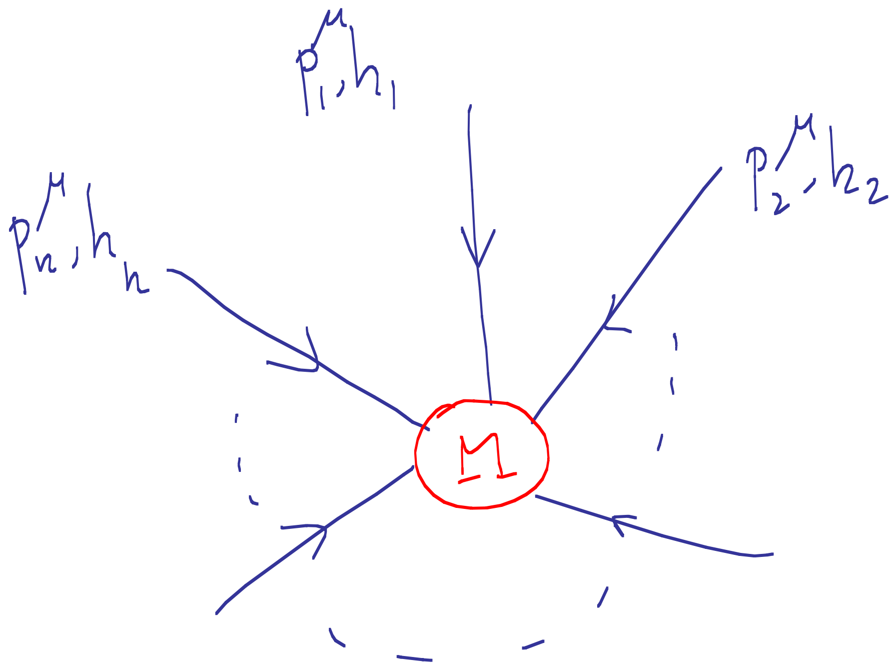


Loops

$+ \cdots$

Restrict to planar limit

Important: Planar Loop Integrand is well-defined



$$\left[p_1^\mu + \cdots + p_n^\mu = 0 \right]$$

$$M_n[p_a, h_a ; l_i] \qquad l_i : \text{loop momenta}$$

More Kinematics

$$p^\mu = (p^0, \vec{p}) \longleftrightarrow p_{A\dot{A}} = \underbrace{\begin{pmatrix} p^0+p^3 & p^1+ip^2 \\ p^1-ip^2 & p^0-p^3 \end{pmatrix}}_{\det p \,=\, p^2 \,=\, 0}$$

$$\Longrightarrow \quad p_{A\dot{A}} = \lambda_A \tilde{\lambda}_{\dot{A}} \, . \quad \text{Lorentz: } SL(2) \times SL(2)$$

Invariants $\langle \lambda_1 \lambda_2 \rangle = \epsilon^{AB} \lambda_{1A} \lambda_{2B}$

$$[\tilde{\lambda}_1 \tilde{\lambda}_2] = \epsilon^{\dot{A}\dot{B}} \tilde{\lambda}_{1\dot{A}} \tilde{\lambda}_{2\dot{B}}$$

[Twistor theorists were using the $\lambda, \tilde{\lambda}$ variables, as a matter of course, 15 years before their importance was appreciated by quantum field theorists.]

Finally – simplest gauge theory of all is $\mathcal{N}=4$ SYM:

"Harmonic Oscillator of the 21st Century"

Unifies helicities.

$$\left.\begin{array}{ll} & +1 \\ Q_{1,2,3,4}\ \downarrow & +\frac{1}{2} \\ & 0 \\ \tilde{Q}_{1,2,3,4}\ \uparrow & -\frac{1}{2} \\ & -1 \end{array}\right\}$$

$$|\tilde{\eta}\rangle = e^{Q\tilde{\eta}}|+1\rangle$$

$$= |+1\rangle + \eta|+\tfrac{1}{2}\rangle + \cdots + \eta^4|-\rangle$$

$$M_n(\lambda_a, \tilde{\lambda}_a, \tilde{\eta}_a) = \sum_{k=0}^{n} M_{n,k}(\lambda_a, \tilde{\lambda}_a, \tilde{\eta}_a)$$

contains amps with k - helicity gluons.

Turns out $M_{n,0} = M_{n,1} = 0$

$M_{n,2}$ = "MHV" amplitude

$M_{n,3}$ = "NMHV" amplitude

$\vdots$

Summary: We are after a theory for

$$\mathcal{M}_{n,k}[\lambda_a, \tilde{\lambda}_a, \tilde{\eta}_a; Z_i]$$

Without Unitary evolution through Spacetime

$$\left\{ \text{Emergent Space-time}, \text{Emergent QM} \right\}$$

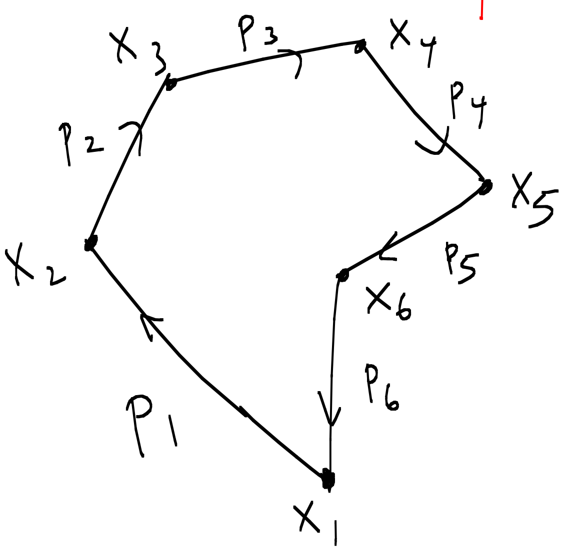
Hidden

Infinite Dimensional

Symmetries

$\mathcal{N}=4$ SYM has an "obvious" (super) conformal symmetry.

Dual (Super) Conformal Symmetry



$$p_a = x_{a+1} - x_a$$

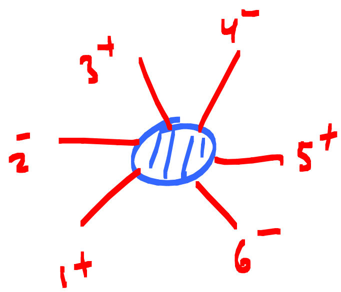
"Experimental" observation

– amplitudes invariant under

Conf. transf. on

this x space!

[Term by term for BCFW form of trees]



$$= \frac{\langle 46\rangle^4 [13]^4}{[12][23]\langle 45\rangle\langle 56\rangle} \frac{1}{(p_1+p_2+p_3)^2} \times \frac{1}{\langle 6|5+4|3]} \frac{1}{\langle 4|5+6|1]}$$

"Spurious" Poles

Are there because these BCFW terms know about both spacetimes!

(Super) Conformal + Dual (Super) Conformal

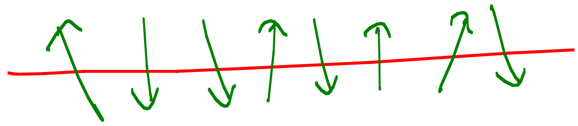
$\Downarrow$ generate

"Yangian Algebra"

Infinite Dimensional Symmetry

Completely Invisible In $\mathcal{L}$

Very striking connection with integrability . . . .



$$H = \sum_i S_i \cdot S_{i+1}$$

$$Q = \sum_{i<j} [S_i, S_j]$$

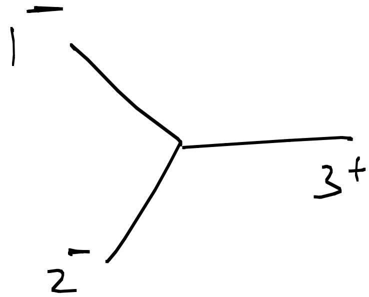
$$\vdots$$

[In particular major breakthroughs in last ~ 5 yrs have solved the problem of determining anom. dimensions in $\mathcal{N}=4$ SYM — again no Feynman diagrams!]

Zero'th Order [Crucial]

Use of Twistors is Kinematical:

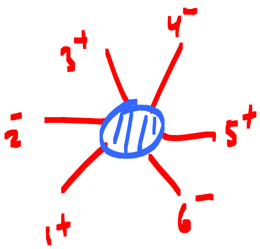
Best variables for amplitudes — linearization of [Super] conformal symmetries!

- 

$$\frac{\langle 12\rangle^3}{\langle 13\rangle\langle 23\rangle}\;\delta^4(\lambda_1\tilde{\lambda}_1+\cdots+\lambda_3\tilde{\lambda}_3)$$

$$\updownarrow$$

$$\mathrm{sgn}\, W_1\cdot Z_3\;\mathrm{sgn}\, W_2\cdot Z_3\;\mathrm{sgn}\, W_1\cdot W_2$$

- 

$$= \frac{\langle 46\rangle^4[13]^4}{[12][23]\langle 45\rangle\langle 56\rangle}\,\frac{1}{(p_1+p_2+p_3)^2} \qquad \longleftrightarrow \qquad \frac{\delta^4[\langle 1234\rangle\eta_5+\cdots]}{\langle 1234\rangle\cdots\langle 5123\rangle}$$

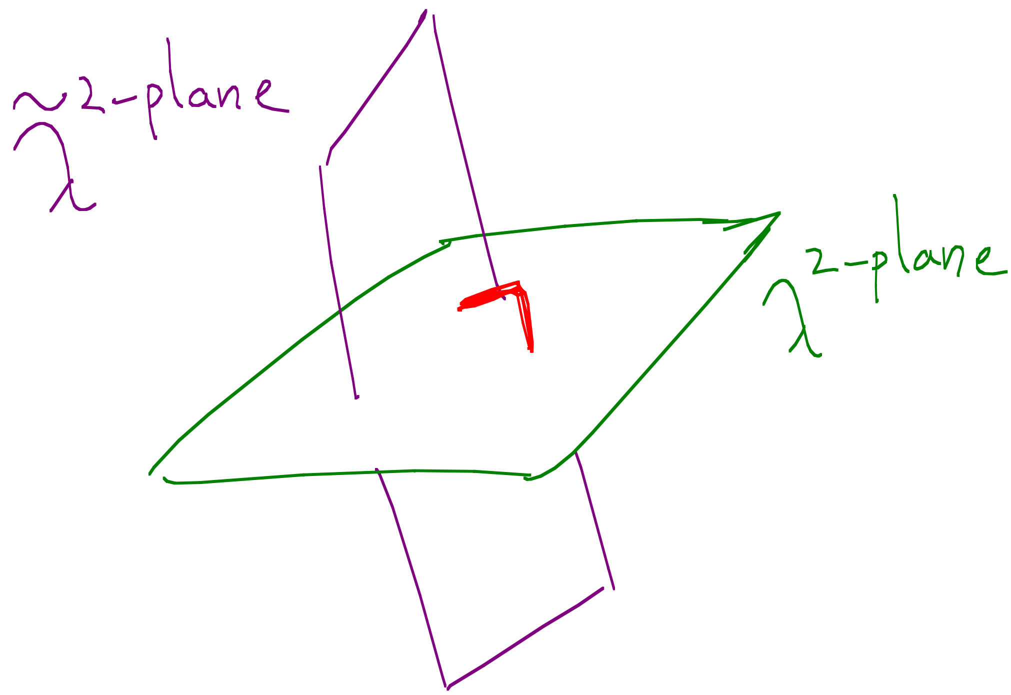
$$\times\;\frac{1}{\langle 6|5+4|3]}\;\frac{1}{\langle 4|5+6|1]}$$

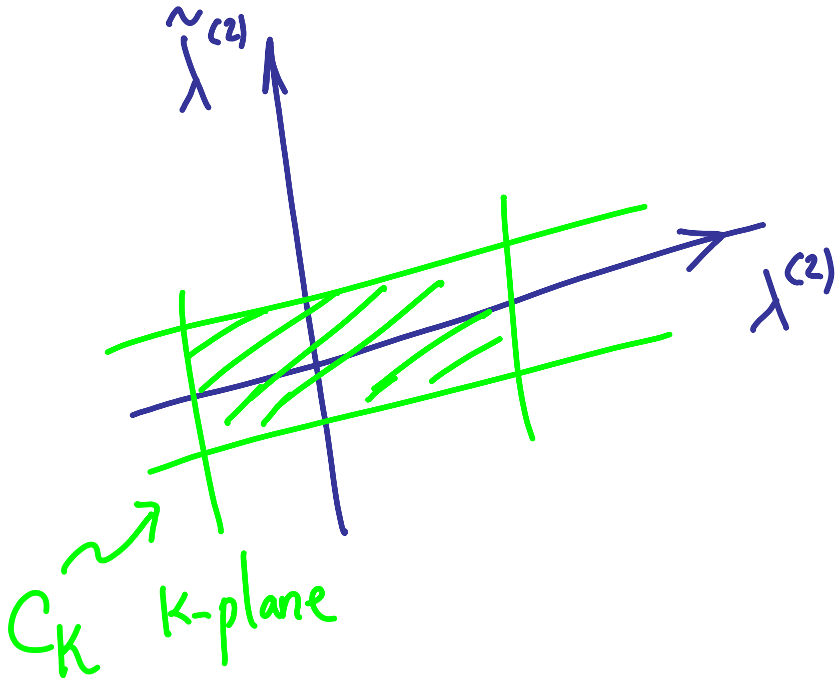
The Grassmannian Formulation

Geometry of Momentum Conservation

$\lambda_a^A, \tilde{\lambda}_a^{\dot{A}}$

$$\sum_a \lambda_a^A \tilde{\lambda}_a^{\dot{A}} = 0$$





Note: parity invariant since

$\lambda \leftrightarrow \tilde{\lambda}$

k plane $\leftrightarrow$ n-k plane

Note: impossible for $k = 0, 1, n-1, n$.

Good!

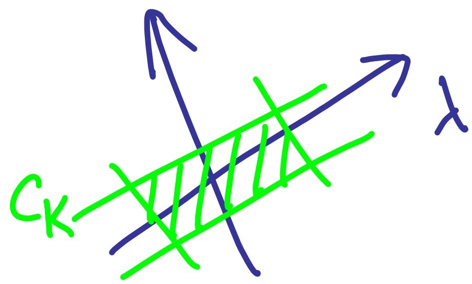
Eqns:

$$C = \begin{array}{c|c} & \xleftarrow{\quad n \quad} \\ \uparrow & \\ k & \begin{bmatrix} \vec{c}_1 \\ \vdots \\ \vec{c}_k \end{bmatrix} \\ \downarrow & \end{array} = C_{\alpha a}$$

Invariance Under GL(k):

$$C_{\alpha a} \longrightarrow L_\alpha^{\ \beta} C_{\beta a}$$

Grassmannian $G(k,n)$, $\dim = k(n-k)$



$$\int d^{2\times k}\rho_\alpha \; \underbrace{\delta^2\left[C_{\alpha a}\rho_\alpha - \lambda_a\right]}_{C \text{ contains } \lambda} \; \underbrace{\delta^2\left[C_{\alpha a}\tilde{\lambda}_a\right]}_{C \text{ orthogonal to } \tilde{\lambda}} \; \underbrace{\delta^4\left[C_{\alpha a}\tilde{\eta}_a\right]}_{\text{SUSY partner}}$$

Preserve GL(k)

This object is very simple in Twistor Space :

$$\prod_{\alpha=1}^{k} \delta^{4|4}\left[C_{\alpha a} W_a \right]$$

Manifests (Super) conformal symmetry

For $k=0,1,n-1,n$, no planes.

For $k=2, n-2$, unique plance

General k – integrate over planes!

$$\frac{1}{\mathrm{vol}\,GL(k)} \int \frac{d^{k\times n} C}{(12\cdots k)\cdots(n1\cdots k-1)} \longleftarrow \underbrace{GL(k)}_{\text{invariance}}$$

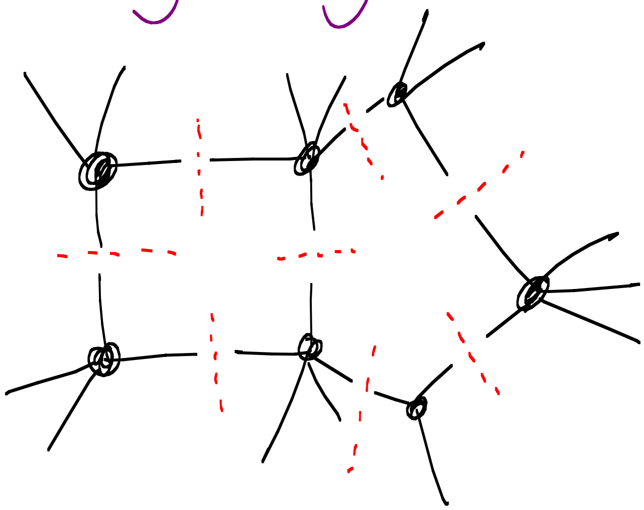
$(m_1 \cdots m_k)$ minor : det of columns $m_1, \dots, m_k$

$$\mathcal{L}_{n,k} = \underbrace{\int \frac{d^{k(n-k)} C_{\alpha a}}{(12 \cdots k) \cdots (n 1 \cdots k-1)}}_{\text{simplest measure}} \times \underbrace{\prod_{\alpha} \delta^{4|4}\left[C_{\alpha a} W_a\right]}_{\substack{\text{simplest} \\ \text{dependence} \\ \text{on kinematics}}}$$

$$\Updownarrow$$

All – Loop Scattering in $\mathcal{N}=4$ SYM!

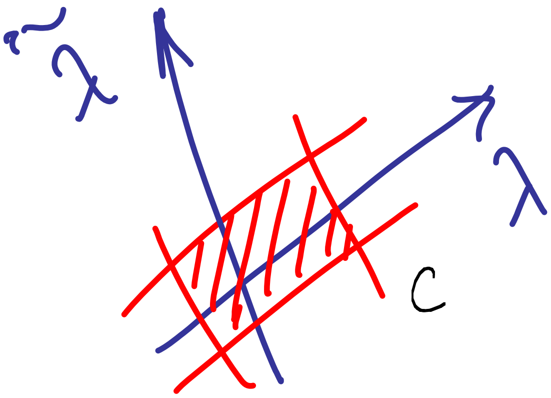
"Leading Singularities"



$\longleftrightarrow$ Residues of $\mathcal{L}_{n,k}$

[Relations giving locality, Unitarity $\longleftrightarrow$ Residue Thms]

Manifest Dual Superconformal Invariance



C contains λ plane:

so really an integral over

$(k-2)$ planes in n dimensions!

Natural linear transformation mapping $k \times k$ minors to $(k-2)\times(k-2)$ minors...

$$\mathcal{L}_{n,k} \to \int \frac{d^{p\times(n-p)} D_{\alpha a}}{(12\cdots p)\cdots(n1\cdots(p-1))} \times \prod_{\alpha=1}^{p} \delta^{4|4}[D_{\alpha a} \mathcal{Z}_a]$$

momentum-twistor variables

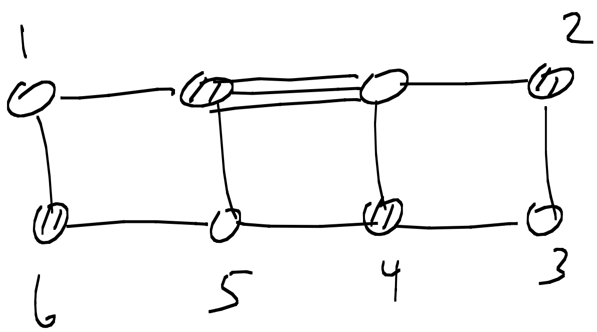
Identical Structure!

Dual superconformal symmetry manifest

The Grassmannian Formulation makes no mention of locality or Unitarity – but makes all symmetries – The Yangian – manifest.

The Grassmannian structure was directly inspired by [+ reflected in] the properties of twistor diagrams for BCFW terms.

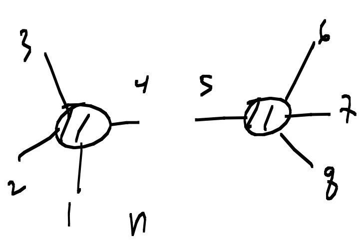
Twistor Diagrams



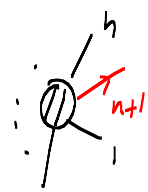
$\longleftrightarrow$ Grassmannian Residues

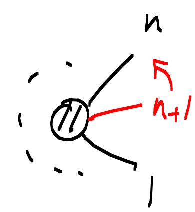
$$
C = \begin{array}{c}
\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
\begin{bmatrix} 1 & * & 0 & * & 0 & * \\ 0 & * & 1 & * & 0 & * \\ 0 & * & 0 & * & 1 & * \end{bmatrix}
\end{array}
$$

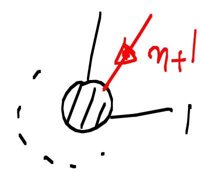
Basic Operations on Yangian Invariants



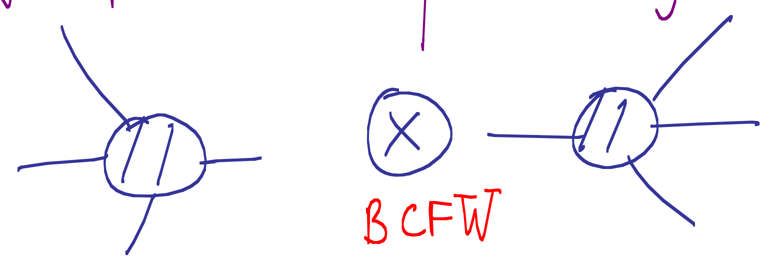
$\int d^{4|4} Z_{n+1}$



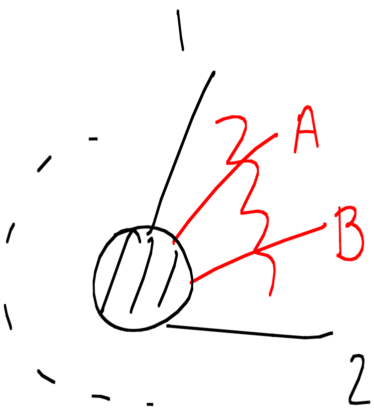




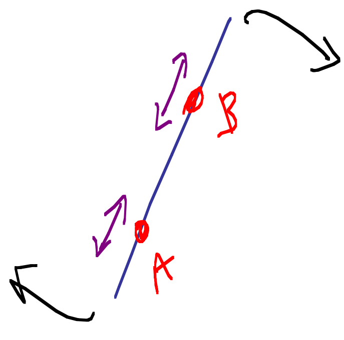
BCFW terms composed from these



Origin of Loops

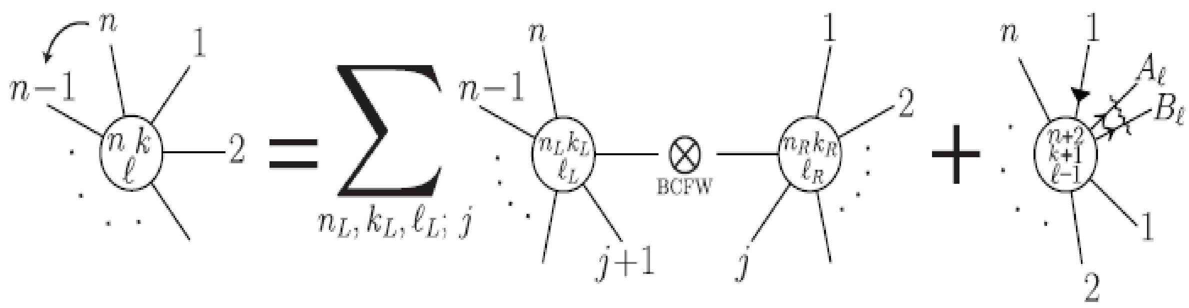


$$\int d^{4|4}Z_A \, d^{4|4}Z_B$$



"Entangled removal" of a pair of particles

All-Loop Recursion



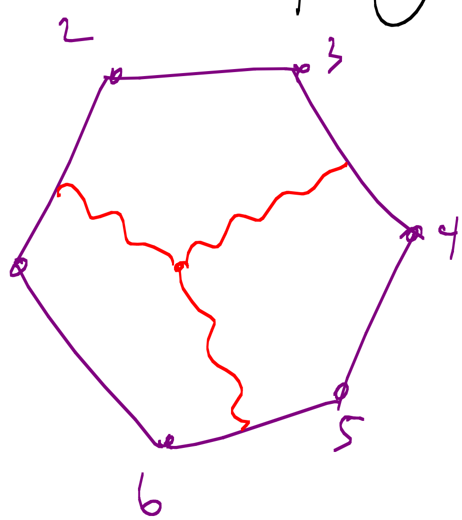
"Classical"

"Quantum"

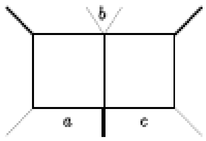
Complete definition with Yangian symmetry manifest.

The words "spacetime", "Lagrangian", "Path Integral", "Gauge Symmetry" ....
make no appearance.

In the dual space-time, this object is interpreted as a certain supersymmetric Wilson loop:

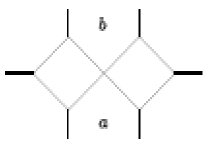


Perfect symmetry has been established between both descriptions.

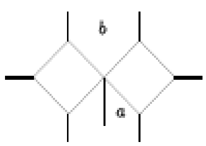


$$-\frac{1}{4}\begin{bmatrix} a & a+1 & b-1 \\ b+1 & c-1 & c \end{bmatrix} \tag{53}$$

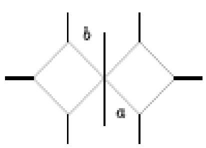
B. Kissing double-box topologies



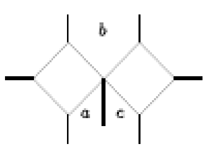
$$-\frac{1}{4}\begin{bmatrix} a & a+1 & b-1 & b \\ b & b+1 & a-1 & a \end{bmatrix} + \frac{1}{4}\begin{bmatrix} a & a+1 \\ b-1 & b \end{bmatrix}\begin{bmatrix} b & b+1 \\ a-1 & a \end{bmatrix} =$$
$$\frac{1}{4}\Big(x^2_{a-1,b+1}x^2_{a+1,b-1}\left(x^2_{ab}\right)^2 - x^2_{a-1,b-1}x^2_{a+1,b+1}\left(x^2_{ab}\right)^2 +$$
$$+ x^2_{a-1,a+1}x^2_{b-1,b+1}\left(x^2_{ab}\right)^2 - x^2_{a-1,b}x^2_{a,b+1}x^2_{a+1,b-1}x^2_{ab} -$$
$$- x^2_{a-1,b+1}x^2_{a,b-1}x^2_{a+1,b}x^2_{ab} + x^2_{a-1,b-1}x^2_{a,b+1}x^2_{a+1,b}x^2_{ab} +$$
$$+ x^2_{a-1,b}x^2_{a,b-1}x^2_{a+1,b+1}x^2_{ab}\Big) \tag{54}$$



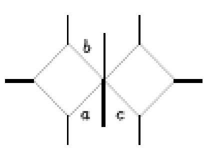
$$-\frac{1}{4}\begin{bmatrix} a+1 & a+2 & b-1 & b \\ b & b+1 & a-1 & a \end{bmatrix} + \frac{1}{4}\begin{bmatrix} a-1 & a \\ b & b+1 \end{bmatrix}\begin{bmatrix} a+1 & a+2 \\ b-1 & b \end{bmatrix} \tag{55}$$



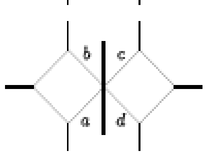
$$-\frac{1}{4}\begin{bmatrix} a+1 & a+2 & b-1 & b \\ b+1 & b+2 & a-1 & a \end{bmatrix} + \frac{1}{4}\begin{bmatrix} a+1 & a+2 \\ b-1 & b \end{bmatrix}\begin{bmatrix} b+1 & b+2 \\ a-1 & a \end{bmatrix} \tag{56}$$



$$-\frac{1}{4}\begin{bmatrix} a & a+1 & b-1 & b \\ b & b+1 & c-1 & c \end{bmatrix} + \frac{1}{4}\begin{bmatrix} a & a+1 \\ b-1 & b \end{bmatrix}\begin{bmatrix} b & b+1 \\ c-1 & c \end{bmatrix} \tag{57}$$



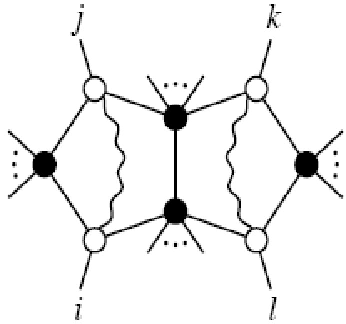
$$-\frac{1}{4}\begin{bmatrix} a & a+1 & b-1 & b \\ b+1 & b+2 & c-1 & c \end{bmatrix} + \frac{1}{4}\begin{bmatrix} a & a+1 \\ b-1 & b \end{bmatrix}\begin{bmatrix} b+1 & b+2 \\ c-1 & c \end{bmatrix} \tag{58}$$



$$-\frac{1}{4}\begin{bmatrix} a & a+1 & b-1 & b \\ c & c+1 & d-1 & d \end{bmatrix} + \frac{1}{4}\begin{bmatrix} a & a+1 \\ b-1 & b \end{bmatrix}\begin{bmatrix} c & c+1 \\ d-1 & d \end{bmatrix} \tag{59}$$

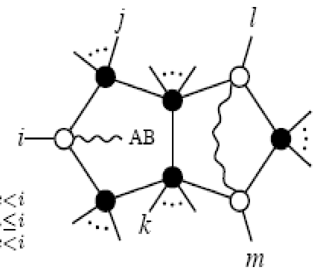
+ 7 more pages....

$$\mathcal{A}^{2-\text{loop}}_{\text{MHV}} = \frac{1}{2}\sum_{i<j<k<l<i}$$

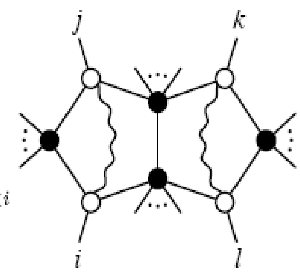


Momentum Twistor Integrals

$$\mathcal{A}^{2-\text{loop}}_{\text{NMHV}} = \sum_{\substack{i<j<l<m\leq k<i\\ i<j<k<l<m\leq i\\ i\leq l<m\leq j<k<i}}$$

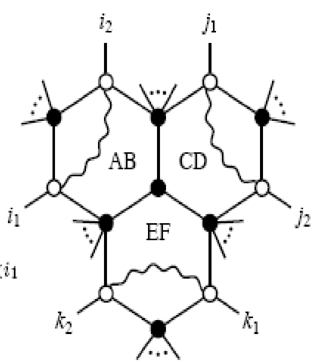


$$\times\, [i, j, j+1, k, k+1] \quad + \frac{1}{2}\sum_{i<j<k<l<i}$$

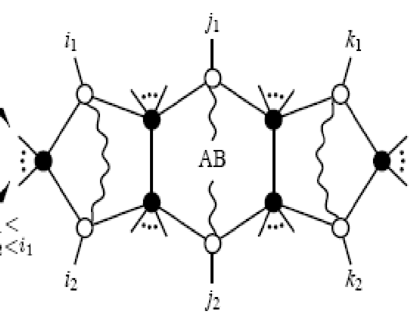


$$\times \left\{ \begin{array}{c} \mathcal{A}^{\text{tree}}_{\text{NMHV}}(j,\ldots,k;\, l,\ldots,i) \\ +\mathcal{A}^{\text{tree}}_{\text{NMHV}}(i,\ldots,j) \\ +\mathcal{A}^{\text{tree}}_{\text{NMHV}}(k,\ldots,l) \end{array} \right\}$$

$$\mathcal{A}^{3-\text{loop}}_{\text{MHV}} = \frac{1}{3}\sum_{\substack{i_1\leq i_2<j_1\leq\\ \leq j_2<k_1\leq k_2<i_1}}$$



$$+\frac{1}{2}\sum_{\substack{i_1\leq j_1<k_1<\\ <k_2\leq j_2<i_2<i_1}}$$



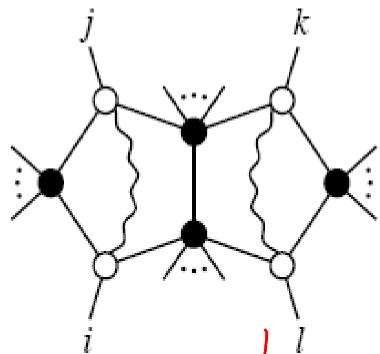
$$
\begin{aligned}
&\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},\frac{1}{1-u_3},1;1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},\frac{1}{1-u_3},\frac{1}{1-u_3};1\right)+\\
&\frac{1}{2}\mathcal{G}\left(v_{123},0,1,\frac{1}{1-u_1};1\right)+\frac{1}{2}\mathcal{G}\left(v_{123},0,\frac{1}{1-u_1},1;1\right)+\frac{1}{2}\mathcal{G}\left(v_{123},1,0,\frac{1}{1-u_1};1\right)-\\
&\frac{5}{4}\mathcal{G}\left(v_{123},1,1,\frac{1}{1-u_1};1\right)+\frac{1}{2}\mathcal{G}\left(v_{123},1,\frac{1}{1-u_1},0;1\right)-\frac{5}{4}\mathcal{G}\left(v_{123},1,\frac{1}{1-u_1},1;1\right)+\\
&\frac{1}{2}\mathcal{G}\left(v_{123},1,\frac{1}{1-u_1},\frac{1}{1-u_1};1\right)+\frac{1}{2}\mathcal{G}\left(v_{123},\frac{1}{1-u_1},0,1;1\right)+\frac{1}{2}\mathcal{G}\left(v_{123},\frac{1}{1-u_1},1,0;1\right)-\\
&\frac{5}{4}\mathcal{G}\left(v_{123},\frac{1}{1-u_1},1,1;1\right)+\frac{1}{2}\mathcal{G}\left(v_{123},\frac{1}{1-u_1},1,\frac{1}{1-u_1};1\right)+\\
&\frac{1}{2}\mathcal{G}\left(v_{123},\frac{1}{1-u_1},\frac{1}{1-u_1},1;1\right)-\frac{1}{4}\mathcal{G}\left(v_{132},1,1,\frac{1}{1-u_1};1\right)-\frac{1}{4}\mathcal{G}\left(v_{132},1,\frac{1}{1-u_1},1;1\right)-\\
&\frac{1}{4}\mathcal{G}\left(v_{132},\frac{1}{1-u_1},1,1;1\right)-\frac{1}{4}\mathcal{G}\left(v_{213},1,1,\frac{1}{1-u_2};1\right)-\frac{1}{4}\mathcal{G}\left(v_{213},1,\frac{1}{1-u_2},1;1\right)-\\
&\frac{1}{4}\mathcal{G}\left(v_{213},\frac{1}{1-u_2},1,1;1\right)+\frac{1}{2}\mathcal{G}\left(v_{231},0,1,\frac{1}{1-u_2};1\right)+\frac{1}{2}\mathcal{G}\left(v_{231},0,\frac{1}{1-u_2},1;1\right)+\\
&\frac{1}{2}\mathcal{G}\left(v_{231},1,0,\frac{1}{1-u_2};1\right)-\frac{5}{4}\mathcal{G}\left(v_{231},1,1,\frac{1}{1-u_2};1\right)+\frac{1}{2}\mathcal{G}\left(v_{231},1,\frac{1}{1-u_2},0;1\right)-\\
&\frac{5}{4}\mathcal{G}\left(v_{231},1,\frac{1}{1-u_2},1;1\right)+\frac{1}{2}\mathcal{G}\left(v_{231},1,\frac{1}{1-u_2},\frac{1}{1-u_2};1\right)+\frac{1}{2}\mathcal{G}\left(v_{231},\frac{1}{1-u_2},0,1;1\right)+\\
&\frac{1}{2}\mathcal{G}\left(v_{231},\frac{1}{1-u_2},1,0;1\right)-\frac{5}{4}\mathcal{G}\left(v_{231},\frac{1}{1-u_2},1,1;1\right)+\frac{1}{2}\mathcal{G}\left(v_{231},\frac{1}{1-u_2},1,\frac{1}{1-u_2};1\right)+\\
&\frac{1}{2}\mathcal{G}\left(v_{231},\frac{1}{1-u_2},\frac{1}{1-u_2},1;1\right)+\frac{1}{2}\mathcal{G}\left(v_{312},0,1,\frac{1}{1-u_3};1\right)+\frac{1}{2}\mathcal{G}\left(v_{312},0,\frac{1}{1-u_3},1;1\right)+\\
&\frac{1}{2}\mathcal{G}\left(v_{312},1,0,\frac{1}{1-u_3};1\right)-\frac{5}{4}\mathcal{G}\left(v_{312},1,1,\frac{1}{1-u_3};1\right)+\frac{1}{2}\mathcal{G}\left(v_{312},1,\frac{1}{1-u_3},0;1\right)-\\
&\frac{5}{4}\mathcal{G}\left(v_{312},1,\frac{1}{1-u_3},1;1\right)+\frac{1}{2}\mathcal{G}\left(v_{312},1,\frac{1}{1-u_3},\frac{1}{1-u_3};1\right)+\frac{1}{2}\mathcal{G}\left(v_{312},\frac{1}{1-u_3},0,1;1\right)+\\
&\frac{1}{2}\mathcal{G}\left(v_{312},\frac{1}{1-u_3},1,0;1\right)-\frac{5}{4}\mathcal{G}\left(v_{312},\frac{1}{1-u_3},1,1;1\right)+\frac{1}{2}\mathcal{G}\left(v_{312},\frac{1}{1-u_3},1,\frac{1}{1-u_3};1\right)+\\
&\frac{1}{2}\mathcal{G}\left(v_{312},\frac{1}{1-u_3},\frac{1}{1-u_3},1;1\right)-\frac{1}{4}\mathcal{G}\left(v_{321},1,1,\frac{1}{1-u_3};1\right)-\frac{1}{4}\mathcal{G}\left(v_{321},1,\frac{1}{1-u_3},1;1\right)-\\
&\frac{1}{4}\mathcal{G}\left(v_{321},\frac{1}{1-u_3},1,1;1\right)-\frac{3}{4}G\left(0,\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)H\left(0;u_1\right)-\\
&\frac{3}{4}G\left(0,\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)H\left(0;u_1\right)-\frac{1}{4}G\left(0,\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)H\left(0;u_1\right)-\\
&\frac{1}{4}G\left(0,\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)H\left(0;u_1\right)-\frac{1}{4}G\left(0,\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3};1\right)H\left(0;u_1\right)+\\
&\frac{1}{4}G\left(0,\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2};1\right)H\left(0;u_1\right)-\frac{3}{4}G\left(\frac{1}{u_1},0,\frac{1}{u_1+u_2};1\right)H\left(0;u_1\right)-\\
&\frac{3}{4}G\left(\frac{1}{u_1},0,\frac{1}{u_1+u_3};1\right)H\left(0;u_1\right)+\frac{1}{2}G\left(\frac{1}{u_1},\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)H\left(0;u_1\right)+\\
&\frac{1}{2}G\left(\frac{1}{u_1},\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)H\left(0;u_1\right)+\frac{1}{4}G\left(\frac{1}{u_1},\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)H\left(0;u_1\right)+\\
&\frac{1}{4}G\left(\frac{1}{u_1},\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)H\left(0;u_1\right)-\frac{1}{4}G\left(\frac{1}{1-u_2},1,\frac{1}{u_1};1\right)H\left(0;u_1\right)+
\end{aligned}
$$

Stunning Simplification

$$R_6^{(2)}(u_1,u_2,u_3) = \sum_{i=1}^{3}\left(L_4(x_i^+,x_i^-) - \frac{1}{2}\mathrm{Li}_4(1-1/u_i)\right)$$
$$-\frac{1}{8}\left(\sum_{i=1}^{3}\mathrm{Li}_2(1-1/u_i)\right)^2 + \frac{1}{24}J^4 + \frac{\pi^2}{12}J^2 + \frac{\pi^4}{72}. \quad (3)$$

[Makes use of "theory of motives"]

This Picture

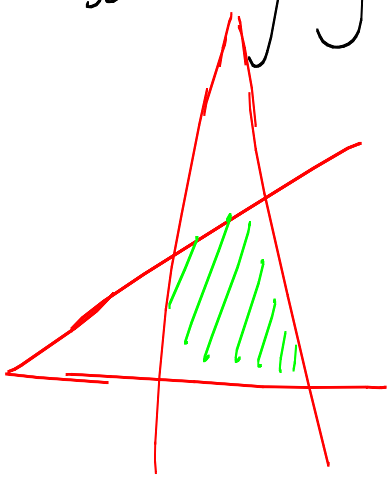
$$\mathcal{A}^{2-\text{loop}}_{\text{MHV}} = \frac{1}{2}\sum_{i<j<k<l<i}$$ 

is telling a geometric story

→ Learn to read off Polylogs from it!

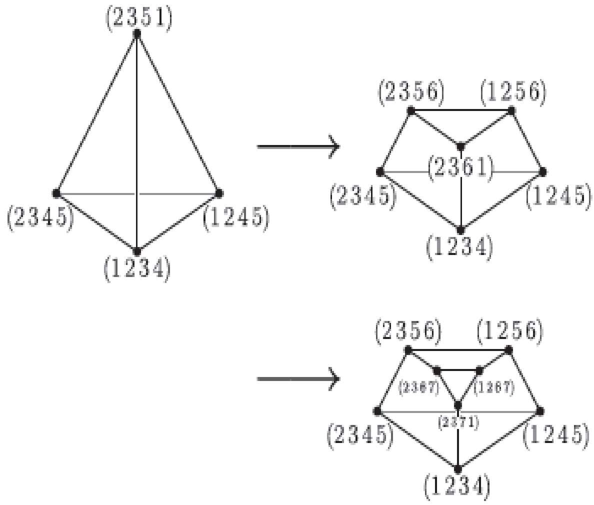
write down the answer...

· In a specific sense, amplitudes are to be thought of as "the volume" of some polytope:

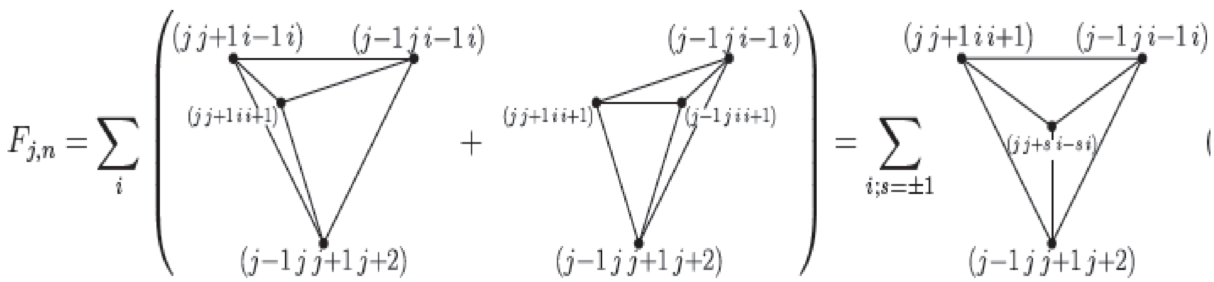


Different triangulations make different properties (Yangian, locality, Unitarity ...) manifest.

Our solution should be thought of as providing one class of triangulations — but we need to more deeply understanding what the object is that is being triangulated!



Understood in Simple Cases
⋮

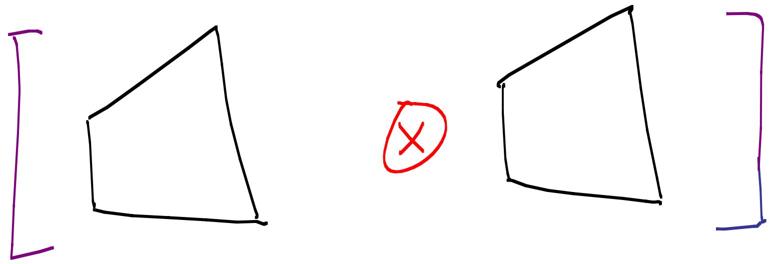


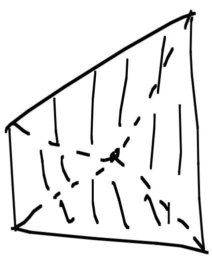
17

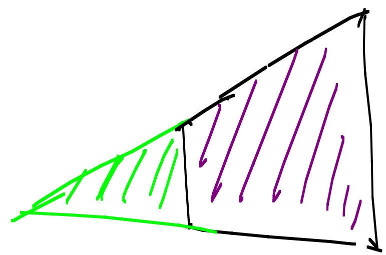
$$M_n^{\mathrm{NMHV}} = \sum_{i,j;s=\pm 1} \frac{\langle \eta_j, \{j{-}1\, j\, j{+}1\, j{+}2\, i\}, \{j{-}1\, j\, j{+}1\, i{-}s\, i\}, \{j\, j{+}s\, i{-}1\, i\, i{+}1\}\rangle}{\langle j{-}1\, j\, j{+}1\, j{+}2\rangle\langle j{-}1\, j\, i{-}1\, i\rangle, \langle j\, j{+}1\, i\, i{+}1\rangle\langle j\, j{+}s\, i{-}s\, i\rangle}$$

NEW LOCAL FORM

Another ex: MHV 1-loop:

"Vol" 

 $\rightarrow$ Locality

 $\rightarrow$ Unitarity

What is the geometry underlying

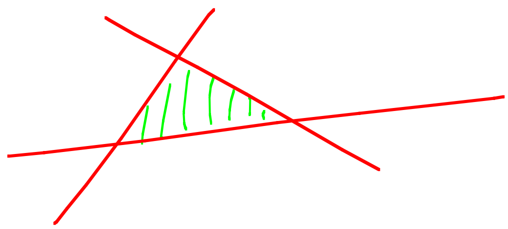
$$\frac{d^{k(n-k)}C}{(12\ldots k)\ \ldots\ (n1\ldots k-1)} \ ?$$

"Positive Part" of Grassmannian

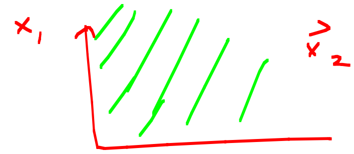
[Lustig, Postnikov ..., Fock + Goncharov]

[many discussions with R. Macpherson, P. Deligne, Goncharov]

Generalize simplices in projective space:



$X = (x_1, x_2, 1)$ $\quad x_i > 0$

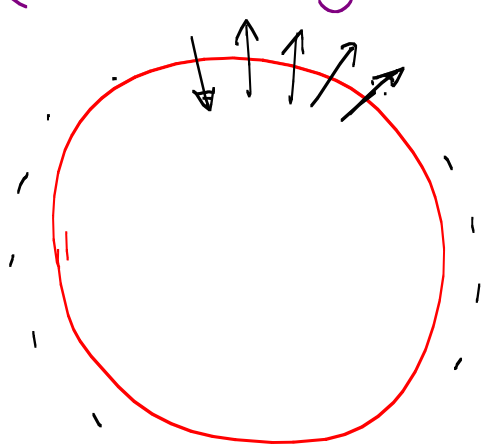


- In e.g. $G(2,n)$:

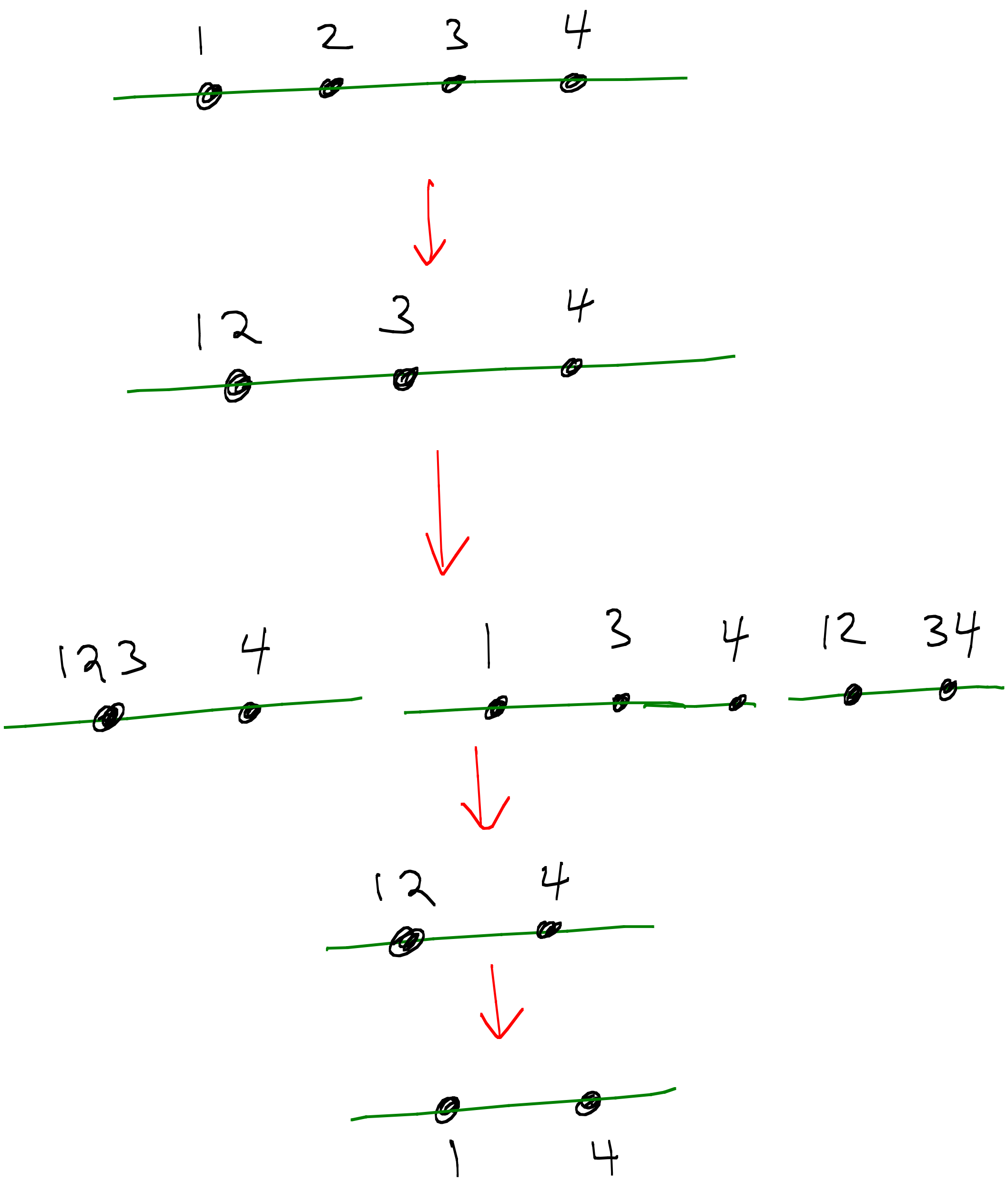
$$\begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}$$

"Positive part" $(ij) > 0$ for $i > j$.

[note (twisted) cyclic structure: $v_1 \to v_2$
$\vdots$
$v_n \to (-1) v_1$]

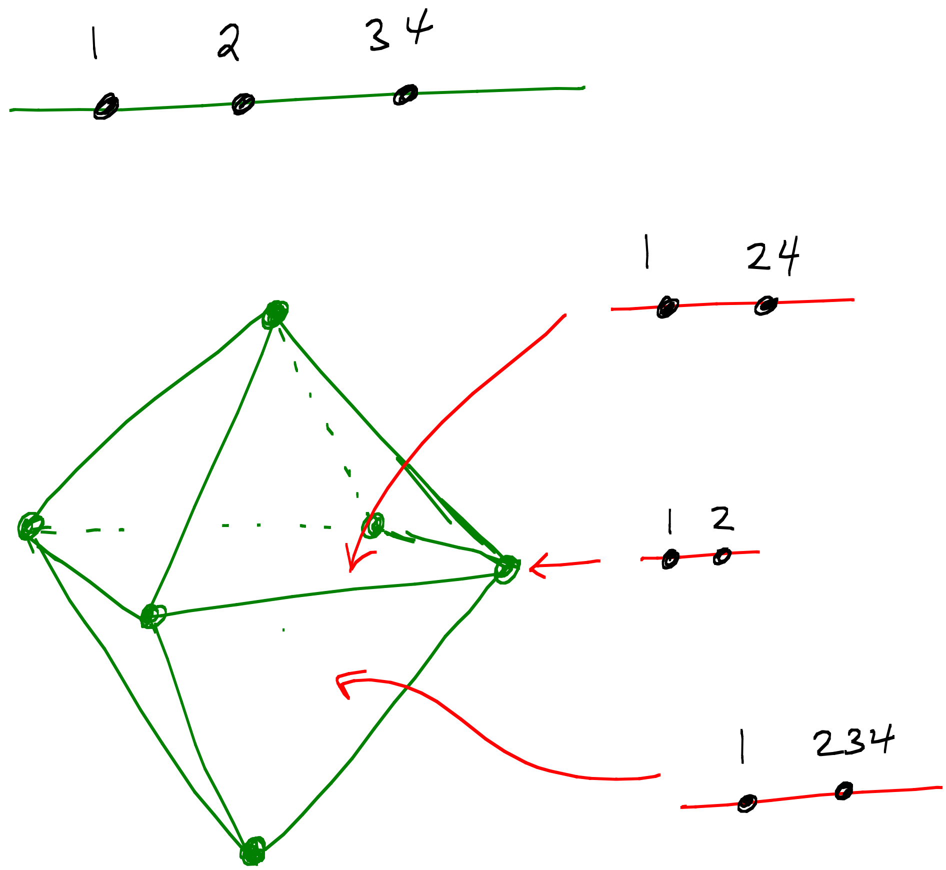


Vastly richer structure: Grassmannian Polytope.

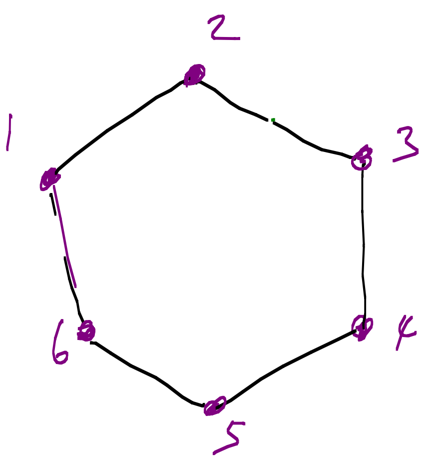


∂ operator is merge + delete

[not just delete!]

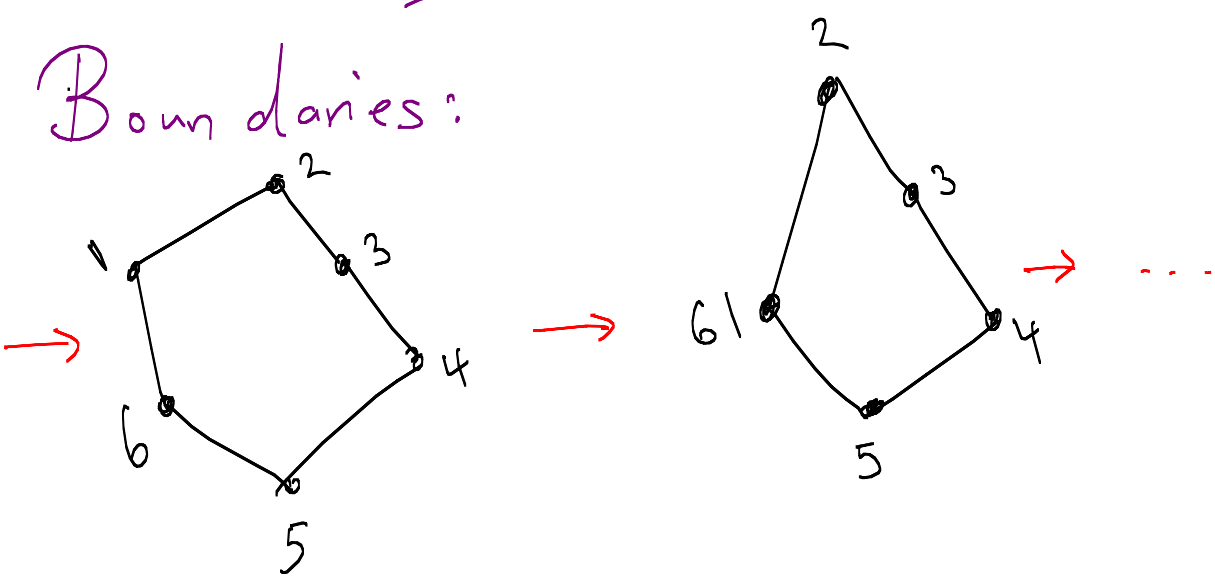

1 2 34
1 24
1 2
1 234

For $G(3,n)$: $(i_1 i_2 i_3) > 0$



$\longrightarrow$ Convex Polygon

Boundaries:



So, boundaries of this polytope associated with linear dependencies of K consecutive columns of $G(K, n)$!

Our measure:

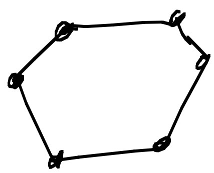
$$\frac{d^{k\times n} C}{(12\cdots k)\cdots(n1\cdots k-1)}$$

is unique one

smooth inside polytope

only sing on boundaries.

Yangian invariance $\longleftrightarrow$ nothing but diffs, preserving positive part!

e.g.  diffs preserving convexity

- Beautiful classification of all facets of this polytope — using "positive co-ordinates" [Postnikov, ... Fock + Goncharov]

$(p_1, \ldots, p_D)$

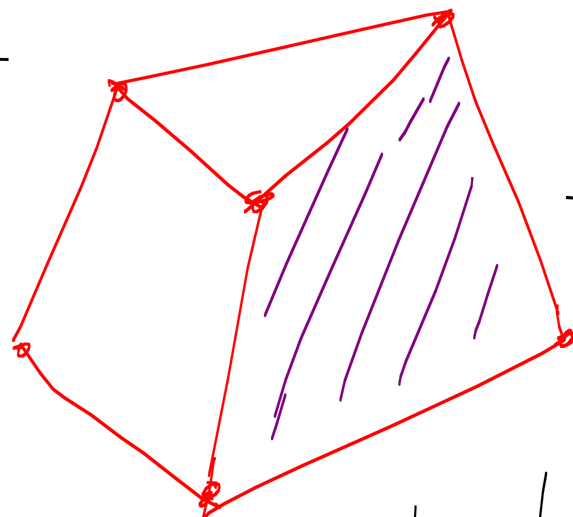
Our measure

$$\frac{d^{k(n-k)}C}{(12\cdots k)\cdots(n1\cdots k-1)} = \frac{dp_1}{p_1}\cdots\frac{dp_D}{p_D}$$

co-ordinate ind. special co-ordinates

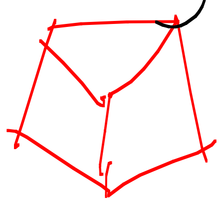
Grassmannian residues / leading singularities are associated with facets of big polytope

$\delta^{4|4}(C\cdot W)$



$\rightarrow$ Explicit expression for all residues.

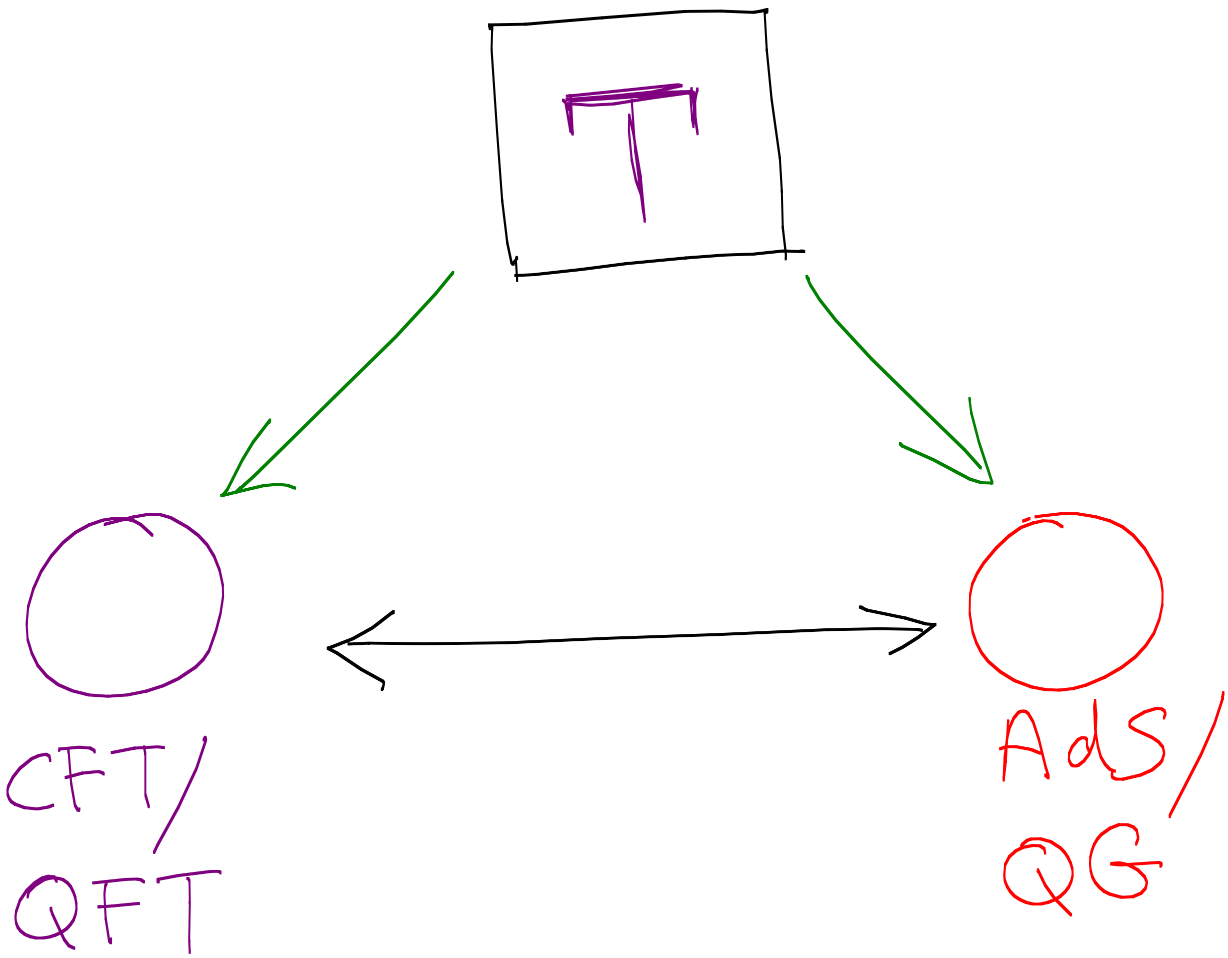
Relations encoding locality + Unitarity:

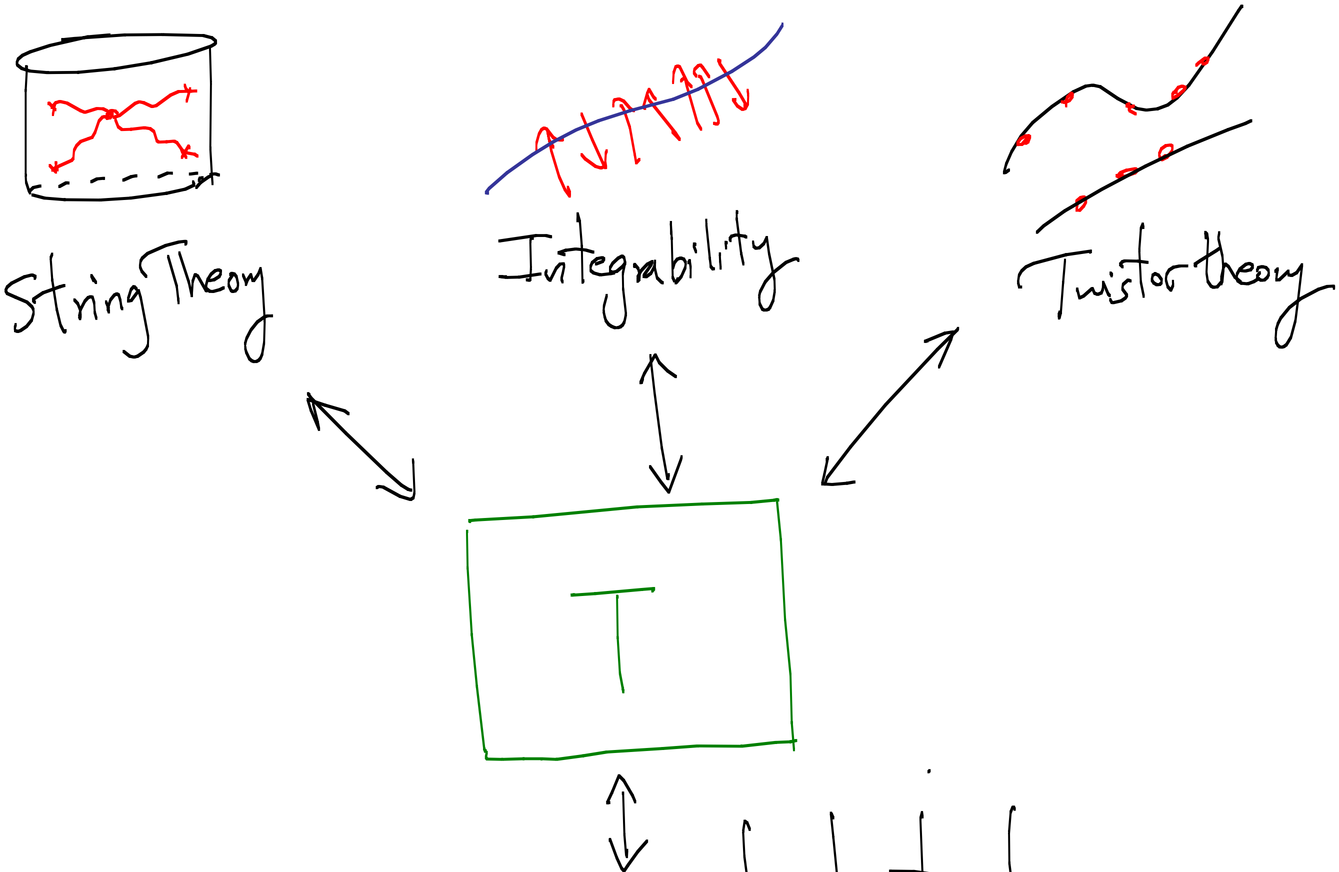
$\partial\Big[$  $\Big] = 0$.

"What is String Theory?"

$$\Big\updownarrow \text{ AdS/CFT}$$

"What is QFT?"


T
CFT/
QFT
AdS/
QG



Amazing new mathematical structures:

(Grassmannians, Polylogs, Motives)

. We are seeing, quite explicitly,
primitive building blocks from which
locality and Unitarity emerge.
This lets us understand physics invariantly,
. without the usual redundancies
obstructing our view. In particular —
hidden symmetries + dualities are
being made manifest.
(Yangian $\Longleftrightarrow$ Fermionic T-duality).

· Amazingly, essentially exactly the same mathematical structures [motives, positive part of $G(k,n) \to$ "cluster varieties"], show up in totally different arena:

Gaiotto $\mathcal{N}=2$ theories coming from compactifying (2,0) theory on a Riemann surface $\longleftrightarrow$ related to BPS Wall-crossing, Hitchin systems... here the physics magic is that of S-duality!

- We are in the middle of an extraordinary period in our understanding of QFT, with possibly deep reprecussions on our picture of Spacetime + QM.
- Grand synthesis yet to come!
- Nice Goal for 85th Celebration!

Happy Birthday Roger!