Mathematical Modelling of Zombies

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HOW LONG CAN WE SURVIVE?

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Abstract

Knowing how long we have before we face off with a zombie could mean the difference between life, death and zombification. Here, we use diffusion to model the zombie population shuffling randomly over a one-dimensional domain. This mathematical formulation allows us to derive exact and approximate interaction times, leading to conclusions on how best to delay the inevitable meeting. Interaction kinetics are added to the system and we consider under what conditions the system displays an infection wave. Using these conditions, we then develop strategies that allow the human race to survive its impending doom.

6.1 Introduction

Humans would not survive a zombie holocaust, or so current work suggests [1]. The fact that reanimated corpses do not stop unless their brain is destroyed, coupled with an insatiable appetite for human flesh, has proven to be a deadly combination [2]. However, the work produced by Munz et al. [1] assumes that zombies and humans are well mixed, meaning that zombies can be found everywhere there are humans. Realistically, the initial horde of zombies will be localized to areas containing dead humans, such as cemeteries and hospitals. In addition, because humans and zombies are not initially separated, humans are not able to run and hide in order to try and preserve themselves. It is a well-documented fact that zombies are deadly but slow-moving [3, 4, 5]. Due to their slow movements, it is quite possible that, given sufficient warning, we would be able to outrun the zombies and produce a defensible blockade where humans could live safely [4]. In order to do so, it would be useful to know just how long an infestation of zombies would take to reach our defences; this would give us an estimate of how long we would have

to scavenge for supplies and weaponry in order that we may protect ourselves from these oncoming, undead predators.

Another useful question to ask would be: 'If a zombie were to infect a group of humans, could we slow down the rate of infection?' This gives a chance for potential survivors to split from the infected group and move to another defensible area.

In order to answer these and other important questions, we add a spatial dimension to a simple infection model. Previous models in this book have looked at time- and population-dependent interactions between humans and the undead. We now add an obvious extension: that of geography, because zombies do not move homogeneously. This allows zombies to shuffle around, giving a much more realistic picture of an invasion.

If the reader currently does not have the time to understand the justification of the mathematics due to their impending doom and would rather skip to useful results, we recommend skipping Section 6.2, where we justify our model; Section 6.3, where we introduce the mathematical formalism; and Section 6.4, in which analytic solutions are provided. Instead, we suggest that the reader jump straight to Section 6.5, which details a quick way of approximating the time left before zombies will be nibbling on their brains. Section 6.6 then looks at adding interaction rules between the populations of the humans and zombies, and can probably be skipped by those looking for immediate answers. The conclusion in Section 6.7 is absolutely vital as we consider all the results we have achieved and interpret them in the context of a strategy for survival.

6.2   Random Walks and Diffusion

To simplify the mathematics, we assume we are working in only one space dimension, denoted x. This is not too restrictive as most results hold in two or even three dimensions. Realistically speaking, this assumption means that there is only one possible access point to our defences and thus the shortest distance between the group of zombies and us is a straight line.

According to Munz et al. [1], "the 'undead' move in small, irregular steps." This makes their individual movement a perfect model of a random walk [6]. The 'drunkard's' or random walk, as you would expect from the name, is a mathematical description of movement in which no direction is favoured. To get an intuitive idea of the motion, consider inebriated people. They stagger backwards and forwards, lurching from one resting place to another.

In Figure 6.1, we illustrate this motion with zombies; since they will move around randomly bumping into one another, they too will spread over the domain. Diffusion has two primary properties: it is random in nature and movement is from regions of high densities to low densities.

We could model the motion of each zombie individually but, since their movement is random, their positions will not be certain unless we track them all exactly [7]. Probabilistic descriptions can be used but, due to the assumed large population
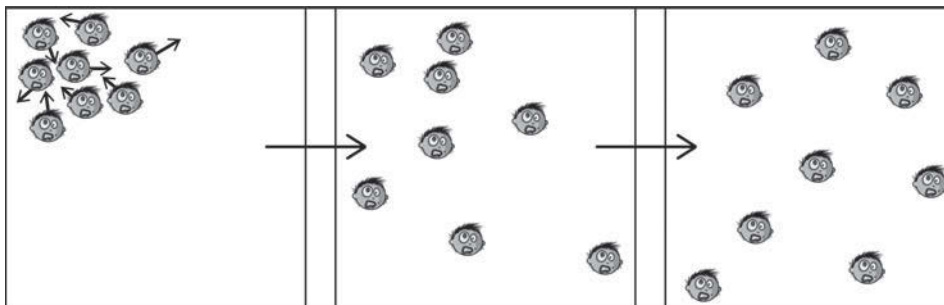
Figure 6.1: An example of two-dimensional diffusion. On the left, an initial group of zombies is placed in the top left corner, close together, their initial directions denoted by the small black arrows. After a time, their random motion will cause them to spread themselves out over the domain.

of zombies, the computational power needed by such simulations would soon get out of hand, even when we are not running for our lives. Instead, we consider the case where the number of zombies is initially large. This produces a continuous model of diffusion that is much easier to use, as it needs less computational power and analytical solutions are available (see Section 6.4). However, what we have gained in simplicity we have lost in accuracy. Realistically the density of zombies in a region will be a discrete integer value: $1, 2, 3, \ldots$ zombies/metre (counting even a dismembered zombie as a whole one). Since we are assuming a large number of zombies, we are forced to allow the density to take any value, including non-integers. Thus the density, originally a discrete function, has been smoothed out to form a continuous function.[1]

6.3   A Mathematical Description of Diffusion

Since time may be running out, we introduce the diffusion equation in an intuitive rather than rigorous way. Further, although word equations are used, a working knowledge of partial derivatives will be useful for the following section.

If we let the density of zombies at a point x and at a time t be $Z(x, t)$, then the density has to satisfy the diffusion equation

$$\frac{\partial Z}{\partial t}(x, t) = D \frac{\partial^2 Z}{\partial x^2}(x, t). \tag{6.1}$$

Explicitly,

$$\frac{\partial Z}{\partial t}(x, t) = \text{rate of change of } Z \text{ over time at a point } x.$$

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[1]The interested reader who would like to learn about the rigorous definitions of continuous functions is first advised to wipe out the walking dead, then rebuild civilization and finally look for any real-analysis textbooks that may have survived intact, such as Bartle and Sherbert [8].

This means that if $\partial Z/\partial t$ is positive, then Z is increasing at that point in time and space. This term allows us to consider how $Z(x,t)$ evolves over time.

The factor D is a positive constant that controls the speed of movement. The units of D are chosen to make the diffusion equation consistent, since the units on each side of the equality must match. The term $\partial Z/\partial t$ has units of [density]/[time] and $\partial^2 Z/\partial x^2$ has units of [density]/[space]^2; thus D must have units of the form [space]^2/[time]. The units could be of the form of kilometres^2/hour or nanometres^2/second. However, for the equation to be most informative we choose scales that are relevant to a zombie invasion; we thus fix the units to be metres^2/minute.

The term on the right-hand side is a little more complex than the left, but essentially it encapsulates the idea that the zombies are moving from high to low densities. This is illustrated with the help of Figure 6.2. Initially, there are many more zombies on the left than the right. Just before the peak in density the arrow (which is the tangent to the curve, or $\partial Z/\partial x$ at this point) is pointing upwards. This means that as x increases, so does the zombie density, Z. At this point,

$$\frac{\partial Z}{\partial x} = \text{rate of change of } Z \text{ as } x \text{ increases} > 0.$$

Just after the peak the arrow is pointing down; thus, at this point,

$$\frac{\partial Z}{\partial x} = \text{rate of change of } Z \text{ as } x \text{ increases} < 0.$$

Hence, at the peak, $\partial Z/\partial x$ is decreasing as x increases. Since $\partial^2 Z/\partial x^2$=rate of change of $\partial Z/\partial x$ as x increases and we have just deduced that $\partial Z/\partial x$ is decreasing at the peak,

$$\frac{\partial^2 Z}{\partial x^2} < 0.$$

From the diffusion equation, we see that, at the peak,

$$\frac{\partial Z}{\partial t} = D\frac{\partial^2 Z}{\partial x^2} < 0.$$

This means that the peak of zombie density is decreasing over time. Similarly, $\partial Z/\partial x < 0$ just before the trough and $\partial Z/\partial x > 0$ after the trough, as shown in Figure 6.2. Thus, at the trough, $\partial^2 Z/\partial x^2 > 0$, implying that the density of zombies is increasing. Overall we see that diffusion causes zombies to move from regions of high density to low density.

It is impossible to overstate the importance of equation (6.1). Whenever the movement of a modelled species can be considered random and directionless, the diffusion equation will be found. This means that, by understanding the diffusion equation, we are able to describe a host of different systems such as heat conduction through solids, gases spreading out through a room, proteins moving around the body, molecule transportation in chemical reactions, rainwater seeping through soil and predator–prey interaction, to name but a few of the great numbers of applications [9, 10, 11].
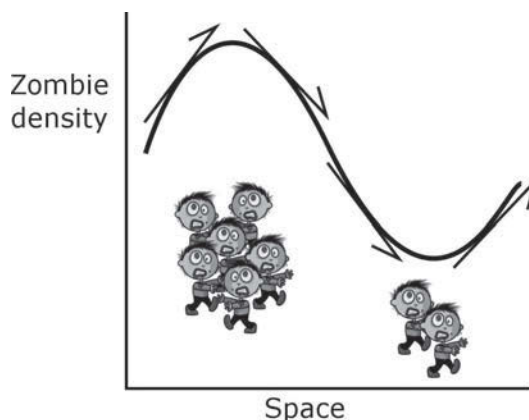
Figure 6.2: An example of how diffusion smoothes out peaks and troughs. At the peak of the density, there are more zombies than the surrounding area. Since the rate of change of number of zombies over space (the tangent shown by the arrows on the curve) is positive before the peak and negative after the peak, the rate of change of number of zombies over space is decreasing. Using this knowledge and equation (6.1), we see that the density at the peak is decreasing. Similarly, at the trough, the density will be increasing.

6.4 Solution to the Diffusion Equation

The diffusion equation gives us a mathematical description of zombie movement. By solving it, we produce the function $Z(x, t)$, the density of the zombies at each position x for all time $t \geq 0$. However, we must add some additional information to the system before we can solve the problem uniquely.[2]

Initially, we have a density of Z_0 zombies/metre. All of the zombies are assumed to start in the region $0 \leq x \leq 1$ (see equation (6.3)). The implication of this is that all of the undead will originate from one place, such as a cemetery or mortuary in a hospital.

The final assumption we make is that the zombies cannot move out of the region $0 \leq x \leq L$. That is, we define L as the length of the domain. This creates a theoretical boundary at $x = 0$ that the population cannot cross; the zombies will simply bounce off this boundary and be reflected back into the domain. We place our defences at $x = L$, thus creating another theoretical boundary here. Thus, at $x = 0$ and $x = L$, the 'flux of zombies' is zero. Various other boundary conditions are possible but for simplicity we chose 'zero flux' conditions so that the boundaries may prevent the zombies from leaving the region but do not alter their population.

The flux of zombies at $x = 0$ and $x = L$ is simply the rate of change of the numbers of zombies through these boundaries. Mathematically, this is the spatial

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[2]Uniqueness is an important concept in mathematics, particularly when solving differential equations. If your solution is not unique, then you cannot be sure which one of the possible solutions is the one needed.

derivative at these points and, since the flux is assumed to be zero, then $\partial Z/\partial x = 0$ at $x = 0, L$. These initial and boundary conditions can be seen in Figure 6.3A.

The system is then fully described as

$$\frac{\partial Z}{\partial t}(x,t) = D\frac{\partial^2 Z}{\partial x^2}(x,t) \qquad \text{(the partial differential equation)} \qquad (6.2)$$

$$Z(x,0) = \begin{cases} Z_0 & \text{for } 0 \leq x \leq 1 \\ 0 & \text{for } x > 1 \end{cases} \qquad \text{(the initial condition)} \qquad (6.3)$$

$$\frac{\partial Z}{\partial x}(0,t) = 0 = \frac{\partial Z}{\partial x}(L,t) \qquad \text{(zero-flux boundary conditions)}. \qquad (6.4)$$

Equation (6.2) with initial condition (6.3) and boundary condition (6.4) can be solved exactly and has the form

$$Z(x,t) = \frac{Z_0}{L} + \sum_{n=1}^{\infty} \frac{2Z_0}{n\pi} \sin\left(\frac{n\pi}{L}\right) \cos\left(\frac{n\pi}{L}x\right) \exp\left(-\left(\frac{n\pi}{L}\right)^2 Dt\right). \qquad (6.5)$$

The interested reader can find more details in Appendix A or in [9].
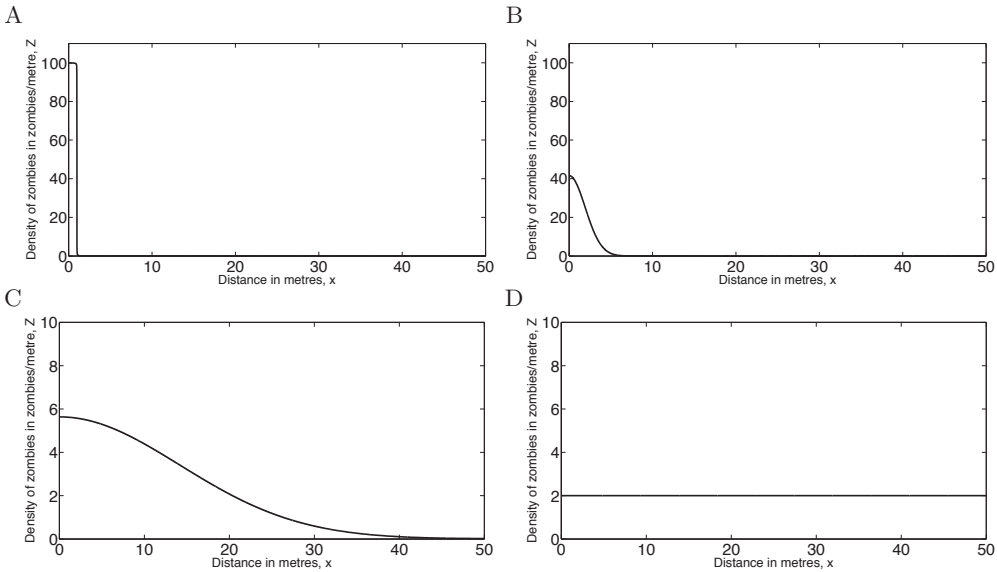

Figure 6.3: Evolution of a system of zombies over 100 minutes. The zombies are assumed to be confined within a domain of length 50 metres and have a diffusion coefficient of $D = 100$ metres^2/minute. A: $t = 0$ minutes. B: $t = 1$ second. C: $t = 1$ minute. C: $t = 100$ minutes.

The sine and cosine functions are those that the reader may remember from their trigonometry courses. The exponential function, 'exp,' is one of the fundamental operators of mathematics, but for now the only property that we are going to make use of is that if $a > 0$ then $\exp(-at) \to 0$ as $t \to \infty$ [8]. Using this fact and equation

(6.5), as t becomes large, most of the right-hand side of the equation becomes very small, approximately zero. Hence, for large values of t, we can approximate

$$Z(x,t) \approx \frac{Z_0}{L}. \tag{6.6}$$

This means that as time increases, zombies spread out evenly across the available space, with average density Z_0/L everywhere.

In Figure 6.3, we have plotted zombie density for a number of times. Figure 6.3A shows the initial condition of a large density of zombies between $0 \le x \le 1$. Figure 6.3B and Figure 6.3C then illustrate how diffusion causes the initial peak to spread out, filling the whole domain. Note that we rescaled the axes in Figures 6.3C and 6.3D in order to show the solution more clearly. Figure 6.3D verifies equation (6.6) since, after 100 minutes, the density of zombies has become uniform throughout the domain.

6.5   Time of First Interaction

Equation (6.5) gives the density of zombies at all places, x, and for all time, t. There are many questions we could answer with this solution; however, the most pressing question to any survivors is, "How long do we have before the first zombie arrives?" If we had proceeded with a stochastic description of the system, we could use the theory of 'first passage time' [7]. However, deterministically, the mathematical formulation of this question is, "For what time, t_z, does $Z(L, t_z) = 1$?" The time t_z is then the time taken (on average) for a zombie to have reached $x = L$.

Unfortunately, this does not have a nice solution that can be plotted like equation (6.5). In Appendix B, we introduce the bisection search technique [12], which enables us to find this solution, approximated to any degree of accuracy we desire.

Using this technique, we can now vary the distance and speed of the zombies to calculate the various interaction times. In Figure 6.4, we plot the interaction time of an initial density of 100 zombies against a diffusion rate ranging from 100m^2/min to 150m^2/min, which (from empirical evidence) is realistic for a slow shuffling motion and for distances between 50 and 90 metres. Notice that, for distances greater than $L = 100$m, the average density of zombies is less than 1 zombie/metre and so there will be no solution, t_z, to $Z(L, t_z) = 1$ since $Z(L, t) < 1$ for all $t \ge 0$. In this situation, the assumption that we have a large number of zombies has failed, making the continuous model an inadequate description.

6.5.1   Diffusive Time Scale

When the apocalypse does happen, we have to ask ourselves the question, "Do we want to waste time computing solutions when we could be out scavenging?" We
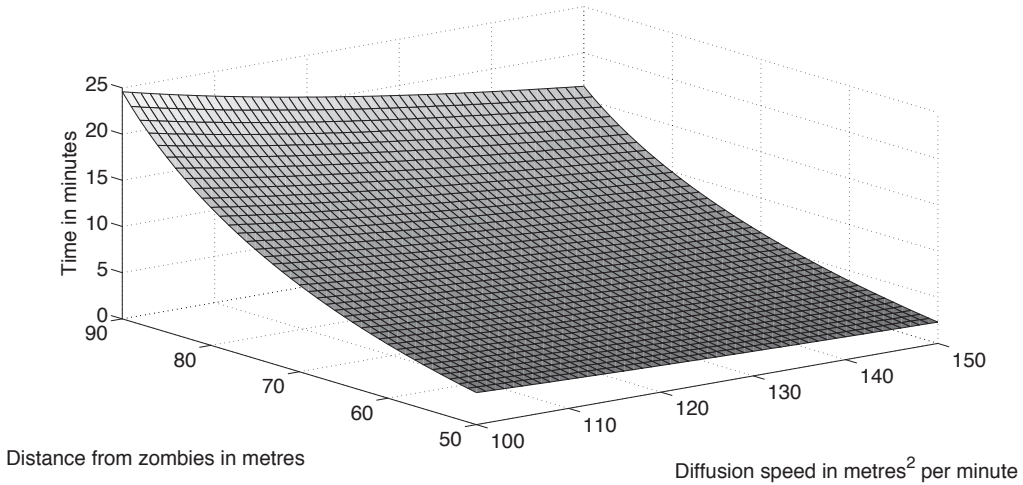
Figure 6.4: Time in minutes until the first zombie arrives for various rates of diffusion and distances.

thus introduce the diffusive time scale

$$t = \frac{1}{D}\left(\frac{L}{\pi}\right)^2 \approx 0.32\frac{L^2}{D}. \tag{6.7}$$

From equation (6.5), we can see that this is the time it takes for the first term of the infinite sum to fall to $\exp(-1)$ of its original value. The factor of $\exp(-1)$ is used due to its convenience.

Only the first term of the expansion is considered, as the exponential function is monotonically increasing. This means that if $a > b$ then $\exp(a) > \exp(b)$ and so as n increases, the contribution from the term

$$\exp\left(-\left(\frac{n\pi}{L}\right)^2 Dt\right)$$

rapidly decreases. Thus the first term gives an approximation to the total solution.

Hence equation (6.7) gives a rough estimate of how quickly the zombies will reach us. For example, if zombies are 90 metres away and have a diffusion rate of 100m^2/min, then $t \approx 26$ minutes, comparable to the solution in Figure 6.4. It also implies a very important result about delaying the human–zombie interaction.

There are two possible ways we could increase the time taken for the zombies to reach us. We could run further away or we could try and slow the zombies down, since the time taken is proportional to the length squared, L^2, and inversely proportional to the diffusion speed, D. This means that if we were to double the distance between ourselves and the zombies, then the time for the zombies to reach us would approximately quadruple. However, if we were to slow the zombies down by half, then the time taken would only double. Since we want to delay interaction

with the zombies for as long as possible, then from the above reasoning, we see that it is much better to expend energy travelling away from the zombies than it is to try and slow them down. Without some form of projectile weaponry or chainsaw, killing zombies is particularly difficult as they do not stop until their brains are destroyed. These conclusions are confirmed in Figure 6.4.

It should be noted that the time derived here is a lower bound. In reality, the zombies would be spreading out in two dimensions and would be distracted by obstacles and victims along the way, so the time taken for the zombies to reach us may be longer. This conservative estimate will keep us safe, especially since the authors would prefer to be long gone rather than chance a few more minutes of scavenging!

6.6 Slowing the Infection

In the previous sections, we assumed that the main source of zombies were the primary infected; that is, the dead who have somehow been brought back to life. However, the primary infected are able to produce a secondary generation of infected individuals through biting. These are victims who, although bitten, survive the initial encounter with a zombie and will eventually turn into zombies themselves [1].

Note that this hijacking of a host species is not restricted to fiction. The natural world has a number of parasites that infect and control their prey [13]. One particularly well-known example is *Cordyceps unilateralis*, a fungus that infects ants and alters their behaviour. These infected ants are then recruited in the effort to distribute fungus spores as widely as possible to other ants.
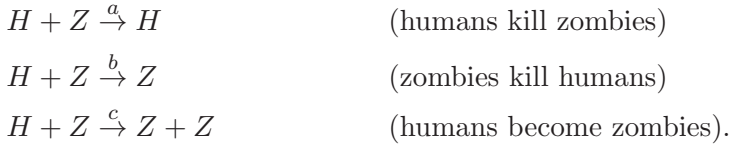
6.6.1 Interaction Kinetics

Initially, we do not consider a spatial model. This simplification allows us to investigate the interactions of the two species more easily. However, in Section 6.6.3, we will use diffusion once again to model movement and thus see how this can affect the dynamics.

To model human–zombie interactions, we suppose that a meeting between the two populations can have three possible outcomes.

1. The human kills the zombie.

2. The zombie kills the human.

3. The zombie infects the human and so the human becomes a zombie.

See Figure 6.5.

Allowing H to stand for the human population and Z to stand for the zombie population, these three rules can be written as though they were chemical reactions:

$$H + Z \xrightarrow{a} H \qquad\qquad \text{(humans kill zombies)}$$

$$H + Z \xrightarrow{b} Z \qquad\qquad \text{(zombies kill humans)}$$

$$H + Z \xrightarrow{c} Z + Z \qquad\qquad \text{(humans become zombies)}.$$

The letters above the arrows indicate the rate at which the transformation happens and are always positive. If one of the rates is much larger than the other two, then this 'reaction' would most likely happen.
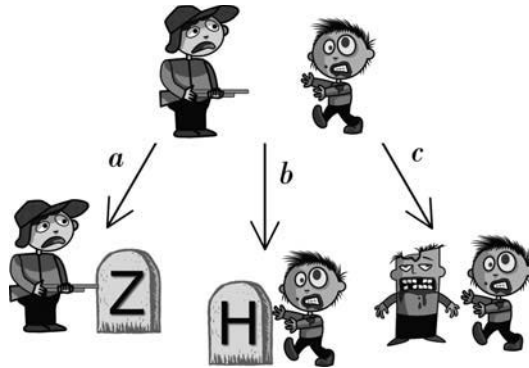

Figure 6.5: The possible outcomes of a human–zombie interaction: (a) humans kill zombies, (b) zombies kill humans, or (c) zombies convert humans.

To transform these reactions into a mathematical equation, we use the Law of Mass Action [10]. This law states that the rate of reaction is proportional to the product of the active populations. Simply put, this means that the above reactions are more likely to occur if we increase the number of humans and/or zombies. Thus we can produce the following equations, which govern the population dynamics:[3]

$$\frac{dH}{dt} = -bHZ - cHZ = -(b+c)HZ \qquad\qquad (6.8)$$

$$\frac{dZ}{dt} = cHZ - aHZ = (c-a)HZ. \qquad\qquad (6.9)$$

6.6.2   Is It Possible to Survive?

Equation (6.8) shows that the population of humans will only ever decrease over time, because $b, c > 0$ and $H, Z \geq 0$. We could add a birth term into this equation, which would allow the population to also increase in the absence of zombies.

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[3]Our use of the symbol d in this case rather than ∂ implies that we are working with one variable dimension—in this case time, t. When dealing with diffusion, we consider both space and time and thus we use ∂.

However, as we have seen from Section 6.5.1, the time scale we are working on is extremely short, much shorter than the nine months it takes for humans to reproduce! Thus we ignore the births since they are not likely to alter the populations a great deal during this period.

Equation (6.9) is not so clear, as the term '$(c - a)$' may either be positive or negative. If $(c-a) > 0$ then the creation rate of zombies, c, must be greater than the rate at which we can destroy them, a. In this case, the humans will be wiped out as our model predicts that the zombie population will grow and the human population will die out. However, there is a small hope for us. If the rate at which humans can kill zombies is greater than the rate at which zombies can infect humans, then $(c - a) < 0$. In this case, both populations are decreasing and thus our survival will come down to a matter of which species becomes extinct first.

If we now let $b + c = \alpha$ then α is the net removal rate of humans; similarly, $c - a = \beta$ is the net creation rate of zombies. We can now write equations (6.8) and (6.9) as

$$\frac{d(\beta H + \alpha Z)}{dt} = \beta \frac{dH}{dt} + \alpha \frac{dZ}{dt} = -\beta \alpha H Z + \alpha \beta H Z \equiv 0.$$

This means that $(\beta H + \alpha Z)$ does not change over time so, although $H(t)$ and $Z(t)$ do change over time (as mentioned above, $H(t)$ is always decreasing), they must change in such a way to keep the value of $(\beta H(t) + \alpha Z(t))$ a constant. Since we can estimate the initial populations of zombies and humans as Z_0 and H_0, respectively, we can define this constant exactly:

$$\beta H(t) + \alpha Z(t) = \beta H_0 + \alpha Z_0. \tag{6.10}$$

As mentioned above, the only way humans can survive is if we are more deadly than zombies[4] and thus $\beta < 0$. To make the following result easier to see, let $\beta = -\gamma$, with $\gamma > 0$. If the humans are to survive, then all of the zombies must be wiped out so $Z(\infty) = 0$ (here $Z(\infty)$ is taken to mean the population of the zombies after a long time has passed; similarly, $H(\infty)$ is the human population after a long time). Upon rearranging equation (6.10), we find that

$$\gamma H(\infty) = \gamma H_0 - \alpha Z_0,$$

and since we desire the human population to still exist, this implies $H(\infty) > 0$. Thus, for our species to survive the war of attrition against the zombies, the initial populations must satisfy

$$\gamma H_0 > \alpha Z_0. \tag{6.11}$$

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[4]If the zombie apocalypse should occur and you are in fact a zombie reading this, please accept the authors' apologies. There is nothing personal against you or your kind. Humans simply do not take kindly to being thought of as lunch. Indeed, if you are reading this, we must assume that you are an intelligent species, in which case you may like to work through these results and see how they might favour zombies instead of humans.

Essentially, the inequality echoes the sentiments we would expect. To survive extinction, the humans need a large enough initial population which is capable of being more deadly than the zombies. If this condition is not satisfied, then the zombie revolution is certain. See Figure 6.6 for an example of this.
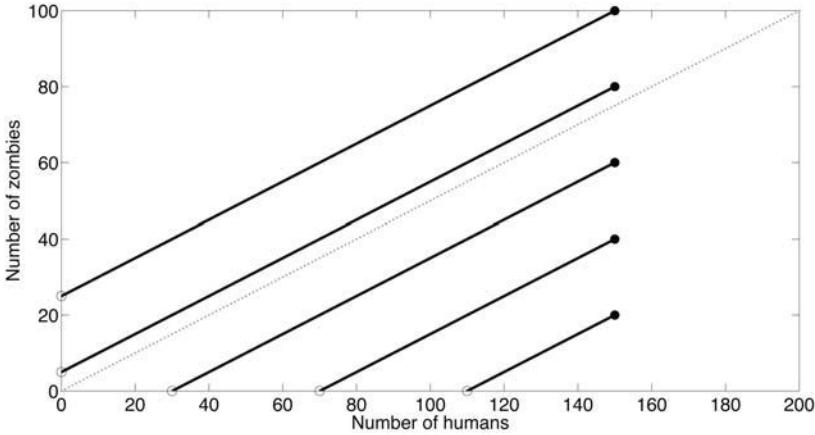

Figure 6.6: The solid lines show the evolution of the populations for a number of different initial zombies. The initial condition is shown as a filled in circle and the final state of the system is shown as a hollow circle. The dotted line, $\gamma H = \alpha Z$, separates the two possible cases, where γ is the net destruction rate of zombies and α is the net human removal rate. Above the dotted line, zombies survive and kill all humans; below the dotted line, humans survive and kill all zombies. If γ were increased (zombies are removed more quickly than they are created) the line would become steeper, meaning that it would become easier for the population to be beneath the line and thus survive. If α were to increase, the line would become shallower, implying that it would be easier to be above the line and thus the zombies would wipe us out. Parameters used are $\gamma = 0.5$ per human per minute and $\alpha = 1$ per zombie per minute.

6.6.3   Infection Wave

The above analysis does not take into account spatial effects. However, the zombies will not be everywhere, but will rather be localized. Thus, in Section 6.5, we considered only the time before the zombies came into contact with the human population. Once this occurs, we then have to consider not only how fast the zombies are moving but also how fast they can infect people. Intuitively, if the infection rate was very small, we would have little to worry about because, even if the zombies managed to reach us, it would be very hard for them to infect us while very easy for us to pick them off one by one. To consider the spatial situation, we now add the diffusion term to each of the equations

$$\frac{\partial H}{\partial t} = D_H \frac{\partial^2 H}{\partial x^2} - \alpha H Z \tag{6.12}$$

$$\frac{\partial Z}{\partial t} = D_Z \frac{\partial^2 Z}{\partial x^2} + \beta H Z. \tag{6.13}$$

Note that the diffusion coefficient of humans (D_H) is likely to be much smaller than the zombies' (D_Z) as, being in control of their mental faculties, humans do not move randomly. Since we are considering the initial stages of the invasion by the undead, we are assuming that humans have not figured out what is going on yet and are not yet running for their lives.

To see how quickly the infection moves through the human population, we look for a wave solution: a particular solution form of a differential equation that does not change its shape as it moves in space and time. This infection wave will move at a certain speed, v, which we hope to find. For this wave solution that we are looking for, we know that space, x, and time, t, will be linked through this speed; we can thus reduce the dimensions of the system from the two coordinates (x, t) to the single coordinate $u = x - vt$. See Appendix C for more justification and the implication of this coordinate change.

The system simplifies to

$$0 = D_H H'' + vH' - \alpha HZ$$
$$0 = D_Z Z'' + vZ' + \beta HZ,$$

where, for the sake of brevity, we have used the notation

$$' = \frac{d}{du}.$$

In Appendix C, we are able to show that the speed has a minimum value of

$$v_{\min}^2 = 4D_Z \beta H_0. \tag{6.14}$$

In order to slow the infection, we should try to reduce the right-hand side of equation (6.14). This in turn reduces the minimum speed for v, causing the infection to move more slowly [14]. To do this, we can reduce either D_Z, β or H_0. We discuss the reality of these decisions in the next section. In Figure 6.7, we show the effect of reducing the zombie diffusion rate from 100 m^2/min to 50 m^2/min. Note that the infection wave is further back in the slower case, as expected.

6.7   Conclusion

In Section 6.5.1, we showed that running away from zombies increases the time before the initial interaction far more than trying to slow the zombies down. Thus fleeing for your life should be the first action of any human wishing to survive. However, we cannot run forever; interaction with zombies is inevitable.

Section 6.6.2 then comes into play. Here we showed that the best long-term strategy is to create a fortified society that can sustain the population, while allowing us to remove the zombies as they approach. This, in effect, reduces α to zero which means that equation (6.11) is easy to satisfy and we will survive.
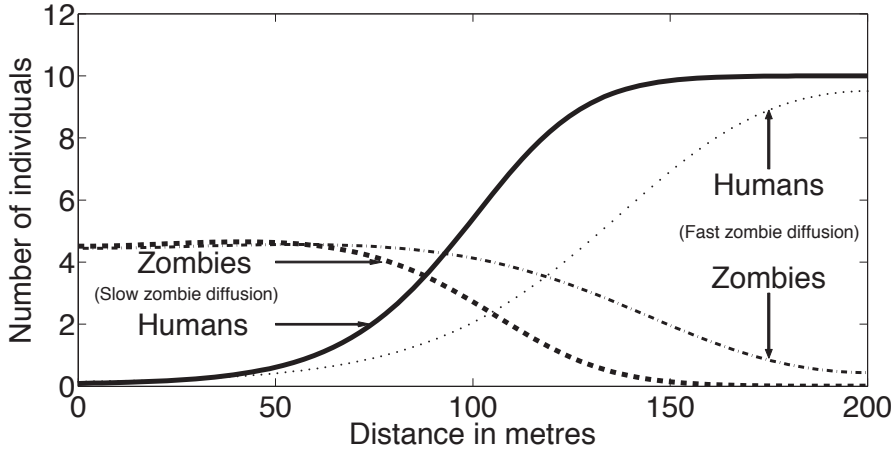
Figure 6.7:   The infective wave solution for two different values of D_Z. The solid and dashed curves are the human and zombie populations respectively, for $D_Z = 100$ m^2 per minute. The dotted and dash-dotted curves are the human and zombie populations respectively, for $D_Z = 50$ m^2 per minute. Other parameters are $D_H = 0.1$ m^2 per minute, $\alpha = 0.1$ per zombie per minute and $\beta = 0.05$ per human per minute.

In the event of the apocalypse, it is unlikely that we would be able to support such a commune without raiding parties scavenging for medicinal, food and fuel supplies [4]. Thus, in this case, we fall back on the maxim of being more deadly than the zombies which increases the left-hand side of equation (6.11).

What if the barricade fails? In Section 6.6.3, we consider this possibility and show that our only hope is to try and reduce the speed of infection. To do this, equation (6.14) implies we must reduce either D_Z, β or H_0. Reducing D_Z amounts to slowing the zombies down; thus an effective fortification should have plenty of obstructions that a human could navigate but a decaying zombie, who may not be so athletic, would find challenging [15]. We have already discussed the case of reducing β; ideally, we want $\beta < 0$. If this is true and we are able to kill the zombies more quickly than they can infect, then the system cannot support a wave solution, thus greatly reducing the speed of zombification. Finally, we consider the possibility of reducing H_0, the population of the commune. This leads to the controversial tactic of removing your fellow survivors: if the zombies are unable to infect them, then their population cannot increase. The authors do not recommend this course of action, as reducing the human population also reduces the number of people able to fight the zombies and humans sacrificing other humans would only speed the extinction of their own species. The population will have enough trouble trying to survive the hordes of undead, without worrying about an attack from their own kind!

Our conclusion is grim, not because we want it to be so but because it is so. It had always been the authors' intention to try and save the human race. So to you,

the reader, who may be the last survivor of the human race, we say: run. Run as far away as you can get; an island would be a great choice.[5] Only take the chance to fight if you are sure you can win and seek out survivors who will help you stay alive.

Good luck. You are going to need it.

Appendices

A   Diffusion Solution Details

In Section 6.4, we claimed

$$Z(x,t) = \frac{Z_0}{L} + \sum_{n=1}^{\infty} \frac{2Z_0}{n\pi} \sin\left(\frac{n\pi}{L}\right) \cos\left(\frac{n\pi}{L}x\right) \exp\left(-\left(\frac{n\pi}{L}\right)^2 Dt\right) \qquad (6.15)$$

as the solution to

$$\frac{\partial Z}{\partial t}(x,t) = D\frac{\partial^2 Z}{\partial x^2}(x,t) \qquad (6.16)$$

$$Z(x,0) = \begin{cases} Z_0 & \text{for } 0 \leq x \leq 1 \\ 0 & \text{for } x > 1 \end{cases} \qquad (6.17)$$

$$\frac{\partial Z}{\partial x}(0,t) = 0 = \frac{\partial Z}{\partial x}(L,t). \qquad (6.18)$$

Given the solution, we can easily check by taking the derivatives that it is indeed a solution to the above constraints, but how do we find such a solution?

Separation of variables

Equation (6.16) depends on the variables space, x, and time, t, so we 'guess' the solution has the form $Z(x,t) = f(x)g(t)$. This is known as separating the variables, as we are assuming that Z can be decomposed into functions each depending on only one of the variables.

This guess is now substituted into equation (6.16) which we use to derive constraints on the forms of f and g. The substituted form of the equation, after rearrangement, is

$$\frac{1}{g}\frac{dg}{dt} = \frac{D}{f}\frac{d^2 f}{dx^2}. \qquad (6.19)$$

Notice that the left-hand side of this equation does not depend of x in any way. Similarly, the right-hand side does not depend on t. This implies that if we change the variable x, then the right-hand side does not change. If it did, then due to the equality, the left-hand side would do too, but we have already noticed that

---

[5]Make sure the bay around the island is steep, as zombies can also cross water since they do not need to breathe [4].

the left-hand side does not depend on x. This logic is exactly the same for the variable t. Due to this argument, we deduce that both sides must be a constant, $-\alpha$, say (the negative sign is arbitrary but useful later on). Equation (6.19) is then split to produce

$$\frac{dg}{dt} = -\alpha g \qquad\qquad (6.20)$$

$$\frac{d^2 f}{dx^2} = -\frac{\alpha}{D} f. \qquad\qquad (6.21)$$

These are standard differential equations and their solution can be found in numerous text books [9]. First, we consider equation (6.20). This has the well-known solution

$$g(t) = C_1 \exp(-\alpha t),$$

where C_1 is a constant that we find later. This is the same exponential function described in Section 6.4. The value of α has yet to be defined exactly; however, this equation implies the sign of α. We want our solutions to not increase, since the zombies are spreading out and therefore $\alpha \geq 0$. This is useful since equation (6.21) has a different solution depending on whether α is positive, negative or zero. Using this information, equation (6.21) has the general solution

$$f(x) = C_2 \cos\left(\sqrt{\frac{\alpha}{D}} x\right) + C_3 \sin\left(\sqrt{\frac{\alpha}{D}} x\right),$$

where C_2, C_3 are, once again, constants that we need to define. To pin down the constants, we appeal to the boundary conditions, equation (6.18). Since we have a separable equation (i.e., $Z(x,t) = f(x)g(t)$), the boundary conditions simplify to

$$\frac{df}{dx}(0) = 0 = \frac{df}{dx}(L).$$

Since

$$\frac{df}{dx} = -C_2 \sin\left(\sqrt{\frac{\alpha}{D}} x\right) + C_3 \cos\left(\sqrt{\frac{\alpha}{D}} x\right),$$

then

$$\frac{df}{dx}(0) = C_3 = 0$$

$$\frac{df}{dx}(L) = -C_2 \sin\left(\sqrt{\frac{\alpha}{D}} L\right) + C_3 \cos\left(\sqrt{\frac{\alpha}{D}} L\right) = 0$$

$$\implies \frac{df}{dx}(L) = -C_2 \sin\left(\sqrt{\frac{\alpha}{D}} L\right) = 0.$$

To satisfy the last equation, we could let $C_2 = 0$. However, this would mean $Z \equiv 0$. Not only is this a pretty dull solution, but also it does not satisfy the initial

condition. In order to satisfy the equation, we use the fact that $\sin(\theta) = 0$ whenever $\theta = n\pi$, where $n = 0, 1, 2, 3, \ldots$, so we define $\alpha = \alpha_n$ through

$$\sqrt{\frac{\alpha_n}{D}} L = n\pi \implies \alpha_n = D \left(\frac{n\pi}{L}\right)^2.$$

Putting f and g back together and defining $C_n = C_1 C_2$ gives

$$Z(x, t) = C_n \cos\left(\frac{n\pi}{L} x\right) \exp\left(-D \left(\frac{n\pi}{L}\right)^2 t\right).$$

Since this is a solution for all values of $n = 0, 1, 2, 3, \ldots$, we can take a linear combination of them and still have a solution. Thus

$$Z(x, t) = \sum_{n=0}^{\infty} C_n \cos\left(\frac{n\pi}{L} x\right) \exp\left(-D \left(\frac{n\pi}{L}\right)^2 t\right). \tag{6.22}$$

A crash course in Fourier series

The form of solution (6.22) is known as a Fourier series expansion. It is a way of representing a function as the sum of different functions. We will not go into the subtleties of Fourier series here; suffice it to say that it is an incredibly powerful technique that allows us to produce solutions to linear differential equations [16, 9].

To solve our specific problem, we must satisfy the initial condition by producing a definite form for the arbitrary constants C_n. To do this, we notice that

$$\int_0^L \cos\left(\frac{m\pi}{L} x\right) dx = \begin{cases} 0 & \text{if } m = 1, 2, 3, \ldots \\ L & \text{if } m = 0 \end{cases}$$

$$\int_0^L \cos\left(\frac{m\pi}{L} x\right) \cos\left(\frac{n\pi}{L} x\right) dx = \begin{cases} 0 & \text{if } m \neq n \\ \frac{L}{2} & \text{if } m = n. \end{cases}$$

Using these, we can deduce that

$$C_0 = \frac{1}{L} \int_0^L Z(x, 0) \, dx$$

$$C_n = \frac{2}{L} \int_0^L Z(x, 0) \cos\left(\frac{n\pi}{L}\right) dx \qquad \text{for } n > 0$$

and, finally, using the initial condition

$$Z(x, 0) = \begin{cases} Z_0 & \text{for } 0 \leq x \leq 1 \\ 0 & \text{for } x > 1, \end{cases}$$

we derive exactly the solution given in equation (6.15).

B   Approximating First Interaction Time

As mentioned in Section 6.5, there is no formula we can use to produce the first interaction time exactly. However, the mathematical problem comes down to finding a solution t_z of equation $Z(L, t_z) = 1$. In this case, the probabilistic method has an advantage over the continuous model since the mean first interaction time can be derived exactly [17].

A simple method of solving $Z(L, t_z) = 1$ would simply be to substitute in a value of t. If $Z(L, t) < 1$ then we double t and consider $Z(L, 2t)$. Notice that if $t_1 < t_2$, then $Z(L, t_1) < Z(L, t_2)$ and so $Z(L, t) < Z(L, 2t)$. Thus we keep doubling t until we reach a value such that $Z(L, 2^n t) > 1$. If $t_0 = 2^{n-1} t$ and $t_1 = 2^n t$, then we know that $Z(x, t) = 1$ for some[6] $t \in [t_0, t_1]$. To gain better approximations to the value of t_z, we then start halving this domain and only keep the half that contains the root, as shown in Figure 6.8.
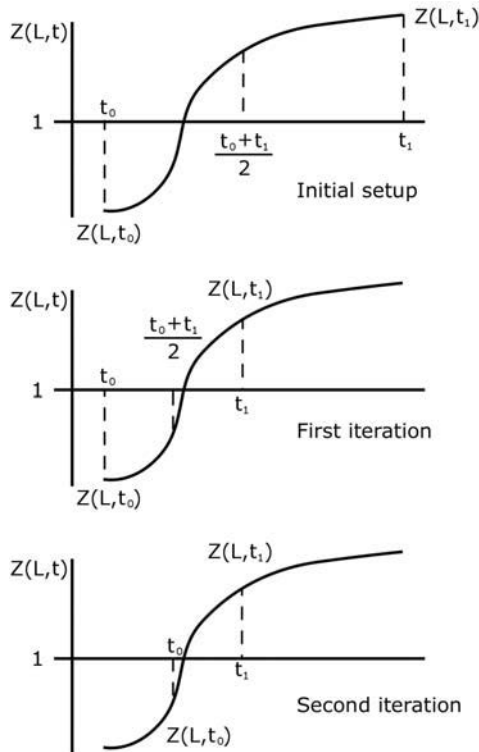

Figure 6.8: Bisection technique. After each iteration, the domain, shown by the dashed lines, becomes half as long as it was before.

---

[6]The symbol $\in$ means 'contained in.' Thus $t \in [t_0, t_1]$ means t is contained in the interval $[t_0, t_1]$ or, simply, $t_0 \leq t \leq t_1$.

Figure 6.8 illustrates this method of halving the interval. In the initial setup, the solution, t_z, is in the left half, so we know that $t_z \in [t_0, (t_0 + t_1)/2]$. We redefine

$$t_1 \equiv \frac{t_0 + t_1}{2},$$

and repeat the process. This time the root is in the right half and so we redefine

$$t_0 \equiv \frac{t_0 + t_1}{2}.$$

After each iteration, we halve the size of the interval and so the interval $[t_0, t_1]$ gets smaller and smaller. Thus, by design, we always have $t_z \in [t_0, t_1]$ and so, by repeating the halving process, we can estimate t_z to any accuracy we like.

The benefit of this method is in its simplicity and reliability; it will always work. However, the cost of this reliability comes at the price of speed. If the initial searching region is very big, it may take a large number of repeats before the process produces an answer to an accuracy with which we are happy. There are quicker methods but these are usually more complex and sometimes they may fail to find a solution altogether. If the zombies are closing in on you and you need to compute their interaction times more quickly, we direct you to consider Newton-Raphson techniques [18] rather than the bisection method; although, if time is really short, we would firmly recommend running away first.

C   Changing Coordinates

Justification

In Section 6.6.3, we made the change of coordinates of $(x, t) \mapsto u = x - vt$. This is because we are looking for a specific form of solution to the equations (6.12) and (6.13) that take the form of 'Fisher waves' [19, 10], which look like those in Figure 6.9. Note that ahead of the wave front the human population is high and the zombie population is low. As the wave invades the human species, it causes the zombie population to increase while the human population decreases.

Specifically, we are looking for a wave that moves at a constant speed and does not change in shape as it moves. Fisher waves are widely known to meet these criteria [19]. Thus if we could define the shape of the wave at each space-time point as $F(x, t)$, then, since the wave does not change shape, it can also be described by $F(u = x - vt) \equiv F(x, t)$, where v is the speed of the wave. This can be seen intuitively, since in any interval of time of length t, a point on the curve at a position $u = x$ will move a distance vt (using distance = speed $\times$ time) so the position of the point on the curve is now $u + vt = x \implies u = x - vt$. By converting to the coordinate of u, we are able to reduce the dimensions of the system and make it simpler.

By looking for Fisher waves, we implicitly assume that the domain is infinite in size with zero-flux boundary conditions. Since we are using a large, finite domain, the derived results represent a good approximation to those actually seen while the wave is not near either of the boundaries.
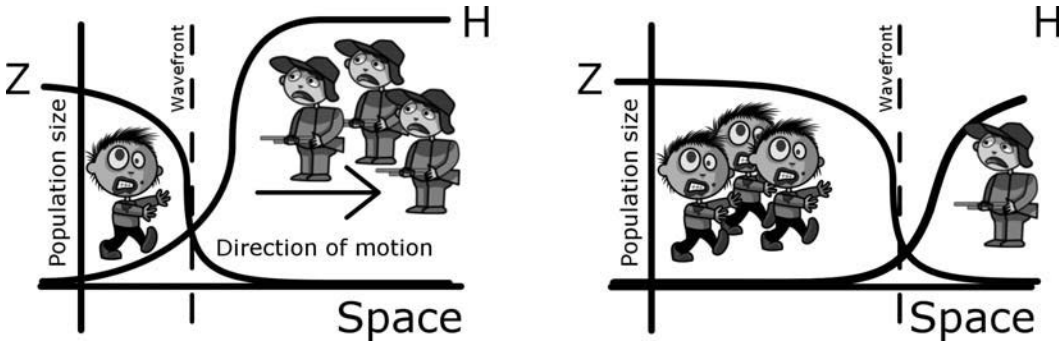
Figure 6.9: Diagram of a 'Fisher wave' form. As time increases, the wave moves to the right with a constant speed, keeping the same shape. The left image evolves to the right image as time increases.

Consequences of changing variables

Since we have changed the dependent variable, we need to change the derivatives as well. This is done using the chain rule [8]:

$$
\frac{\partial}{\partial x} = \frac{\partial}{\partial u}\frac{\partial u}{\partial x} = \frac{\partial}{\partial u}
$$
$$
\frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial u^2}
$$
$$
\frac{\partial}{\partial t} = \frac{\partial}{\partial u}\frac{\partial u}{\partial t} = -v\frac{\partial}{\partial u}.
$$

These are then used in equations (6.12) and (6.13) to produce the form

$$
0 = D_H H'' + vH' - \alpha HZ \tag{6.23}
$$
$$
0 = D_Z Z'' + vZ' + \beta HZ, \tag{6.24}
$$

where we have used the notation

$$
' = \frac{d}{du}.
$$

Speed bound

By considering Figure 6.9, we can see that, in front of the wave, $H = H_0$ (the initial population) and $Z = 0$, as the infection has not reached that point yet. We use this information to consider what would happen to a small perturbation of the population ahead of the wave. Either the perturbation will die out or it will increase. To capture the effects of the perturbation, we substitute in the following forms for H and Z:

$$
H(u) = H_0 + h\exp(\lambda u) \tag{6.25}
$$
$$
Z(u) = z\exp(\lambda u), \tag{6.26}
$$

where h and z are the small initial perturbations. The outcome of the perturbation is then determined by the sign of λ. If $\lambda < 0$, the perturbations die out and the populations go back to their initial values, but if $\lambda > 0$ then the perturbations grow and so the infection is able to take hold. Most importantly, λ should be a real value. If it were complex (i.e., if it had the form $\lambda = X + iY$, where X and Y are any real numbers and $i = \sqrt{-1}$), then the system would predict that the solution would oscillate like a *sine* or *cosine* function and this would cause the $Z(u)$ function to become negative, which is unrealistic. To stop this situation from occurring, we need λ to be real.

We now substitute equations (6.25) and (6.26) into equation (6.24) and ignore the product hz since h and z are both small factors, so $hz \ll h$ or z. Upon simplification, we are left with $0 = D_z\lambda^2 + v\lambda + \beta H_0$. This can be simply solved with the quadratic formula [20], which means

$$\lambda = \frac{-v \pm \sqrt{v^2 - 4D_z\beta H_0}}{2D_z}.$$

Because λ must be real, we need to make sure the term inside the square root is non-negative and so inequality (6.14) follows.

D   Acknowledgements

E   Glossary

In this glossary of terms, we have, at times, sacrificed rigour for readability.

Continuous. A function is continuous if a small change in the input results in a small change in the output.

Diffusion equation. A partial differential equation that describes a population's random movement from regions of high density to low density.

Discrete. A function is discrete if it is defined only on isolated points.

First passage time. The first time an individual of the population reaches a specific location.

Fisher wave. A solution that travels in time, moving with constant speed and shape. The solution connects a low population state to a high population.

Flux. The rate of flow of material at a specific point.

Fourier series. An infinite series representation of a function that uses trigonometric functions sine and cosine.

Law of Mass Action. The rate of a reaction is proportional to the density of reactants multiplied together.

Monotonically decreasing. A function is monotonically decreasing if it never increases as its variable increases.

Random walk. Movement of an individual in which each step is independent of any previous step.

Stochastic. Random.

Wave solution. A particular solution form of a differential equation that does not change its shape as it moves in space and time.

Zero flux boundary condition. A specific form of boundary condition that assumes that the solution domain is insulated and thus individuals can neither leave nor enter.

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