Appl. Math. Lett. Vol. 7, No. 1, pp. 47–51, 1994
Printed in Great Britain.
0893-9659/94 $6.00 + 0.00

An Approximation to a Sharp Type Solution of a Density-Dependent Reaction-Diffusion Equation

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(Received June 1993; accepted August 1993)

Abstract—In this paper, we use a perturbation method to obtain an approximation to a saddle-saddle heteroclinic trajectory of an autonomous system of ordinary differential equations (ODEs) arising in the equation $u_t = [(u + \varepsilon u^2)u_x]_x + u(1-u)$ in the case of travelling wave solutions (t.w.s.): $u(x,t) = \phi(x - ct)$. We compare the approximate form of the solution profile and speed thus obtained with the actual solution of the full model and the calculated speed, respectively.

Keywords—Travelling waves, Wavespeed, Perturbation method, Sharp solutions, Density-dependent diffusion.

1. INTRODUCTION

A wide range of problems in the natural sciences can be formulated in terms of looking for heteroclinic and/or homoclinic trajectories of autonomous systems of ODEs in a n-dimensional phase space [1–4]. Several techniques have been developed and used to investigate the existence and uniqueness (when it makes sense) of these trajectories [5–8], and a number of analytical and numerical methods have been applied to determine approximately the resultant heteroclinic and homoclinic trajectories [5,9,10].

In this paper, we use the perturbation method to obtain an approximation of the saddle-saddle connection which appears in a two-dimensional system of ODEs arising from transforming a one-dimensional nonlinear degenerate diffusion equation into travelling wave coordinates, and compare it with the numerical solution of the full model.

2. THE ANALYSIS

We consider t.w.s. connecting the homogeneous steady states $u(x,t) \equiv 1$ and $u(x,t) \equiv 0$ for the nonlinear degenerate diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[(u + \varepsilon u^2)\frac{\partial u}{\partial x}\right] + u(1-u), \tag{1}$$

for $(x,t) \in R \times R^+$, where $0 \le \varepsilon \ll 1$, with initial condition $u(x,0) = u_0(x) \in C_{\mathbf{R}}$ and $0 \le u_0(x) \le 1$. We also impose the restriction $0 \le u(x,t) \le 1$. This equation is a particular generalisation of Fisher's equation [11].

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Substituting $u(x,t) = \phi(x - ct) \equiv \phi(\xi)$ into (1), we get an ODE for ϕ which can be written as a singular (at $\phi = 0$) system of ODEs. By introducing the parameter τ such that

$$\frac{d\tau}{d\xi} = \frac{1}{\phi + \varepsilon\phi^2},$$

and defining $\phi(\tau) \equiv \phi(\xi(\tau))$ and $v(\tau) \equiv v(\xi(\tau))$, we obtain the following nonsingular system (full details can be seen in [12]):

$$\begin{aligned} \dot{\phi} &= (\phi + \varepsilon\phi^2)v \\ \dot{v} &= -cv - (1 + 2\varepsilon\phi)v^2 - \phi(1 - \phi), \end{aligned} \tag{2}$$

corresponding to t.w.s. of (1). The equilibrium points of (2) are:$P_0 = (0,0)$, $P_1 = (1,0)$ and $P_c = (0,-c)$. The first one of these is a nonhyperbolic point. The other two are hyperbolic saddle points. By using higher order terms and the Centre Manifold Theorem we conclude that P_0 is a saddle-node. The global analysis of the behaviour of the trajectories of a more general system (containing (2) as a special case) has been carried out in [12,13]. In particular, it has been shown that there exists a unique value, c^*, of the wavespeed c for which a saddle-saddle connection from P_1 to P_c exists. Here we calculate explicitly an approximation to the wavespeed and solution profile of (2). We re-state (2) as the boundary value problem:

$$\frac{dv}{d\phi} = \frac{-cv - (1 + 2\varepsilon\phi)v^2 - \phi(1 - \phi)}{(\phi + \varepsilon\phi^2)v}, \tag{3}$$

with $v(1) = 0$ and $v(0) = -c$.

Problem (3) for the case $\varepsilon = 0$ has an unique solution for the critical value $c_0 = 1/\sqrt{2}$ of c given by (see [1,14,15]):

$$v_0(\phi) = \frac{1}{\sqrt{2}}(\phi - 1), \tag{4}$$

and no solution for other values of c. For small enough values of ε we seek, to a first order approximation (in terms of a power series of ε), a solution to (3) of the form

$$v(\phi; \varepsilon) = v_0(\phi) + v_1(\phi)\varepsilon, \tag{5}$$

and

$$c(\varepsilon) = c_0 + c_1\varepsilon, \tag{6}$$

where the function v_1 and the constant c_1 are unknown. We also require that $v(\phi; \varepsilon)$ satisfies the following conditions:

$$v(0; \varepsilon) = -c(\varepsilon) \quad \text{and} \quad v(1; \varepsilon) = 0, \tag{7}$$

which implies the following boundary conditions for v_1

$$v_1(0) = -c_1 \quad \text{and} \quad v_1(1) = 0. \tag{8}$$

Substituting (5) and (6) into (3) and equating powers of ε we obtain: at $O(\varepsilon^0)$,

$$v_0 v_0' \phi = -c_0 v_0 - v_0^2 - \phi(1 - \phi). \tag{9}$$

At $O(\varepsilon)$,

$$v_0 v_0' \phi^2 + v_0 v_1' \phi + v_0' v_1 \phi = -c_0 v_1 - c_1 v_0 - 2 v_0 v_1 - 2\phi v_0^2, \tag{10}$$

where $' \equiv d/d\phi$. Note that (9) gives us the equation of the nonperturbed problem ($\varepsilon = 0$) with solution given by (4).

Substituting (4) and v_0' into (10), we get the following first order ODE for v_1

$$(\phi - 1)\phi v_1'(\phi) + (3\phi - 1)v_1(\phi) + \frac{3}{\sqrt{2}}\phi^3 - \frac{5}{\sqrt{2}}\phi^2 + \left[\frac{2}{\sqrt{2}} + c_1\right]\phi - c_1 = 0. \tag{11}$$

The general solution of (11) is given by

$$v_1(\phi)(1-\phi)^2\phi = -\frac{3}{5\sqrt{2}}\phi^5 + \frac{2}{\sqrt{2}}\phi^4 + \left[-\frac{c_1}{3} - \frac{7}{3\sqrt{2}}\right]\phi^3 + \left[\frac{1}{\sqrt{2}} + c_1\right]\phi^2 - c_1\phi + A, \tag{12}$$

where A is an arbitrary constant. By using the boundary conditions (8) in the above equality, we obtain

$$A = 0 \quad \text{and} \quad c_1 = \frac{1}{5\sqrt{2}}. \tag{13}$$

Conversely, we can easily verify that for these values of A and c_1, $v_1(\phi)$ (given by (12)) satisfies the boundary conditions (8). In fact, the first condition is immediate and the second condition follows from L'Hospital's Rule.

Therefore, to order ε, we have that (2) possesses a saddle-saddle connection for c given by

$$c(\varepsilon) = \frac{1}{\sqrt{2}} + \frac{1}{5\sqrt{2}}\varepsilon. \tag{14}$$

To find an approximation to the corresponding trajectory, we first note that substituting (13) into (12) greatly simplifies the latter to the form

$$v_1(\phi)(1-\phi)^2\phi = \frac{1}{5\sqrt{2}}(\phi - 1)^3(1 - 3\phi)\phi. \tag{15}$$

Hence, we can write the approximation to the heteroclinic trajectory as

$$v(\phi;\varepsilon) = \frac{1}{\sqrt{2}}(\phi - 1)\left[1 + \frac{(1-3\phi)}{5}\varepsilon\right]. \tag{16}$$

To calculate the corresponding solution profile, we substitute (16) into the first equation of (2) and use the reparameterisation to obtain the following ODE for ϕ (up to first order in ε):

$$\frac{d\phi}{d\xi} = \frac{3\varepsilon}{5\sqrt{2}}(\phi - 1)(a - \phi), \tag{17}$$

where $a = (5+\varepsilon)/3\varepsilon$. We can verify that $a > 1$ for $0 < \varepsilon < 5/2$. Solving this equation, we obtain the approximation to the sharp front of (1) in travelling wave coordinates as

$$\phi(\xi) = a\left[1 - \exp\left[-\frac{3\varepsilon}{5\sqrt{2}}(a-1)(\xi - \xi^*)\right]\right]\left[1 - a\exp\left[-\frac{3\varepsilon}{5\sqrt{2}}(a-1)(\xi - \xi^*)\right]\right]^{-1}, \tag{18}$$

for all $\xi < \xi^*$ and $\phi(\xi) = 0$ for all $\xi > \xi^*$. Note that for $\varepsilon = 0$ this reduces to the sharp solution of (2) for linear diffusion.

Figure 1 shows a typical sharp solution for (1) and Figure 2 shows the phase portrait of system (2) for different values of ε. In Figure 3, we compare the analytic approximation (18) to the solution profile of the numerical simulation of the nonlinear model (1). Clearly, for small values of ε, our analytic approximations agree very closely with the actual numerically computed solution. Hence, we have found an analytic approximation to the sharp type solution of (1) in the case of nonlinear density-dependence.

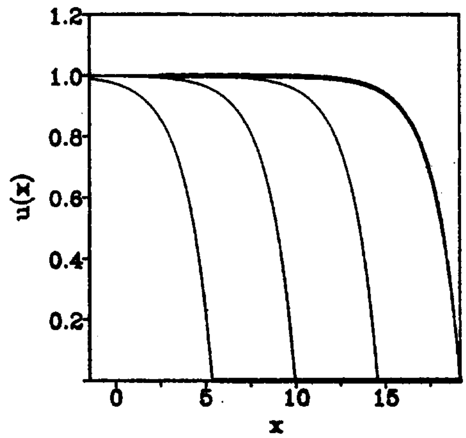
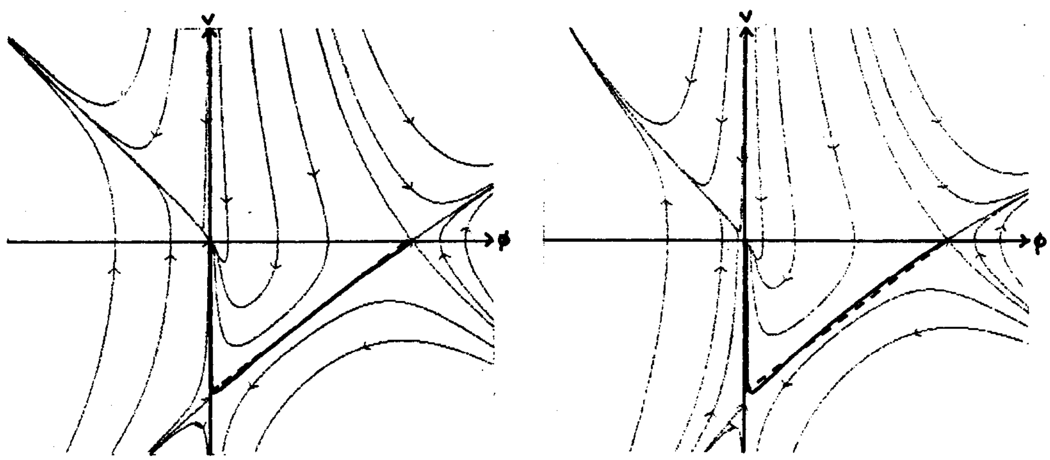


Figure 1. A typical sharp solution of (1) for $\varepsilon = 0.2$, with a step function initial condition.



(a) $\varepsilon = 0.2$; the computed wavespeed is $c(0.2) \approx 0.7354$, which agrees exactly (to four decimal places) with the analytical approximation.

(b) $\varepsilon = 0.5$; the computed wavespeed is $c(0.5) \approx 0.7753$, the analytical approximation is 0.7778.

Figure 2. Phase portrait of (2) for different values of ε. The broken line represents the heteroclinic connection for $\varepsilon = 0$, with computed wavespeed $c(0) \approx 0.7071$; the phase trajectories for $\epsilon > 0$ are plotted with continuous lines, and the heteroclinic trajectory for this case is indicated by the bold line.

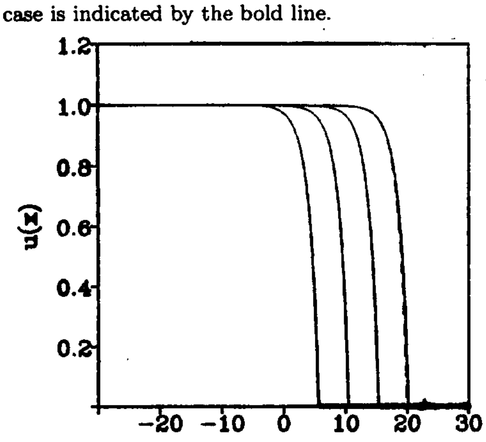


Figure 3. Numerically computed travelling wave solution of (1) for $\varepsilon = 0.5$ (solid lines) for four different times. The typical shape of the analytical approximation (18) to the profile (dotted line) agrees very closely with the numerical solution.

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