# NON-REALIZABILITY OF THE TORELLI GROUP AS AREA-PRESERVING HOMEOMORPHISMS

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Abstract. Nielsen realization problem for the mapping class group $\mathrm{Mod}(S_g)$ asks whether the natural projection $p_g : \mathrm{Homeo}_+(S_g) \to \mathrm{Mod}(S_g)$ has a section. While all the previous results use torsion elements in an essential way, in this paper, we focus on the much more difficult problem of realization of torsion-free subgroups of $\mathrm{Mod}(S_g)$. The main result of this paper is that the Torelli group has no realization inside the area-preserving homeomorphisms.

## 1. Introduction

Let $S_g$ be a surface of genus $g$. Let $p_g : \mathrm{Homeo}_+(S_g) \to \mathrm{Mod}(S_g)$ be the natural projection where $\mathrm{Homeo}_+(S_g)$ denotes the group of orientation-preserving homeomorphisms of $S_g$ and $\mathrm{Mod}(S_g) := \pi_0(\mathrm{Homeo}_+(S_g))$. In 2007, Markovic [Mar07] answered a well-known question of Thurston that $p_g$ has no section for $g \geq 5$. The proof in [Mar07] uses both torsions and the braid relations in an essential way, which both disappear in most finite index subgroups of $\mathrm{Mod}(S_g)$. Motivated by this, Farb [Far06, Question 6.6] asked the following question:

**Problem 1.1** (Sections over finite index subgroups)**.** *Does the natural projection $p_g$ have a section over every finite index subgroup of* $\mathrm{Mod}(S_g)$*, or not?*

This problem presents two kinds of difficulties: the lack of understanding of finite index subgroups of $\mathrm{Mod}(S_g)$ and the lack of understanding of relations in $\mathrm{Homeo}_+(S_g)$. To illustrate the latter, we state the following problem ([MT18][Problem 1.2]).

**Problem 1.2.** *Give an example of a finitely-generated, torsion free group $\Gamma$, and a surface $S$, such that $\Gamma$ is not isomorphic to a subgroup of* $\mathrm{Homeo}_+(S)$.

Motivated by the above problems and difficulties, we study the section problem for the Torelli group $\mathcal{I}(S_g)$, which is torsion free (e.g., [FM12, Theorem 6.8]). Recall that $\mathcal{I}(S_g)$ is the subgroup of $\mathrm{Mod}(S_g)$ that acts trivially on $H_1(S_g;\mathbb{Z})$. For any area form on $S_g$, let $\mathrm{Homeo}^a_+(S_g)$ be the group of orientation-preserving, area-preserving homeomorphisms of $S_g$. In this paper, we prove the following:

**Theorem 1.3.** *The Torelli group cannot be realized as a group of area-preserving homeomorphisms on $S_g$ for $g \geq 6$. In other words, the natural projection $p^a_g : \mathrm{Homeo}^a_+(S_g) \to \mathrm{Mod}(S_g)$ has no section over $\mathcal{I}(S_g)$.*

The property we use about the Torelli group is that it is generated by simple bounding pair maps [Joh83]. To extend the method of this paper to study the Nielsen realization problem for all finite index subgroups of $\mathrm{Mod}(S_g)$, we need to study the subgroup generated by powers of simple bounding pair maps or powers of simple Dehn twists. In most cases, this subgroup is an infinite index subgroup of $\mathrm{Mod}(S_g)$ called the power group; see [Fun14] for discussions of power groups.

**Previous work.** Nielsen posed the realization problem for finite subgroups of $\mathrm{Mod}(S_g)$ in 1943 and Kerckhoff [Ker83] showed that a lift always exists for finite subgroups of $\mathrm{Mod}(S_g)$. The first result on Nielsen realization problem for the whole mapping class group is a theorem of Morita [Mor87] that there is no section for the projection $\mathrm{Diff}^2_+(S_g) \to \mathrm{Mod}(S_g)$ when $g \geq 18$. Then Markovic [Mar07] (further extended by Markovic–Saric [MS08] on the genus bound; see also [Cal12] for simplification of the proof and [Che18] for the proof in the braid group case) showed that $p_g$ does not have a section for $g \geq 2$. Franks–Handel [FH09], Bestvina–Church–Suoto [BCS13] and Salter–Tshishiku [ST16] also obtained non-realization theorems for $C^1$ diffeomorphisms. Notice that MoritAs result also extends to all finite index subgroups of $\mathrm{Mod}(S_g)$, but all the other results that we mention above do not extend to the case of finite index subgroups. We refer the readers to the survey paper by Mann–Tshishiku [MT18] for more history and previous ideas.

We remark that the Nielsen realization problem for the Torelli group is also connected with another well-known MoritAs conjecture on the non-vanishing of the even MMM classes. Morita showed that most MMM classes vanish on $\mathrm{Diff}^2_+(S_g)$ and he conjectured that the even MMM classes do not vanish on the Torelli group [Mor99, Conjecture 3.4]. Therefore, if one can prove MoritAs conjecture, one also gives a proof that the Torelli group cannot be realized in $\mathrm{Diff}^2_+(S_g)$.

**Ingredients of the paper.** The proof in this paper is essentially a local argument by considering the action on a sub-annulus. We use the following key ingredients:

(1) Markovic's theory on minimal decomposition, extending it to the pseudo-Anosov case;
(2) Poincaré-Birkhoff's theorem on existence of periodic orbits;
(3) Handel's theorem on the closeness of the rotation interval.
(4) Matsumoto's theorem about prime ends rotation numbers.

Let $c$ be a separating simple closed curve and $T_c$ be the Dehn twist about $c$. The goal of the argument is to find an invariant subsurface with the frontier homotopic to $c$ such that the action of $T_c$ on the frontier has an irrational rotation number. Then by studying the action on the frontier, the fact that $T_c$ has an irrational rotation number is incompatible with the group structure. The main work of the paper is to obtain the invariant subsurface. This is done by using Poincaré-Birkhoff's theorem and Handel's theorem. Matsumoto's theorem is used to reduce the problem into a one-dimensional problem.

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## 2. Rotation number of annulus homeomorphisms

In this section, we discuss the properties of rotation numbers on annuli.

2.1. **Rotation number of an area-preserving homeomorphism of an annulus.** Firstly, we define the rotation number for geometric annuli. Let

$$N = N(r) = \{w \in \mathbb{C} : \frac{1}{r} < |w| < r\}$$

be the geometric annulus in the complex plane $\mathbb{C}$. Denote the geometric strip in $\mathbb{C}$ by

$$P = P(r) = \{x + iy = z \in \mathbb{C} : |y| < \frac{\log r}{2\pi}\}.$$

The map $\pi(z) = e^{2\pi iz}$ is a holomorphic covering map $\pi : P \to N$. The deck transformation on $P$ is $T(x, y) = (x + 1, y)$.

Denote by $p_1 : P \to \mathbb{R}$ the projection to the $x$-coordinate, and by $\mathrm{Homeo}_+(N)$ the group of homeomorphisms of $N$ that preserves orientation and the two ends. Fix $f \in \mathrm{Homeo}_+(N)$, and $x \in N$, and let $\widetilde{x} \in P$ and $\widetilde{f} \in \mathrm{Homeo}_+(P)$ denote lifts of $x$ and $f$ respectively. We define the translation number of the lift $\widetilde{f}$ at $\widetilde{x}$ by

$$\rho(\widetilde{f}, \widetilde{x}, P) = \lim_{n\to\infty}(p_1(\widetilde{f}^n(\widetilde{x})) - p_1(\widetilde{x}))/n. \tag{1}$$

The rotation number of $f$ at $x$ is then defined as

$$\rho(f, x, N) = \rho(\widetilde{f}, \widetilde{x}, P) \qquad (\mathrm{mod}\ 1). \tag{2}$$

The rotation number is not defined everywhere (see, e.g., [Fra03] for more background on rotation numbers). The closed annulus $N_c$ is

$$N_c = \{\omega \in \mathbb{C} : \frac{1}{r} \leq |\omega| \leq r\},$$

For $f \in \mathrm{Homeo}_+(N_c)$, the rotation and translation numbers are defined analogously.

Let $A$ be an open annulus embedded in a Riemann surface (in particular this endows $A$ with the complex structure). By the Riemann mapping theorem, there is a unique $N(r) = N$ and a conformal map $u_A : A \to N$. For any $f \in \mathrm{Homeo}_+(A)$ (the group of end-preserving homeomorphisms), we define the rotation number of $f$ on $A$ by

$$\rho(f, x, A) := \rho(g, u_A(x), N),$$

where $g = u_A \circ f \circ u_A^{-1}$.

We have the following theorems of Poincaré-Birkhoff and Handel about rotation numbers [Han90] (See also Franks [Fra03]).

**Theorem 2.1** (Properties of rotation numbers)**.** *If $f : N_c \to N_c$ is an orientation preserving, boundary component preserving, area-preserving homeomorphism and $\widetilde{f} : P_c \to P_c$ is any lift, then:*

- *(Handel) The translation set*

$$R(\widetilde{f}) = \bigcup_{\widetilde{x}\in P_c} \rho(\widetilde{f}, \widetilde{x}, P_c)$$

*is a closed interval.*

- *(Poincaré-Birkhoff) If $r \in R(\widetilde{f})$ is rational, then there exists a periodic orbit of $f$ realizing the rotation number $r$ mod 1.*

2.2. **Separators and its property.** We let $A$ continue to denote an open annulus embedded in a Riemann surface. Then $A$ has two ends and we choose one of them to be the left end and the other one to be the right end. We call a subset $X \subset \mathrm{Int}(A)$ *separating* (or essential) if every arc $\gamma \subset A$ which connects the two ends of $A$ must intersect $X$.

**Definition 2.2** (Separator)**.** We call a subset $M \subset A$ a *separator* if $M$ is compact, connected and separating.

The complement of $M$ in $A$ is a disjoint union of open sets. We have the following lemma.

**Lemma 2.3.** *Let $M$ be a separator. Then there are exactly two connected components $A_L(M)$ and $A_R(M)$ of $A - M$ which are open annuli homotopic to $A$ and with the property that $A_L(M)$ contains the left end of $A$ and $A_R(M)$ contains the right end of $A$. All other components of $A - M$ are simply connected.*

*Proof.* We compactify the annulus $A$ by adding points $p_L$ and $p_R$ to the corresponding ends of $A$. The compactifications is a two sphere $S^2$. Moreover, $M$ is a compact and connected subset of $S^2 - \{p_L, p_R\}$.

Now, we observe that every component of $S^2 - M$ is simply connected. Denote by $\Omega_L$ and $\Omega_R$ the connected components of $S^2 - M$ containing $p_L$ and $p_R$ respectively. Since $M$ is separating we conclude that these are two different components. We define $A_L(M) = \Omega_L - p_L$ and $A_R(M) = \Omega_R - p_R$. It is easy to verify that these are required annuli. □

We now prove another property of a separator. Let $\pi : \widetilde{A} \to A$ be the universal cover.

**Proposition 2.4.** *Let $M \subset A$ be a separator. Then $\pi^{-1}(M)$ is connected.*

*Proof.* Let $M_n \subset A$ be a decreasing sequence of separators such that each $M_n$ is a compact domain with smooth boundary, and

$$\bigcap M_n = M.$$

(It is elementary to construct such $M_n$'s). Then

$$\bigcap \pi^{-1}(M_n) = \pi^{-1}(M),$$

and $\pi^{-1}(M_n)$ is decreasing. If each $\pi^{-1}(M_n)$ is connected then $\pi^{-1}(M)$ is the intersection of a decreasing sequence of connected sets, and it is connected as such. Therefore, it suffices to prove that $\pi^{-1}(M)$ is connected assuming $M$ is a separator which is a compact domain with smooth boundary. We do this in the remainder of the proof.

Since $M$ is a compact domain with boundary which separates the two ends of $A$, we can find a circle $\gamma \subset M$ which is essential in $A$ (i.e. $\gamma$ is a separator itself) (note that $M$ has only finitely many boundary components). Denote by $T$ the deck transformation of $\widetilde{A}$. Thus, the lift $\pi^{-1}(\gamma)$ is a $T$ invariant, connected subset of $\widetilde{A}$. Let $C$ be the component of $\pi^{-1}(M)$ which contains $\pi^{-1}(\gamma)$. Then $C$ is $T$ invariant. We show $\pi^{-1}(M) = C$.

Let $p \in M$. Since $M$ is a compact domain with smooth boundary, we can find an embedded closed arc $\alpha \subset M$ which connects $p$ and $\gamma$. Let $\widetilde{p}$ be a lift of $p$ and let $\widetilde{\alpha}$ be the corresponding lift of $\alpha$ such that $\widetilde{p}$ is one of its endpoints. Then, the other endpoint of $\widetilde{\alpha}$ is in $\pi^{-1}(\gamma)$, and this shows that $\widetilde{p} \in C$. This concludes the proof.

□

Now we discuss an ordering on the set of separators.

**Proposition 2.5.** *Suppose $M_1, M_2 \subset A$ are two disjoint separators. Then either $M_1 \subset A_L(M_2)$ or $M_1 \subset A_R(M_2)$. Moreover, $M_1 \subset A_L(M_2)$ implies $M_2 \subset A_R(M_1)$.*

*Proof.* Since $M_1$ is connected it follows that $M_1$ is a subset of a connected component $C$ of $A - M_2$. If $C$ is simply connected, the cover $\pi^{-1}(C) \to C$ is a trivial cover. Let $\widetilde{C}$ be a connected component of $\pi^{-1}(C)$. By Proposition 2.4, the set $\pi^{-1}(M)$ is connected so it is contained in a single connected component of $\pi^{-1}(C)$. However, this contradicts the fact that $\pi^{-1}(M)$ is translation invariant. Thus, either $M_1 \subset A_L(M_2)$ or $M_1 \subset A_R(M_2)$.

Suppose $M_1 \subset A_L(M_2)$. Then $A_L(M_1) \subset A_L(M_2)$ as well. On the other hand, by the first part of the proposition we already know that either $M_2 \subset A_L(M_1)$ or $M_2 \subset A_R(M_1)$. If $M_2 \subset A_L(M_1)$, then $A_L(M_2) \subset A_L(M_1)$. This shows that $A_L(M_1) \subset A_L(M_2)$ which implies that $M_2 \subset A_L(M_2)$. This is absurd so we must have $M_2 \subset A_R(M_1)$. □

**Definition 2.6.** The inclusion $M_1 \subset A_L(M_2)$ is denoted as $M_1 < M_2$.

2.3. **The rotation interval of an annular continuum and prime ends.** Let $K \subset A$ be a separator (in literature also known as an *essential continuum*). We call $K$ an essential *annular continuum* if $A - K$ has exactly two components. Observe that an essential annular continuum can be expressed as a decreasing intersection of essential closed topological annuli in $A$.

It is possible to turn any separator $M \subset A$ into an essential annular continuum. Let $M$ be a separating connected set. By Lemma 2.3, we know that $A - M$ has exactly two connected annular components $A_L(M)$ and $A_R(M)$, and all other components of $A - M$ are simply connected. We call a simply connected component of $A - M$ a *bubble component.* Then the *annular completion* $K(M)$ of $M$ is defined as the union of $M$ and the corresponding bubble components of $A - M$.

**Proposition 2.7.** *Let $M \subset A$ be a separator. Then the annular competition $K(M)$ is an annular continuum.*

*Proof.* We can again compactify $A$ by adding the points $p_L$ and $p_R$, one at each end. The compactification is the two sphere $S^2$. Then $A_L(M)$ and $A_R(M)$ are two disjoint open discs in $S^2$, and $K(M) = S^2 - (A_L(M) \cup A_R(M))$. But the complement of two disjoint open discs in $S^2$ is connected. This proves the proposition.

□

Now let $f$ be a homeomorphism of $A$ that leaves an annular continuum $K$ invariant. If $\mu$ is an invariant Borel probability measure, we define the $\mu$-rotation number

$$\sigma(f,\mu) = \int_A \phi d\mu$$

where $\phi : A \to \mathbb{R}$ is the function which lifts to the function $p_1 \circ f - p_1$ on $\widetilde{A}$ (recall that $p_1 : \widetilde{A} \to \mathbb{R}$ is the projection onto the first coordinate).

The set of $f$ invariant Borel probability measures on $K$ is a non empty, convex, and compact set (with respect to the weak topology on the space of measures). We define the *rotation interval* of $K$

$$\sigma(f,K) = \{\sigma(f,\mu) | \mu \in M(K)\}$$

which is a non-empty segment $[\alpha,\beta]$ of $\mathbb{R}$. The interval is non empty because there exists at least one $f$ invariant measure, and it is an interval because the set of $f$ invariant measures is convex.

The following is a classical result of Franks–Le Calvez [FC03, Corollary 3.1].

**Proposition 2.8.** *If $\sigma(f,K) = \{\alpha\}$, the sequence*

$$\frac{p_1 \circ f^n(x) - p_1(x)}{n}$$

*converges uniformly for $x \in \pi^{-1}(K)$ to the constant function $\alpha$. This implies that points in $K$ all have the rotation number $\alpha$.*

The following theorem of Franks–Le Calvez [FC03, Proposition 5.4] is a generalization of the Poincaré-Birkhoff Theorem.

**Theorem 2.9.** *If $f$ is area-preserving and $K$ is an annular continuum, then every rational number in $\sigma(f,K)$ is realized by a periodic point in $K$.*

The theory of prime ends is an important tool in the study of 2-dimensional dynamics which can be used to transform a 2-dimensional problem into a 1-dimensional problem. Recall that we assume that $A$ is an open annulus embedded in a Riemann surface $S$. Suppose that $f$ is a homeomorphism of $S$ which leaves $A$ invariant. Furthermore, let $K \subset A$ be an annular continuum and suppose that $f$ leaves $K$ invariant. Then both $A_L(K)$ and $A_R(K)$ are $f$ invariant.

Since $A$ is embedded in $S$, we can define the frontiers of $A$, $A_L(K)$, and $A_R(K)$. By Carathéodory's theory of prime ends (see, e.g., [Mil06, Chapter 15]), the homeomorphism $f$ yields an action on the frontiers of $A_L(K)$ and $A_R(K)$. Consider the right hand frontier of $A_L(K)$ (the one which is contained in $A$). Then the set of prime ends on this frontier is homeomorphic to the circle, and we denote by $f_L$ the induced homeomorphism this circle. Likewise, the set of prime ends on left hand frontier of $A_R(K)$ is homeomorphic to the circle, and we denote by $f_R$ the induced homeomorphism this circle.

The rotation number of a circle homeomorphism (defined by Equation (2)), is well defined everywhere and is the same number for any point on the circle. The rotation numbers of $f_L$ and $f_R$ are called $r_L$ and $r_R$. We refer to them as the left and right prime end rotation numbers of $f$. We have the following theorem of Matsumoto [Mat12].

**Theorem 2.10** (Matsumoto's theorem)**.** *If $K$ is an annular continuum, then its left and right prime ends rotation numbers $r_L, r_R$ belong to the rotation interval $\sigma(f, K)$.*

## 3. Minimal decompositions and the Torelli group theory

3.1. **Minimal decompositions.** We recall the theory of minimal decompositions of surface homeomorphisms. This is established in [Mar07]. Firstly we recall the upper semi-continuous decomposition of a surface; see also Markovic [Mar07, Definition 2.1]. Let $M$ be a surface.

**Definition 3.1** (Upper semi-continuous decomposition)**.** Let $\mathbf{S}$ be a collection of closed, connected subsets of $M$. We say that $\mathbf{S}$ is an upper semi-continuous decomposition of $M$ if the following holds:

- If $S_1, S_2 \in \mathbf{S}$, then $S_1 \cap S_2 = \emptyset$.
- If $S \in \mathbf{S}$, then $E$ does not separate $M$; i.e., $M - S$ is connected.
- We have $M = \bigcup_{S \in \mathbf{S}} S$.
- If $S_n \in \mathbf{S}, n \in \mathbb{N}$ is a sequence that has the Hausdorff limit equal to $S_0$ then there exists $S \in \mathbf{S}$ such that $S_0 \subset S$.

Now we define acyclic sets on a surface.

**Definition 3.2** (Acyclic sets)**.** Let $S \subset M$ be a closed, connected subset of $M$ which does not separate $M$. We say that $S$ is *acyclic* if there is a simply connected open set $U \subset M$ such that $S \subset U$ and $U - S$ is homeomorphic to an annulus.

The simplest examples of acyclic sets are a point, an embedded closed arc and an embedded closed disk in $M$. Let $S \subset M$ be a closed, connected set that does not separate M. Then $S$ is acyclic if and only if there is a lift of $S$ to the universal cover $\widetilde{M}$ of $M$, which is a compact subset of $\widetilde{M}$. The following theorem is a classical result called Moore's theorem; see, e.g., [Mar07, Theorem 2.1].

**Theorem 3.3** (Moore's theorem)**.** *Let $M$ be a surface and $\mathbf{S}$ be an upper semi-continuous decomposition of $M$ so that every element of $\mathbf{S}$ is acyclic. Then there is a continuous map $\phi : M \to M$ that is homotopic to the identity map on $M$ and such that for every $p \in M$, we have $\phi^{-1}(p) \in \mathbf{S}$. Moreover $\mathbf{S} = \{\phi^{-1}(p) | p \in M\}$.*

We call the map $M \to M/\sim$ the *Moore map* where $x \sim y$ if and only if $x, y \in S$ for some $S \in \mathbf{S}$. The following definition is [Mar07, Definition 3.1]

**Definition 3.4** (Admissible decomposition)**.** Let $\mathbf{S}$ be an upper semi-continuous decomposition of $M$. Let $G$ be a subgroup of Homeo$(M)$. We say that $\mathbf{S}$ is admissible for the group $G$ if the following holds:

- Each $f \in G$ preserves setwise every element of $\mathbf{S}$.
- Let $S \in \mathbf{S}$. Then every point, in every frontier component of the surface $M - S$ is a limit of points from $M - S$ which belong to acyclic elements of $\mathbf{S}$.

If $G$ is a cyclic group generated by a homeomorphism $f : M \to M$ we say that $\mathbf{S}$ is an admissible decomposition of $f$.

An admissible decomposition for $G < \mathrm{Homeo}(M)$ is called *minimal* if it is contained in every admissible decomposition for $G$. We have the following theorem [Mar07, Theorem 3.1].

**Theorem 3.5** (Existence of minimal decompositions)**.** *Every group $G < \mathrm{Homeo}(M)$ has a unique minimal decomposition.*

Denote by $\mathbf{A}(G)$ the sub collection of acyclic sets from $\mathbf{S}(G)$. By a mild abuse of notation, we occasionally refer to $\mathbf{A}(G)$ as a subset of $S_g$ (the union of all sets from $\mathbf{A}(G)$). To distinguish the two notions we do the following. When we refer to $\mathbf{A}(G)$ as a collection then we consider it as the collection of acyclic sets. When we refer to as a set (or a subsurface of $S_g$) we have in mind the other meaning.

We have the following result [Mar07, Proposition 2.1].

**Proposition 3.6.** *Every connected component of $\mathbf{A}(G)$ (as a subset of $S_g$) is a proper subsurface of $M$ with finitely many ends.*

**Lemma 3.7.** *For $H < G < \mathrm{Homeo}(M)$, we have that $\mathbf{A}(G) \subset \mathbf{A}(H)$.*

*Proof.* $\mathbf{A}(G) \subset \mathbf{A}(H)$ because the minimal decomposition of $G$ is also an admissible decomposition of $H$ and the minimal decomposition of $H$ is finer than that of $G$. □

3.2. **The Torelli group and simple BP maps.** From this point one, we fix a closed surface $S_g$ and a separating simple closed curve $c$ which divides the surface $S_g$ into a genus $k$ subsurface and a genus $g-k$ subsurface for $k > 2$. We call the genus $k$ part the *left subsurface* $S_L$ and the genus $g-k$ part the *right subsurface* $S_R$. For any simple closed curve $a$, denote by $T_a$ the Dehn twist about $a$.

**Definition 3.8.** For $a, b$ two disjoint non-separating curves on $S_L$ bounding a genus 1 subsurface, we call the bounding pair map $T_aT_b^{-1}$ a *simple bounding pair map*, which is shortened as a simple BP map.

Let $\mathcal{LI}(c) \subset \mathcal{I}(S_g)$ be the subgroup generated by simple BP maps on the left subsurface $S_L$. About $\mathcal{LI}(c)$, we have the following proposition.

**Proposition 3.9.** *We have that $T_c^{2-2k} \in \mathcal{LI}(c)$ and $T_c^{2-2k}$ is a product of commutators in $\mathcal{LI}(c)$.*

*Proof.* The Birman exact sequence for the mapping class group of $S_L$ fixing the boundary component has the following form

$$1 \to \pi_1(UTS_k) \xrightarrow{\text{Push}} \mathrm{Mod}(S_k^1) \to \mathrm{Mod}(S_k) \to 1.$$

Here, $UTS_k$ denotes the *unit tangent bundle* of $S_k$; i.e., the $S^1$-subbundle of the tangent bundle $TS_k$ consisting of unit-length tangent vectors (relative to an arbitrarily-chosen Riemannian metric). In this context, the kernel $\pi_1(UTS_k)$ is known as the *disk-pushing subgroup*. Let $e$ be the generator of the center of $\pi_1(UTS_k)$, which satisfies that $T_c = \mathrm{Push}(e)$. We have the following $\mathbb{Z}$-extension

$$1 \to \mathbb{Z} \to \pi_1(UTS_k) \to \pi_1(S_k) \to 1. \tag{3}$$

For every central $\mathbb{Z}$-extension

$$1 \to \mathbb{Z} \to \widetilde{Q} \to Q \to 1,$$

there is an associated Euler number which is evaluated on $H_2(Q;\mathbb{Z})$. The Euler number of (3) is $2-2k$ on the generator of $H_2(\pi_1(S_k);\mathbb{Z})$. This means that for a standard generating set $a_1, b_1, ..., a_k, b_k$ of $\pi_1(S_k)$, their lifts $\widetilde{a_1}, \widetilde{b_1}, ..., \widetilde{a_k}, \widetilde{b_k}$ in $\pi_1(UTS_k)$ satisfies the following:

$$e^{2-2k} = [\widetilde{a_1}, \widetilde{b_1}]...[\widetilde{a_k}, \widetilde{b_k}].$$

Therefore we have the relation

$$T_c^{2-2k} = [\mathrm{Push}(\widetilde{a_1}), \mathrm{Push}(\widetilde{b_1})]...[\mathrm{Push}(\widetilde{a_k}), \mathrm{Push}(\widetilde{b_k})].$$

Up to multiplying a power of $T_c$, the map $\mathrm{Push}(\widetilde{a_i})$ or $\mathrm{Push}(\widetilde{b_i})$ is a single BP map (see [FM12, Fact 4.7]). Any BP map $T_a T_b^{-1}$ on $S_L$ is a product of simple BP maps since we can find simple closed curves $c_0 = a, ..., c_{k+1} = b$ such that $c_i, c_{i+1}$ bounds a genus 1 subsurface. Then

$$T_a T_b^{-1} = \Pi_{i=0}^{k} T_{c_i} T_{c_{i+1}}^{-1}$$

Thus $T_c^{2-2k}$ can be written as a product of simple BP maps. □

In this paper, we choose $k = 4$ for the rest of the paper. The reason for this is that we need some room for the existence of pseudo-Anosov Torelli elements in $\mathcal{L}\mathcal{I}(c)$, which is essential for applying the minimal decomposition theory.

## 4. Characteristic annuli and Rotation numbers

4.1. **Minimal decomposition for a realization.** From now on, we work with the assumption that there exists a realization of the Torelli group

$$\mathcal{E} : \mathcal{I}(S_g) \to \mathrm{Homeo}_+^a(S_g).$$

For an element $f \in \mathcal{I}(S_g)$, or a subgroup $F < \mathcal{I}(S_g)$, we shorten $\mathbf{A}(\mathcal{E}(f))$ as $\mathbf{A}(f)$, and $\mathbf{A}(\mathcal{E}(F))$ as $\mathbf{A}(F)$, to denote the corresponding collections of acyclic components. Recall that $c \subset S_g$ is a fixed simple closed curve that divides $S_g$ into subsurfaces $S_L$ and $S_R$ so that $S_L$ has genus 4 (see the definition in the previous section). We have the following theorem about the minimal decompositions of $\mathcal{E}(T_c^{-6})$.

**Theorem 4.1.** *The set $\mathbf{A}(T_c^{-6})$ has a component $\mathbf{L}(c)$ which is homotopic to $S_L$ and a component $\mathbf{R}(c)$ homotopic to $S_R$.*

*Remark.* We use the same argument as in [Mar07]. Since we are working with the Torelli group which contains no Anosov elements, we need to use pseudo-Anosov elements. The argument is almost the same as [Mar07]. For this reason, we postpone the proofs to Section 6.

4.2. **Invariant annuli.** In the remainder of the paper we let

$$\mathbf{B} = S_g - \mathbf{L}(c) - \mathbf{R}(c).$$

Since $\mathbf{L}(c)$ and $\mathbf{R}(c)$ are open (as subset of $S_g$), it follows that $\mathbf{B}$ is compact. Moreover, $\mathbf{L}(c)$ and $\mathbf{R}(c)$ are disjoint, each having exactly one end (and this end is homotopic to $c$), so it follows that $\mathbf{B}$ is connected. By definition, we know that $\mathbf{B}$ is $\mathcal{E}(\mathcal{L}\mathcal{I}(c))$-invariant since homeomorphisms from $\mathcal{E}(\mathcal{L}\mathcal{I}(c))$ commute with $f$.

To simplify the notation, we set

$$f = \mathcal{E}(T_c^{-6}).$$

**Definition 4.2.** We say that $A \subset S_g$ is an invariant annulus if

(1) $A$ is an open annulus, homotopic to the curve $c$,
(2) $\mathbf{B} \subset A$,
(3) $A$ is invariant under $f$.

Next, we prove the lemma which says that the rotation numbers of points from $\mathbf{B}$ (under the action of $f$) do not depend on which invariant the annulus we use.

**Lemma 4.3.** *Let $A_1, A_2$ be two invariant annuli. Then*

$$\rho(f, x, A_1) = \rho(f, x, A_2), \qquad x \in \mathbf{B}.$$

*Proof.* We let $\pi_i : P_i \to A_i$, $i = 1, 2$, denote the universal cover, where $P_i$ is the infinite strip in the complex plane such that $A_i$ is (as a Riemann surface) isomorphic to $P_i/\langle T \rangle$, where $T(x, y) = (x + 1, y)$. The height of the strip $P_i$ depends on the modulus of $A_i \subset S_g$. We let

$$\mathbf{B}_i = \pi_i^{-1}(\mathbf{B}).$$

Since $A_1$ and $A_2$ are open annuli containing the compact set $\mathbf{B}$, there is a homeomorphism $g : A_1 \to A_2$, such that $g|_{\mathbf{B}} = \mathrm{Id}$. Choose a lift $\widetilde{g} : P_1 \to P_2$ of $g$. Then $\widetilde{g}(\mathbf{B}_1) = \mathbf{B}_2$. Moreover, since both $P_1$ and $P_2$ live in the same complex plane, and have the same group of deck transformations, and since $\widetilde{g}$ conjugates the deck transformation to itself, it follows that

$$d\big(y, \widetilde{g}(y)\big) < d_0, \quad \text{for every} \quad y \in \mathbf{B}_1, \tag{4}$$

for some constant $d_0 > 0$ (here $d$ stands for the Euclidean distance on $P_1$).

Let $\widetilde{f}_1 : P_1 \to P_1$ be a lift of $f$ to $P_1$. We then choose $\widetilde{f}_2 : P_2 \to P_2$, a lift of $f$ to $P_2$, such that

$$\widetilde{f}_2 = \widetilde{g} \circ \widetilde{f}_1 \circ \widetilde{g}^{-1}, \qquad \text{on} \qquad \mathbf{B}_2. \tag{5}$$

Recall the definition of the rotation number of $f$ at $x \in A_i$

$$\rho(f, x, A_i) = \lim_{n \to \infty} (p_1(\widetilde{f}_i^n(\widetilde{x}_i)) - p_1(\widetilde{x}_i))/n \qquad (\mathrm{mod}\ 1),$$

where $\widetilde{x}_i$ is a lift of $x$ to $P_i$. Replacing (4) and (5) into this definition shows that $\rho(f,x,A_1) = \rho(f,x,A_2)$. The lemma is proved.

$\square$

4.3. **Characteristic annuli and rotation numbers.** Let $p_L : \mathbf{L}(c) \to \mathbf{L}(c)/\sim$ and $p_R : \mathbf{R}(c) \to \mathbf{R}(c)/\sim$ be the Moore maps of $\mathbf{L}(c)$ and $\mathbf{R}(c)$ corresponding to the decomposition $\mathbf{S}(c)$. Let $\mathbf{L} \subset \mathbf{L}(c)/\sim$ be an open annulus bounded by the end of $\mathbf{L}(c)'$ on one side, and by a simple closed curve on the other. The open annulus $\mathbf{R} \subset \mathbf{R}(c)/\sim$ is defined similarly. We have the following definition (see [Mar07, Chapter 5]).

**Definition 4.4.** An annulus of the form $A = p_L^{-1}(\mathbf{L}) \cup \mathbf{B} \cup p_R^{-1}(\mathbf{R})$ is called a *characteristic annulus*.

Every characteristic annulus is an invariant annulus. We observe that $\mathbf{B}$ is a separator in $A$, that is, $\mathbf{B}$ is an essential, compact, and connected subset of $A$. Note that a characteristic annulus $A$ is invariant under $f$, but it may not be invariant under homeomorphisms which are lifts (with respect to $\mathcal{E}$) of other elements from the Torelli group. However, $\mathbf{B}$ is invariant under these lifts of elements from $\mathcal{L}\mathcal{I}(c) \subset \mathcal{I}(S_g)$ (the subgroup generated by simple BP maps on the left subsurface $S_L$). As we see from the next lemma, the dynamical information about $f$ is contained in $\mathbf{B}$.

**Lemma 4.5.** *Fix a characteristic annulus $A$. Then*

(1) *every number $0 < r < 1$ appears as the rotation number $\rho(f,x,A)$, for some $x \in A$,*
(2) *if $0 < \rho(f,x,A) < 1$, then $x \in B$.*

*Proof.* The idea is show that the translation numbers of the restrictions of $f$ to the frontiers of $A$ differ by 6. We then apply Handel's theorem.

Define a new upper-semicontinuous decomposition $\mathbf{S}_{new}$ of $S_g$ as follows. Outside of $A$, the decomposition consists of elements of $\mathbf{S}(c)$ (note that the outside of $A$ is contained in $\mathbf{A}(c)$). Inside of $A$ the decomposition $\mathbf{S}_{new}$ consists of points. By definition, this is an upper-semicontinuous decomposition which consists of acyclic components only.

Let $p : S_g \to S_g' := S_g/\sim$ be the Moore map of $\mathbf{S}_{new}$ as in Figure 1. By definition, $p|_A$ is a homeomorphism. The action of $f$ on $S_g$ is semi-conjugated by $p$ to a homeomorphism $f'$ on $S_g'$. Since $f$ preserves each components of $\mathbf{S}(c)$, we know that $f'|_{S_g'-p(A)} = id$. The action $f'$ on $p(A)$ is conjugate to $f$ on $A$ since $p|_A$ is a homeomorphism. Since $p|_{\mathbf{L}(c)} = p_L|_{\mathbf{L}(c)}$ and $p|_{\mathbf{R}(c)} = p_R|_{\mathbf{R}(c)}$, the boundary components of $p(A)$ in $S_g'$ are two simple closed curves $\partial_L$ and $\partial_R$.

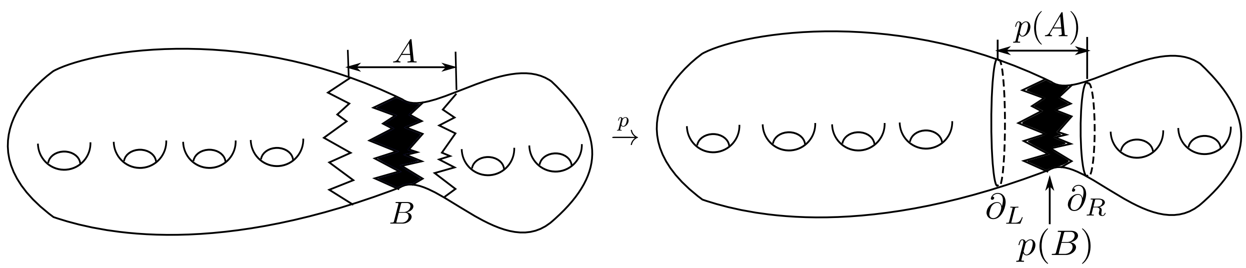


FIGURE 1. the Moore map $p$ for $\mathbf{S}_{new}$

Recall that $\mathcal{E}$ is a realization. Thus, $f'$ is homotopic to the standard Dehn twist map $T_c^{-6}$ on $S'_g$. Since $f'$ is identity map outside of $A'$, the homeomorphism $f'$ is homotopic to the standard Dehn twist map $T_c^{-6}$ on $A'$. By Handel's Theorem 2.1, the collection of all translation numbers $\rho(\widetilde{f'}, x, \widetilde{A'}_c)$ is a closed interval (here $\widetilde{f'} : P_c \to P_c$ is any lift of $f'$ to the infinite trip $P_c$ which is the universal cover of $A'$). Since this set contains points 0 and $-6$, we conclude that every $0 < r < 1$ appears as the rotation number $\rho(f', x, A')$ for some $x \in A'$. The statement (1) is proved.

For (2), recall that $A - \mathbf{B} \subset \mathbf{A}(c)$. Let $C(x) \in \mathbf{A}(c)$ be the corresponding acyclic set that contains $x \in A - \mathbf{B}$. Denote by $\pi : P \to A$ the universal cover. Since $C(x)$ is acyclic we conclude that any connected component of $\pi^{-1}\big(C(x)\big)$ is a compact set. Let $\widetilde{f} : P \to P$ be a lift of $f$. Since $f$ preserves the set $C(x)$, the homeomorphism $\widetilde{f}$ permutes the connected components of $\pi^{-1}\big(C(x)\big)$. Therefore, the translation number of $\widetilde{f}$ at points in $\pi^{-1}\big(C(x)\big)$ must be an integer (if it exists). Therefore, the rotation number of $f$ at $x$ (if it exists) is 0. This implies that $E_r \subset \mathbf{B}$. $\square$

4.4. **A special characteristic annulus $A_h$.** In this subsection, we define a special characteristic annulus $A_h$ with respect to a BP map $h$ and study its properties. Firstly, we have the following theorem about minimal decompositions of $\mathcal{E}(h)$ and $\mathcal{E}(\langle T_c^{-6}, h\rangle)$ which will be proved in Section 6. Here $\langle T_c^{-6}, h\rangle$ denotes the group generated by these two elements.

**Theorem 4.6.** *For a simple BP map $h = T_a T_b^{-1}$, the set $\mathbf{A}(h)$ contains a component $\mathbf{R}(h)$ with two ends homotopic to $a, b$ respectively. Further more, the set $\mathbf{A}(\langle T_c^{-6}, h\rangle)$ has a component $\mathbf{M}_1(c, h)$ with ends homotopic to $a, b, c$ respectively and a component $\mathbf{M}_2(c, h)$ with ends homotopic to $c$.*

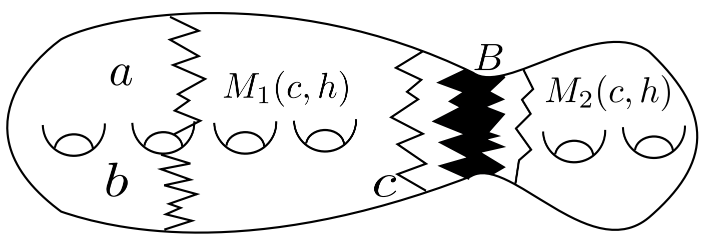


FIGURE 2. Location of $\mathbf{M}_1(c, h)$ and $\mathbf{M}_2(c, h)$

Let $\mathbf{M}(c, h) \subset S_g$ be the connected subsurface which contains $\mathbf{M}_1(c, h)$, which has two ends, and which shares its two ends with $\mathbf{M}_1(c, h)$ (these are the two ends of $\mathbf{M}_1(c, h)$ homotopic to $a$ and $b$ respectively). We let

$$\mathbf{B_h} = \mathbf{M}(c, h) - \mathbf{M}_1(c, h) - \mathbf{M}_2(c, h).$$

Since the decomposition of $f$ is finer than that of $\mathcal{E}(\langle T_c^{-6}, h\rangle)$, we know that $\mathbf{M}_1(c, h) \subset \mathbf{L}(c)$ and $\mathbf{M}_1(c, h) \subset \mathbf{R}(c)$. This also implies that $\mathbf{B} \subset \mathbf{B_h}$. Similarly to how we defined the Moore map $p$ above, we define the Moore map $p_h : S_g \to S''_g = S_g/\sim$, with respect to the new upper-semi continuous decomposition of $S_g$ defined as follows: on $\mathbf{M}_1(c, h) \cup \mathbf{M}_2(c, h)$ we use the (acyclic) components of $\mathbf{S}(c)$, and on the rest of the surface $S_g$ each point is one component. We let $A''$ be any sub-annulus of $p_h\big(\mathbf{M}(c, h)\big)$, bounded by two simple closed curve curves, and which contains $p_h\big(\mathbf{B}_h\big)$.

Set

$$A_h = p_h^{-1}(A'').$$

The minimal decomposition for $f$ is finer than the one of $\mathcal{E}(\langle T_c^{-6}, h\rangle)$. The following claim and proposition follow from this observation.

**Claim 4.7.** *Each $A_h$ is a characteristic annulus.*

*Remark.* We call $A_h$ a special characteristic annulus with respect to $h$.

*Proof.* The quotient map $p : S_g \to S'_g$ factors through the quotient map $p_h : S_g \to S''_g$. That is, $p = \xi \circ p_h$, where $\xi : S''_g \to S'_g$ is another quotient map. Each $A_h$ is given by $A_h = p_h^{-1}(A'')$. We let $A' = \xi(A'')$, and observe that $A_h = p^{-1}(A')$. This proves the claim. □

**Proposition 4.8.** *Fix a simple BP map $h$, and a special characteristic annulus $A_h$. Then*

(1) $\mathbf{B} \subset \mathbf{B_h}$,

(2) $\mathbf{B_h} \subset \mathbf{A}(h)$,

(3) $C(x) \subset \mathbf{B}_h$, *for every $x \in \mathbf{B}_h$, where $C(x) \in \mathbf{A}(h)$ is the corresponding acyclic component containing $x$.*

*Proof.* The proof of (1) is very similar to the proof of the previous claim and we leave it to the reader. To prove (2) we recall the set $\mathbf{R}(h)$ from Theorem 4.6. Since the minimal decomposition of $\mathcal{E}(h)$ is finer than the one of $\mathcal{E}(\langle T_c^{-6}, h\rangle)$, it follows that $\mathbf{M}(c,h) \subset \mathbf{R}(h)$. Together with $\mathbf{B}_h \subset \mathbf{M}(c,h)$, this yields (2).

It remains to prove (3). Let $x \in \mathbf{B_h}$. Then by (2) of this proposition we know there exists $C(x) \in \mathbf{A}(h)$ containing $x$. We argue by contradiction. Suppose that $C(x)$ is not contained in $\mathbf{B}_h$. Then there exists $y \in C(x)$ such that $y \in \mathbf{M}(c,h) - \mathbf{B}_h$. But then $y$ belongs to an acyclic component $D(y) \in \mathbf{A}(c,h)$. However, the minimal decomposition of $\mathcal{E}(h)$ is finer than the one of $\mathcal{E}(\langle T_c^{-6}, h\rangle)$, which implies that $C(x) = C(y) \subset D(y)$. This means that $x \in D(y)$, and thus $x \in \mathbf{A}(c,h)$. But by the definition we know that $\mathbf{B}_h \cap \mathbf{A}(c,h) = \emptyset$. This contradiction proves the proposition.

□

## 5. The proof of Theorem 1.3

In this section, we prove Theorem 1.3 which states that the natural projection $p_g^a : \mathrm{Homeo}_+^a(S_g) \to \mathrm{Mod}(S_g)$ has no section over $\mathcal{I}(S_g)$. Again, assume that $\mathcal{E} : \mathcal{I}(S_g) \to \mathrm{Homeo}_+^a(S_g)$ is a section of $p_g^a$. Equip $S_g$ with a Riemann surface structure.

5.1. **Outline of the proof.** Recall that $c$ is a separating simple closed curve that divides the surface $S_g$ into a genus 4 subsurface and a genus $g-4$ subsurface. Fix a characteristic annulus $A$. Let $E_r$ be the set of points in $A$ that have rotation numbers equal to $r$ under $\mathcal{E}(T_c^{-6})$. Lemma 4.5 states that the set $E_r$ is not empty when $0 < r < 1$.

The key observation of the proof lies in the analysis of connected components of $E_r$. Let $E$ be a component of $E_r$. We show the following:

(1) $E$ is $\mathcal{E}(h)$-invariant for every simple BP map $h$

(2) $\overline{E}$ is a separator in $A$,

(3) if $E$ contains a periodic orbit, then $E$ contains a separator.

Denote by $K(\overline{E})$ the annular completion of $\overline{E}$, and let $\rho(\mathcal{E}(T_c^{-6}), K(\overline{E}))$ be the rotation interval of $K(\overline{E})$. We claim that $\rho(K(\overline{E})) = \{r\}$. First of all, we know that $r \in \rho(\mathcal{E}(T_c^{-6}), K(\overline{E}))$. If $\rho(\mathcal{E}(T_c^{-6}), K(\overline{E})) \neq \{r\}$, then $\rho(\mathcal{E}(T_c^{-6}), K(\overline{E}))$ contains infinitely many rational numbers. By Theorem 2.9, there exist three periodic points $x_1, x_2, x_3 \in K(\overline{E})$ with different rational rotation numbers $r_1, r_2, r_3$. Let $F_i$ denote the connected component of $E_{r_i}$ containing $r_i$, and let $M_i \subset F_i$ be a separator.

By Proposition 2.5, there is an ordering on disjoint separators. Without loss of generality, we assume that $M_1 < M_2 < M_3$. Based on a discussion about the position $E$ with respect to $M_i$'s, we obtain a contradiction. Thus, $\rho(\mathcal{E}(T_c^{-6}), K(E))$ is the singleton $\{r\}$.

We know from Theorem 2.10 that the left and right prime ends rotation numbers of $K(\overline{E})$ are both $r$. But in the group of circle homeomorphisms, the centralizer of an irrational rotation is essentially an abelian group. This contradicts the fact that a power of $\mathcal{E}(T_c^{-6})$ is a product of commutators in its centralizer as in Proposition 3.9.

5.2. **The set $E_r$.** Once again we use abbreviation $f = \mathcal{E}(T_c^{-6})$. For a characteristic annulus $A$, we let

$$E_r = \{x \in A : \rho\big(f, x, A\big) = r\}.$$

By Lemma 4.3, we know that the definition of $E_r$ does not depend on the choice of the characteristic annulus. By Lemma 4.5, if $0 < r < 1$, we know that $E_r$ is nonempty and $E_r \subset \mathbf{B}$.

Next, we prove the following key lemmas.

**Lemma 5.1.** *Fix $0 < r < 1$, and let $E$ denote a connected component of $E_r$. Fix a simple BP map $h$. For $x \in E$, let $C(x) \in \mathbf{A}(h)$ be the corresponding acyclic set. Then $C(x) \subset E$. In particular, $E$ is $\mathcal{LI}(c)$-invariant.*

*Proof.* To prove that $E$ is $\mathcal{LI}(c)$ invariant, we only need to show that $E$ is invariant under simple BP maps since simple BP maps generate $\mathcal{LI}(c)$ by Proposition 3.9. Let $h$ denote a fixed simple BP map.

Recall the special characteristic annulus $A_h$ and the set $\mathbf{B_h}$ that we defined in Section 4.4. Then, for each $x \in \mathbf{B_h}$ we have $C(x) \subset \mathbf{B_h}$, where $C(x) \in \mathbf{A}(h)$ is the corresponding acyclic set (we can do this by Proposition 4.8).

**Claim 5.2.** *There exists $d_0 > 0$ with the following properties. For $x \in \mathbf{B_h}$, we let $C(x) \in \mathbf{A}(h)$ denote the corresponding acyclic set. Then every connected component of the lift $\pi^{-1}(C(x))$ has the diameter at most $d_0$, for every $x \in \mathbf{B_h}$ (the diameter is computed with respect to the Euclidean metric on the infinite strip $\widetilde{A_h}$).*

*Proof.* Let $\mathbf{d} : \mathbf{B_h} \to \mathbb{R}$ be the function such that $\mathbf{d}(x)$ is the diameter of a connected component of $\pi^{-1}(C(x))$. This definition does not depend on the choice of the connected component of $\pi^{-1}(C(x))$

because different components are images of each other by the deck group of translations (and they are isometries for the Euclidean metric on $\widetilde{A_h}$).

Moreover, from the upper-semicontinuity of the acyclic decomposition $\mathbf{A}(h)$, it follows that $\mathbf{d}$ is an upper-semicontinuous function. Thus, it achieves its maximum on the compact set $\mathbf{B}_h$. We let $d_0$ be the maximum value of the function $\mathbf{d}$. $\square$

For $x \in E$, It remains to show that $C(x) \subset E$. Since $C(x)$ is compact and connected, and $C(x) \subset \mathbf{B_h}$, it suffices to show that the rotation number of each $y \in C(x)$ is equal to $r$.

Fix $\widetilde{f} : \widetilde{A_h} \to \widetilde{A_h}$, a lift of $f$, and $\widetilde{x} \in \pi^{-1}(x)$. Let $\widetilde{y} \in \pi^{-1}(y)$ be the point which belongs to the same connected component of $\pi^{-1}(C(x))$ as $\widetilde{x}$. We denote this connected component of $\pi^{-1}(C(x))$ by $D$. Since $f$ permutes the acyclic sets $C(z)$, for $z \in \mathbf{B_h}$, from the previous claim we conclude that the Euclidean distance between $\widetilde{f}^k(\widetilde{x})$ and $\widetilde{f}^k(\widetilde{y})$ is at most $d_0$, for any integer $k$. Therefore, the translation numbers $\rho(\widetilde{f}, \widetilde{x}, \widetilde{A})$ and $\rho(\widetilde{f}, \widetilde{y}, \widetilde{A})$ are equal (since $x \in E \subset E_r$ we already know that $\rho(\widetilde{f}, \widetilde{x}, \widetilde{A})$ exists). Thus, $y \in E$, and we are done.

$\square$

5.3. **Further properties of connected components of $E_r$.** In this subsection, we show that the closure of each connected component of $E_r$ is a separator when $0 < r < 1$. Let $E$ be one connected component of $E_r$. Fix any characteristic annulus $A$. Denote by $\pi : \widetilde{A} \to A$ the universal cover and recall the $x$-coordinate function $p_1 : \widetilde{A} \to \mathbb{R}$, on the infinite strip $\widetilde{A}$. (The function $p_1$ is what we use to define translation numbers on $\widetilde{A}$.)

**Lemma 5.3.** *The closed set $\overline{E}$ is a separator (as defined in Section 2).*

*Proof.* By Lemma 3.9, the left Torelli group $\mathcal{LI}(c)$ is generated by simple BP maps. Write $T_c^{-6}$ as the product of simple BP maps

$$T_c^{-6} = h_k \cdots h_1.$$

For simplicity, we let $g_i = \mathcal{E}(h_i)$. Then

$$f = g_k \cdots g_1. \tag{6}$$

Fir $x_1 \in E$, and let $C_1$ be the element of $\mathbf{A}(h_1)$ that contains $x_1$. Inductively for $i \in \mathbb{Z}$, we let $C_{i+1}$ be the element of $\mathbf{A}(h_{i+1})$ which contains $x_{i+1}$, where $x_{i+1} = g_i(x_i)$. By Lemma 5.1 we have $x_i \in E$. Moreover, from (6) we find that

$$x_{i+k} = f(x_i), \tag{7}$$

for every $i \in \mathbb{Z}$.

Now, we lift everything to the universal cover $\pi : \widetilde{A} \to A$. The corresponding lift of $C_i$ is also denoted by $C_i$, while the corresponding lift of the point $x_i$ is denoted by $\widetilde{x}_i$. Once we fix a lift $\widetilde{x}_1$ of $x$, the remaining lifts are uniquely determined. Moreover, there exists a unique lift $\widetilde{f} : \widetilde{A} \to \widetilde{A}$ such that $\widetilde{f}(\widetilde{x}_i) = \widetilde{x}_{i+k}$.

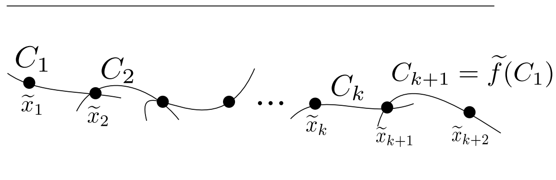


Figure 3

The sequence of subsets $C_i$ satisfies the following properties as in Figure 3:

- $C_i$ and $C_{i+1}$ contain a common point for every $i \in \mathbb{Z}$ (by definition);
- $C_i$ is connected and $C_i \subset \pi^{-1}(E)$ (by Lemma 5.1).

Define
$$C = C_1 \cup ... \cup C_k,$$
and
$$K_0 := \bigcup_{n\in\mathbb{Z}} \widetilde{f}^n(C).$$

We now prove that $\overline{E}$ is a separator. Since $E$ is connected (and compactly contained in $A$), it follows that $\overline{E}$ is compact and connected in $A$. It remains to show it separates the two ends of $A$. Let $\gamma$ denotes a simple closed arc in $\overline{E}$ connecting the two ends of $A$ (we choose $\gamma$ so it connects two accessible points of the two frontiers of $A \subset S_g$). It suffices to show that $\overline{E}$ intersects any such $\gamma$. In fact, we show a stronger statement that $E$ intersects any such $\gamma$.

We argue by contradiction, and assume that there is such a $\gamma$ which $E$ does not intersect. Choose a lift $\widetilde{\gamma}$ of $\gamma$. Then $\widetilde{\gamma}$ divides $\widetilde{A}$ into two sides $\Omega_-$ and $\Omega_+$, such that $p_1(\Omega_-)$ is bounded above and $p_1(\Omega_+)$ is bounded below (recall that $p_1$ is the $x$-coordinate map). Then $\pi^{-1}(E)$ does not intersect $\widetilde{\gamma}$. In the rest of the proof we show that $K_0$ intersects $\widetilde{\gamma}$, which is a contradiction.

Since the translation number $\rho\big(\widetilde{f}, \widetilde{x}_1, \widetilde{A}\big)$ is not equal to zero, we conclude that
$$\lim_{n\to+\infty} p_1\big(f^n(\widetilde{x}_1)\big) = +\infty,$$
and
$$\lim_{n\to+\infty} p_1\big(f^n(\widetilde{x}_1)\big) = +\infty,$$
Together with (7), this implies that $K_0$ intersects both $\Omega_+$ and $\Omega_-$. Define a function $H : \Omega_+ \cup \Omega_- \to \mathbb{R}$ by letting $H(x) = -1$ for $x \in \Omega_-$ and $H(x) = 1$ for $x \in \Omega_+$. Then $H$ is a continuous function on $\Omega_+ \cup \Omega_-$. If we assume that $K_0$ does not intersect $\widetilde{\gamma}$, then $K_0 \subset \Omega_+ \cup \Omega_-$ and the restriction of $H$ to $K_0$ is continuous well. However, $K_0$ is connected so $H(K_0)$ is a connected subset of $\mathbb{R}$. But this $H(K_0) - \{0, 1\}$, which is not connected. It follows that $K_0$ intersects $\widetilde{\gamma}$ and we are finished. $\square$

Building on the construction from the previous proof we show that for a connected component $E$ that contains a periodic orbit the following stronger property holds.

**Lemma 5.4.** *Let $x$ be a periodic orbit of $f$ such that $\rho(f, x, A) = p/q$ and $0 < p/q < 1$. Then, the connected component $E$ of $E_{p/q}$ which contains $x$, also contains a separator (as a subset).*

*Proof.* By construction of the set $K_0$, we have

$$\widetilde{f}(K_0) = K_0.$$

Since $x = x_1$ is a periodic point for $f$, there exists an integer $l$ such that $f^l(x_1) = x_1$. Then by (7) we have $x_{1+kl} = x_1$. This implies that for some integer $m$ the equality

$$\widetilde{x}_{1+kl} = T^m(\widetilde{x}_1),$$

holds, where $T^m$ is the translation by $m$.

Thus, $K_0$ is invariant under $T^m$. This shows that $\pi(K_0) \subset E$ is compact. Furthermore $\pi(K_0)$ is connected since $K_0$ is connected, and we proved in the previous lemma that $\pi(K_0)$ separates the ends of $A$. Thus, $\pi(K_0)$ is a separator. $\square$

5.4. **Finishing the proof.** Fix an irrational number $r \in (0, 1)$. By Lemma 4.5, we know that $E_r$ is not empty. Let $E$ be a connected component of $E_r$. By Lemma 5.1, we know that $E$ is invariant under $\mathcal{LI}(c)$. By Lemma 5.3, we know that $\overline{E}$ is a separator. The annular completions $K(\overline{E})$ of $\overline{E}$ is also $\mathcal{LI}(c)$-invariant since the definition is canonical. The following claim is at the heart of the entire construction.

**Claim 5.5.** *Let $r_L$ and $r_R$ be the left and right prime ends rotation numbers of $f$ on $K(\overline{E})$. Then $r_L = r_R = r$.*

*Proof.* We prove that the rotation interval $\sigma(f, K(\overline{E}))$ is a singleton $\{r\}$. Then by Theorem 2.10, we know that $r_L = r_R = r$.

Since $K(\overline{E})$ is an annular continuum, and $f$ is area-preserving, we have Theorem 2.9 saying that every rational number in the translation interval $\sigma(f, K(\overline{E}))$ is realized by a periodic orbit of $f$. We argue by contradiction. Suppose $\sigma(f, K(\overline{E}))$ is not a singleton, there exist periodic points $x_1, x_2, x_3 \in K(\overline{E})$ with three different rotation numbers $r_i \in \sigma(f, K(\overline{E}))$. Denote by $M_i$ a separator contained the connected component of $E_{r_i}$ containing $x_i$ (such $M_i$ exists by Lemma 5.4). Without loss of generality, we assume that $M_1 < M_2 < M_3$ by Proposition 2.5.

Since $E$ consists of points with irrational rotation number $r$, we know that $E$ is disjoint from the separators $M_1, M_2$, and $M_3$. On the other hand, each $M_i$ is contained in $K(\overline{E})$. We show this yields a contradiction. We break the discussion into the following two cases. (Recall that for an annular continuum $K \subset A$, by $A_R$ and $A_L$ we denote the two annuli in the complement of $K$.)

Since $E$ is disjoint from $M_2$ and $A_R(M_2)$ is a connected component of $A - M_2$, one of the following must happen.

- $E \cap A_R(M_2) = \emptyset$: We claim that $K(\overline{E}) \cap A_R(M_2) = \emptyset$ which contradicts $x_3 \in A_R(M_2) \cap K(\overline{E})$. Since $E \cap A_R(M_2) = \emptyset$ and that $A_R(M_2)$ is open, we know that $\overline{E} \cap A_R(M_2) = \emptyset$. Since $A_R(M_2)$ is connected, disjoint from $\overline{E}$ and contains the left end of $A$, we know $A_R(M_2) \subset A_R(\overline{E})$ by Proposition 2.5. Therefore $A_R(M_2) \cap K(\overline{E}) = \emptyset$.

- $E \subset A_R(M_2)$ which means $E \cap A_L(M_2) = \emptyset$: With the same argument above, we show that $K(\overline{E}) \cap A_L(M_2) = \emptyset$ which contradicts $x_1 \in A_L(M_2) \cap K(\overline{E})$.

$\square$

We conclude that $\mathcal{LI}(c)$ acts on the left prime ends of $K(\overline{E})$, where the action of $f$ has irrational rotation number $r$. However, $f$ is a product of commutators in $\mathcal{LI}(c)$ by Lemma 3.9, the following lemma gives us a contradiction.

**Lemma 5.6** (Centralizer of an irrational rotation)**.** *If $\phi \in \mathrm{Homeo}_+(S^1)$ has an irrational rotation number, then $\phi$ cannot be written as a product of commutators in its centralizer.*

*Proof.* Since the rotation number of $\phi$ is irrational, $\phi$ has no periodic orbit. Let $\mathbf{M} \subset S^1$ denote the minimal set of $\phi$ (in particular, the orbits of points in $\mathbf{M}$ are dense in $\mathbf{M}$ by [Ghy01, Proposition 5.6]). Then either $\mathbf{M}$ is equal to $S^1$ or it is a Cantor set. In the latter case, the complement of $\mathbf{M}$ is a countable union of open interval. Collapsing these intervals to points we obtain the quotient space $S^1/\sim$ which is homeomorphic to $S^1$. Note that every homeomorphism which belongs to the centralizer of $\phi$ descends to a homeomorphism of the quotient.

Therefore, we may assume that the minimal set of $\phi$ is the circle. Then by a theorem of Poincaré (see e.g., [Ghy01, Theorem 5.9]), we know $\phi$ is conjugate to an actual irrational rotation. It suffices to show that an irrational rotation $\phi$ can not be written as a product of commutators in its centralizer. However the centralizer of $\phi$ is the Abelian group $SO(2)$, thus any commutator in the centralizer of $\phi$ is the identity map of $S^1$. The proof is complete.

□

## 6. Pseudo-Anosov analysis and proof of Theorem 4.1 and 4.6

Let $b \in S_2$ be a base-point. Let $\mathcal{Z} : S_2 \to S_2$ be a pseudo-Anosov map on $S_2$ such that $b$ is one singularity. We can "blow up" the base-point $b$ to a circle. Let $M$ be a surface of genus $g \geq 2$. We decompose $M$ into the union of a genus 2 surface missing one disk $L$, a closed annulus $N$ and a genus $g-2$ surface missing one disk $R$. We construct a map $\mathcal{Z}_M : M \to M$ as the following: $\mathcal{Z}_M$ is the identity map on $R$, the blow up of $\mathcal{Z}$ on $L$ and any action on $N$.

Let $\mathcal{P} : M \to S_2$ be the map that collapses points in $N \cup R$ to a point. Then let $\widetilde{M}$ be the cover of $M$ that is a pull back of the universal cover $\mathbb{H}^2 \to S_2$ where $\mathbb{H}^2$ denotes the hyperbolic plane. We pick a lift of the homeomorphism $\mathcal{Z}$ to the universal cover $\widetilde{\mathcal{Z}} : \mathbb{H}^2 \to \mathbb{H}^2$. There is a projection $\widetilde{\mathcal{P}} : \widetilde{M} \to \mathbb{H}^2$. Geometrically it is the pinching map that pinches each copy of lifts of $R$ on $\widetilde{M}$. The map $\mathcal{Z}_M$ can also be lifted to $\widetilde{M}$ as $\widetilde{\mathcal{Z}_M} : \widetilde{M} \to \widetilde{M}$.

Let $\mathcal{F}$ be a homeomorphism that is homotopic to $\mathcal{Z}_M$. Since $\mathcal{F}$ and $\mathcal{Z}_M$ are homotopic, we could lift $\mathcal{F}$ to $\widetilde{\mathcal{F}} : \widetilde{M} \to \widetilde{M}$ such that $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{Z}_M}$ have bounded distance.

**Definition 6.1.** For $\widetilde{x} \in \widetilde{M}$ and $\widetilde{y} \in \mathbb{H}^2$, we say that $(\widetilde{\mathcal{F}}, \widetilde{x})$ *shadows* $(\widetilde{\mathcal{Z}}, \widetilde{y})$ if there exists $C$ such that

$$d_{\mathbb{H}^2}(\widetilde{\mathcal{P}}(\widetilde{\mathcal{F}}^n(\widetilde{x})), \widetilde{\mathcal{Z}}^n(\widetilde{y})) < C$$

We call that a sequence of points $\{x_n\}$ in $\mathbb{H}^2$ is a an $\widetilde{\mathcal{Z}}$ pseudo-orbit if the set $\{d_{\mathbb{H}^2}(\widetilde{\mathcal{Z}}(x_n), x_{n+1})\}$ is bounded.

**Lemma 6.2.** *The sequence $\{\widetilde{\mathcal{P}}(\widetilde{\mathcal{F}}^n(x))\}$ is an $\widetilde{\mathcal{Z}}$-pseudo-orbit for every $x \in \widetilde{M}$.*

*Proof.* This lemma follows from the following inequality:

$$d_{\mathbb{H}^2}(\widetilde{\mathcal{Z}}(\widetilde{\mathcal{P}}(\widetilde{\mathcal{F}}^n(\widetilde{x}))), \widetilde{\mathcal{P}}(\widetilde{\mathcal{F}}^{n+1}(\widetilde{x}))) = d_{\mathbb{H}^2}(\widetilde{\mathcal{P}}(\widetilde{\mathcal{Z}_M}(\widetilde{\mathcal{F}}^n(\widetilde{x}))), \widetilde{\mathcal{P}}(\widetilde{\mathcal{F}}(\widetilde{\mathcal{F}}^n(\widetilde{x})))) \leq C. \qquad \square$$

There is a difference between a pseudo-Anosov map and an Anosov map: for an Anosov homeomorphism, every pseudo-orbit has a uniformly bounded distance to a unique actual orbit; however for a pseudo-Anosov homeomorphism, we do not have such nice relation. The stable and unstable foliations have singularities and their leaf spaces $L^s$ and $L^u$ are more complicated. Namely, $L^s$ and $L^u$, and their metric completions $\overline{L^s}$ and $\overline{L^u}$ have the structure of $\mathbb{R}$-trees [FH07]. Then $\widetilde{\mathcal{Z}}$ induces a map $\widetilde{\mathcal{Z}}^s : \overline{L^s} \to \overline{L^s}$ that uniformly expands distance by a factor $\lambda > 1$ and a map $\widetilde{\mathcal{Z}}^u : \overline{L^u} \to \overline{L^u}$ that uniformly contracts distance by a factor $1/\lambda$. There is an embedding of $\mathbb{H}^2$ in $\overline{L^s} \times \overline{L^u}$. Denote by $\overline{\mathcal{Z}} = (\widetilde{\mathcal{Z}}^s, \widetilde{\mathcal{Z}}^u)$. As discussed in [FH07], every $\widetilde{\mathcal{Z}}$ pseudo-orbit has a uniformly bounded distance to a unique actual orbit of $\overline{\mathcal{Z}}$ on $\overline{L^s} \times \overline{L^u}$.

Using this property, there exists a unique map $\Theta : \widetilde{M} \to \overline{L^s} \times \overline{L^u}$ such that $\{\widetilde{\mathcal{P}}(\widetilde{\mathcal{F}}^n(x))\}$ is shadowed by the orbit of $\Theta(x)$. We have the following two theorems from [Mar07, Lemma 4.14] and [FH07, Theorem 1.2].

**Theorem 6.3.** *Let $\mathcal{F}, \mathcal{G} \in \mathrm{Homeo}(M)$ such that $\mathcal{F}, \mathcal{G}$ commute, $\mathcal{G}$ is homotopic to the identity map on the component $L$ and $\mathcal{F}$ is homotopic to $\mathcal{Z}_M$. We have that $\mathcal{G}$ preserves each connected component of $\Theta^{-1}(c, w)$ for $(c, w) \in \overline{L^s} \times \overline{L^u}$.*

Let $\widetilde{\mathbf{S}}$ be the collection of all components of the sets $\Theta^{-1}(c, w)$. Set $\mathbf{S} = \pi_M(\widetilde{\mathbf{S}})$ where $\pi_M : \widetilde{M} \to M$ be the covering map. The following is [Mar07, Proposition 4.1].

**Proposition 6.4.** *The set $\mathbf{S}$ is a proper upper semi-continuous decomposition of $M$. Moreover, there exists a simple closed curve $\gamma$, which is homotopic to the boundary of $L$ (the surface of genus 2 minus a disc) such that if $p \in M$ belongs to the component of $M - \gamma$ that is homotopic to $L$, then the component of $\mathbf{S}$ that contains $p$ is acyclic.*

Let $\mathcal{E} : \mathcal{I}(M) \to \mathrm{Homeo}_+(M)$ be a section of $p_g$. Using the above ingredients, we can prove the following lemma the same way as [Mar07, Theorem 4.1] by the existence of pseudo-Anosov elements in $\mathcal{I}(S_2)$ (see, e.g., [FM12, Corollary 14.3]).

**Theorem 6.5.** *Let $M$ be a surface of genus $g > 2$ and $\alpha \subset M$ be a simple closed curve such that $\alpha$ separates $M$ into a genus 2 surface minus a disc and a compact surface of genus $g - 2$ with one end. For an element $h \in \mathcal{I}(M)$ that can be realized as a homeomorphism that is the identity inside the corresponding subsurface of $M$ that is homeomorphic to a genus 2 surface minus a disc, there exists an admissible decomposition of $M$ for $\mathcal{E}(h)$ with the following property: there exists a simple closed curve $\beta$, homotopic to $\alpha$ such that if $p \in M$ belongs to the genus 2 surface minus a disc (which is one of the two components obtained after removing $\beta$ from $M$), then the component of the decomposition that contains $p$ is acyclic.*

We now use the above to proof Theorem 4.1 and 4.6.

*Proof of Theorem 4.1 and 4.6.* For simplicity, we prove the existence of $\mathbf{M}_1(c, h)$ only. The proof of others are similar. Define

$$H = \langle \mathcal{E}(T_c^{-6}), \mathcal{E}(h) \rangle < \mathrm{Homeo}(M)$$

for $h = T_a T_b^{-1}$. The other cases can be proved the same way. Let $\mathbf{S}(H)$ be the minimal decomposition of $H$ and $\mathbf{A}(H)$ be the union of acyclic components of $\mathbf{S}(H)$, which is a subsurface by Proposition 3.6. Let $\alpha$ be the curve as in the following figure.

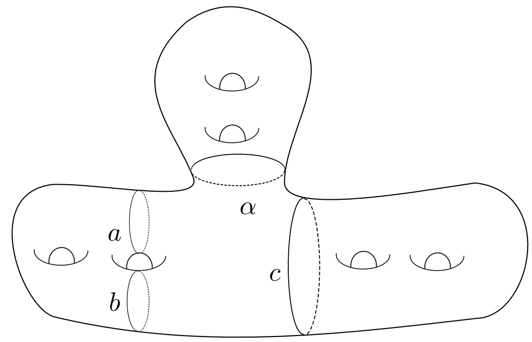


FIGURE 4

By Theorem 6.5, there exists a simple closed curve $\beta$ homotopic to $\alpha \subset M$ such that there exists a component $W$ of $\mathbf{A}(H)$ that contains the component of $M - \beta$ that is homotopic to the genus 2 surface minus a disc. Since $H$ is not homotopic to the identity map, we know that $W \neq M$. This implies that $W$ has at least one end. We plan to prove that $W$ has exactly three ends and they are homotopic to $a, b, c$ respectively.

Let $\beta_n$ be a nested sequence that determines one end $E$ of $W$. Firstly, we claim that $\beta_n$ cannot intersect $a, b, c$ (as isotopy classes of simple closed curves). If $\beta_n$ intersects $a$ geometrically, then $\mathcal{E}(h)(\beta_n)$ intersects $\beta_n$. This contradicts the fact that $\mathcal{E}(h)(\beta_n) \subset W$. If $\beta_n$ is not homotopic to $a, b$ or $c$, then there exists a separating curve $\delta$ such that $\delta$ intersects $\beta_n$ and $\delta$ does not intersect $a, b, c$. Then since $\mathcal{E}(T_\delta)(\beta_n)$ intersect $\beta$, it has to intersect $E$. However since $\mathcal{E}(T_\delta)$ commutes with $H$, we know that $\mathcal{E}(T_\delta)$ permutes components of $\mathbf{A}(H)$. This contradicts the fact that $\mathcal{E}(T_\delta)(\beta_n)$ intersect $\beta$ and that $\beta$ is a nested sequence that determines one end of $W$.

We now need to show that $a, b, c$ are all homotopic to some end of $W$. We prove this by contradiction. If the frontier of $W$ does not contain one end homotopic to $a$, then $W$ contains at most two ends, homotopic to $b, c$. Let $p_W : W \to W/\sim$ be the Moore map for components of $\mathbf{S}(H)$ in $W$. Let $A_b$ be an open annulus that is bounded by the end of $W/\sim$ homotopic to $b$ and a simple closed curve homotopic to $b$. We define $A_c$ similarly. Then $U = p_W^{-1}(A_b) \cup (M - W) \cup p_W^{-1}(A_c)$ is an open set. We define a new upper semi continuous decomposition $\mathbf{S}'$ that consists of elements of $\mathbf{S}(H)$ in $M - U$ and points in $U$. Let $p : M \to M/\sim$ be the Moore map for $\mathbf{S}'$. Therefore $H$ is semi-conjugate to a new action $H'$ that is the identity on $M - U$. This contradicts the fact that $\mathcal{E}(h)$ is homotopic to $T_a T_b^{-1}$ in $M - U$, which is not homotopic to identity. The proof that $b, c$ are also homotopic to ends of $W$ is similar. $\square$

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