The logarithmic spiral: a counter-example to the $K = 2$ conjecture

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Abstract

Given a non-empty compact connected subset $X \subset \mathbb{S}^2$ with complement a simply-connected open subset $\Omega \subset \mathbb{S}^2$, let $\mathrm{Dome}(\Omega)$ be the boundary of the hyperbolic convex hull in $\mathbb{H}^3$ of X. We show that if X is a certain logarithmic spiral, then we obtain a counter-example to the conjecture of Thurston and Sullivan that there is a 2-quasiconformal homeomorphism $\Omega \to \mathrm{Dome}(\Omega)$ which extends to the identity map on their common boundary in $\mathbb{S}^2$. This leads to related counter-examples when the boundary is real analytic, or a finite union of intervals (straight intervals, if we take $\mathbb{S}^2 = \mathbb{C} \cup \{\infty\}$. We also show how this counter-example enables us to construct a related counter-example which is a domain of discontinuity of a torsion-free quasifuchsian group with compact quotient. Another result is that the average long range bending of the convex hull boundary associated to a certain logarithmic spiral is approximately $.98\pi/2$, which is substantially larger than that of any previously known example.

1 Introduction

Let $\Omega \subset \mathbb{C}$, with $\Omega \neq \mathbb{C}$, be a simply connected region. Let $X = \mathbb{S}^2 \setminus \Omega$ and let $\mathcal{CH}(X)$ be the corresponding hyperbolic convex hull. The relative boundary $\partial\mathcal{CH}(X) \subset \mathbb{H}^3$ faces Ω. It is helpful to picture a domed stadium—see Figure 2—such as one finds in Minneapolis, with Ω its *floor* and the *dome* given by $\mathrm{Dome}(\Omega) = \partial\mathcal{CH}(X)$.

The dome is canonically associated with the floor, and gives a way of studying problems concerning classical functions of a complex variable de-

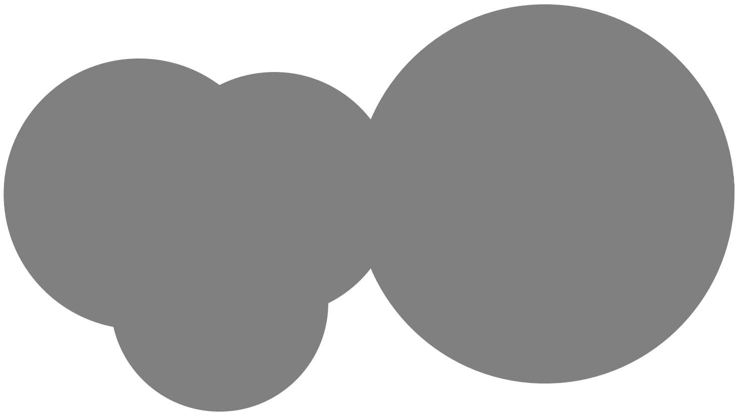

Figure 1. Ω is the union of four disks. $\mathrm{Dome}(\Omega)$ is the union of five flat pieces as can be seen in Figure 2. The fifth piece is a hyperbolic triangle in the hyperbolic plane represented by a circle lying in the union of three of the original disks. The dome has four bending lines, as shown in Figure 2.

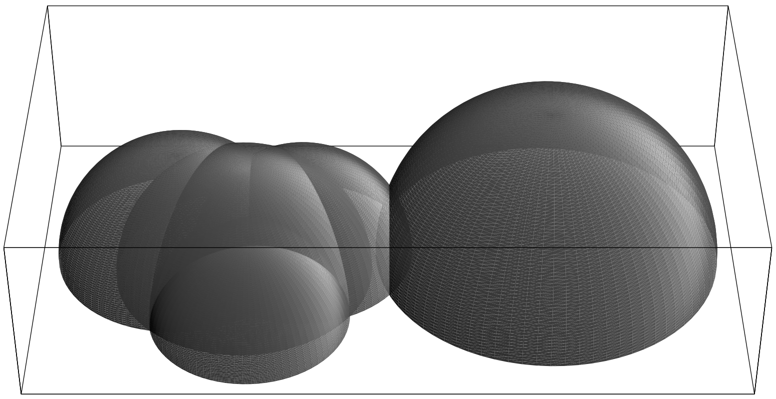

Figure 2. $\mathrm{Dome}(\Omega)$, where Ω is shown in Figure 1. The dome is placed in the upper halfspace model, and is viewed from inside the convex hull of the complement of Ω, using Euclidean perspective. The space under the dome lies between Ω and $\mathrm{Dome}(\Omega)$. Since the upper halfspace model is conformal, the angle between disks in Figure 1 is equal to the angle between flat pieces shown in Figure 2.

fined on Ω by using methods of two and three-dimensional hyperbolic geometry. In this direction the papers of C.J. Bishop (see [6], [3], [4] and [5]) were particularly significant in stimulating us to do the research reported on here.

Conversely, the topic was developed in the first place (see [16] and [17]) in order to use methods of classical complex variable theory to study 3-dimensional manifolds.

The discussion begins with the following result of Bill Thurston.

1.1 Theorem. *The hyperbolic metric in $\mathbb{H}^3$ induces a path metric on the dome, which we will refer to as its hyperbolic metric. There is an isometry of the dome with its hyperbolic metric onto $\mathbb{D}^2$ with its hyperbolic metric.*

A proof of this can be found in [9].

The main result (Theorem 1.2) in the theory is due to Sullivan (see [16] and [9]).

1.2 Theorem (Sullivan, Epstein-Marden). *There exists $K > 1$ such that, for any simply connected $\Omega \neq \mathbb{C}$, there is a K-quasiconformal map $\Psi : \Omega \to \mathrm{Dome}(\Omega)$, which extends continuously to the identity map on the common boundary $\partial\Omega$.*

1.3 Definition. Given an open simply connected region in the 2-sphere, whose complement contains at least two points, we denote by $K(\Omega)$ the infimum of values of K for which there exists a K-quasiconformal homeomorphism $\phi : \Omega \to \mathrm{Dome}(\Omega)$ which extends to the identity on the boundary of Ω. We denote by $K_{eq}(\Omega)$ the infimum when we vary over all K-quasiconformal homeomorphisms which commute with the group of Möbius transformations preserving Ω. Then, for each Ω, $K_{eq}(\Omega) \geq K(\Omega)$. It is unknown whether there are cases where the inequality is strict. We define $K = \sup_\Omega K(\Omega)$ and $K_{eq} = \sup_\Omega K_{eq}(\Omega)$, where the supremum is taken over all simply connected open subsets of $\mathbb{S}^2$. (However, in this paper K will also be used to denote the quasiconformal constant of some particular quasiconformal homeomorphism. Which use is intended will be clear from the context.) So $K_{eq} \geq K$, and it is not known whether we have equality. □

In [9] Epstein and Marden showed that $K \leq K_{eq} \leq 82.7$. Using some of the same methods, but dropping the requirement of equivariance, Bishop [3] improved this to $K \leq 7.82$.

By explicit computation in the case where Ω is a slit plane, one can see that $K \geq K(\Omega) = 2$ and $K_{eq} \geq K_{eq}(\Omega) = 2$. In [17, p7], Thurston, discussing the equivariant form of the problem, wrote *The reasonable conjecture seems to be that the best K is 2, but it is hard to find an angle for proving a sharp constant.* This has since become known as Thurston's $K = 2$ Conjecture. The conjecture has also been ascribed to Sullivan.

In this paper, we will show that $K(\Omega_c) > 2.1$ for the complement of a certain logarithmic spiral L_c. We will deduce that the best constant is also greater than 2.1 in the quasifuchsian case with Ω one of the domains of discontinuity and the quotient of Ω by the group compact. We will also deduce an example of an Ω with boundary a non-singular real-analytic curve and $K(\Omega) > 2.1$. In a previous paper [10], we produced a specific quasifuchsian group with the quotient of Ω a punctured torus and with $K_{eq}(\Omega) > 2$.

1.4 Remark. If one is being completely precise, the condition in Theorem 1.2 that Ψ extends continuously to the identity on the boundary is not sufficient to pin down what we really intend. The condition is good enough unless $X = \mathbb{S}^2 \setminus \Omega$ is a topological interval. If X is a straight interval in the plane, the reflection in the line containing this interval is anti-conformal. For any embedded topological interval X, we have a similar orientation reversing homeomorphism. We could compose Ψ with such a homeomorphism, but we don't want to consider such homeomorphisms on a par with Ψ.

There are different ways to improve the definition to avoid such peculiar examples. The method we will adopt is to orient Ω as a subset of $\mathbb{C}$ and to orient Dome(Ω) so that the nearest point retraction $r : \Omega \to \text{Dome}(\Omega)$ is orientation preserving. To make sense of the phrase "orientation preserving", note that r is a proper map and therefore induces a homomorphism of compactly supported cohomology in dimension two. We will require that Ψ preserves orientation in addition to extending to the identity on the boundary.

This is equivalent to requiring that Ψ should have the same effect on prime ends as r. Yet another equivalent formulation is to require that Ψ be homotopic to r, with the homotopy fixing boundary values.

In the special case of a logarithmic spiral, the "reflection" in the spiral can be made quasiconformal in the metric sense, though orientation reversing. □

The calculations in this paper were originally much more complex than the versions appearing here. It was only possible to carry them out by using a package such as Mathematica. In the current version the calculations have been very substantially simplified, and we believe they are now checkable by hand, though the checking would be time-consuming. Hand calculations are unreliable, and any reader who wants to follow the calculations in detail is advised to have open a version of Mathematica or Maple while reading the paper. We are grateful to Allan Hayes who gave technical help with Mathematica.

2 An overview of the calculation

In this paper we are going to show that the complement of a certain logarithmic spiral gives a counter-example to the $K = 2$ conjecture of Thurston and Sullivan. The proof requires some complicated and intricate computations. In this section we provide the reader with an overview of the computation and an explanation of why it is possible to make precise computations in this situation. First we recall the definition of a logarithmic spiral.

2.1 Definition. Let $c > 0$. The *logarithmic spiral* associated to c is the curve $\sigma_c : \mathbb{R} \to \mathbb{C}$ given by $\sigma_c(t) = \exp(t(c + i))$. We often consider the logarithmic spiral to be embedded in $\mathbb{S}^2 = \mathbb{C} \cup \{\infty\}$. We obtain a compact subset $L_c \subset \mathbb{S}^2$ by adjoining 0 and ∞ to the image of σ_c. Let $\Omega_c = \mathbb{S}^2 \setminus L_c$. We will often suppress the dependence on c and write simply σ, Ω and L. $\square$

The logarithmic spiral is also known as the *equiangular spiral.* The angle θ between the oriented tangent vector at a point $z \in L_c$ and the oriented line from 0 through z satisfies $c \tan(\theta) = 1$. Some logarithmic spirals are shown in Figure 3.

By the Riemann Mapping Theorem, any open simply connected region of $\mathbb{S}^2$ is complex analytically isomorphic to any other simply connected region. We can fix the isomorphism by specifying an oriented triple of ideal points (prime ends) for each region, and insisting that the isomorphism should send the triple attached to one region to the triple attached to the other.

2.2 Definition. By an *augmented region* we will mean a two dimensional disk with a conformal structure together with an ordered triple of prime ends.

Throughout this paper we will work with augmented regions. All our maps between augmented regions are homeomorphisms which are required to send the triple of prime ends for the domain region to the triple of prime ends for the image region. The triple fixes an orientation of the region, so that all our maps are orientation preserving homeomorphisms. Putting matters

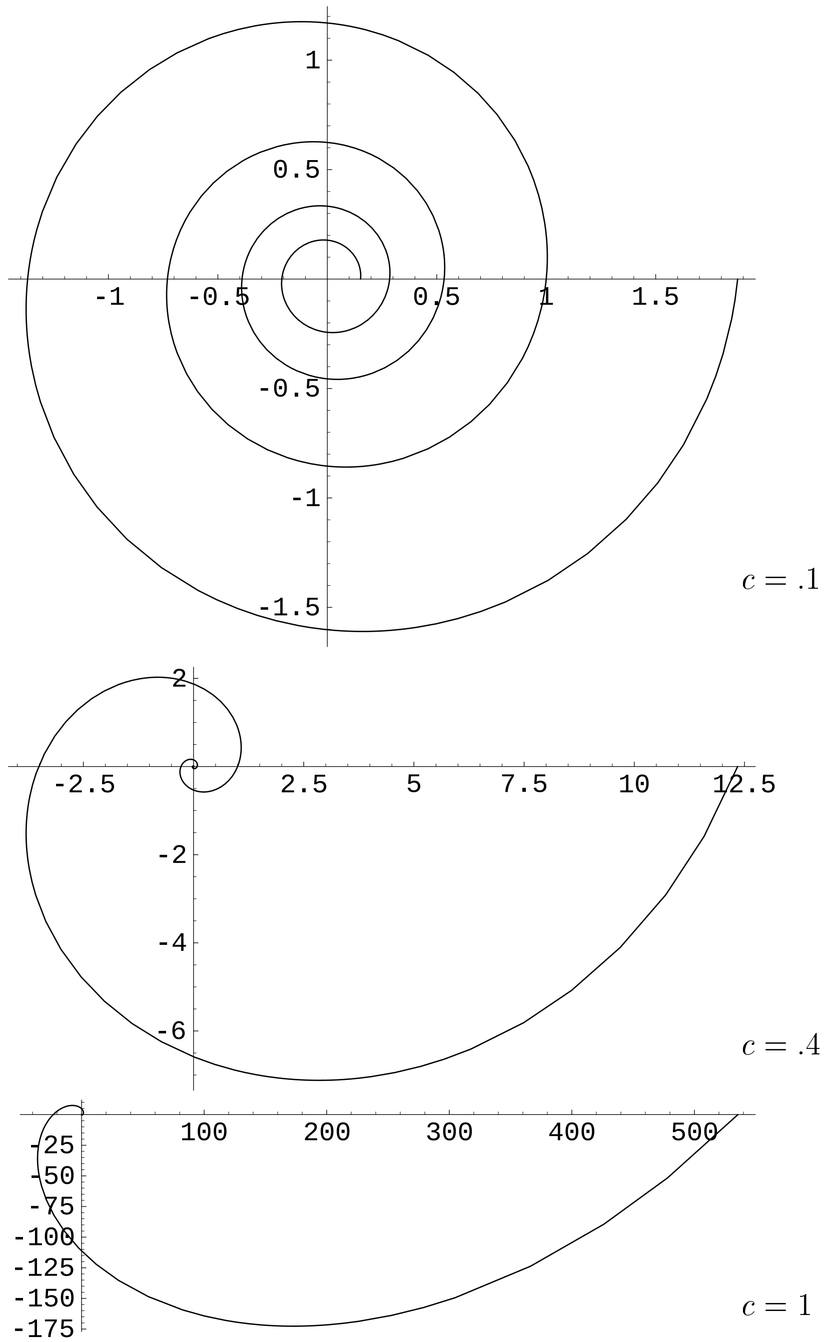

Figure 3. Some logarithmic spirals

another way, if the orientation for the region is specified in some natural way, then the ordered triple of prime ends must run counter-clockwise. The conformal structure together with the orientation is equivalent to a complex analytic structure.

The upper halfplane $\mathbb{U}^2$ is augmented by the triple $(\infty, 0, 1)$ of prime ends. This carries the usual hyperbolic metric. Given any augmented region V, the Riemann mapping $V \to \mathbb{U}^2$ induces a Riemannian metric on V. We call this the *hyperbolic metric* of V.

Each augmented region has a unique oriented geodesic which we call the *main geodesic* from the first prime end of the triple to the second. The main geodesic for $\mathbb{U}^2$ is the vertical geodesic from ∞ to 0. For every augmented region, the third prime end lies on the left of the main geodesic. $\square$

2.3 Definition. The complement Ω of the logarithmic spiral L has special prime ends at 0 and ∞. Each other point $z \in L$ gives rise to two prime ends for Ω, corresponding to the two different local sides of L. The *positive* prime end at z is defined by a curve in Ω ending at z, such that the absolute value $|z|$ decreases strictly along the curve. The other prime end is said to be the *negative* prime end.

We make Ω an augmented region by specifying the triple $(0, \infty, 1^+)$. Here we mean by 1^+ the positive prime end at $1 \in L$. To check that the orientations have been chosen appropriately, note that the main geodesic passes 1^+ with 1^+ on its left.

Let $S \subset \mathbb{C}$ be the infinite open horizontal strip with lower boundary the x-axis $y = 0$ and with upper boundary $y = \pi$. S becomes an augmented region by specifying the triple $(\infty, -\infty, 0)$ of prime ends. The main geodesic is the line $y = \pi/2$ oriented from right to left. (We know this because reflection in this line is an anti-conformal homeomorphism of S which preserves the prime ends $-\infty$ and ∞. It therefore corresponds to an orientation reversing isometry of the hyperbolic plane.) The main geodesic has 0 on its left, so the orientations are consistent. $\square$

The following result is obvious.

2.4 Lemma. *The map* $\exp : S \to \mathbb{U}^2$ *is a complex analytic isomorphism preserving the augmented structure. On the main geodesic of* S*, the euclidean metric is equal to the hyperbolic metric.*

Group actions help a lot in understanding the situation and in making our computations. We therefore start by making the group actions explicit.

Given $s \in \mathbb{R}$, we have two isometries of $\mathbb{R}$, namely the translation $\tau_s : \mathbb{R} \to \mathbb{R}$ given by $\tau_s(t) = t + s$, and the reflection $\rho_s : \mathbb{R} \to \mathbb{R}$ given by $\rho_s(t) = -t + 2s$. Every isometry of $\mathbb{R}$ is equal to one of these. Let G be the group of isometries of $\mathbb{R}$ and let T be the subgroup of translations. Then T is isomorphic to $\mathbb{R}$. Sometimes we will write $\mathbb{R}$ instead of T and s instead of τ_s.

We make G act on $\mathbb{S}^2$, preserving L and Ω, as follows.

$$\tau_s(z) = \exp(s(c+i))z, \quad \rho_s(z) = \exp(2s(c+i))/z. \tag{2.4.1}$$

It is immediate to check that this defines an action of G. Note that τ_s acts on $\mathbb{S}^2$ as a loxodromic transformation fixing 0 and ∞, while ρ_s acts as an elliptic transformation with fixed points $\exp(s(c+i)) \in L$ and $\exp(-s(c+i)) \in \Omega$.

The following result is easy and its proof is left to the reader.

2.5 Theorem. *The logarithmic spiral $\sigma : \mathbb{R} \to \mathbb{C}$ is G-equivariant. G is the full group of complex analytic isomorphisms of Ω preserving setwise the pair of prime ends $\{0, \infty\}$. Each translation τ_s preserves each of the special prime ends 0 and ∞. Each element $\rho_s \in G \setminus T$ has a unique fixed point in Ω which lies on the main geodesic of Ω and interchanges the prime ends 0 and ∞. Conversely, each point on the main geodesic is the unique fixed point of a unique element of $G \setminus T$.*

G is isomorphic to the group of complex analytic isomorphisms of S which preserve setwise the pair $\{\infty, -\infty\}$ of prime ends of S. The element $\tau_s \in T$ acts by horizontal translation by s. We define the action by $\rho_s \in G \setminus T$ on S to be rotation of π about $s + i\pi/2$. Such a transformation interchanges the two endpoints of the main geodesic and sends the main geodesic to itself.

G is isomorphic to the group of complex analytic isomorphisms of $\mathbb{U}^2$ preserving setwise the pair $\{\infty, 0\}$. The element τ_s acts by scalar multiplication $\exp(s) : \mathbb{U}^2 \to \mathbb{U}^2$. We define the action by $\rho_s \in G \setminus T$ to be the elliptic involution with fixed point $i\exp(s)$. Each element of $G \setminus T$ acts as an elliptic involution interchanging the two endpoints of the main geodesic and fixing a unique point of the main geodesic.

As indicated above, these group actions are fundamental for our computation. Using appropriate automorphisms of G (NOT the identity), we can make all our constructions equivariant. This means that the computation can be carried out at individual points, rather than at every point of the regions concerned.

Let $h : \Omega \to \mathrm{Dome}(\Omega)$ be any quasiconformal homeomorphism which extends to the identity on the logarithmic spiral L (and which has the same effect on prime ends as the nearest point retraction $r : \Omega \to \mathrm{Dome}(\Omega)$—see Remark 1.4). Let $R : S \to \Omega$ be the complex analytic isomorphism which sends the triple $(\infty, -\infty, 0)$ of prime ends of S to the triple $(0, \infty, 1^+)$ of Ω. Let ι be the isometry given by Theorem 1.1. We choose ι so that it sends the triple $(0, \infty, 1^+)$ of prime ends of $\mathrm{Dome}(\Omega)$ to the triple $(\infty, 0, 1)$ of prime ends of $\mathbb{U}^2$. The prime end 1^+ of $\mathrm{Dome}(\Omega)$ is, by definition, the image of the prime end 1^+ of Ω under the nearest point retraction.

We consider the composite $H : S \to S$ given by $H = \log \circ \iota \circ h \circ R$ where

$$S \xrightarrow{R} \Omega \xrightarrow{h} \mathrm{Dome}(\Omega) \xrightarrow{\iota} \mathbb{U}^2 \xrightarrow{\log} S. \tag{2.5.2}$$

We have to find a lower bound for the dilatation of H.

All the maps in the definition of H are equivariant for appropriate automorphisms of G, except for h. Since h has the correct behaviour on prime ends, $H : \partial S \to \partial S$ is G-equivariant for an appropriate automorphism of G. It follows that there is an affine map $A : S \to S$ such that the restrictions of H and A to ∂S are equal. Our objective is to compute A.

We have the following result, due independently to Strebel in [15] and to Beurling and Ahlfors in [2].

2.6 Theorem. *Let $A : \mathbb{R}^2 \to \mathbb{R}^2$ be any orientation preserving affine homeomorphism sending S to itself and sending the lower boundary of S to itself. Then the dilatation of A is stricter less than the dilatation of any other homeomorphism of the closed strip to itself which agrees with A on ∂S.*

A very short proof of this important result (but with weak inequality instead of strict inequality) will be provided in Theorem 3.1.

It follows that we can compute a lower bound for the dilatation of H, once we can compute $H|\partial S$. For $c = 1$, we will show that the dilatation is greater than 2.1. It follows that the dilatation of h, which is equal to the dilatation of H, is also greater than 2.1, thus disproving the $K = 2$ conjecture.

In the sections which follow, we will provide details for the above argument.

3 Strips and quasiconformal homeomorphisms

In this section we give an elementary proof of a fundamental result due independently to Strebel in [15] and to Beurling and Ahlfors in [2].

3.1 Theorem. *Let S be the horizontal strip in the plane with height 1 and lower boundary the x-axis. Let A be an orientation preserving affine homeomorphism of $\mathbb{R}^2$ which sends S to S and sends the lower boundary of S to itself. Let $H : S \to S$ be a quasiconformal homeomorphism such that $H|\partial S = A|\partial S$. Let $K(H)$ be the quasiconformal dilatation of H and $K(A)$ the quasiconformal dilatation of A. Then $K(A) \leq K(H)$.*

Proof: Composing with a translation, we may assume that $A(0,0) = H(0,0) = (0,0)$. This means that $A : \mathbb{R}^2 \to \mathbb{R}^2$ becomes a linear map, and $A(x,y) = (ax + by, y)$, where a and b are constants. We define the map $H_n : S \to S$ by

$$H_n(x,y) = \frac{H(nx, ny - [ny]) + A(0,[ny])}{n}.$$

It is easy to check that this is continuous, that it is a homeomorphism, and that it is $K(H)$-quasiconformal. The compactness theorem for quasiconformal homeomorphisms (see Ahlfors [1, Theorem 2, page 51]) shows that a subsequence of $(H_{n!})_{n\in\mathbb{N}}$ converges, uniformly on compact subsets, to a $K(H)$-quasiconformal limit. This limit must be A, because it is eventually equal to A on every horizontal line of rational height. Hence $K(A) \leq K(H)$.

∎

4 The Riemann mapping

In this section we compute the Riemann mapping of the complement of a logarithmic spiral. We use the notation of Section 2.

The Riemann mapping $\Omega \to \mathbb{H}^2$ can be found explicitly. Note that $\log : \Omega \to \mathbb{C}$ is a complex analytic isomorphism with an infinite strip at slope $1/c$. We use the branch of log which is equal to its usual value on the x-axis just to the right of $z = 1$, and we extend by analytic continuation to Ω. The boundary of the strip consists of two straight lines at slope $1/c$. The upper boundary passes through 0 and the lower boundary passes through $-2\pi i$. This is shown in Figure 4. The end of the spiral at 0 is mapped by log to infinity in the lower left quadrant, and the end of the strip at infinity is mapped by log to infinity in the upper right quadrant.

Recall the definition of $R : S \to \Omega$ from just before 2.5.2. Explicitly,

$$R(z) = \exp\left(\frac{2cz}{-c+i}\right)$$

and this is the unique complex analytic isomorphism which sends the ordered triple $(\infty, -\infty, 0)$ of prime ends of S to the ordered triple $(0, \infty, 1^+)$ of prime ends of Ω. The Riemann mapping $\mathbb{U}^2 \to \Omega$ is the composition $R \circ \log$.

We already have an action of G on Ω, described in Equation 2.4.1. We also have an action of G on S. There is an automorphism $\phi_R : G \to G$ with respect to which R is G-equivariant. The formulas are easier if we restrict to the subgroup T of translations, and our proof only requires us to know the restriction of ϕ_R to T-equivariant, so we will adopt the less general approach.

We are looking for the automorphism ϕ_R of $T = \mathbb{R}$ which makes the map R T-equivariant. Such an automorphism can be identified with multiplication by a non-zero real number. We see immediately that we have $\phi_R(s) = -2cs/(c^2+1)$, or $\phi_R = -2c/(c^2+1)$.

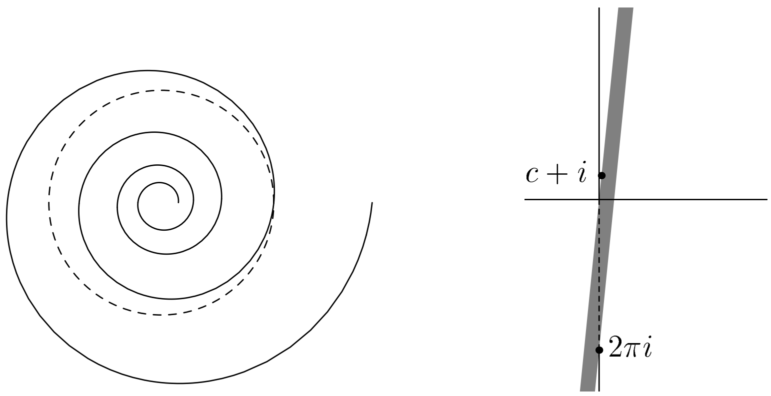

Figure 4. The image under $\log$ of Ω is shown on the right as a shaded strip. The unit circle on the left is shown dashed. The image of the unit circle on the left traversed clockwise is the dashed interval on the right traversed from 0 to $-2\pi i$ on the y-axis. The pictures are drawn with $c = 0.1$, so that $c+i$ is just to the right of the y-axis.

5 Maximal circles and the dome

In this section we find Dome(Ω). First we compute the maximal open disks in Ω. Each maximal disk defines an open half hyperbolic 3-space. In the upper half 3-space $\mathbb{U}^3$ of the hyperbolic 3-space $\mathbb{H}^3$, this half space is a solid hemisphere without its boundary. The union of these solid hemispheres is the complement of the hyperbolic convex hull of the logarithmic spiral L. The dome is the boundary of the union in $\mathbb{U}^3$—a picture can be seen in Figure 6.

Using the action of the group $T = \mathbb{R}$, we can determine all maximal circles once we have found one of them. We will find the maximal circle tangent to L at $z = 1$. In terms of the parameter t, $z = 1$ is the point $t = 0$. There are two maximal circles tangent at this point, one on each side of the spiral. We work out the maximal circle tangent at $t = 0$ which defines a positive prime end (see Definition 2.3).

5.1 Lemma. *Let s_0 be the t-parameter of the second tangent point of the maximal circle which is tangent at $t = 0$ and which defines a positive prime end at $t = 0, z = 1$ (see Figure 5). Then $\tan(s_0/2) = -c\tanh(s_0 c/2)$, and $\pi < s_0 < 2\pi$. Moreover s_0 is the unique solution of this equation in $(0, 2\pi]$.*

5.2 Definition. For the remainder of this section, s_0 will have the meaning assigned to it in Lemma 5.1. □

Proof (Lemma 5.1): Let $r > 0$ be the radius of our maximal circle. We must solve the equation

$$\sigma(0) - \frac{ir\sigma'(0)}{|\sigma'(0)|} = \sigma(s_0) + \frac{ir\sigma'(s_0)}{|\sigma'(s_0)|}$$

for s_0, given $c > 0$, with $\sigma(t) = \exp(t(c + i))$. We regard r as a complex number (following a suggestion of Allan Hayes) and solve, obtaining

$$r = \frac{\sigma(0) - \sigma(s_0)}{i\left(\frac{\sigma'(s_0)}{|\sigma'(s_0)|}\right) + \left(\frac{\sigma'(0)}{|\sigma'(0)|}\right)}. \tag{5.2.3}$$

The first requirement for the solution is that r should be real. This condition is preserved if we multiply by any non-zero scalar, which makes the computation easy. The condition becomes

$$c\tanh(s_0 c/2) = -\tan(s_0/2). \tag{5.2.4}$$

With $c > 0$, the graphs of $-c\tanh(ct/2)$ and $\tan(t/2)$ cross exactly once in the interval $(0, 2\pi]$, and this crossing point is in the interval $(\pi, 2\pi)$. It follows that the smallest strictly positive solution of Equation 5.2.4 satisfies $\pi < s_0 < 2\pi$. We simplify Equation 5.2.3 using Equation 5.2.4 and see that

$$r = \sqrt{1+c^2}\left(-1+e^{s_0 c}\right)/2. \tag{5.2.5}$$

This is positive if $s_0 > 0$ and $c > 0$, so we obtain the desired solution. ∎

From the maximal disk we obtain a solid hemisphere $M \subset \mathbb{U}^3$, which is the hyperbolic convex hull of the maximal disk. The group $T = \mathbb{R}$ acts and the translates of M under G give a snail shape, as shown in Figure 6.

Dome(Ω) is foliated by bending lines. Each bending line is a hyperbolic geodesic whose endpoints are the points of contact between some maximal circle and the spiral L. Such a geodesic is a semicircle in $\mathbb{U}^3$ orthogonal to $\mathbb{C}$. In Figure 7, we draw the bending line β whose endpoints are at $b_\pm = \exp(\pm s_0(c+i)/2)$, where $\tan(s_0/2) = -c\tanh(s_0 c/2)$. The proof that β is a bending line is contained in the proof of Proposition 5.3. The corresponding maximal circle is also shown in Figure 7.

It is a standard and easy result that the path metric on a smooth submanifold of a Riemannian manifold is induced by the induced Riemannian

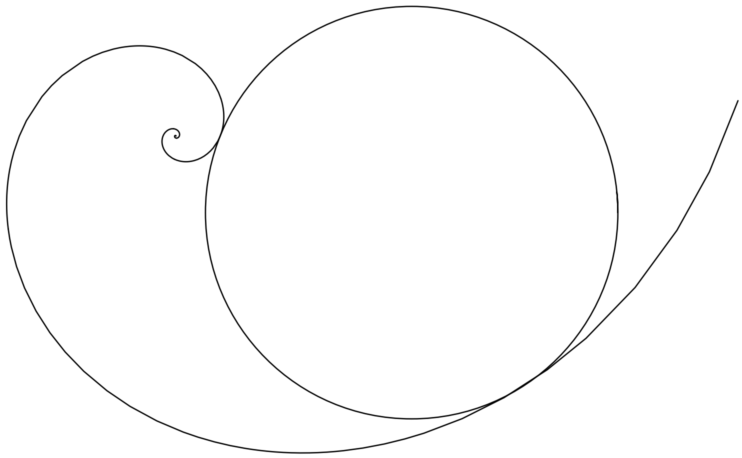

Figure 5. This shows the logarithmic spiral with $c = 0.4$ and the maximal circle which is tangent at $z = 1$. The parameter values at which the circle touches are $t = 0$ and $s_0 = 0.900104$. We find s_0 using Newton's method following Lemma 5.1.

structure. In our case this result can be applied to the dome embedded in $\mathbb{U}^3$.

The main geodesic (see Definition 2.3) is the unique oriented geodesic γ in the Riemannian metric of the dome going from the prime end $0 \in \partial\mathbb{H}^3$ to the prime end $\infty \in \partial\mathbb{H}^3$. Both γ and the foliation by bending lines are preserved by the group G. It follows that γ meets the foliation at a constant angle. Next we determine this angle (in fact, its cosine, which is equivalent) in terms of c.

5.3 Proposition. *The speed of* γ

$$\|\gamma'(t)\| = \sqrt{\frac{1 + c^2 + \cos(s_0) + c^2 \cosh(s_0 c)}{\cosh(s_0 c) - \cos(s_0)}}.$$

Here the length of the tangent vector $\gamma'(t)$ *is measured using the hyperbolic metric. It is independent of* t.

Let θ *be the angle between the geodesic* γ *and a bending line. Then*

$$\cos(\theta) = \frac{\sin(s_0) + c \sinh(s_0 c)}{\sqrt{(-\cos(s_0) + \cosh(s_0 c))\ (1 + c^2 + \cos(s_0) + c^2 \cosh(s_0 c))}}.$$

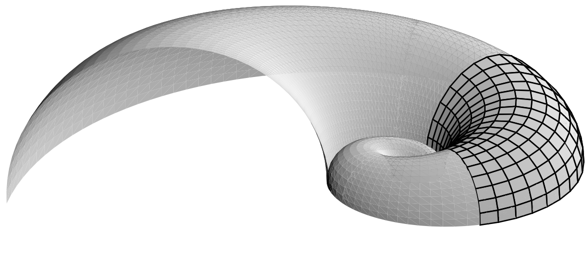

Figure 6. This shows $\mathrm{Dome}(\Omega) \subset \mathbb{U}^3$. The picture is drawn for $c = 0.4$. Part of the spiral is covered with a mesh of bending lines and constant distance curves. These are curves which are at constant distance from the main geodesic with respect to the induced Riemannian metric on the dome.

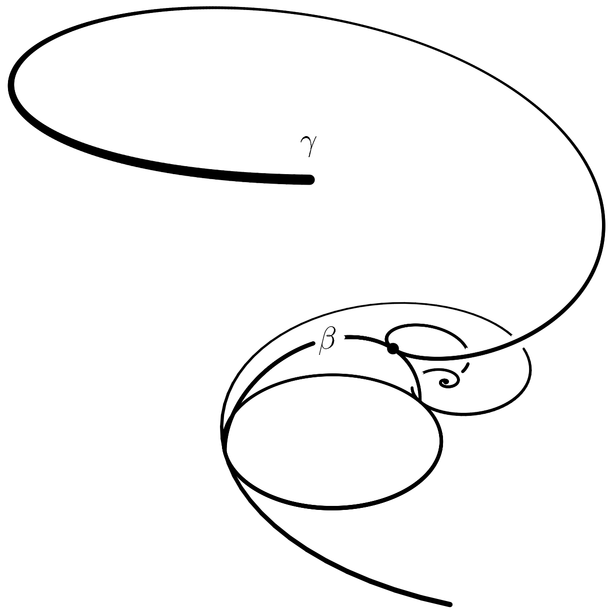

Figure 7. Taking $c = .3$, we show part of the spiral in the plane, and the maximal circle which is tangent at $t = \pm s_0/2$. The bending line joining the two tangency points is denoted by β. The geodesic from 0 to ∞ on the dome with respect to the induced hyperbolic metric on the dome is denoted by γ—only part of this geodesic is drawn. The intersection of γ and β is marked with a black dot. The thickness of the curves indicates the distance from the eye in euclidean perspective.

(The direction of γ is from 0 to ∞. The direction of the bending line is from the endpoint β_- with smaller absolute value to the endpoint β_+ with larger absolute value. The direction of γ' at $p = \beta \cap \gamma$ is obtained by rotating β' about p in a counter-clockwise direction. We use the natural orientation of $\mathrm{Dome}(\Omega)$ *given by its triple* $(0, \infty, 1^+)$ *of prime ends to understand the meaning of "counter-clockwise".)*

Proof: Consider the Möbius transformation $f \in G$ defined by $f(z) = 1/z$. The fixed points are ± 1. Here $-1 \in \Omega$ and $1 \in L$. We extend f to upper half three-space, obtaining an elliptic involution with fixed point set the set of points on the semicircular geodesic α with endpoints ± 1. Then f interchanges $\exp(\pm s_0(c+i)/2)$. The geodesic β with endpoints

$$\beta_\pm = \exp(\pm s_0(c+i)/2) \in L$$

is a bending line. To see this, we apply the element of T which is multiplication by $\exp(s_0(c+i)/2) \in L$ and use Lemma 5.1 to see that the image is a bending line.

We want to work out the intersection $\beta \cap \gamma$, and then determine the angle between γ' and β' at the intersection point. Now the elliptic involution f preserves Ω and $\mathrm{Dome}(\Omega)$. Therefore the geodesic α, which is fixed under f, must be orthogonal to the dome at the unique point p of intersection of α and the dome. (Another way of viewing α is to take the geodesic from $(-1, 0, 0)$ to its image under the nearest point retraction. If we extend this to a bi-infinite geodesic, we obtain α.) Moreover f preserves both γ and β setwise, while reversing the direction of each, and so p is the intersection of α, β and γ.

We will now compute p. We can parameterize the points of the semicircle β in terms of $\lambda \in (-1, 1)$. An arbitrary point on β is given by

$$(\beta_+ + \beta_-)/2 + \lambda(\beta_+ - \beta_-)/2 + r\sqrt{1-\lambda^2}.(0,0,1)$$

where $r = |\beta_+ - \beta_-|/2$. We can solve for the intersection with the plane $y = 0$, finding

$$\lambda = \frac{1 - e^{s_0 c}}{1 + e^{s_0 c}}.$$

The intersection point is

$$p = \frac{1}{1 + e^{s_0 c}} \left(2e^{s_0 c/2} \cos(s_0/2), 0, \sqrt{1 + e^{2 s_0 c} - 2 e^{s_0 c} \cos(s_0)} \right)$$

To help locate p, note that, since $\pi < s_0 < 2\pi$, we have $\cos(s_0/2) < 0$.

The derivative at p of β with respect to λ is

$$\big(\cos(s_0/2)\sinh(s_0c/2), \quad \cosh(s_0c/2)\,\sin(s_0/2), \\ \sqrt{\frac{-\cos(s_0)+\cosh(s_0c)}{2}}\,\sinh(s_0c/2)\big)$$

We can also work out the derivative at p of γ with respect to the parameter t of the logarithmic spiral. The parameter t is some constant multiple of hyperbolic distance in the dome, since G acts on the dome by isometries. We find that the derivative of γ' at p is equal to

$$\frac{\Big(2\,c\,e^{s_0c/2}\,\cos(s_0/2), 2\,e^{s_0c/2}\,\cos(s_0/2), c\,\sqrt{1+e^{2s_0c}-2\,e^{s_0c}\,\cos(s_0)}\Big)}{1+e^{s_0c}}.$$

The hyperbolic length of this vector is equal to its euclidean norm divided by the third coordinate of p. We thus obtain the expression in the statement of Proposition 5.3 for the hyperbolic speed of γ.

We determine $\cos(\theta)$ from the inner products of the unit vectors tangent to γ and β at p, and obtained the expression claimed in the statement of Proposition 5.3. ■

6 The uniquely extremal quasiconformal map

Recall that $T = \mathbb{R}$ acts on Ω by $t.z = \exp(t(c+i))z$ for $z \in \Omega$ and $t \in \mathbb{R}$, as described in Equation 2.4.1. This transformation also acts on $\mathbb{C}$ and on $\mathbb{U}^3$ and there is therefore an action on $\mathrm{Dome}(\Omega)$. Recall from the end of Section 4 that multiplication by $-2c/(c^2+1) : \mathbb{R} \to \mathbb{R}$ makes $R : S \to \Omega$ $\mathbb{R}$-equivariant.

In Formula 2.5.2 we defined the composite $H = \log \circ \iota \circ h \circ R$. In order to study this composite, it is more convenient to replace h by $r : \Omega \to \mathrm{Dome}(\Omega)$, the nearest point retraction. The nearest point retraction has the same boundary values as h, and has the advantage that it is equivariant for the action of $\mathbb{R}$, the associated automorphism of $\mathbb{R}$ being the identity. We write $Q = \log \circ \iota \circ r \circ R$.

In order to describe ι more explicitly, and to determine the associated automorphism of $\mathbb{R}$ which makes ι equivariant, we need to examine bending lines in more detail. Let $b : \mathbb{R} \to \mathbb{U}^3$ be the bending line from $b(-\infty) = 1$

to $b(\infty) = \exp(s_0(c+i))$. Here s_0 is defined in Definition 5.2. Just before Formula 2.5.2, we normalized ι by requiring that the triple $(0, \infty, 1^+)$ of prime ends of Dome(Ω) be sent to the triple $(\infty, 0, 1)$ of prime ends of $\mathbb{U}^2$.

The oriented bending line meets the oriented main geodesic in the dome at an angle θ, where $\cos(\theta)$ is computed in Proposition 5.3. The image semicircle in $\mathbb{U}^2$ under ι of the bending line and the main geodesic is shown in Figure 8 for $c = .4$. The righthand endpoint is at 1 and the lefthand endpoint is at $(1-\cos(\theta))/(1+\cos(\theta))$.

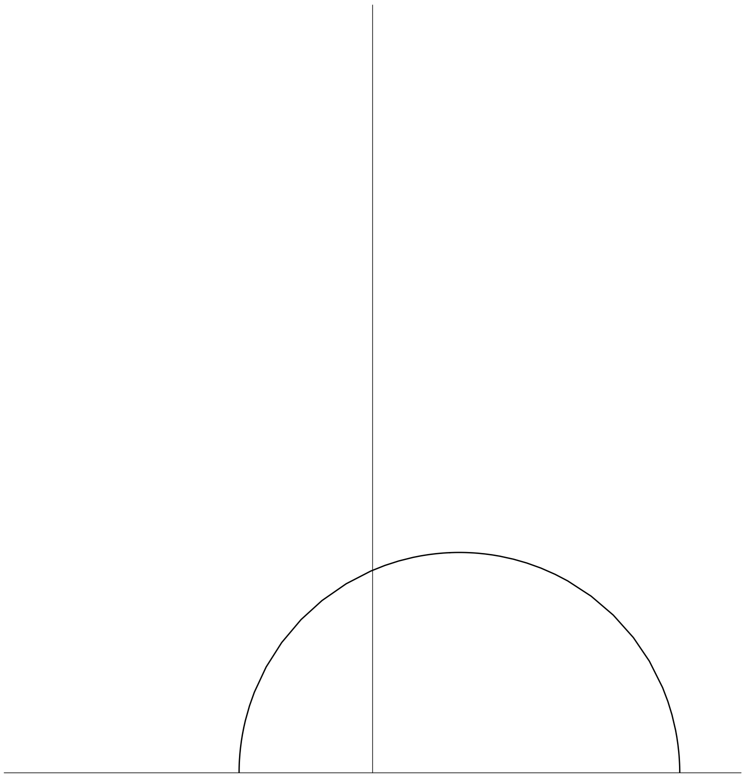

Figure 8. This shows the main geodesic, pointing downwards, and the image of the bending line b, pointing from right to left. This is the picture for $c = .4$.

The automorphism of $\mathbb{R}$ associated with ι can be read off from Proposition 5.3. It is multiplication by

$$- \|\gamma'(t)\| = -\sqrt{\frac{1 + c^2 + \cos(s_0) + c^2\cosh(s_0 c)}{\cosh(s_0 c) - \cos(s_0)}}.$$

Composing the homomorphisms we have already found, it follows that the composite map $Q : S \to S$ is $\mathbb{R}$-equivariant. The associated automorphism of $\mathbb{R}$ is multiplication by

$$a(c) = \frac{2c}{1+c^2}\sqrt{\frac{1+c^2+\cos(s_0)+c^2\cosh(s_0 c)}{\cosh(s_0 c)-\cos(s_0)}}$$

where s_0 is given by Definition 5.2. In particular, on the lower boundary of S, $H(x) = Q(x) = a(c)x$, since 0 is fixed because of the way we have chosen ι.

To see what happens on the upper boundary, we need to find the image of the endpoints of the bending line b in the domain and range copies of S.

In Ω, the endpoints are at 1^+ and at $\exp(s_0(c+i))^-$, where the significance of the superscript $\pm$ is explained in Definition 2.3. We have to apply log and then multiply by $(-c+i)/2c$. The first point goes to 0. In order to be on the correct branch of log, we need to add $-2\pi i$ to the exponent. The second point therefore goes to

$$b_1 = \frac{-c+i}{2c}(s_0(c+i) - 2\pi i) = \frac{-s_0 - c^2 s_0 + 2\pi}{2c} + \pi i.$$

The image of the ends of the bending line under ι are at 1 and at $-\frac{1-\cos(\theta)}{1+\cos(\theta)}$. The images under $\log : \mathbb{U}^2 \to S$ are 0 and $b_2 = \log(1-\cos(\theta)) - \log(1+\cos(\theta)) + i\pi$. We define the affine map $A : S \to S$ by $A(x,y) = (a(c)x + b(c)y, y)$ in such a way that $A(0,0) = (0,0)$ and $A(b_1,\pi) = (b_2,\pi)$. It follows that $b(c) = (-a(c)b_1 + b_2)/\pi$.

Then $H|\partial S = A|\partial S$. In view of Theorem 2.6, we need only compute the dilatation of A. The complex dilatation $k(c)$ and the dilatation $K(c)$ are given by

$$k(c) = \frac{a(c)-1+ib(c)}{a(c)+1-ib(c)} \quad \text{and} \quad K(c) = \frac{1+|k(c)|}{1-|k(c)|}.$$

The above expressions enables us to graph $K(c)$ against c, as in Figure 9, and we see that we get values greater than 2.

The maximum value of the dilation is obtained for $c \approx 1.08968$. We obtain $K(c) \approx 2.111$. For $c = 1$, we find $K(c) \approx 2.109$. Carrying out the computation using interval arithmetic, one can show that all these digits are correct.

7 Approximations

7.1 Question. Suppose we have two open simply-connected regions Ω_1 and Ω_2 which are "almost the same". Suppose we have a K_1-quasiconformal homeomorphism $\psi_1 : \Omega_1 \to \text{Dome}(\Omega_1)$ and which extends to the identity on the boundary. Is there a K_2-quasiconformal homeomorphism $\psi_2 : \Omega_2 \to \text{Dome}(\Omega_2)$ which extends to the identity on the boundary, with $K_2 \leq K_1 + \epsilon$, where $\epsilon > 0$ is small. $\square$

The objective of this section is to make this question precise and to answer it in the affirmative. This will enable us to construct counter-examples to the $K = 2$ conjecture with various desirable additional properties.

Let $\Omega \subset \mathbb{S}^2$ be an open simply-connected region such that the complement $X = \mathbb{S}^2 \backslash \Omega$ is non-empty and contains more than one point. Compact subsets of $\mathbb{S}^2$ are quite complicated, and X may display some unexpected properties, so our discussion needs to be careful.

7.2. The folded case. In a general discussion, we sometimes need a special discussion for the *folded case*. By this we mean that $X \subset \mathbb{S}^2$ is a closed circular arc. In this case, $\mathcal{CH}(X)$ is a hyperbolic halfplane, rather than a 3-dimensional object, and Theorem 1.1 needs to be more carefully stated.

If we are in the folded case and p does not lie on the unique bending line, then p has to be split into two distinct points, one corresponding to each

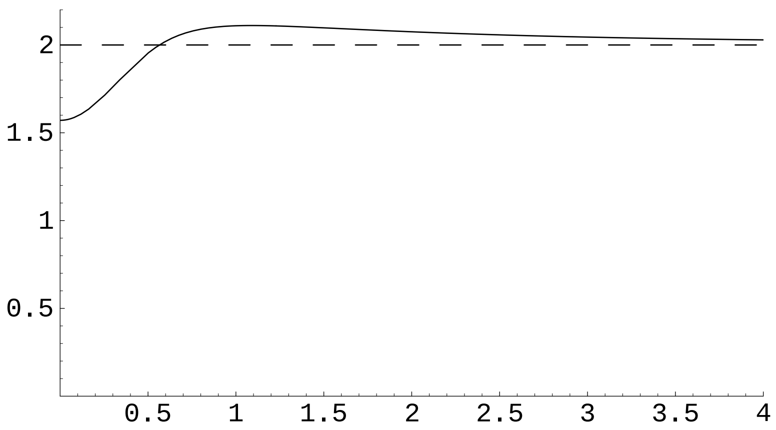

Figure 9. The graph of the dilatation $K(c)$ against c.

oriented support plane for $\mathcal{CH}(X)$ through p. With this modified definition of Dome(Ω), Theorem 1.1 once again becomes true.

7.3. The space of oriented planes. In view of § 7.2, it is sometimes useful to use the metric on the space of pairs (p, P), where $p \in \mathbb{H}^3$ and P is an oriented plane through p. This is the same as the unit tangent bundle of $\mathbb{H}^3$ with the tangent vector orthogonal to P. The unit tangent bundle, considered as a smooth manifold in its own right, has a standard Riemannian metric defined in terms of the Riemannian connection on $\mathbb{H}^3$. Let $U \subset \mathbb{H}^3$ be a small ball. Then topologically, the unit tangent bundle over U is $U \times \mathbb{S}^2$. As a Riemannian manifold it is very near to being a product. In practical terms, this means that the distance between (p_1, P_1) and (p_2, P_2) is determined more by the large angle at which P_1 and P_2 meet (if they do meet at a large angle) than by the small distance $d(p_1, p_2)$.

7.4. Notation. The basic framework of our discussion will be a sequence X_n of compact connected subsets of $\mathbb{S}^2$ with connected non-empty complement Ω_n. We assume that none of the X_n are singletons. Then each Ω_n is complex analytically isomorphic to the open unit disk. We will assume that $(X_n)_{n \in \mathbb{N}}$ converges to X_∞, where X_∞ is neither a point nor the whole of $\mathbb{S}^2$ and that $\Omega_\infty = \mathbb{S}^2 \setminus X_\infty$ is therefore complex analytically isomorphic to the open unit disk. We sometimes write Ω instead of Ω_∞ and X instead of X_∞. We denote the induced path metric on Dome(Ω_n) by d_n and the induced path metric on Dome$(\Omega)_\infty$ by d_∞. We denote the metric in $\mathbb{H}^3$ by d. Let $r_\infty : \Omega_\infty \to \text{Dome}(\Omega_\infty)$ and $r_n : \Omega_n \to \text{Dome}(\Omega_n)$ denote the nearest point retractions. We will use these notations throughout this section.

The main aim of this section is to prove the following result.

7.5 Theorem (The Approximation Theorem). *Let $K \geq 1$. Given any sequence*

$$\left(\phi_n : \Omega_n \to \text{Dome}(\Omega_n) \subset \overline{\mathbb{D}^3}\right)_{n \in \mathbb{N}}$$

of K-quasiconformal homeomorphisms which extend continuously to the identity on the common boundary, there is a subsequence, converging uniformly on compact subsets, to a K-quasiconformal $\phi_\infty : \Omega_\infty \to \text{Dome}(\Omega_\infty) \subset \overline{\mathbb{D}^3}$ which extends continuously to the identity on the common boundary.

7.6 Remark. As discussed in Remark 1.4, if X is an arc we further require that each ϕ_n $(n \in \mathbb{N} \cup \{\infty\})$ should preserve orientation. □

The following corollary to Theorem 7.5 is a useful reformulation. We use the same notation as in Theorem 7.5 and also the notation introduced in Definition 1.3.

7.7 Corollary. *We have* $K(\Omega_\infty) \leq \liminf_n K(\Omega_n)$

7.8. Normalizing the Riemann mappings. Without loss of generality, we may assume that, for each n, $0 \in \Omega_n$, and that $0 \in \Omega_\infty$. For each n, we have the Riemann mapping $R_n : \mathbb{D}^2 \to \Omega_n$. We normalize R_n by insisting that $R_n(0) = 0$ and that the derivative $R_n'(0)$ is real and positive. We normalize the Riemann mapping $R_\infty : \mathbb{D}^2 \to \Omega_\infty$ in the same way. It is then standard that $(R_n)_{n\in\mathbb{N}}$ converges to R_∞ uniformly on compact subsets. Also $(R_n^{-1})_{n\in\mathbb{N}}$ converges to R_∞^{-1} uniformly on compact subsets.

7.9. Normalizing the isometries. We also need to normalize the isometries $\iota_n : \mathrm{Dome}(\Omega_n) \to \mathbb{D}^2$ of Theorem 1.1 in such a way that the sequence $(\iota_n)_{n\in\mathbb{N}}$ converges in an appropriate sense to $\iota_\infty : \mathrm{Dome}(\Omega)_\infty \to \mathbb{D}^2$.

We do this as follows. Choose a basepoint b_∞ and another reference point c_∞ with $b_\infty, c_\infty \in \mathrm{Dome}(\Omega_\infty)$ and $d_\infty(b_\infty, c_\infty) > 4\pi$. For each pair (b_∞, P_∞), where P_∞ is an oriented support plane for $\mathcal{CH}(X_\infty)$ at b_∞, and each pair (b_n, P_n), where P_n is an oriented support plane for $\mathcal{CH}(X_n)$ at $b_n \in \mathrm{Dome}(\Omega_n)$, we have the distance $d\left((b_n, P_n), (b_\infty, P_\infty)\right)$ described in § 7.3. We select b_n, P_n and P_∞ such that this distance is minimized. This defines b_n. Similarly we define $c_n \in \mathrm{Dome}(\Omega_n)$. Since $d_\infty(b_\infty, c_\infty) > 4\pi$, we see that, for n large enough, $b_n \neq c_n$. We will assume that, for each n, $b_n \neq c_n$.

For $1 \leq n \leq \infty$ we define $\iota_n : \mathrm{Dome}(\Omega_n) \to \mathbb{D}^2$ to be the unique orientation preserving isometry (see Remark 1.4) that sends b_n to 0 and c_n to a point on the positive x-axis.

7.10. Choosing points. Whenever we have a point in $\mathrm{Dome}(\Omega_\infty)$ and we have to choose a nearby point in $\mathrm{Dome}(\Omega_n)$, we will use the technique of § 7.9, choosing at the same time a support plane at each of the two points, and minimizing the distance of § 7.3 between such pairs. We will proceed similarly if we are given a point in $\mathrm{Dome}(\Omega_n)$ and have to choose a nearby point in $\mathrm{Dome}(\Omega_\infty)$.

To see why we choose nearby points in the way just described, take $X_n = \{(x,y) : \ x \geq 0, \ |y - 1/n| \leq e^{-n}\}$. Then the sequence

$$\left((1, (1/n) - e^{-n}, 1) \in \mathrm{Dome}(\Omega_n)\right)_{n\in\mathbb{N}}$$

converges to $(1,0,1) \in \mathrm{Dome}(\Omega_\infty)$ and, if we are not paying attention to support planes, we are likely to take the wrong oriented support plane in the limit. Even if we thicken X_∞ a little, so there is no chance of getting the orientation wrong in the limit, we are still likely to come across situations where nearest points, defined without taking the oriented support plane into account, have counter-intuitive behaviour that one can track down to the corresponding oriented support planes pointing in unexpected directions.

The plan of the proof of Theorem 7.5 is to investigate the sequence $(\iota_n \circ \phi_n \circ R_n : \mathbb{D}^2 \to \mathbb{D}^2)_{n\in\mathbb{N}}$ of K-quasiconformal homeomorphisms. We will show that the desired map $\phi_\infty : \Omega_\infty \to \mathrm{Dome}(\Omega_\infty)$ with the desired properties arises from various limiting processes. There are a number of technical problems, of which the most severe is that we cannot take the limit of the sequence $(\phi_n : \Omega_n \to \mathrm{Dome}(\Omega_n))_{n\in\mathbb{N}}$ in a simple way because both domain and range keep on changing. In fact, as n varies, none of the image sets $\mathrm{Dome}(\Omega_n)$ need have any points in common at all.

The discussion will need an appropriate language and some basic results. Given a locally compact separable metric space X, we use the Chabauty topology on the set of closed subsets of X (see [8]). This is a compact metrizable topology having the following properties.

7.11 Lemma. *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of closed subsets of X and let $A \subset X$ be a closed subset of X. Then $(A_n)_{n\in\mathbb{N}}$ converges to A in the Chabauty topology if and only if:*
1. For each subsequence $(A_{n_k})_{k\in\mathbb{N}}$ and each choice of $x_k \in A_{n_k}$, if $(x_k)_{k\in\mathbb{N}}$ converges to $x \in X$, then $x \in A$.
2. If $x \in A$, then there is a sequence $(x_n \in A_n)_{n\in\mathbb{N}}$ which converges to x.

Chabauty convergence is equal to Hausdorff convergence if X is compact.

7.12. Maximal disks and support planes. If $X = \mathbb{S}^2 \setminus \Omega$, the concepts of a support plane P for the hyperbolic convex hull $\mathcal{CH}(X)$ and a maximal disk in Ω seem, at first glance, to be equivalent. But is this really correct?

It is immediate that every oriented support plane for $\mathcal{CH}(X)$ gives rise to a maximal open disk $D \subset \Omega$. Then $\partial D \cap X$ contains at least two points. Let U be a component of $\partial D \setminus X$. Then the endpoints of U are the endpoints of a bending line. Conversely, any bending line in $\mathrm{Dome}(\Omega)$ lies in some oriented support plane P corresponding to some maximal open disk D, and the two endpoints of the bending line are also endpoints of a component U of $\partial D \setminus X$.

(If $\partial D \cap X$ contains exactly two points, then there are exactly two choices for U. Otherwise there is only one choice for U.)

However, not every maximal disk corresponds to a support plane. For an example, take $\Omega = \{(x, y) : y > x^2\}$. Then the disk of radius $1/2$ which is tangent at $(0, 0)$ to $\partial\Omega$ is maximal, but the corresponding plane is not a support plane for $\mathcal{CH}(X)$—see Figure 10.

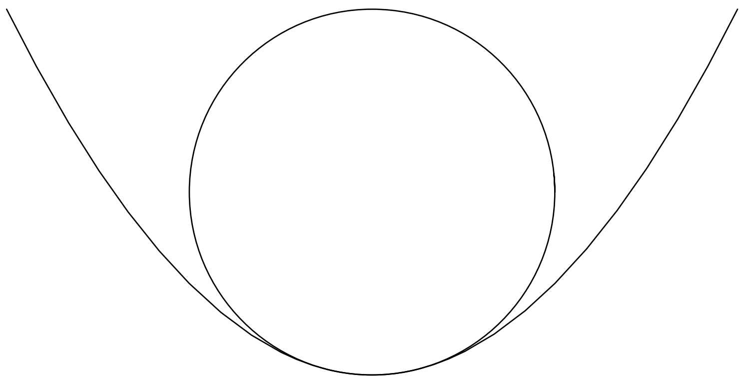

Figure 10. The parabola gives an example of a maximal circle which is not the boundary of a support plane for the convex hull.

If the boundary of a maximal disk meets X in at least two points, then it is the boundary of a support plane for the convex hull.

7.13 Lemma. *As a sequence of closed subsets of the closed unit ball,* $\overline{\mathcal{CH}(X_n)}$ *converges to* $\overline{\mathcal{CH}(X_\infty)}$. *As a sequence of subsets of the open unit ball,* $\mathrm{Dome}(\Omega_n)$ *converges to* $\mathrm{Dome}(\Omega_\infty)$.

Proof: To check the first condition of Lemma 7.11 for the convergence of the convex hulls, let x be the limit of a sequence $(x_k)_{k\in\mathbb{N}}$, with $x_k \in \mathcal{CH}(X_{n_k})$. Suppose, by contradiction, that $x \notin \overline{\mathcal{CH}(X)}$. Let H be a closed half 3-space containing $\overline{\mathcal{CH}(X)}$ but not x, and let H' be a slightly larger halfspace with the same properties. Then $x_k \in X_{n_k} \subset H'$ for k large. This is a contradiction.

To check the second condition of Lemma 7.11 for the convergence of the convex hulls, let $x \in \overline{\mathcal{CH}(X)}$. If $x \in \mathbb{S}^2$, then $x \in X$ and the result is immediate. Otherwise, one can find four points in $X \subset \mathbb{S}^2$, whose convex hull contains x. Nearby points in X_n will have a convex hull containing a point x_n near x. This completes the proof of the convergence statement about convex hulls.

Since $\operatorname{Dome}(\Omega_n)$ is the frontier of $\mathcal{CH}(X_n)$ in $\mathbb{D}^3$, the convergence of domes follows. ∎

It is standard that any K-quasiconformal homeomorphism from the open unit disk to itself extends continuously to the closed unit disk. We want to show that the sequence $(\iota_n \circ \phi_n \circ R_n : \mathbb{D}^2 \to \mathbb{D}^2)_{n\in\mathbb{N}}$ has a subsequence which converges uniformly on the closed unit disk (when extended to the boundary). By Mori's Theorem (see [1]), we only need to find a compact subset $C \subset \mathbb{D}^2$, such that, for each n, $\iota_n \phi_n R_n(0) \in C$.

We will depend on the fact that every K-quasiconformal homeomorphism is a $(K, \epsilon(K))$-quasi-isometry, where $\epsilon(K)$ depends only on K—see [10]. In particular, $\iota_n \circ \phi_n \circ R_n : \mathbb{D}^2 \to \mathbb{D}^2$ is a $(K, \epsilon(K))$-quasi-isometry. Since we know little about ϕ_n, we compare this composition with $\iota_n \circ r_n \circ R_n$, where $r_n : \Omega_n \to \operatorname{Dome}(\Omega_n)$ is the nearest point retraction. We have a much better understanding of the second composite, but how can this understanding be used?

We have the following lemma was implicit in [9].

7.14 Lemma. *Let $\Omega \subset \mathbb{S}^2$ be any open simply-connected region whose complement contains at least two points. Let $r : \Omega \to \operatorname{Dome}(\Omega)$ be the nearest point retraction. Then r is a (K_1, ϵ_1)-quasi-isometry, where $K_1 > 1$ and $\epsilon_1 > 0$ are universal constants, independent of Ω.*

Proof: Let $X = \mathbb{S}^2 \setminus \Omega$ and let ϵ satisfy $0 < \epsilon < \log 2$. We factor r through S, where S is the surface at distance ϵ from $\mathcal{CH}(X)$. We use the path metric d_S on S and the path metric d_∞ on $\operatorname{Dome}(\Omega)$. The nearest point map $\Omega \to S$ is bi-lipschitz, as shown in [9, Theorem 2.3.1]. To see that the nearest point map $r : S \to \operatorname{Dome}(\Omega)$ is a quasi-isometry, note that, for $x, y \in S$,

$$d_\infty(rx, ry) \le d_S(x,y) \le \cosh(\epsilon) d_\infty(x,y) + c_1 \sinh(\epsilon)(d_\infty(x,y) + 1). \quad (7.14.6)$$

Here the term containing $\cosh(\epsilon)$ corresponds to distances in S on the inverse images of flat pieces and the term containing $\sinh(\epsilon)$ corresponds to distances in S on the inverse image of bending lines. The universal constant c_1 is the maximum amount of bending in any unit length, and $c_1 \le 2\pi$, as shown by Bridgeman in [7]. ∎

We also have the following standard results about quasi-isometries from $\mathbb{D}^2$ to itself, where $\mathbb{D}^2$ has the hyperbolic metric. Recall that a quasi-isometry does not need to be continuous.

7.15 Lemma. *Every quasi-isometry extends to a continuous homeomorphism (in fact a quasisymmetric homeomorphism) of* $\mathbb{S}^1$*. For each pair* (K, ϵ)*, with* $K \geq 1$ *and* $\epsilon \geq 0$*, there is a constant* $L > 0$*, such that, for any* (K, ϵ)*-quasi-isometry* f*, and any hyperbolic geodesic* $\gamma \subset \mathbb{D}^2$*,* $f(\gamma)$ *lies within a hyperbolic Hausdorff distance* L *of the geodesic connecting its endpoints.*

We also have the following standard result.

7.16 Lemma. *Let* $K \geq 1$ *and let* $\epsilon \geq 0$*. Let* $f : \mathbb{D}^2 \to \mathbb{D}^2$ *be a* (K, ϵ)*-quasi-isometry. Then the cross-ratio of* $(f(1), f(i), f(-1), f(-i))$ *is bounded away from 0, 1 and* ∞*, with the bound depending only on* K *and* ϵ *and not on* f*. (This result can easily be improved to give the quasisymmetry result mentioned in Lemma 7.15.)*

Proof: By composing with a Möbius transformation, we may assume that $f(1) = 1$, $f(i) = i$ and $f(-1) = -1$. There are uniform bounds depending only on K and ϵ for the distances between any two geodesics joining any two points of $\{f(1), f(i), f(-1), f(-i)\}$. This proves the required results. ∎

7.17 Lemma. *For each pair* (K, ϵ)*, with* $K \geq 1$ *and* $\epsilon \geq 0$*, there is a constant* $L_1 > 0$*, such that, for any pair* $f_1, f_2 : \mathbb{D}^2 \to \mathbb{D}^2$ *of* (K, ϵ)*-quasi-isometries with the same boundary values, the hyperbolic distance* $d(f_1, f_2) \leq L_1$*. Conversely, if a pair of quasi-isometries are a bounded hyperbolic distance apart, then they have the same boundary values.*

Proof: Given a point $z \in \mathbb{D}^2$, choose two orthogonal geodesics α and β through z. Let α_1 and β_1 be the two geodesics connecting their endpoints under the f_i. By Lemma 7.15, $d(f_1(z), \alpha_1) \leq L$, and similarly if we replace α_1 by β_1 or f_1 by f_2. By Lemma 7.16, the angle between α_1 and β_1 is bounded away from 0 and π. The result follows. ∎

In particular, we have the following result.

7.18 Corollary. *Let* $K \geq 1$ *be given. There is a number* L_1 *depending only on* K*, with the following property. Let* $\Omega \subset \mathbb{S}^2$ *be any open simply-connected region, whose complement* X *contains at least two points, and let* $\phi : \Omega- >$ $\mathrm{Dome}(\Omega)$ *be a* K*-quasiconformal orientation preserving homeomorphism (see Remark 1.4) which extends to the identity on the boundary. Let* $R : \mathbb{D}^2 \to \Omega$ *be a Riemann mapping and* $\iota : \mathrm{Dome}(\Omega) \to \mathbb{D}^2$ *be an orientation preserving* *isometry. Let $r : \Omega \to \mathrm{Dome}(\Omega)$ be the nearest point retraction, and let d denote the hyperbolic metric in $\mathbb{H}^2$. Then, for all $z \in \mathbb{D}^2$, $d(\iota\phi R(z), \iota r R(z)) < L_1$.*

It follows from Corollary 7.18 that, as $n \in \mathbb{N}$ varies, $\iota_n \phi_n R_n(0)$ varies within a bounded distance of $\iota_n r_n R_n(0) = \iota_n r_n(0)$. In order to show that some subsequence of $(\iota_n \circ \phi_n \circ R_n)_{n\in\mathbb{N}}$ converges to a K-quasiconformal homeomorphism, we need to show that, as n varies, $\iota_n r_n(0)$ varies inside a compact subset of $\mathbb{D}^2$. Since ι_n is an isometry, and, by § 7.9, $\iota_n(b_n) = 0$, we need to show that $d_n(r_n(0), b_n)$ is bounded as n varies. We will prove this by showing that it converges to $d_\infty(r_\infty(0), b_\infty)$ as n tends to ∞. This is the task of the next lemmas.

The first of these lemmas is easy and we omit the proof.

7.19 Lemma. *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence with $x_n \in \mathrm{Dome}(\Omega_n)$ with limit $x_\infty \in \mathrm{Dome}(\Omega_\infty)$. For each n, let P_n be an oriented support plane for $\mathcal{CH}(X_n)$ at x_n. Suppose that $(P_n)_{n\in\mathbb{N}}$ converges to an oriented plane P_∞. Then P_∞ is an oriented support plane for $\mathcal{CH}(X_\infty)$ at x_∞.*

7.20 Lemma. *Let $C \subset \mathbb{H}^3$ be a fixed compact set. Given $n \in \mathbb{N}$ and $x \in C \cap \mathrm{Dome}(\Omega_n)$, let*

$$f_n(x) = \min_{P,y,Q} \{d\left((x,P),(y,Q)\right)\}$$

where $y \in \mathrm{Dome}(\Omega_\infty)$, P is an oriented support plane for $\mathcal{CH}(X_n)$ at x, Q is an oriented support plane for $\mathcal{CH}(X)$ at y, and d is the metric of § 7.3. If $C \cap \mathrm{Dome}(\Omega_n) = \emptyset$, we set $\epsilon_n = 0$. Otherwise we set $\epsilon_n = \sup\{f_n(x) : x \in C \cap \mathrm{Dome}(\Omega_n)\}$. Then $(\epsilon_n)_{n\in\mathbb{N}}$ converges to 0.

Proof: The proof is by contradiction. If it is false, we find $\epsilon > 0$ and a sequence $((x_i, P_i, y_i, Q_i))_{i\in\mathbb{N}}$ with the following properties. For each i, $x_i \in \mathrm{Dome}(\Omega_{n_i})$, where $(n_i)_{i\in\mathbb{N}}$ is a strictly increasing sequence of positive integers. P_i is an oriented support plane for $\mathcal{CH}(X_{n_i})$ containing x_i, $y_i \in \mathrm{Dome}(\Omega_\infty)$ and Q_i is an oriented support plane for $\mathcal{CH}(X_\infty)$ containing y_i. For each i, the quadruple (x_i, P_i, y_i, Q_i) realizes $f_{n_i}(x_i)$ and $f_{n_i}(x_i) \geq \epsilon$. The sequence $(x_i)_{i\in\mathbb{N}}$ converges to some point $y \in C$, $(P_i)_{i\in\mathbb{N}}$ converges to some oriented plane Q. Then $y \in \mathrm{Dome}(\Omega_\infty)$, by Lemma 7.13. By Lemma 7.19, Q is an oriented support plane for $\mathcal{CH}(X_\infty)$. But this means that $(d((x_i, P_i), (y, Q)))_{i\in\mathbb{N}}$ converges to 0, which contradicts our choice of P_i, y_i and Q_i. ∎

The next lemma has a statement which is almost the same as that of Lemma 7.20: however, the proof is different.

7.21 Lemma. *Let $C \subset \mathbb{H}^3$ be a fixed compact set which meets* $\mathrm{Dome}(\Omega_\infty)$*. Given $n \in \mathbb{N}$ and $y \in C \cap \mathrm{Dome}(\Omega_\infty)$, let*

$$f_n(y) = \min_{Q,x,P} \{d((x,P),(y,Q))\}$$

where $x \in \mathrm{Dome}(\Omega_n)$, P is an oriented support plane for $\mathcal{CH}(X_n)$ at x, Q is an oriented support plane for $\mathcal{CH}(X_\infty)$ at y, and d is the metric of § 7.3. We set $\epsilon_n = \sup\{f_n(y) : y \in C \cap \mathrm{Dome}(\Omega_\infty)\}$. Then $(\epsilon_n)_{n\in\mathbb{N}}$ converges to 0.

Proof: As in the proof of Lemma 7.20, we have a sequence with $d((x_i,P_i),(y_i,Q_i))$ minimal for fixed y_i, and greater than or equal to ϵ, where $x_i \in \mathrm{Dome}(\Omega_{n_i})$ and $y_i \in \mathrm{Dome}(\Omega_\infty)$. We may assume that $(y_i)_{i\in\mathbb{N}}$ converges to $y \in C \cap \mathrm{Dome}(\Omega_\infty)$ and that $(Q_i)_{i\in\mathbb{N}}$ converges to an oriented support plane Q at y.

Let S be the ϵ-distant surface from $\mathcal{CH}(X_\infty)$ and let $p \in S$ be the point such that the geodesic yp is orthogonal to Q. Let Q' be an oriented plane near to Q, orthogonal to yp, and very near y. For large n, all points of $\mathcal{CH}(X_n)$ are on the negative side of Q'. By Lemma 7.13, for n very large, there is a point $z_n \in \mathrm{Dome}(\Omega_n)$ which is extremely near to y. We fix a large i, and take $n = n_i$, with n_i as in the previous paragraph. Let S be the sphere with centre p passing through z_n (see Figure 11 for a suitable picture, except that the labels are different). Then the portion of $\mathbb{H}^3$ which is inside S and on the negative side of Q' has very small diameter. It contains the point of $z' \in \mathrm{Dome}(\Omega_n)$ which is nearest to p, and there is an oriented support plane P' for $\mathcal{CH}(X_n)$ through z' which is orthogonal to pz'. By taking Q' sufficiently near to Q and then i sufficiently large, we can make

$$d((z',P'),(y_i,Q_i)) \le d((z',P'),(y,Q)) + d((y,Q),(y_i,Q_i))$$

arbitrarily small. This is a contradiction, and the required result is proved. ∎

7.22 Lemma. *Let $C \subset \mathbb{H}^3$ be compact. For each $n \in \mathbb{N}$, let $x_n, y_n \in C \cap$* $\mathrm{Dome}(\Omega_n)$*. Using the method of § 7.10, we choose $u_n, v_n \in \mathrm{Dome}(\Omega)$ near x_n and y_n respectively. Then $d_n(x_n,y_n) - d_\infty(u_n,v_n)$ tends to 0 uniformly as n tends to infinity, independently of the choice of sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$. In particular, $d_n(x_n,y_n)$ is uniformly bounded.*

Proof: Let $\alpha(\epsilon) = \cosh(\epsilon) + c_1 \sinh(\epsilon)$ and $\beta(\epsilon) = c_1 \sinh(\epsilon)$ be the functions which appear in Inequality 7.14.6. We fix $\epsilon > 0$, and then choose n very large. The use of § 7.10 means that we have chosen specific nearby oriented support planes P_n at x_n and Q_n at u_n. Lemma 7.20 enables us to be sure that $d\left((x_n, P_n), (u_n, Q_n)\right)$ is small in a way that depends only on n and not on the particular sequences.

Let S be the ϵ distant surface from $\mathcal{CH}(X_n)$, and let $p_n \in S$ be the point where the orthogonal geodesic to P_n through x_n meets S. Then $d(p_n, x_n) = \epsilon$. Let $s_n = r_\infty(p_n) \in \mathrm{Dome}(\Omega)$.

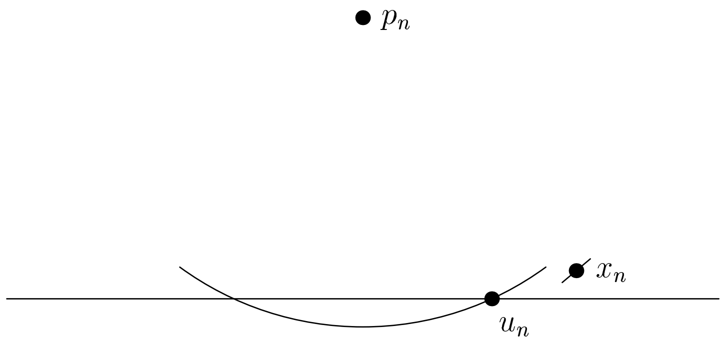

Figure 11. This illustrates the proof of Lemma 7.22.

The situation is illustrated in Figure 11. Since $s_n \in \mathrm{Dome}(\Omega)$ is the nearest point to p_n, s_n must lie inside the sphere S centred at p_n and passing through u_n. Since the orthogonal to the support plane at x_n passes through p_n, and the oriented support plane P at u_n is nearby, P must be nearly tangent to S. Also s_n must lie on the negative side of P. It follows that s_n must lie in the very small portion of $\mathbb{H}^3$ lying on the negative side of P and inside S, and that the support plane at s_n orthogonal to the interval $s_n p_n$ is near to P. We deduce that $d_\infty(s_n, u_n)/d(s_n, u_n)$ is uniformly near 1, so that $d_\infty(s_n, u_n)$ is uniformly small.

Let $q_n \in S$ and $t_n = r_\infty(q_n)$ be defined from v_n in a manner analogous to the definition of p_n and s_n, starting with u_n. For n large enough, we have

$$d_\infty(u_n, v_n) - \epsilon < d_\infty(s_n, t_n) \leq d_S(p_n, q_n) \leq \alpha(\epsilon) d_n(x_n, y_n) + \beta(\epsilon).$$

In the same way, this time taking S to be the ϵ-distant surface from $\mathcal{CH}(X_\infty)$, we can show that $d_n(x_n, y_n) < \alpha(\epsilon) d_\infty(x_\infty, y_\infty) + \beta(\epsilon) + \epsilon$.

Here ϵ is arbitrarily small, and there is an n_0 depending on ϵ and on C, but not on the particular sequences of points, such that the inequalities above are true for $n \geq n_0$. This completes the proof. ∎

The proof of the next lemma is almost identical with the proof of Lemma 7.22 and we leave it to the reader.

7.23 Lemma. *Let $C \subset \mathbb{H}^3$ be compact. For each $n \in \mathbb{N}$, let $x_n, y_n \in C \cap \mathrm{Dome}(\Omega)$. Using the method of § 7.10, we choose $u_n, v_n \in \mathrm{Dome}(\Omega_n)$ near x_n and y_n respectively. Then $d_\infty(x_n, y_n) - d_n(u_n, v_n)$ tends to 0 uniformly as n tends to infinity, independently of the choice of sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$.*

There is yet another version of this lemma, proved by the same methods.

7.24 Lemma. *Let $C \subset \mathbb{H}^3$ be compact. For each $n \in \mathbb{N}$, let $x_n \in C \cap \mathrm{Dome}(\Omega)$ and $y_n \in C \cap \mathrm{Dome}(\Omega_n)$. Using the method of § 7.10, we choose $u_n \in \mathrm{Dome}(\Omega_n)$ near x_n and $v_n \in \mathrm{Dome}(\Omega)$ near y_n. Then $d_\infty(x_n, v_n) - d_n(u_n, y_n)$ tends to 0 uniformly as n tends to infinity, independently of the choice of sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$.*

We are now in a position to understand the isometries ι_n much better. Recall the definitions given in § 7.9. We see from Lemma 7.23 that $(d_n(b_n, c_n))_{n\in\mathbb{N}}$ converges to $d_\infty(b_\infty, c_\infty)$.

7.25 Lemma. *$d_n(r_n(0), b_n)$ is bounded as n varies.*

Proof: Proving the lemma by contradiction, we may assume that $(d_n(r_n(0), b_n))_{n\in\mathbb{N}}$ tends to ∞ as n tends to ∞. Now the sequence $(r_n(0))_{n\in\mathbb{N}}$ can easily be seen to converge to $r_\infty(0)$ and also $(b_n)_{n\in\mathbb{N}}$ converges to b_∞. This means that the various sequences are all contained in a compact set. This contradicts the last sentence of the statement of Lemma 7.22 as required. ∎

This completes the proof that, after going to a subsequence, we may assume that $(\iota_n \circ \phi_n \circ R_n)_{n\in\mathbb{N}}$ converges to a K-quasiconformal homeomorphism, $\zeta : \mathbb{D}^2 \to \mathbb{D}^2$. In fact, by Mori's Theorem, we may assume that all the homeomorphisms are extended to the closed unit disk, and the convergence is then uniform on the closed unit disk with respect to the euclidean metric. We define $\phi_\infty : \Omega_\infty \to \mathrm{Dome}(\Omega_\infty)$ by $\phi_\infty = \iota_\infty^{-1} \circ \zeta \circ R_\infty^{-1}$. Then ϕ_∞ is clearly orientation-preserving and K-quasiconformal. In order to complete the proof of Theorem 7.5, we still need to show that ϕ_∞ can be continuously extended to the identity on $\partial\Omega_\infty$—this is the only outstanding issue.

For the next lemma we drop the ∞ suffixes.

7.26 Lemma. *The map ϕ extends continuously to the identity on $\partial\Omega$.*

Proof: We denote by d_e the euclidean metric in the closed unit ball. The nearest point retractions can be extended to continuous maps $r : \overline{\mathbb{D}^3} \to \mathcal{CH}(X)$ and $r_n : \overline{\mathbb{D}^3} \to \mathcal{CH}(X_n)$ by defining them to equal the identity on their images. The sequence $(r_n)_{n\in\mathbb{N}}$ converges uniformly to r as maps into $\overline{\mathbb{D}^3}$ with respect to the euclidean metric on the closed unit 3-ball.

Let $x \in \Omega$ be near $\partial\Omega$. For n large enough, $x_n = R_n R^{-1}(x)$ is extremely near x. Now $y_n = \phi_n(x_n)$ is at a d_n-bounded distance from $r_n(x_n)$, according to Lemma 7.17. This implies that $d(y_n, r_n(x_n))$ is bounded, where d is the hyperbolic distance in $\mathbb{H}^3$. Now $d_e(r_n(x_n), x_n)$ is small, since x_n is near $\partial\Omega_n$—the geometry of the situation ensures that this distance is comparable with $d_e(x_n, \partial\Omega_n)$. Bounded hyperbolic distances become very small near $\mathbb{S}^2$. It follows that $d_e(x, y_n)$ is small.

By Lemma 7.23, $d_n(b_n, c_n) - d_\infty(b_\infty, c_\infty)$ tends to zero. Since ι_n and ι are isometries, it follows that $d(0, \iota_n c_n) - d(0, \iota c_\infty)$ tends to zero. Since both $\iota_n c_n$ and ιc_∞ have been chosen on the positive x-axis in $\mathbb{D}^2$, it follows that $(\iota_n c_n)_{n\in\mathbb{N}}$ converges to ιc_∞.

Let $z_n \in \mathrm{Dome}(\Omega)$ be defined by $\iota(z_n) = \iota_n(y_n)$. Let $u_n \in \mathrm{Dome}(\Omega)$ be chosen as near as possible to y_n, using the method of § 7.10. According to Lemma 7.24, $d_n(b_n, y_n) - d_\infty(b_\infty, u_n)$ and $d_n(c_n, y_n) - d_\infty(c_\infty, u_n)$ each tend to zero as n tends to infinity. It follows that $d(0, \iota_n y_n) - d(0, \iota u_n)$ and $d(\iota_n c_n, \iota_n y_n) - d(\iota c_\infty, \iota u_n)$ both tend to zero. Therefore $d(\iota c_\infty, \iota_n y_n) - d(\iota c_\infty, \iota u_n)$ tends to zero. From this we deduce that $d(\iota_n y_n, \iota_n u_n)$ tends to zero. (Normally knowing the (approximate) distances of a point p from two distinct points 0 and ιc_∞ in the hyperbolic plane is not sufficient to determine the approximate position of p. In our case, since ι_n and ι preserve orientation, the information is sufficient.) It follows that $d_\infty(z_n, u_n)$ tends to zero, and so $d_e(y_n, z_n)$ tends to zero. Therefore $d_e(x, z_n)$ is small.

Now

$$\iota^{-1}\iota_n\phi_n R_n R^{-1}(x) = \iota^{-1}\iota_n\phi_n(x_n) = \iota^{-1}\iota_n(y_n) = z_n.$$

It follows that $d_e(\iota^{-1}\iota_n\phi_n R_n R^{-1}(x), x)$ is small for large n. Letting n tend to infinity, we find that $d_e(\phi(x), x)$ is small. If we check carefully through this computation, we see that all the estimates are uniform in x, depending only on its euclidean distance from $\partial\Omega$. Therefore ϕ extends continuously to the identity on $\partial\Omega$. ∎

8 Counter-examples

8.1 Theorem. *There is an example of a Jordan domain U whose boundary is a non-singular real analytic curve, and for which the best quasiconformal constant is greater than* 2.1 *for any homeomorphism $U \to \mathrm{Dome}(U)$ which extends to the identity on the boundary.*

Proof: Let $R : \mathbb{D}^2 \to \Omega$ be the Riemann mapping of the complement Ω of a logarithmic spiral for which $K(\Omega) > 2.1$. We define Ω_n to be the image under R of the open disk with radius $1-1/n$. By Theorem 7.5, $2.1 < \liminf K(\Omega_n)$. It follows that for large enough n, $K(\Omega_n) > 2.1$. ∎

Corollary 7.7 also gives the following counter-example.

8.2 Theorem. *There is an example of a Jordan domain U whose boundary is a piecewise linear curve, and for which $K(U) > 2.1$.*

We will now establish a group invariant form of the counter-example.

8.3 Theorem. *There is an example of a quasifuchsian group corresponding to a compact surface, for which one of the domains of discontinuity Ω has $K_{eq}(\Omega) \geq K(\Omega) > 2.1$.*

8.4. Comparison with previous example In [10], we gave an example of a quasifuchsian group Γ_1, with a domain of discontinuity Ω_1, such that Ω_1/Γ_1 is a punctured torus. We denote by Γ_2 the quasifuchsian group of Theorem 8.3 and by Ω_2 the relevant domain of discontinuity. Here is a list of comparisons of these two examples.

8.4.1. Ω_1/Γ_1 is a non-compact surface of finite area, in fact a punctured torus. Ω_2/Γ_2 is a compact surface, but we don't know its genus.

8.4.2. Γ_1 is defined explicitly and Ω_1 reasonably explicitly. On the other hand Γ_2 is unknown. We know that Ω_2 approximates the complement of a logarithmic spiral, but we don't know anything about the local properties of the approximation.

8.4.3. We know only that $K_{eq}(\Omega_1) > 2$. We have the more precise information $K_{eq}(\Omega_2) \geq K(\Omega_2) > 2.1$. This is a stronger result both because we have replaced 2 by 2.1 and because we are using all quasiconformal homeomorphisms from Ω_2 to $\mathrm{Dome}(\Omega_2)$ and not only equivariant quasiconformal homeomorphisms.

Proof (of Theorem 8.3): We start with the output from Theorem 8.1, namely a Jordan domain U with a non-singular analytic boundary, such that the best quasiconformal constant is greater than 2.1. In fact, we take U to be one of the domains Ω_n from the proof of Theorem 8.1. We have a Riemann mapping $R : D^2 \to U$ which can be extended to a non-singular analytic map of the closed disk. We regard the closed unit disk as a standard subspace of $\mathbb{S}^2$, and extend $R|\mathbb{D}^2$ to a C^∞-diffeomorphism $R : \mathbb{S}^2 \to \mathbb{S}^2$. R^{-1} has a Beltrami differential μ which is equal to zero on U. Also $|\mu|$ is bounded away from 1.

Let E be the interior of $\mathbb{S}^2 \setminus \mathbb{D}^2$. This is an open disk centred at ∞. We choose a large regular convex hyperbolic polygon $P_n \subset E$, centred at $\infty \in \mathbb{S}^2$ in such a way that this is the fundamental domain for a cocompact fuchsian group Γ_n acting on $\mathbb{S}^2$ and preserving $\mathbb{D}^2$ and E.

We define a Beltrami differential μ_n by setting it equal to 0 on $\mathbb{D}^2$ and to μ on P_n. Elsewhere we define μ_n by insisting that μ_n should be equivariant for the action of Γ_n. Since each $\gamma \in \Gamma_n$ is complex analytic, the transform of μ under γ has the same L_∞-norm as μ. It follows that $|\mu_n|$ has an essential supremum which independent of n and strictly smaller than 1. Let $R_n : \mathbb{S}^2 \to \mathbb{S}^2$ be the solution to the Beltrami equation for μ_n, normalized so that $R_n(z) = R(z)$ for $z = 0$, 1 or ∞.

Now $(\mu_n)_{n\in\mathbb{N}}$ converges pointwise to μ. It follows from [13, Theorem 5.2, page 187] that $(R_n)_{n\in\mathbb{N}}$ converges uniformly to R. This enables us to apply Theorem 7.5. We deduce that, for some n, the region $R_n(\mathbb{D}^2)$ has best quasiconformal constant greater than 2.1. Since μ_n is invariant under Γ_n, R_n conjugates Γ_n to a group of Möbius transformations. This conjugated group is a quasifuchsian group associated to a compact surface. One of the regions of discontinuity is $R_n(\mathbb{D}^2)$. ■

8.5. Boundary dilatation. Given a quasiconformal map f of Ω and a point $p \in \partial\Omega$, the notion of *substantial boundary dilatation* $H_p(f)$ of f at p is defined as follows. Denote the disk of radius r about p by $B_r(p)$. Suppose we have a quasiconformal map g of Ω with the same boundary values as f (write $g \sim f$). We recall that we are not talking here of the boundary of

Ω, but of boundary values after composing with conformal maps such that domain and range become equal to $\mathbb{D}^2$. We denote by $K_{p,r}(g)$ the maximal dilatation of g restricted to $B_r(p) \cap \Omega$. We make the following sequence of definitions:

$$\tilde{H}_p(g) = \lim_{r \searrow 0} K_{p,r}(g), \quad H_p(f) = \inf_{g \sim f} \tilde{H}_p(g),$$
$$H(f) = \max_{p \in \partial\Omega} H_p(f).$$

The number $H(f)$ is called the *boundary dilatation* of f. It is known [11, 12] that the maximum is attained.

Automatically $H(f) \leq K_0$ where K_0 is the maximal dilatation of an extremal quasiconformal map with the boundary values of f. Typically, there is strict inequality. If $\partial\Omega$ is smooth and f has C^1-boundary values, then $H = 1$. See [14] for an exposition of this topic.

In the case of the logarithmic spiral, the boundary dilatation is 1 at each point of the spiral. At either endpoint, the boundary dilatation is equal to $K(\Omega_c)$. This can be proved by the same method as in the proof of Lemma 9.1, by expanding about one of the two singularities. In Lemma 9.1 one expands by a linear change of scale. Here one expands by multiplication by $e^{t(c+i)}$ where t gets larger and larger. It follows that one gains nothing, in the case of the logarithmic spiral, by examining the finer invariant boundary dilatation rather than the usual dilatation.

9 Limiting values for quasiconformal constants

In this section we discuss the limiting values of $K(\Omega_c)$ as c tends to 0 or to ∞. Visual inspection of Figure 9 appears to give limiting values of $\pi/2$ and 2 respectively. We will see how these apparent results fit with Theorem 7.5.

In a discussion which is independent of the limiting values of $K(\Omega_c)$, we will show how local features of Ω can give estimates from below of $K(\Omega)$.

We first look at the logarithmic spiral as c tends to ∞. We normalize by sending 1 to 0, $\exp(2\pi c)$ to 1 and ∞ to ∞. Under this normalization, the logarithmic spiral L_c converges to the positive x-axis. An easy calculation shows that, for the split plane Ω_∞, we have $K(\Omega_\infty) = 2$. By Corollary 7.7, $2 \leq \liminf_{c \to \infty} K(\Omega_c)$. To see how this fits in with the previous calculations, note that s_0 converges to π as c tends to infinity. Here s_0 is defined in Definition 5.2. We can work out the limits explicitly from the explicit formulas in Section 5 and Section 6, finding that $K(\Omega_c)$ tends to 2.

There are presumably conditions which would enable one to deduce that the sequence $(K(\Omega_n))_{n\in\mathbb{N}}$ converges to $K(\Omega_\infty)$. For example, in the case of the limit of Ω_c as c tends to infinity, the limiting behaviour of the regions is good enough. In general, we can have $K(\Omega_\infty) < \liminf K(\Omega_n)$. To see this, we first prove a lemma.

9.1 Lemma. *Suppose that there is a point $p \in \partial\Omega$ and a neighbourhood U of p, such that $(U, U \cap \partial\Omega, p)$ is equivalent by a Möbius transformation to $(V, V \cap A, 0)$, where V is a disk neighbourhood of 0 in $\mathbb{C}$ and A is the positive x-axis. Then $K(\Omega) \geq 2$.*

Proof: In this proof Ω and Ω_∞ do not denote the same open subset of $\mathbb{C}$. We may assume, without loss of generality, that $U = V$, $U \cap \partial\Omega = V \cap A$ and $p = 0$. Using a change of scale, we define Ω_n to be the image of Ω under multiplication by n. Then $\Omega_\infty = \lim \Omega_n$ is the slit plane and so $K(\Omega_\infty) = 2$. (The proof of this last fact proceeds by converting both the region and its dome to infinite strips, and then applying Theorem 3.1.) For each n we have $K(\Omega_n) = K(\Omega)$. By Corollary 7.7, $2 = K(\Omega_\infty) \leq \liminf K(\Omega_n) = K(\Omega)$. ∎

Now let Ω_n be a open unit disk from which the interval $(1 - 1/n, 1)$ has been removed. Then $K(\Omega_n) \geq 2$ by Lemma 9.1. But the limit Ω_∞ is the open unit disk with $K(\Omega_\infty) = 1$.

Theorem 7.5 can be generalized, though we will not bother with the details of the proof. We give the generalization in the same form as we gave Definition 1.3. Suppose $(X_n)_{n\in\mathbb{N}}$ is a sequence of compact, connected subsets of $\mathbb{S}^2$, such that, for each n, X_n has at least two points and $\Omega_n = \mathbb{S}^2 \setminus X_n$ is connected. Then Ω_n is simply connected. Suppose $(X_n)_{n\in\mathbb{N}}$ tends to a limit X_∞, which is not a singleton and not the whole 2-sphere. Then X_∞ is connected, but its complement may consist of a countable union of components, each of which is simply connected.

9.2 Theorem. *Let Ω be a component of $\mathbb{S}^2 \setminus X_\infty$. Then $K(\Omega) \leq \liminf K(\Omega_n)$*

Proof: The proof is much the same as the proof of Theorem 7.5, but we need to be careful to ensure that $0 \in \Omega$ and that b_∞ and c_∞ lie in $\mathrm{Dome}(\Omega)$. The Riemann maps $R_n : \mathbb{D}^2 \to \Omega_n$ converge to the Riemann map $R : \mathbb{D}^2 \to \Omega$, and the isometries ι_n^{-1} converge to ι^{-1}. We omit the details. ∎

Theorem 9.2 is relevant to the limit of the complement Ω_c of the logarithmic spiral as c tends to 0. We normalize so that 1 goes to 0, $\exp(2\pi c)$

goes to 1, and $\exp(\pi c)$ goes to ∞. Let X_c be the normalized version of the logarithmic spiral L_c. Then X_c converges to the countable union of all the vertical lines $x = m$, $(m \in \mathbb{Z})$ in $\mathbb{C}$. If we fix attention on any vertical strip Ω in the complement, we have $K(\Omega) = \pi/2$, as can be found from explicit calculation, using Theorem 3.1. Once again we can work out the limits explicitly from the explicit formulas in Section 5 and Section 6, finding that $K(\Omega_c)$ tends to $\pi/2$. This fits with Theorem 9.2.

10 Long range bending

Recall from Figure 7 that γ is the geodesic from 0 to ∞ in the path metric of the dome $\mathrm{Dome}(\Omega_c)$ for Ω_c, the complement of the logarithmic spiral L_c. In this section we work out the long range bending, that is the amount of bending per unit hyperbolic length along the geodesic γ. We show that, for $c = 0.485897$, we have bending approximately $1.53922 \approx 0.979896\pi/2$ per unit hyperbolic length along γ. Previously the best result known was 1, which is the long range bending of a strip in $\mathbb{C}$.

Let M_t be the maximal circle in Ω_c which is tangent on the positive side (see Definition 2.3) of the logarithmic spiral L_c at $\exp(t(c+i))$. It is also tangent at $\exp((s_0+t)(c+i))$ on the negative side. Here s_0 is defined in Definition 5.2. Let r_t be the radius of M_t and σ_t its centre. By Equation 5.2.5,

$$r_t = e^{tc}\sqrt{1+c^2}\left(-1+e^{s_0 c}\right)/2.$$

From this, we easily deduce that

$$\sigma_t = e^{t(c+i)}\left(1 + \frac{1-ic}{2}\left(e^{s_0 c}-1\right)\right).$$

Let β_t be the bending line with endpoints at $e^{t(c+i)}$ and $e^{(t+s_0)(c+i)}$. This is the bending line contained in the hyperbolic plane with boundary M_t. For small $t > 0$, the bending between the bending lines β_0 and β_t is approximately equal to the angle $\theta(t)$ between the circles M_0 and M_t. We can calculate $\theta(t)$ from the usual euclidean cosine rule:

$$\cos(\theta(t)) = \frac{r_0^2 + r_t^2 - |\sigma_t - \sigma_0|^2}{2r_0 r_t}.$$

Substituting the values we know for r_t, σ_t, r_0 and σ_0 we find that

$$\cos(\theta(t)) = \cos(t) + \frac{2(\cos(t) - \cosh(ct))}{(1+c^2)(\cosh(s_0 c) - 1)}.$$

Expanding in powers of t, we find

$$\cos(\theta(t)) = 1 - \frac{\coth(s_0c/2)^2t^2}{2} + O(t^3).$$

Therefore $\theta(t) = \coth(s_0c/2)t + O(t^2)$.

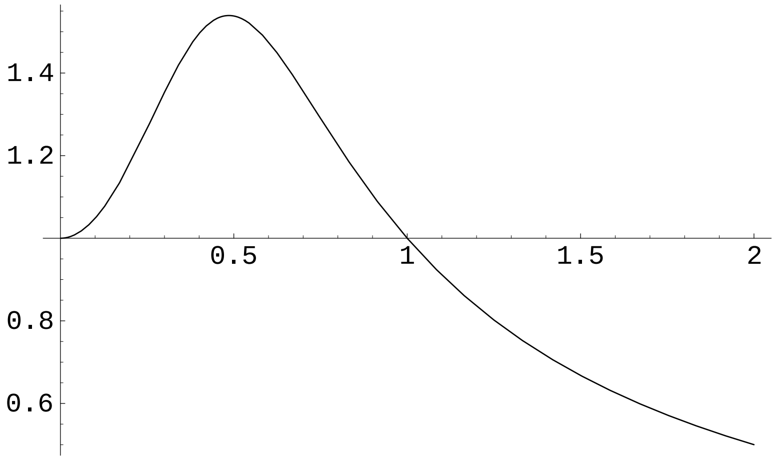

Figure 12. This is a graph of long range bending per unit hyperbolic length, against the parameter c. One observes that there is a maximum near $c = 0.5$.

The long range bending is equal to $\theta'(0)/\gamma'(0)$, which we can compute using Proposition 5.3, to show that the long range bending per unit length is equal to

$$\coth(s_0c/2)\sqrt{\frac{\cosh(s_0c) - \cos(s_0)}{1 + c^2 + \cos(s_0) + c^2\cosh(s_0c)}}$$

In Figure 12 we see a graph of the long range bending per unit length against the parameter c. Using a bisection method, one sees that the maximum actually occurs at $c \approx 0.485897$, where the long range bending is approximately 1.53922. This is just less than $\pi/2$. We record this as a theorem.

10.1 Theorem. *For $c = 0.485897$, the long range bending along a certain geodesic in* $\mathrm{Dome}(\Omega_c)$ *is approximately* 1.53922.

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