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Complex angle scaling

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Abstract

We introduce the method of complex angle scaling to study deformations of hyperbolic structure. We show how the method leads to a construction of counterexamples to the equivariant $K = 2$ conjecture for hyperbolic convex hulls.

1. Introduction

Our purpose is to give an exposition of the theory of *complex angle scaling*[1]. Complex angle scaling mappings are quasiconformal homeomorphisms or locally injective quasiregular developing mappings of $\mathbb{H}^2$ into or onto $\mathbb{S}^2$. In many cases, complex angle scalings are *continous* analogues of the *complex earthquakes* described by McMullen [McM98]. The mappings are also closely related to "quakebends" introduced in [EM87]; these are mappings $\mathbb{H}^2 \to \mathbb{H}^3$.

In the form described here our theory applies to finite or infinite discrete measured laminations in hyperbolic 2-space $\mathbb{H}^2$. For certain parameter values they can be defined to be quasiconformal on all $\mathbb{S}^2$ and they even have a natural extension to quasiconformal mappings of $\mathbb{H}^3$.

In contrast, McMullen's complex earthquakes are defined on a finite area hyperbolic surface S as maps $\mathscr{ML}(S) \to \mathbf{Teich}(S)$. However for finite laminations on S, namely those which lift to infinite discrete laminations in $\mathbb{H}^2$, both give exactly the same boundary values on $\partial\mathbb{H}^2$ and therefore determine the same deformation of a fuchsian group covering S. Complex angle scalings do not naturally extend to the lifts of general measured laminations on S.

We will apply our theory to prove that Thurston's equivariant $K = 2$ conjecture does not hold. We have proven this in [EMM01]; the proof presented here is expressed in terms of angle scaling, which is how we discovered it. The angle scaling method also yields a new and perhaps more natural proof of Sullivan's Theorem 5.1 that the Teichmüller distance between the conformal structure on a simply connected region

[1]We want to thank the referee for helpful comments.

and on the convex hull boundary that faces it differs by a universally bounded amount. This requires an elaboration of the theory presented here.

For the full details of the work discussed, we refer to [EMM01] and the work in progress [EMM02].

2. Complex angle scaling

2.1. The mappings on a wedge

We begin with the wedge

$$W = \{z \in \mathbb{C} : 0 < \arg z < \alpha < \pi\}.$$

The *complex angle scaling map* on W is defined in terms of the complex parameter $t = u + iv \in \mathbb{C}$ as

$$f_+(z) = e^{t\theta} z = re^{u\theta} e^{i(1+v)\theta}, \quad \text{for } z = re^{i\theta} \in W.$$

The boundary values of f_+ are $f_+(\zeta) = \zeta$, if $\zeta \geq 0$, and $f_+(\zeta) = e^{t\alpha}\zeta$, if $\arg \zeta = \alpha$.

In parallel we define

$$f_-(z) = e^{-t\theta} z = re^{-u\theta} e^{i(1-v)\theta}, \quad \text{for } z \in W.$$

We list the properties of these mappings:

- For all $-1 < \mathrm{Im}(t) \leq +1$, the map f_+ is a K_t-quasiconformal homeomorphism[2] and for $-1 < \mathrm{Im}(t)$ is K_t-quasiregular[3] where

$$K_t = \frac{1 + |\kappa(t)|}{1 - |\kappa(t)|}, \quad \kappa(t) = \frac{-t}{2i + t} \cdot \frac{z}{\bar{z}}. \tag{2.1}$$

- For all $-1 \leq \mathrm{Im}(t) = v < +1$, the map f_- is K_t-quasiconformal and for $\mathrm{Im}(t) < +1$ is K_t-quasiregular where

$$K_t = \frac{1 + |\kappa(t)|}{1 - |\kappa(t)|}, \quad \kappa(t) = \frac{t}{2i - t}. \tag{2.2}$$

[2] A K-quasiconformal mapping is an orientation preserving homeomorphism whose maximal dilatation (or distortation) is $\leq K$.

[3] A K-quasiregular mapping has the form $G \circ F$ where F is K-quasiconformal and G is holomorphic on the range of F; it is locally injective if and only if G is.

- The boundary values of f_+, f_- on each edge of the wedge W is the restriction of a Möbius transformation.
- f_+ and f_- commute with any Möbius transformation that maps W onto itself (that is, any hyperbolic transformation with fixed points at the vertices 0 and ∞).

For larger angles, we must think of the image of W as spread over $\mathbb{S}^2$.

Important special cases are:

(i) $\mathrm{Re}t = u = 0$. Then $f_+(z) = e^{iv}z = re^{i(1+v)\theta}$. This is a smoothed[4] version of *grafting*. The wedge W of angle α is mapped onto a wedge of angle $(1+v)\alpha$.

(ii) $t = i$; $v = 1$. f_+ is called the *angle doubling map* whereas f_- is called the *angle collapsing map*.

(iii) $\mathrm{Im}(t) = v = 0$. Then $f_+(z) = e^{u\theta}z$. This is a smoothed version of an earthquake. The wedge W is mapped onto itself but it is stretched (or contracted if $u < 0$) continuously from its right edge to its left.

Among all quasiconformal mappings of W, f_+, f_- are extremal for their boundary values [Str76]. When $t = i$, $f_+(z)$ is 2-quasiconformal in W.

2.2. Extension of the action to space

Here we will work in the upper halfspace model of $\mathbb{H}^3$ which we will parameterize over the complex plane as

$$\mathbb{H}^3 = \{(z,s) : z \in \mathbb{C},\ s > 0\}.$$

Let $W^* = \{(z,s) : z \in W,\ s > 0\} \subset \mathbb{H}^3$ denote the wedge over W. The mappings f_+, f_- have natural continuous extensions respectively as

$$f_+ : (z,s) \mapsto (ze^{t\theta}, s|e^{t\theta}|), \quad f_- : (z,s) \mapsto (ze^{-t\theta}, s|e^{-t\theta}|).$$

Their boundary values on the faces of W^* are the extensions to space of the Möbius transformations that preserve the corresponding edges of W. The extensions to space of f_+, f_- also commute with the extensions to W^* of the Möbius transformations that preserve W. They are 3-dimensional quasiconformal when they are homeomorphisms.

[4]Here smoothed means C^0, not C^∞.

2.3. Action on a pair of symmetric wedges

Consider now the situation that we have the symmetric wedges in $\mathbb{R}$,

$$W_+ = \{z : 0 \le \beta < \arg z < \alpha + \beta \langle \pi \rangle\}; \quad W_- = \{z : -\alpha - \beta < \arg z < -\beta \le 0\}.$$

Define f_+ in W_+ by conjugating W_+ to angle scaling on our standard W above by the rotation $z \mapsto e^{-i\beta} z$.

Let $\overline{W}$ denote the reflected image of the standard wedge W in $\mathbb{R}$. Define f_- in W_- by conjugating it via the rotation $z \mapsto e^{i\beta} z$ to the angle scaling map $z \mapsto e^{-t\theta} z$, $z \in \overline{W}$.

Interpolating by the identity and a loxodromic Möbius transformation, we obtain the following continuous map $E_t(z) : \mathbb{C} \to \mathbb{C}$:

- $E_t(z) = f_+(z) \quad \text{if } z \in W_+$,
- $E_t(z) = f_-(z), \quad \text{if } z \in W_-$,
- $E_t(z) = z$ for z in the component of $\mathbb{C} \setminus W_+ \cup W_-$ containing the positive real axis,
- $E_t(z) = e^{t\alpha} z$ for z in the component of $\mathbb{C} \setminus W_+ \cup W_-$ containing the negative real axis.

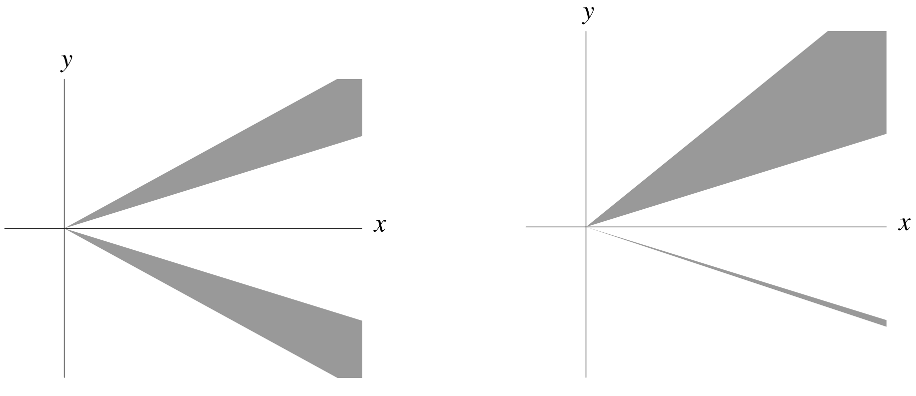

Figure 1: The lefthand picture shows two symmetric wedges of angle $\alpha = 0.2$ radians. The righthand picture shows the result of angle scaling by $t \in \mathbb{C}$ lying on the positive y-axis, with a value just less than $t = i$. So the angle of the upper wedge is almost doubled, and the angle of the lower wedge is almost collapsed. The corners in the shaded wedges are due to clipping in the production of the pictures.

Note that the E_t-image of the negative real axis is inclined at angle $\pi + \nu\alpha$ to the positive real axis.

The map as described on $\mathbb{C}$ extends to a continuous map of all $\mathbb{H}^3$. It will become quite messy as $|\mathrm{Im}(t)|$ becomes large.

The construction we have described is the basis of our angle scaling method.

2.4. Crescents associated with a finite lamination

By a *crescent* we mean an open region bounded by two circular arcs. A wedge is a special case. Any two crescents with the same interior vertex angle are Möbius equivalent in such a way that a designated initial edge of one is sent to a designated initial edge of the other.

Suppose Λ is a finite or infinite discrete geodesic lamination in $\mathbb{H}^2$. An *associated set of crescents* is a set of crescents in one-to-one correspondence with the leaves of Λ with the two properties

- Each leaf $\ell \in \Lambda$ corresponds to a crescent in $\mathbb{H}^2$ whose vertices are the endpoints of ℓ,
- The interiors of the crescents are mutually disjoint.

If Λ is invariant under a fuchsian group, we require that the set of associated crescents be invariant as well.

In turn, associated with the crescents, is the positive measured lamination

$$(\Lambda, \mu) = \{(\ell, \alpha_\ell)\}, \quad 0 < \alpha_\ell < \pi,$$

where α_ℓ is the vertex angle of the crescent associated with ℓ. Here we are ignoring the degenerate (but important) special case of one leaf with associated angle π.

We are aiming at using the associated set of crescents as a domain for angle scaling mappings. But first we will address the following converse question: Given a finite or infinite discrete measured lamination $(\Lambda, \mu) = \{(\ell, \alpha_\ell)\}$, $0 < \alpha_\ell < \pi$, can we find an associated set of crescents with exactly the angles $\{\alpha_\ell\}$?

3. The ortho-condition

Suppose $(\Lambda, \mu) = \{(\ell, \alpha_\ell)\}$ is a finite measured lamination in $\mathbb{H}^2$. We will present a necessary and sufficient condition that there exists an associated set of nonoverlapping

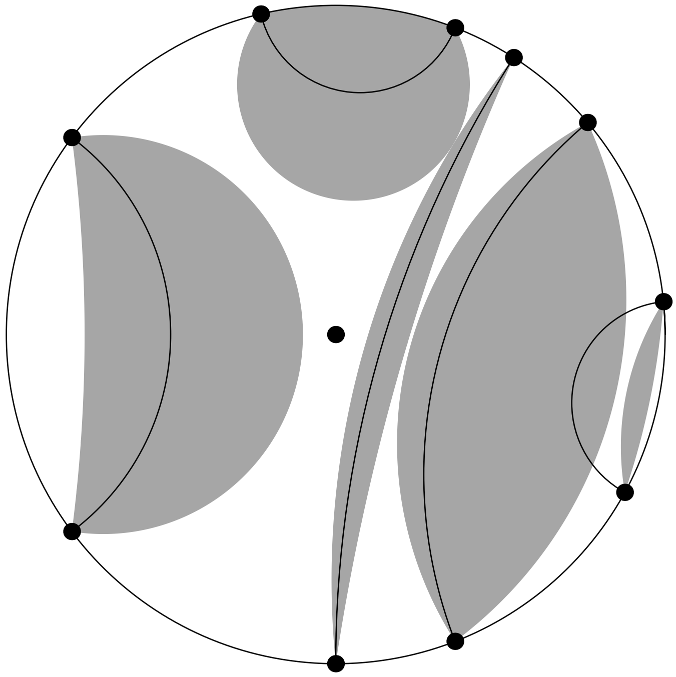

Figure 2: This picture shows five geodesics in the hyperbolic plane, and five associated crescents. The crescents have to be disjoint from each other and lie inside the unit circle. They do not need to be disjoint from the geodesics. A crescent is allowed to have an edge lying on the unit circle, as is the case for one of the crescents shown here. Note that the central gap containing the centre of the unit circle has three boundary pieces lying on the unit circle. That is, the gap is not separated, even though two crescents touch, because a crescent is open, by definition.

crescents such that the crescent with vertices at the endpoints of a leaf ℓ has vertex angle α_ℓ.

First we will examine the case that Λ is a ladder: By this term we mean that there is a geodesic γ transverse to and intersecting all the leaves of Λ. Choosing a direction along γ we can then order the leaves $\ell_1, \ldots, \ell_n$. Denote the associated angles by $\alpha_1, \ldots, \alpha_n$.

First assume that the leaves have no common endpoints. Find the hyperbolic distances $d_1, \ldots, d_{n-1}$ between successive leaves. If γ is orthogonal to all the leaves, then these distances are just the distances between the successive points of intersection of the leaves with γ.

Then with the data $\{\alpha_i, d_i\}$ construct the following model piecewise linear arc in say the unit disk model $\mathbb{D}$ of $\mathbb{H}^2$. Start with the ray $a_0 = (-1, 0]$ along the negative real axis. At $z = 0$ construct the arc a_1 of length d_1 with exterior bending angle α_1 with respect to a_0. Then construct a_2 of length d_2 with exterior bending angle α_2 with respect to a_1. And so on. The last arc a_n has exterior bending angle α_n with respect to a_{n-1} and will extend to a point on $\partial\mathbb{D}$. Our model arc is called the *ortho-curve* σ determined by the ladder.

If σ has no self-intersections, we say that the ladder (Λ, μ) satisfies the *ortho-condition*. If in addition the endpoints of σ on $\partial\mathbb{D}$ are distinct, we say the *strong ortho-condition* is satisfied. When the strong ortho-condition is satisfied, it will remain satisfied under all small enough changes in $\{\alpha_i\}$, or in the position of the leaves.

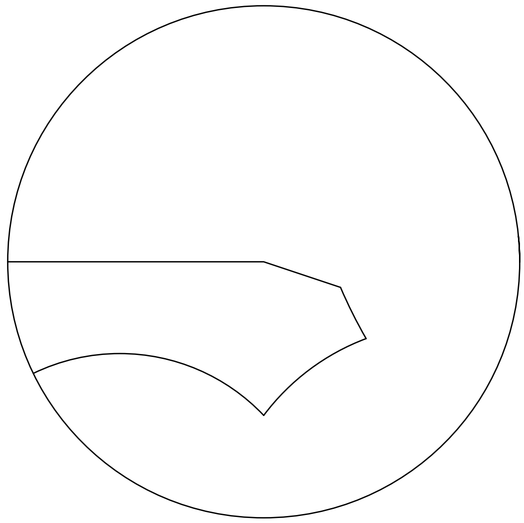

Figure 3: This ortho-curve is associated to a ladder (Λ, μ). Each vertex in the ortho-curve corresponds to a geodesic in Λ (or to several geodesics in Λ with common endpoints). The bending angles of the ortho-curve at a vertex is equal to the measure of the corresponding geodesic in Λ. The length of each finite side of the ortho-curve is equal to the distance between corresponding geodesics.

We must modify the construction if the leaves have common endpoints. Suppose the leaves $\ell_i, \dots, \ell_{i+k}$ share the same endpoint $\zeta \in \partial\mathbb{D}$, but neither ℓ_{i-1} nor ℓ_{i+k+1} share ζ. Set $\beta = \sum_{j=i}^{i+k} \alpha_j$. In the construction replace the set of leaves $\{\ell_j : i \le j \le i+k\}$ by just ℓ_i and replace α_i by β. Apply this fix at all common fixed points until we return to the situation considered above. Then construct the ortho-curve.

Now return to a general finite positive measured lamination (Λ, μ). We say that it satisfies the *ortho-condition* (respectively, *strong ortho-condition*) if *every* ladder in Λ

satisfies the ortho-condition (respectively, strong ortho-condition).

Exactly the same analysis applies to infinite discrete laminations. Such a system arises as follows. Suppose $S = \mathbb{H}^2/G$ is a surface of finite area. Choose a simple geodesic $\gamma \subset S$ and a maximal area tubular neighborhood A about γ. The collection of lifts of A to $\mathbb{H}^2$ will be a system of crescents with the lifts of γ. The system will satisfy the ortho- but not the strong ortho-condition. In this situation, all the vertex angles are the same—see section 6 for an example.

Theorem 3.1 ([EMM02]). *Suppose $(\Lambda,\mu) = \{(\ell,\alpha_\ell)\}$ is a finite or infinite discrete positive measured lamination in $\mathbb{H}^2$. There is an associated system of nonoverlapping crescents if and only if it satisfies the ortho-condition. If (Λ,μ) is invariant under the action of a fuchsian group G, the associated crescents can be chosen to be invariant as well.*

4. Angle scaling on systems of crescents

In this section we will work in the disk model $\mathbb{D}$ of $\mathbb{H}^2$. Refer to Figure 2 as we describe the construction of the mappings.

Assume that $(\Lambda,\mu) = \{(\ell,\alpha_\ell)\}$ is a finite or infinite discrete positive measured lamination with an associated set of nonoverlapping crescents $\{C\}$ in $\mathbb{D}$. Reflect the set of crescents in $\partial\mathbb{D}$ to get a set of nonoverlapping crescents $\{C'\}$ in the exterior $\mathbb{D}^{ext}$. We may assume the measured laminations and the sets of crescents are invariant under G where G is a fuchsian group without elliptics—or simply $G = \{\mathrm{id}\}$ (this will be the situation in the finite case).

Given (Λ,μ) as above, for each $t = u + iv \in \mathbb{C}$ we will define a continuous map $\mathbb{S}^2$ into or onto $\mathbb{S}^2$, $E_t(z) = E_t(z;\Lambda,\mu)$, as follows.

Start by choosing a crescent C_1, an edge e_1, its reflected partner C_1' in the exterior $\mathbb{D}^{ext}$, and the reflected edge e_1'. For simplicity we will assume that the region bounded by the closed curve $e_1 \cup e_1'$ contains no crescents.

On the boundary component $e_1 \cup e_1'$ of $C_1 \cup C_1'$, $E_t(z)$ is the identity. On $C_1 \cup C_1'$, E_t is angle scaling f_t. On the union of the other pair of boundary edges, $\varepsilon_1 \cup {\varepsilon_1}'$, $E_t(z)$ is the restriction of a Möbius transformation, say T_1. We will describe how to continue f_t to the other side of $\varepsilon_1 \cup \varepsilon_1'$.

Suppose the crescents $C_2, C_2', C_3, \ldots$ are adjacent on the right to the curve $\varepsilon_1 \cup \varepsilon_1'$ in the sense that no crescent separates it from any of these C_i. Let e_2^* denote the union of the boundary components of the $\{C_i, C_i'\}$, $i > 1$, which likewise are not separated from $\varepsilon_1 \cup \varepsilon_1'$. (We have to allow for tangencies between crescents.) Correspondingly

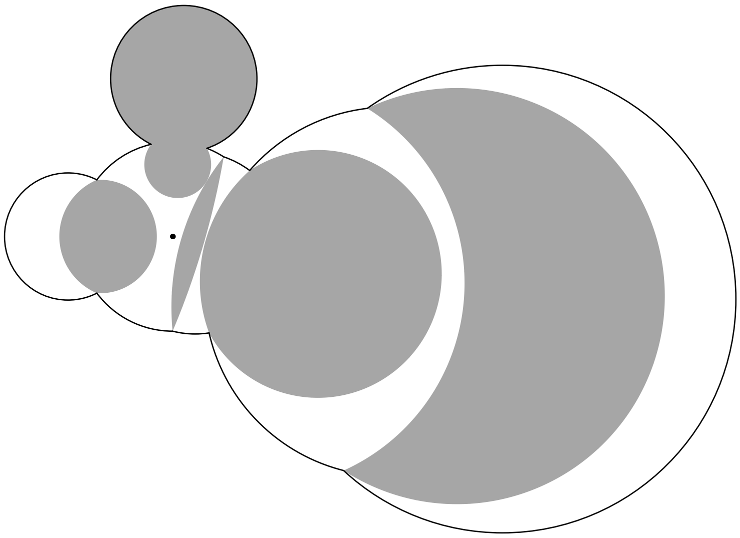

Figure 4: This is what we get by applying angle doubling to the crescents shown in Figure 2. Note that the image crescents may well have angles that are bigger than π. It is also possible (not shown) for one the image crescent to contain ∞, so that, in the plane, the image crescent surrounds the rest of the picture. We have marked the origin here and in Figure 2 with a black dot. The gap containing the origin is unchanged by angle doubling, though this may not be clear because the two pictures are drawn to different scales.

let D_1 be the union of those components (called *gaps*) of $\overline{\mathbb{D}} \cup \mathbb{D}^{ext} \setminus C_2 \cup C_2' \cup \cdots$ with the property that the boundary of each component is a union of arcs in $e_2^* \cup (\varepsilon_1 \cup \varepsilon_1')$.

For z in the closure $\overline{D}_1$ define $E_t(z) = T_1(z)$. In particular on the appropriate edge of each $C_i \cup C_i'$, $i > 1$, E_t is now well defined. In each $C_i \cup C_i'$ now define $E_t(z) = (f_t)_i(z)$, where $(f_t)_i$ is the angle scaling map in $C_i \cup C_i'$ whose boundary values are normalized to be T_1 on the appropriate edges.

Continuing in this pattern we end up with the following result.

Lemma 4.1. *Let (Λ, μ) be a finite or infinite discrete positive G-invariant measured lamination satisfying the ortho-condition. For each $t = u + iv \in \mathbb{C}$, the complex angle scaling maps on a set of associated crescents, interpolated in the gaps by Möbius transformations, fit together to give a continuous map, or G-equivariant continuous map, $E_t(z;\Lambda,\mu) : \mathbb{S}^2 \to \mathbb{S}^2$.*

The geometry of the map will in general be quite complicated as the images of the individual crescents may expand, contract, or fold over.

The proof in the case of an infinite lamination requires a convergence argument [EMM02].

4.1. The geometry of complex scaling: earthquaking and grafting/bending

The map $E_t(z;\Lambda,\mu)$ may be called again a *complex angle scaling map* or *complex earthquake*. It is often useful to regard it as a composition,

$$E_t = E_{iv} \circ E_u, \quad t = u + iv.$$

Here $E_u : \mathbb{S}^2 \to \mathbb{S}^2$ is a homeomorphism, its actions in $\mathbb{D}$ and $\mathbb{D}^{ext}$ being symmetric. In particular its action in $\mathbb{D}$ is a (smoothed out) earthquake sending the lamination Λ to a new lamination Λ_u; we write $E_u : (\Lambda,\mu) \to (\Lambda_u,\mu)$ and understand that each leaf of Λ_u takes its measure from its preimage. Equally E_u sends a crescent system for (Λ,μ) to one for (Λ_u,μ). The values of E_u on $\partial\mathbb{D}$ are exactly the boundary values of a classical earthquake. By reflection, E_u acts as well in $\mathbb{D}^{ext}$.

In contrast the action of E_{iv} is a (smoothed out) grafting on the crescents associated with (Λ_u,μ). When $v > 0$ each crescent in $\mathbb{D}$ is expanded $\alpha \mapsto (1+v)\alpha$ while each crescent in $\mathbb{D}^{ext}$ is contracted, collapsed, or folded over according to the sign of $(1-v)\alpha$.

The action of E_{iv} can also be described as *bending*. We will describe this operation on $\mathbb{D}$; there is a parallel action in $\mathbb{D}^{ext}$.

Thurston taught us that given a measured lamination (Λ,μ), there is an isometry $\mathbb{H}^2 \to \mathbb{H}^3$ taking $\mathbb{D}$ onto a pleated surface. Each leaf $\ell \subset \Lambda$ is sent a geodesic, and each component of $\mathbb{D} \setminus \Lambda$ is sent to a geodesic polygon. The exterior bending angles are given by μ (for complete details see [EM87]). We could equally well construct, instead of the mapping E_{iv} into $\mathbb{S}^2$, a mapping E^*_{iv} onto a (normalized) pleated surface in $\mathbb{H}^3$. From this point of view, associated with each *bending line* $\ell \subset E^*_{iv}(\mathbb{D})$ is the exterior bending angle $v\alpha_\ell$; the two flat pieces that border ℓ have exterior bending angle $v\alpha_\ell$.

The most interesting special case is *angle doubling*, the case that $t = i$.

We summarize our results (concentrating on the case $\mathrm{Im}(t) > -1$) as follows.

Theorem 4.2 ([EMM02]). *Suppose (Λ,μ) is a finite or infinite discrete G-invariant measured lamination that satisfies the ortho-condition. There exists a G-equivariant complex angle scaling mapping $E_t(z) = E_t(z;\Lambda,\mu)$ for every $t \in \mathbb{C}$. It has the following properties.*

(i) *For* $\{-1 < Im(t) < +1\}$, $E_t : \mathbb{S}^2 \to \mathbb{S}^2$ *is a* K_t*-quasiconformal homeomorphism. Furthermore* $\{E_t(z)\}$ *is a holomorphic motion of* $\mathbb{S}^2$.

(ii) *For* $\{-1 < Im(t)\}$, $E_t : \mathbb{D} \to \mathbb{S}^2$ *is a* K_t*-quasiregular developing mapping. For each* $z \in \overline{\mathbb{D}}$, $E_t(z)$ *is defined and holomorphic in* t.

(iii) *When* $t = i$, *the angle doubling map* $E_i : \mathbb{D} \to \Omega_i = E_i(\mathbb{D})$ *is a* 2*-quasiconformal homeomorphism. It has a continuous extension to* $\overline{\mathbb{D}}$.

If $\partial\Omega_i$ *is* not *a Jordan curve, there exists a ladder* $\{\ell_j\} \subset \Lambda$ *which has a common orthogonal and which does* not *satisfy the strong ortho-condition. Conversely if there is such a ladder,* Ω_i *is not a Jordan region.*

A quasiregular developing mapping is locally quasiconformal but not in general a global homeomorphism. The quasiconformal factor from Equation(2.1) is

$$K_t = \frac{|t+2i| + |t|}{|t+2i| - |t|}, \quad \mathrm{Im}(t) > -1.$$

5. Simply connected regions and their domes

Suppose $\Omega \subset \mathbb{C}$ is a simply connected region, but not $\mathbb{C}$ itself. The hyperbolic convex hull in $\mathbb{H}^3$ of the closed set $V = \mathbb{S}^2 \setminus \Omega$ is the intersection of all hyperbolic halfspaces whose closures on $\partial\mathbb{H}^3$ contain V. The convex hull has a single boundary component in $\mathbb{H}^3$. The boundary lies over Ω as the dome lies over the floor in a domed stadium. Accordingly we will denote the convex hull boundary component by $\mathrm{Dome}(\Omega)$.

The dome is the union of bending lines and flat pieces. The bending lines form a closed set of geodesics which have their endpoints in $\partial\Omega$. In the present exposition the bending lines will form a finite or infinite discrete set. The complement is the union of flat pieces, in general an ideal hyperbolic polygon. The bending angle at an isolated bending line is the exterior angle formed by the two flat pieces with the line as a common edge. Thus the limiting case of bending angle zero means the two flat pieces are not bent with respect to each other at all.

The key to analyzing the dome is Thurston's theorem that the hyperbolic metric in $\mathbb{H}^3$ restricts to the dome to give a path metric, which is then called the hyperbolic metric of the dome. In terms of this metric and the hyperbolic metric, say in $\mathbb{D}$, there is an isometry

$$\iota : \mathrm{Dome}(\Omega) \to \mathbb{D}$$

that carries the set of bending lines with their bending measure to a measured lamination in $\mathbb{D}$. For the full details see [EM87].

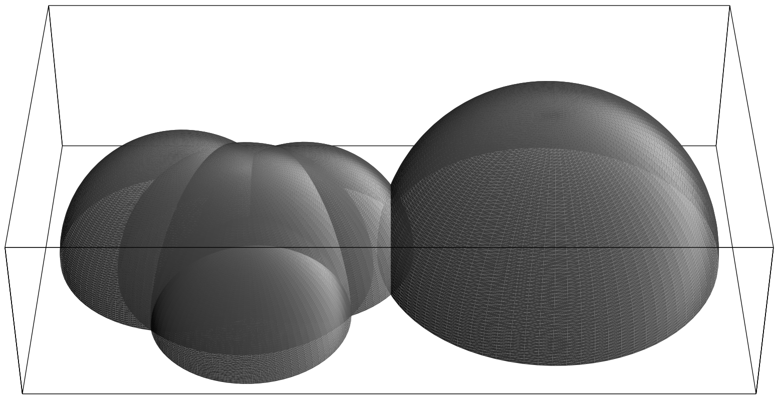

Figure 5: A dome

Sullivan stated the following theorem in [Sul81b] with half a page indicating how one might set about proving it. A complete (but very complicated) proof was later given in [EM87]. The theorem says that the hyperbolic metrics on Ω and $\mathrm{Dome}(\Omega)$ are of uniformly bounded Teichmüller distance apart.

Theorem 5.1 ([Sul81b, EM87]). *There exists a universal constant $K > 1$ with the following property. For any simply connected $\Omega \in \mathbb{C}$, $\Omega \neq \mathbb{C}$, there exists a K-quasiconformal mapping $\Phi : \mathrm{Dome}(\Omega) \to \Omega$ that extends continuously to be the identity on the common boundary $\partial\Omega$. If Ω is invariant under a discrete group Γ of Möbius transformations, then Φ is equivariant under Γ.*

The boundary values pin down the homotopy class of Φ.

It is natural to ask: What is the least value K for which Sullivan's theorem holds? The simple example of the plane slit along the positive real axis shows that it cannot be that $K < 2$. The best result is by Bishop [Bis01], who showed that $K \leq 7.82$.

Conjecture 5.2. $K = 2$.

There are actually two possible constants: the universal K as stated, but also a constant $K_{eq} \geq K$ which is an upper bound for the maximal dilatation of maps Φ with the additional property that Φ must be equivariant whenever Ω is preserved by a discrete group Γ such that Ω/Γ has finite hyperbolic area. It is the latter constant that Thurston seems to have in mind because of its possible application to hyperbolic 3-manifolds.

Conjecture 5.3 ([Thu86b]). $K_{eq} = 2$.

To verify either of these conjectures it must be shown that for *any* simply connected region with the required properties, the maximal dilatation of *some* appropriate quasiconformal mapping does not exceed two.

Teichmüller space theory implies that $K \le K_{eq}^3$ [Leh87, Thm. 4.7].

Actually *both* of these conjectures are false.

Later in section 6 we will outline a proof that $K_{eq} > 2$. The counterexamples are regular sets of groups in an open set of the Teichmüller space of once-punctured tori quasifuchsian groups.

More recently it has been shown that $K > 2$ without any requirement of equivariance. The counterexample is the complement of an infinite logarithmic spiral [EMk]. For the example the maximal dilatation of the extremal quasiconformal map is approximately 2.1.

On the other hand, there are many regions such that the corresponding constant satisfies $K = 2$. A large class of such regions can be constructed by using complex angle scaling. Moreover, $K = 2$ for all regions "sufficiently close" to a round disk (Theorem 5.3) and for all euclidean convex regions [EMM02].

To suggest the relationship between the image region $\mathbb{D} \to \Omega$ obtained by angle scaling and $\mathrm{Dome}(\Omega)$ observe the following. Suppose $C \subset \mathbb{D}$ is a crescent of angle $0 < \alpha < \pi$. Under angle scaling C becomes a crescent C' of angle $(1+v)\alpha$. If $0 < (1+v)\alpha < 2\pi$, C' is embedded. $\mathrm{Dome}(\Omega)$ consists of two flat pieces with exterior bending angle $v\alpha$. When $v = 1$ (angle doubling) the bending angle is α.

The following statement augments Theorem 4.3.

Theorem 5.4 ([EMM02]). *Suppose $(\Lambda, \mu) \subset \mathbb{D}$ is a finite or infinite discrete positive measured lamination invariant under a fuchsian group G, that satisfies the ortho-condition. Set $t = i$ (angle doubling). The map $E_i : \mathbb{D} \to \Omega = E_i(\mathbb{D})$ has the following properties:*

(i) *E_i is 2-quasiconformal and equivariant with respect to G.*

(ii) *E_i has a continuous extension to $\overline{\mathbb{D}}$.*

(iii) *The isometry $\iota^{-1} : \mathbb{D} \to \mathrm{Dome}(\Omega)$ takes $(\Lambda, \mu) \subset \mathbb{D}$ to the bending measure of $\mathrm{Dome}(\Omega)$ and has a continuous extension to $\partial\Omega$. Moreover if E_i and ι are normalized in the same way, the map*

$$\iota^{-1} \circ E_i^{-1} : \Omega \to \mathbb{D} \to \mathrm{Dome}(\Omega)$$

is 2-quasiconformal and has a continuous extension to $\partial\Omega$ where it is the identity.

We remark that if instead we take $t \in \mathbb{C}$ with $\mathrm{Im}(t) = +1$ while $\mathrm{Re} t \neq 0$, E_t can again be interpreted as angle doubling. Namely, it is angle doubling on the lamination $(\Lambda_u, \mu) \in \mathbb{H}^2$ which results from applying the earthquake $E_u : \mathbb{D} \to \mathbb{D}$ to (Λ, μ), $t = u + iv$. Thus property (ii) holds for E_t and (i), (iii) hold for E_i with respect to (Λ_u, μ).

5.1. Nearly round domains

We digress to raise the following issue. Given a general positive Borel measured lamination $(\Lambda, \mu) \subset \mathbb{D}$, it is a hard question to determine whether or not it is the bending measure of the dome of a simply connected region Ω. Of course in the discrete case, we know the answer is yes if it satisfies the ortho-condition. But a much more general result is possible if we stick to regions Ω which are “close” to round circles.

The precise statement is in terms of the norm

$$||\mu|| = ||(\Lambda, \mu)|| = \sup_{\gamma} \mu(\gamma) \in [0, \infty]$$

as γ varies over all transverse geodesic intervals γ of unit length. In particular, $||\mu|| = 0$ if and only if Ω is a round disk. One may therefore use the norm as a measure of roundness of Ω.

Theorem 5.5 ([EMM01]). *Suppose* (Λ, μ) *is a positive Borel measured lamination. There exists a constant*

$$0 < c \leq 2 \arcsin(\tanh(1/2)) \approx 0.96$$

with the following property: if $||\mu|| < c$*, then* (Λ, μ) *is the bending measure of* $\mathrm{Dome}(\Omega)$ *for a simply connected region* Ω*, which is a quasidisk. There exists a* 2*-quasiconformal map* $\Phi : \Omega \to \mathrm{Dome}(\Omega)$ *which extends continuously to the identity on* $\partial\Omega$*. Moreover, if* Ω *is invariant under a discrete group* Γ*, then* Φ *is equivariant under* Γ*.*

It is known that $c = 0.73$ will work, and that any constant which is greater than $2 \arcsin(\tanh(1/2))$ will not work. It is conjectured that the latter value is the correct upper bound for c.

In the other direction, it is known that if a pleated surface is embedded, its bending measure necessarily satisfies $||\mu|| < 4.88$, [Bri98], [Bri02]. Actually the estimate in [Bri02] is stated to hold in the presence of a group action, but according to Bridgeman (personal communication), a small modification of its proof shows that it holds independently of any group structure.

6. Earthquake disks and $K_{eq} > 2$

We will present the examples that show $K_{eq} > 2$. This also gives the opportunity of further illuminating the method of angle scaling.

The starting point is a square, once punctured torus $T = \mathbb{D}/G$. That T is square means that there are simple closed geodesics $\mathbf{a}, \mathbf{b} \subset T$ which are of equal length and orthogonal at their point of intersection. Let A be a maximal collar about $\mathbf{a}$. Lift $\mathbf{a}$ and A to $\mathbb{D}$. The lifts $\{\mathbf{a}^*\}$ of $\mathbf{a}$ are arranged in ladders each with a common orthogonal which is a lift of $\mathbf{b}$. The lifts $\{A^*\}$ of A form an associated system of crescents, all with the same vertex angle which we will designate as α. The crescents satisfy the ortho-condition but not the strong condition. Let (Λ, μ) denote the associated measured lamination.

Now bring in the angle scaling map

$$E_t(z) = E_t(z; \Lambda, \mu), \quad t = u + iv.$$

We know that $E_u : \mathbb{D} \to \mathbb{D}$ is an earthquake. Denote the length of $\mathbf{a}$ by L. Replacing the projection $E_{u*} : T \to T$ by $E_{(u+L/\alpha)_*}$ is akin to replacing E_{u*} by its composition with a Dehn twist about $\mathbf{a}$.

Consider the following three sets in the parameter t-space.

$$S = \{t \in \mathbb{C} \mid -1 < \mathrm{Im}(t) < 1\}.$$

This strip is the set for which E_t extends to a quasiconformal mapping of $\mathbb{S}^2$.

$$\mathfrak{T}_\circ = \{t \mid E_t : \mathbb{D} \to \Omega_t \text{ is injective}\}_0 \supset S.$$

Here the subscript 0 indicates that we must take the component containing $t = 0$, as the indicated set is not connected. Since all bending angles are less than π, for $t \in \mathfrak{T}_\circ$, $|\mathrm{Im}(t)| < \pi/\alpha$. Like S, $\mathfrak{T}_0$ is symmetric about $\mathbb{R}$.

$$\mathfrak{T} = \mathfrak{T}_\circ \cup \{t \mid \mathrm{Im}(t) > 0\}.$$

The set $\mathfrak{T}$ is called an *earthquake disk*. It is the following theorem that shows it is simply connected.

Theorem 6.1 (McMullen [McM98]). *$\mathfrak{T}$ is biholomorphically equivalent to the Teichmüller space of T.*

In this realization, Teichmüller space is parameterized by earthquakes on Λ then graftings on the images $E_u(\Lambda)$.

For $t \in S$, E_t is quasiconformal in $\mathbb{D}$. As $t \in \mathfrak{T}$ goes beyond S, E_t becomes instead a developing mapping; it sends $\mathbb{D}$ to a simply connected region spread over $\mathbb{S}^2$, and sends the initial fuchsian group G to a group Γ acting on the image so that the quotient surface is a quasiconformally equivalent to T. These surfaces fill out Teichmüller space. However Γ itself will in general not be discrete.

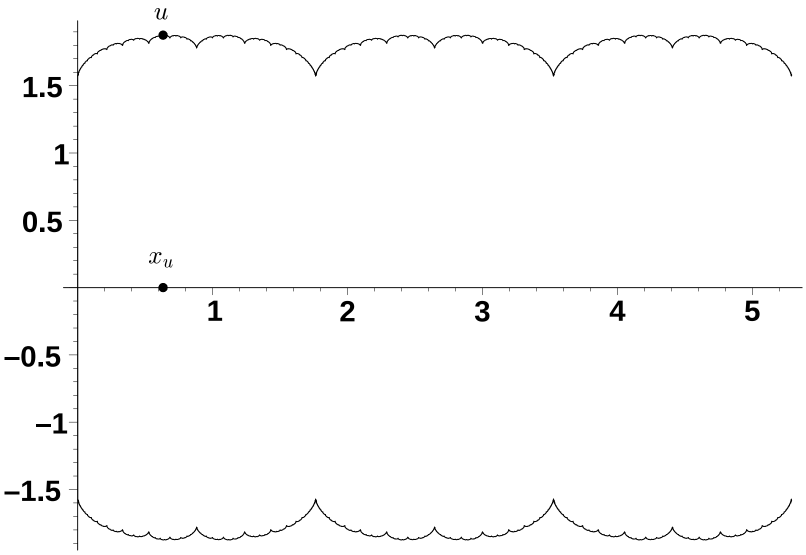

Figure 6: This figure from [EMM01] is the trace $A = 2\sqrt{2}$ slice of quasifuchsian space with basepoint the square torus with $\mathbf{a}, \mathbf{b}$ of length arccosh 3. On this torus, the vertex angle of the maximal collar symmetrical about $\mathbf{a}$ is $\alpha = \pi/2$. The horizontal strip bounded by the upper and lower curves shown represents $\mathfrak{T}_0$. The union of this strip and the upper halfplane is $\mathfrak{T}$. However the scaling indicated in the figure is not according to our conventions here. Instead the top and bottom curves should cut the imaginary axis in the points $\pm i$ and the periodicity of the strip $\mathfrak{T}_0$ is $2(\operatorname{arccosh} 3)/\pi$. Thus S is bounded by the horizontal lines that go through the cusps $2n(\operatorname{arccosh} 3)/\pi \pm i$, $-\infty < n < \infty$, on the upper and lower curves. The point $u = x_u + iy_u$ is a highest point on the upper curve. The halfplane $H \supset \mathfrak{T}$ is bounded by the horizontal line through the lowest points $2n(\operatorname{arccosh} 3)/\pi - y_u i$ of the lower curve. This figure was drawn by David Wright.

Properties:

- $S, \mathfrak{T}_\circ, \mathfrak{T}$ are invariant under the translations $t \mapsto t + nL/\alpha$,
- $\{t = i\} \in \partial\mathfrak{T}_\circ \cap \partial S$ and for all small $\varepsilon > 0$, $\{t = \pm\varepsilon + i\} \in \mathfrak{T}_\circ$.
- $E_i : \mathbb{D} \to E_i(\mathbb{D}) = \Omega_i$ is 2-quasiconformal,
- $E_i : \mathbb{D}^{ext} \to \Omega_i' = \mathbb{S}^2 \setminus \overline{\Omega_i} \neq \emptyset$ (Angle collapse on exterior crescents),
- $\partial\Omega_i$ is not a Jordan curve,
- $\Omega_i \cup \Omega_i'$ is the ordinary set of Γ, where $E_{i*} : G \to \Gamma$ is the induced isomorphism,
- Ω_i'/Γ is the 3-punctured sphere.

The point $t = i$ is a minimal point on the top curve because after any nonperiodic earthquake, the geodesics $\mathbf{a}, \mathbf{b}$ are no longer orthogonal, allowing larger crescent angles. This in turn permits larger values of $\operatorname{Im}(t)$ appear in $\mathfrak{T}_\circ$.

Set

$$\chi = \sup\{\operatorname{Im}(t) \mid t \in \mathfrak{T}_\circ\} < \pi/\alpha \ ,$$

$$-\chi = \inf\{\operatorname{Im}(t) \mid t \in \mathfrak{T}_\circ\} \ ,$$

and introduce the halfplane

$$H = \{t \in \mathbb{C} \mid \operatorname{Im}(t) > -\chi\} \supset \mathfrak{T}.$$

The hyperbolic distances satisfy

$$\rho_H(z,w) < \rho_{\mathfrak{T}}(z,w), \quad z,w \in \mathfrak{T}.$$

Here we have made essential use of the fact that $\mathfrak{T} \neq H$. In particular,

$$\log 2 = \rho_H(u, u + i\chi) < \rho_{\mathfrak{T}}(u, u + i\chi).$$

Choose $u + iv \in \mathfrak{T}_\circ$ such that $\rho_{\mathfrak{T}}(u, u + iv) > \log 2$.

If K^* denotes the maximal dilatation of the extremal quasiconformal map between the Teichmüller points $u, u + iv$ then since Teichmüller distance coincides with hyperbolic distance in $\mathfrak{T}$,

$$\log K^* > \log 2, \text{ therefore, } K^* > 2.$$

What does this have to do with the relation between the floor and the dome? Using the point $t = u + iv \in \mathfrak{T}_\circ$ we know that

- The earthquake E_u sends the lamination $\Lambda \to \Lambda_u$, the initial fuchsian group G to a fuchsian deformation G_u where the surface $\mathbb{D}/G_u$ corresponds to the Teichmüller point u (the marking is given by the isomorphism induced by E_u).
- $E_{iv}(z;\Lambda_u,\mu) : \mathbb{D} \to \Omega_t$ induces an isomorphism $G_u \to \Gamma_t$. Ω_t is a Jordan region and Ω_t/Γ_t corresponds to the Teichmüller point $t = u+iv$.
- A Riemann map $\Phi : \Omega_t \to \mathbb{D}$ sends Γ_t to a fuchsian group G_t where $\mathbb{D}/G_t \equiv \Omega_t/\Gamma_t$.
- The isometry $\iota : \mathrm{Dome}(\Omega_t) \to \mathbb{D}$ chosen so that $\iota \circ E_{iv}\big|_{\partial\mathbb{D}} = \mathrm{id}$ takes Γ_t back to G_u.

Now we can complete the proof. Suppose $f : \mathrm{Dome}(\Omega_t) \to \Omega_t$ is a K-quasiconformal, Γ_t-equivariant mapping which extends continuously to the identity on $\partial\Omega_t$. Then

$$\Phi \circ f \circ \iota^{-1} : \mathbb{D} \to \mathbb{D}$$

is K-quasiconformal. It induces an isomorphism

$$G_u \xrightarrow{\iota^{-1}} \Gamma_t \xrightarrow{f} \Gamma_t \xrightarrow{\Phi} G_t.$$

That is

$$u \equiv \mathbb{D}/G_u \mapsto \mathbb{D}/G_t \equiv u+iv.$$

We conclude that

$$\log K \geq d_{\mathfrak{T}}(u, u+iv) > \log 2.$$

Since $K > 2$ for any f associated with Ω_t, in particular for the extremal f, it follows that we necessarily have $K_{eq} > 2$.

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