BARYCENTRIC EXTENSION AND THE BERS EMBEDDING FOR ASYMPTOTIC TEICHMÜLLER SPACE

CLIFFORD J. EARLE, VLADIMIR MARKOVIC, AND DRAGOMIR SARIC

ABSTRACT. In this paper we show that the Bers map of the asymptotic Teichmüller space $AT(X)$ of an arbitrary hyperbolic Riemann surface X is injective. We prove further that $AT(X)$ and the fibers of the quotient map from $T(X)$ to $AT(X)$ are contractible and that every point in the fiber over the basepoint of $AT(X)$ is represented by a quasiconformal map that is an asymptotic hyperbolic isometry. The barycentric extension operator plays a central role in our proofs.

INTRODUCTION

We call a Riemann surface hyperbolic if its universal covering surface is conformally equivalent to the upper half plane $\mathbf{H}^+$. The Teichmüller space $T(X)$ of an arbitrary hyperbolic Riemann surface X was defined many years ago by Lipman Bers, and the spaces $T(X)$ have been extensively studied (see the books [12], [16], [8] and the literature cited there).

A quotient space of $T(X)$, called the asymptotic Teichmüller space and denoted by $AT(X)$, was introduced by Gardiner and Sullivan (see [9]) for $X = \mathbf{H}^+$ and by Earle, Gardiner and Lakic for arbitrary hyperbolic X (see [6], [7], and Chapter 14 of [8]). They proved that $AT(X)$ has a complex manifold structure such that the quotient map $\pi\colon T(X) \to AT(X)$ is holomorphic, with a surjective Fréchet derivative at every point. They showed further that Bers's embedding of $T(X)$ in a Banach space induces a holomorphic and locally biholomorphic map of $AT(X)$ into a quotient Banach space. We call this induced map the asymptotic Bers map.

Several people ([10], [14], and [8]) have shown that the asymptotic Bers map is globally one-to-one (and therefore a biholomorphic embedding) when X is the upper half plane. The question whether the map is injective for general X was posed in [7] and [8]. Our first theorem answers that question positively.

Theorem 1. *For any hyperbolic Riemann surface X, the asymptotic Bers map is a biholomorphic map of $AT(X)$ onto a bounded open set in a complex Banach space.*

As we shall see in §10, Theorem 1 follows readily from our main result, Theorem 4, which we state and prove in §9. Additional consequences of Theorem 4, including the following two theorems, are derived in §§11, 12, and 13, where a more precise statement of Theorem 3 can be found (see Lemma 13.1).

Theorem 2. *The fibers of the quotient map $\pi\colon T(X) \to AT(X)$ and the space $AT(X)$ are all contractible.*

1991 *Mathematics Subject Classification.* Primary 30F60; Secondary 30C62.

Theorem 3. *Let $T_0(X)$ be the fiber of $\pi\colon T(X) \to AT(X)$ that contains the basepoint of $T(X)$. Every point of $T_0(X)$ can be represented by a quasiconformal map that is an asymptotic hyperbolic isometry.*

We prove Theorem 4 by making arguments from [14] appropriately invariant. For that purpose we use the barycentric extension operator introduced by Douady and Earle in [5]. However, as the referee has encouraged us to point out, the proof will work equally well if barycentric extensions are replaced by some other set of quasiconformal mappings with appropriate invariance properties and dependence on their boundary values. For example, the Beurling-Ahlfors extension operator (see [9] or Chapter 16 of [8]) can be used if the Riemann surface X has a cyclic fundamental group.

The quasiconformal (hyperbolically) harmonic diffeomorphisms of the upper half plane also have appropriate invariance properties and dependence on their boundary values. Markovic has investigated them (see for example [15]) with the goal of using them to prove Theorem 1, but barycentric extensions have served that purpose better since at present it is not known whether all points of an infinite dimensional $T(X)$ can be represented by these harmonic maps.

However, facts about these maps can be proved by suitably adapting the proof of Theorem 4. For example, Guowu Yao (private communication) has used that method to show that the classical Hopf map has an asymptotic analogue that maps an appropriate Banach space into $AT(X)$ injectively.

We thank the referee for his helpful comments. We also thank Fred Gardiner for suggesting that Theorem 3 should follow from our main theorem and should be included in this paper.

1. The spaces $T(X)$, $AT(X)$, and $T_0(X)$

This is the first of eight sections of preparatory material, mostly well known, about quasiconformal and asymptotically conformal maps, $T(X)$ and $AT(X)$, barycentric extensions, and related matters.

Let $\varpi\colon \mathbf{H}^+ \to X$ be a holomorphic universal covering of the hyperbolic Riemann surface X. Every quasiconformal self-mapping g of X lifts to a quasiconformal self-mapping $\widetilde{g}$ of $\mathbf{H}^+$. We say that g is Teichmüller trivial if it has a lift $\widetilde{g}$ whose continuous extension to the closed half plane equals the identity on the extended real axis. The Teichmüller trivial quasiconformal self-mappings of X form a group under composition.

We say that a quasiconformal map f between two Riemann surfaces is asymptotically conformal if for every $\varepsilon > 0$ there is a compact set E in the domain of f such that the restriction of f to the complement of E is $(1+\varepsilon)$-quasiconformal.

The classical Teichmüller space $T(X)$ is defined as a quotient space of the set of all quasiconformal maps with domain X in the following way. The quasiconformal maps f_1 and f_2 of X onto Y_1 and Y_2 respectively are said to be Teichmüller equivalent if there is a Teichmüller trivial map g such that the map $f_2 \circ g \circ f_1^{-1}$ of Y_1 onto Y_2 is conformal. This is an equivalence relation, and $T(X)$ is by definition the set of Teichmüller equivalence classes. The basepoint of $T(X)$ is the Teichmüller equivalence class of the identity self-mapping of X.

The asymptotic Teichmüller space $AT(X)$ is defined in the introductions to [7] and Chapter 14 of [8] as the quotient space of the same set of quasiconformal maps with respect to a weaker equivalence relation, which we shall call asymptotic equivalence. The quasiconformal maps f_1 and f_2 of X onto Y_1 and Y_2 respectively are said to be asymptotically equivalent if there is a Teichmüller trivial map g such that the map $f_2 \circ g \circ f_1^{-1}$ of Y_1 onto Y_2 is asymptotically conformal. The set of these asymptotic equivalence classes is the asymptotic Teichmüller space $AT(X)$. Its basepoint is the asymptotic equivalence class of the identity self-mapping of X.

The definitions of Teichmüller and asymptotic equivalence are word-for-word the same except that the word "conformal" in the first is replaced in the second by the expression "asymptotically conformal." Since conformal maps are asymptotically conformal, there is a well-defined quotient map $\pi\colon T(X) \to AT(X)$ that maps each Teichmüller equivalence class $[f]_{T(X)}$ to the asymptotic equivalence class $[f]_{AT(X)}$ that contains it. Notice that π maps basepoint to basepoint.

A class $[f]_{T(X)}$ in $T(X)$ is called asymptotically conformal if $\pi([f]_{T(X)})$ is the basepoint of $AT(X)$. This happens if and only if $[f]_{T(X)}$ contains an asymptotically conformal map. The set of asymptotically conformal classes in $T(X)$ is denoted by $T_0(X)$. It is proved in [7] and Chapter 14 of [8] that $T_0(X)$ is a closed complex submanifold of $T(X)$.

Remark 1. In [9], [6], and [7], $T_0(X)$ was defined as the closure in $T(X)$ of the set of classes $[f]_{T(X)}$ that contain an asymptotically conformal map. The simpler definition given here is justified by the nontrivial theorem that the classes containing asymptotically conformal maps already form a closed set.

2. $T(X)$ AND $AT(X)$ AS QUOTIENT SPACES OF $M(\mathbf{H}^+,\Gamma)$

As above, let $\varpi\colon \mathbf{H}^+ \to X$ be a holomorphic universal covering of the hyperbolic Riemann surface X. Let Γ be the Fuchsian group of covering transformations for ϖ. By definition the set $L^\infty(\mathbf{H}^+,\Gamma)$ of Beltrami differentials for Γ is the closed subspace of $L^\infty(\mathbf{H}^+)$ consisting of the μ in $L^\infty(\mathbf{H}^+,\Gamma)$ that satisfy the Γ-invariance condition

$$(1) \qquad (\mu \circ \gamma)\overline{\gamma'}/\gamma' = \mu \qquad \text{for all } \gamma \in \Gamma$$

The functions in the open unit ball $M(\mathbf{H}^+,\Gamma)$ of $L^\infty(\mathbf{H}^+,\Gamma)$ are called the Beltrami coefficients for Γ. Condition (1) implies that they project to the Beltrami coefficients on X.

There is a classical map $\mu \mapsto [\mu]_{T(X)}$ of $M(\mathbf{H}^+,\Gamma)$ onto $T(X)$ that is defined as follows. For each μ in $M(\mathbf{H}^+,\Gamma)$ let f^μ be the quasiconformal self-mapping of the Riemann sphere $\overline{\mathbb{C}}$ that maps $\mathbf{H}^+$ onto itself, fixes 0, 1, and ∞, has the Beltrami coefficient μ in $\mathbf{H}^+$, and satisfies $f^\mu(\bar z) = \overline{f^\mu(z)}$ for all z. The invariance condition (1) implies that

$$(2) \qquad \gamma^\mu = f^\mu \circ \gamma \circ (f^\mu)^{-1}$$

is a Möbius transformation for all γ in Γ.

The group $\Gamma^\mu = \{\gamma^\mu : \gamma \in \Gamma\}$ acts on $\mathbf{H}^+$, producing a quotient Riemann surface $X^\mu = \mathbf{H}^+/\Gamma$ such that the quotient map $\varpi^\mu\colon \mathbf{H}^+ \to X^\mu$ is a holomorphic universal covering. By (2), the map f^μ projects to a quasiconformal map w^μ of X onto X^μ that satisfies $w^\mu \circ \varpi = \varpi^\mu \circ f^\mu$. By definition, $[\mu]_{T(X)}$ is the Teichmüller equivalence class of w^μ.

Similarly, the map $\mu \mapsto [\mu]_{AT(X)}$ is defined by setting $[\mu]_{AT(X)}$ equal to the asymptotic equivalence class of w^μ.

The next proposition lists some well known properties of these maps.

Proposition 1. *The map $\mu \mapsto [\mu]_{T(X)}$ from $M(\mathbf{H}^+, \Gamma)$ to $T(X)$ is surjective and, for μ and ν in $M(\mathbf{H}^+, \Gamma)$, $[\mu]_{T(X)} = [\nu]_{T(X)}$ if and only if $f^\mu = f^\nu$ on $\widehat{\mathbb{R}}$.*

The map $\mu \mapsto [\mu]_{AT(X)}$ from $M(\mathbf{H}^+, \Gamma)$ to $AT(X)$ is surjective and, for μ and ν in $M(\mathbf{H}^+, \Gamma)$, $[\mu]_{AT(X)} = [\nu]_{AT(X)}$ if and only if there exist $\widetilde{\mu}$ and $\widetilde{\nu}$ in $M(\mathbf{H}^+, \Gamma)$ such that $[\widetilde{\mu}]_{T(X)} = [\mu]_{T(X)}$, $[\widetilde{\nu}]_{T(X)} = [\nu]_{T(X)}$, and $w^{\widetilde{\mu}} \circ (w^{\widetilde{\nu}})^{-1} \colon X^{\widetilde{\nu}} \to X^{\widetilde{\mu}}$ is asymptotically conformal.

Proof. The statements about $\mu \mapsto [\mu]_{T(X)}$ are proved in the previously cited books about Teichmüller spaces. The statements about $\mu \mapsto [\mu]_{AT(X)}$ follow readily from the definitions and are proved in [6], [7], and [8]. We shall prove them here for the reader's convenience.

The map $\mu \mapsto [\mu]_{AT(X)}$ is surjective because it is the composition of the surjective maps $\mu \mapsto [\mu]_{T(X)}$ and $\pi \colon T(X) \to AT(X)$.

Let $\widetilde{\mu}$ and $\widetilde{\nu}$ in $M(\mathbf{H}^+, \Gamma)$ satisfy $[\widetilde{\mu}]_{T(X)} = [\mu]_{T(X)}$ and $[\widetilde{\nu}]_{T(X)} = [\nu]_{T(X)}$. If the map $w^{\widetilde{\mu}} \circ (w^{\widetilde{\nu}})^{-1} \colon X^{\widetilde{\nu}} \to X^{\widetilde{\mu}}$ is asymptotically conformal, then obviously

$$[\mu]_{AT(X)} = [\widetilde{\mu}]_{AT(X)} = [\widetilde{\nu}]_{AT(X)} = [\nu]_{AT(X)}.$$

Conversely, if $[\mu]_{AT(X)} = [\nu]_{AT(X)}$ then there is a Teichmüller trivial quasiconformal map g of X onto itself such that the map $w^\mu \circ g \circ (w^\nu)^{-1} \colon X^\nu \to X^\mu$ is asymptotically conformal. Let $\widetilde{g}$ be a lift of g that fixes $\widehat{\mathbb{R}}$ pointwise. It is well known and easy to prove that $\widetilde{g} \circ \gamma = \gamma \circ \widetilde{g}$ for all γ in Γ. Therefore the Beltrami coefficient $\widetilde{\mu}$ of $f^\mu \circ \widetilde{g}$ in $\mathbf{H}^+$ belongs to $M(\mathbf{H}^+, \Gamma)$. Since $\widetilde{g}$ fixes $\widehat{\mathbb{R}}$ pointwise, $f^\mu \circ \widetilde{g} = f^{\widetilde{\mu}}$ in $\mathbf{H}^+$ and $f^\mu = f^{\widetilde{\mu}}$ on $\widehat{\mathbb{R}}$. Therefore $[\widetilde{\mu}]_{T(X)} = [\mu]_{T(X)}$. Set $\widetilde{\nu} = \nu$. Then $[\widetilde{\nu}]_{T(X)} = [\nu]_{T(X)}$ and $w^{\widetilde{\mu}} \circ (w^{\widetilde{\nu}})^{-1}$ is the asymptotically conformal map $w^\mu \circ g \circ (w^\nu)^{-1}$. $\square$

There is another useful criterion for equality of $[\mu]_{AT(X)}$ and $[\nu]_{AT(X)}$. To state it we require a definition.

Definition 1. For any μ in $L^\infty(\mathbf{H}^+)$ and any measurable subset S of $\mathbf{H}^+$ we denote the L^∞ norm of the restriction of μ to S by $||\mu||_S$. We say that μ in $L^\infty(\mathbf{H}^+, \Gamma)$ vanishes at infinity on $X (= \mathbf{H}^+/\Gamma)$ if and only if for every $\varepsilon > 0$ there is a compact set E in X such that $||\mu||_{\varpi^{-1}(X \setminus E)}$ is less than ε.

The set of μ that vanish at infinity on X is a closed subspace of $L^\infty(\mathbf{H}^+, \Gamma)$. We shall denote it by $L_0^\infty(\mathbf{H}^+, \Gamma)$.

Lemma 2.1. *For μ and ν in $M(\mathbf{H}^+, \Gamma)$, the map $w^\mu \circ (w^\nu)^{-1} \colon X^\nu \to X^\mu$ is asymptotically conformal if and only if $\mu - \nu$ belongs to $L_0^\infty(\mathbf{H}^+, \Gamma)$.*

Proof. Let λ be the Beltrami coefficient of $f^\mu \circ (f^\nu)^{-1}$ in $\mathbf{H}^+$. Then λ is a Beltrami coefficient for Γ^ν and $f^\mu \circ (f^\nu)^{-1} = f^\lambda$. By definition, $w^\lambda (= w^\mu \circ (w^\nu)^{-1})$ is asymptotically conformal if and only λ vanishes at infinity on X^ν.

By a well known chain rule calculation, which we shall use again in §9 and §11,

$$|\lambda \circ f^\nu| = \frac{|\mu - \nu|}{|1 - \overline{\nu}\mu|},$$

so λ vanishes at infinity on X^ν if and only if $\mu - \nu$ vanishes at infinity on X. $\square$

Combining Proposition 1 and Lemma 2.1 we obtain another known result (see Lemma 2.9 in [7]).

Corollary 2.1. *For μ and ν in $M(\mathbf{H}^+,\Gamma)$, $[\mu]_{AT(X)} = [\nu]_{AT(X)}$ if and only if there exist $\widetilde{\mu}$ and $\widetilde{\nu}$ in $M(\mathbf{H}^+,\Gamma)$ such that $[\widetilde{\mu}]_{T(X)} = [\mu]_{T(X)}$, $[\widetilde{\nu}]_{T(X)} = [\nu]_{T(X)}$, and $\widetilde{\mu} - \widetilde{\nu}$ vanishes at infinity on X.*

Remark 2. Corollary 2.1 implies that the map $\mu \mapsto [\mu]_{AT(X)}$ from $M(\mathbf{H}^+,\Gamma)$ to $AT(X)$ factors through the natural projection of $M(\mathbf{H}^+,\Gamma)$ onto the open unit ball of the quotient Banach space $L^\infty(\mathbf{H}^+,\Gamma)/L_0^\infty(\mathbf{H}^+,\Gamma)$.

Remark 3. If X equals $\mathbf{H}^+$, then the map $\varpi\colon \mathbf{H}^+ \to X$ is bijective and the group Γ of covering transformations is trivial. In this case $L^\infty(\mathbf{H}^+,\Gamma)$ is all of $L^\infty(\mathbf{H}^+)$, and $M(\mathbf{H}^+,\Gamma)$ is simply the open unit ball of $L^\infty(\mathbf{H}^+)$, which we shall denote by $M(\mathbf{H}^+)$.

Remark 4. For any Γ, the ball $M(\mathbf{H}^+,\Gamma)$ is a subset of $M(\mathbf{H}^+)$, so $M(\mathbf{H}^+,\Gamma)$ can be mapped into $T(\mathbf{H}^+)$ and $AT(\mathbf{H}^+)$ as well as to $T(X)$ and $AT(X)$. The classical first part of Proposition 1 implies that if μ and ν belong to $M(\mathbf{H}^+,\Gamma)$ then $[\mu]_{T(\mathbf{H}^+)} = [\nu]_{T(\mathbf{H}^+)}$ if and only if $[\mu]_{T(X)} = [\nu]_{T(X)}$. We can therefore, as is well known, regard $T(X)$ as a subspace of the "universal" Teichmüller space $T(\mathbf{H}^+)$.

The situation for the AT spaces is quite different. If X is not simply connected, then the group Γ has infinite order, so $L^\infty(\mathbf{H}^+,\Gamma)\cap L_0^\infty(\mathbf{H}^+)$ is the trivial subspace of $L^\infty(\mathbf{H}^+)$. Corollary 2.1 therefore implies that if μ and ν belong to $M(\mathbf{H}^+,\Gamma)$ then $[\mu]_{AT(\mathbf{H}^+)} = [\nu]_{AT(\mathbf{H}^+)}$ if and only if $[\mu]_{T(X)} = [\nu]_{T(X)}$. Therefore the quotient map $\pi\colon T(\mathbf{H}^+) \to AT(\mathbf{H}^+)$ maps $T(X)$ (viewed as a subspace of $T(\mathbf{H}^+)$) into $AT(\mathbf{H}^+)$ injectively.

3. Bers's Schwarzian derivative map

For each μ in $M(\mathbf{H}^+,\Gamma)$ let f_μ be the quasiconformal mapping of the Riemann sphere $\overline{\mathbb{C}}$ onto itself that fixes the points 0, 1, and ∞, is conformal in the lower half plane, and has the Beltrami coefficient μ in $\mathbf{H}^+$.

Let $\mathbf{H}^-$ be the lower half plane. Since f_μ is conformal in $\mathbf{H}^-$ we can form its Schwarzian derivative

$$\phi_\mu(z) = S(f_\mu)(z) = \frac{f_\mu'''(z)}{f_\mu'(z)} - \frac{3}{2}\left(\frac{f_\mu''(z)}{f_\mu'(z)}\right)^2, \qquad z \in \mathbf{H}^-.$$

It is well known (see for example Section 3.3 in Chapter 3 of [16]) that ϕ_μ belongs to the Banach space $B(\mathbf{H}^-,\Gamma)$ of holomorphic functions ϕ on $\mathbf{H}^-$ that have finite norm

$$\|\phi\|_B = \sup\{|z-\bar{z}|^2|\phi(z)| : z \in \mathbf{H}^-\}$$

and satisfy the Γ-invariance condition

$$(\phi\circ\gamma)(\gamma')^2 = \phi \qquad \text{for all } \gamma \in \Gamma$$

in $\mathbf{H}^-$. The elements of $B(\mathbf{H}^-,\Gamma)$ are called bounded holomorphic quadratic differentials for Γ. They project to quadratic differentials on the Riemann surface $X^* = \mathbf{H}^-/\Gamma$, which is conjugate to X.

A proof of the following fundamental theorem of Bers can be found in [2] or in Chapter 3 of [16].

Proposition 2 (Bers). *The formula*

$$\mathcal{S}(\mu) = \phi_\mu, \qquad \mu \in M(\mathbf{H}^+,\Gamma),$$

defines a holomorphic map $\mathcal{S}$ of $M(\mathbf{H}^+,\Gamma)$ onto a bounded open subset of $B(\mathbf{H}^-,\Gamma)$, and $\mathcal{S}(\mu)=\mathcal{S}(\nu)$ if and only if $[\mu]_{T(X)}=[\nu]_{T(X)}$. In addition, for each μ in $M(\mathbf{H}^+,\Gamma)$ there exist a neighborhood V_μ of $\mathcal{S}(\mu)$ and a holomorphic map s_μ of V_μ into $M(\mathbf{H}^+,\Gamma)$ such that $s_\mu(\mathcal{S}(\mu))=\mu$ and $\mathcal{S}(s_\mu(\phi))=\phi$ for all ϕ in V_μ.

Remark 5. By the implicit function theorem, the final statement in Proposition 2 is equivalent to the statement that for each μ in $M(\mathbf{H}^+,\Gamma)$ the $\mathbb{C}$-linear map $\mathcal{S}'(\mu)$ from $L^\infty(\mathbf{H}^+,\Gamma)$ to $B(\mathbf{H}^-,\Gamma)$ is surjective and has a continuous $\mathbb{C}$-linear right inverse. Any holomorphic map whose derivative at each point has such a right inverse is called a holomorphic split submersion.

Remark 6. For each ϕ in $B(\mathbf{H}^-,\Gamma)$ the function

$$\mu_\phi(z)=\phi(\bar z)|z-\bar z|^2,\qquad z\in\mathbf{H}^+, \tag{3}$$

is a Beltrami differential for Γ. The map $\phi\mapsto\mu_\phi$ is an isometry of $B(\mathbf{H}^-,\Gamma)$ onto a closed subspace of $L^\infty(\mathbf{H}^+,\Gamma)$.

Remark 7. When the group Γ of covering transformations is trivial we write $B(\mathbf{H}^-)$ instead of $B(\mathbf{H}^-,\Gamma)$. For any Γ the space $B(\mathbf{H}^-,\Gamma)$ is a closed subspace of $B(\mathbf{H}^-)$, and $\mathcal{S}\colon M(\mathbf{H}^+,\Gamma)\to B(\mathbf{H}^-,\Gamma)$ is simply the restriction to $M(\mathbf{H}^+,\Gamma)$ of the map $\mathcal{S}\colon M(\mathbf{H}^+)\to B(\mathbf{H}^-)$.

4. Barycentric extension

In [5] Douady and Earle defined a "barycentric extension" operator that extends homeomorphisms of the unit circle to homeomorphisms of the closed unit disk. To define the barycentric extension of a homeomorphism of the extended real axis $\widehat{\mathbb{R}}$ we shall map $\widehat{\mathbb{R}}$ and $\mathbf{H}^+$ to the unit circle and open unit disk by the Möbius transformation $A(z)=(z-i)/(z+i)$.

Definition 2. Let h be any homeomorphism of $\widehat{\mathbb{R}}$. Set $\varphi=A\circ h\circ A^{-1}$ and let $\widehat{\varphi}$ be the barycentric extension of φ defined in [5]. The *barycentric extension* of h is the homeomorphism $A^{-1}\circ\widehat{\varphi}\circ A$ of $\mathbf{H}^+\cup\widehat{\mathbb{R}}$ onto itself.

Definition 2 makes it easy to transfer many results of [5] to the $\widehat{\mathbb{R}}$ setting. The following proposition lists some of them. The conformal naturality condition (c), which says that barycentric extension respects composition with members of the group G of all Möbius transformations that map $\mathbf{H}^+$ onto itself, is crucial to our arguments.

Proposition 3. *Let* ex *be the map that sends each homeomorphism of $\widehat{\mathbb{R}}$ to its barycentric extension, and let φ be any homeomorphism of $\widehat{\mathbb{R}}$. Then*
(a) $\mathrm{ex}(\varphi)=\varphi$ *on* $\widehat{\mathbb{R}}$,
(b) $\mathrm{ex}(\varphi)$ *and its inverse are real-analytic in* $\mathbf{H}^+$,
(c) *if g and h belong to the group G, then* $\mathrm{ex}(g\circ\varphi\circ h)=g\circ\mathrm{ex}(\varphi)\circ h$,
(d) *if φ has a quasiconformal extension to $\overline{\mathbb{C}}$, then* $\mathrm{ex}(\varphi)$ *is quasiconformal in* $\mathbf{H}^+$.

As in [5], we obtain a "barycentric mapping" $\beta\colon M(\mathbf{H}^+)\to M(\mathbf{H}^+)$ from the barycentric extension operator in the following way. For any μ in $M(\mathbf{H}^+)$ the restriction to $\widehat{\mathbb{R}}$ of the quasiconformal map f^μ (see §2) is a homeomorphism φ^μ of $\widehat{\mathbb{R}}$. By Proposition 3(d), the barycentric extension $\mathrm{ex}(\varphi^\mu)$ of φ^μ is a quasiconformal map of $\mathbf{H}^+$ onto itself. We define $\beta(\mu)$ to be the Beltrami coefficient of $\mathrm{ex}(\varphi^\mu)$ in $M(\mathbf{H}^+)$. Observe that $f^{\beta(\mu)} = \mathrm{ex}(\varphi^\mu)$ in $\mathbf{H}^+ \cup \widehat{\mathbb{R}}$.

The following proposition was proved in [5] for the unit disk case. The statement that the map β is real-analytic means that each μ in $M(\mathbf{H}^+)$ has a neighborhood in which β is represented by a uniformly convergent power series

$$\beta(\mu+\nu) = \sum_{m=0}^{\infty} A_m(\nu, \nu, \ldots, \nu),$$

where the map A_m is continuous, symmetric, and m-linear (over $\mathbb{R}$) for each m. See Chapters 11 and 12 of [4].

Proposition 4. *The barycentric mapping $\beta\colon M(\mathbf{H}^+) \to M(\mathbf{H}^+)$ is real-analytic and satisfies the following additional conditions for all Beltrami coefficients μ and ν in $M(\mathbf{H}^+)$:*

(a) *the function $\beta(\mu)$ on $\mathbf{H}^+$ is real-analytic,*

(b) *$\beta(\mu) = \beta(\nu)$ if and only if $f^\mu = f^\nu$ on $\widehat{\mathbb{R}}$,*

(c) *$f^{\beta(\mu)} = f^\mu$ on $\widehat{\mathbb{R}}$,*

(d) *$\beta(\beta(\mu)) = \beta(\mu)$.*

Proof. Condition (a) follows from the fact that $f^{\beta(\mu)}$ and its inverse are real-analytic functions on $\mathbf{H}^+$. Conditions (b), (c), and (d) are trivial consequences of the definitions. To prove that β is a real-analytic map we shall express it in terms of the corresponding map in the unit disk case, which was denoted by σ in [5].

Let Δ be the open unit disk and let $M(\Delta)$ be the open unit ball in $L^\infty(\Delta)$. Given any μ in $M(\Delta)$, let F^μ be the unique homeomorphism of the closed unit disk that fixes the points -1. $-i$, and 1 and is quasiconformal in Δ with Beltrami coefficient μ.

The formula $A^*(\mu) = (\mu \circ A)\overline{A'}/A'$, μ in $L^\infty(\Delta)$, defines a $\mathbb{C}$-linear isometry of $L^\infty(\Delta)$ onto $L^\infty(\mathbf{H}^+)$. An elementary chain rule calculation shows that

$$f^{A^*(\mu)} = A^{-1} \circ F^\mu \circ A \tag{4}$$

in $\mathbf{H}^+$ for all μ in $M(\Delta)$. It follows readily from (4) and the definitions of β, σ, and the barycentric extensions that

$$f^{\beta(A^*(\mu))} = A^{-1} \circ F^{\sigma(\mu)} \circ A = f^{A^*(\sigma(\mu))}$$

in $\mathbf{H}^+$ for all μ in $M(\Delta)$. Therefore $\beta \circ A^* = A^* \circ \sigma$. Since σ is real-analytic (see [5]), so is β. □

The map $\mathcal{S}$ from $M(\mathbf{H}^+)$ to $B(\mathbf{H}^-)$ has a "barycentric section" s that is defined using the map β.

Lemma 4.1. *There is an injective real-analytic map s from the bounded open subset $\mathcal{S}(M(\mathbf{H}^+))$ of $B(\mathbf{H}^-)$ to $M(\mathbf{H}^+)$ such that $s \circ \mathcal{S} = \beta$ in $M(\mathbf{H}^+)$ and $\mathcal{S} \circ s$ is the identity in $\mathcal{S}(M(\mathbf{H}^+))$.*

Proof. For any μ and ν in $M(\mathbf{H}^+)$, Propositions 1, 2, and 4 imply that $\mathcal{S}(\mu) = \mathcal{S}(\nu)$ if and only if $\beta(\mu) = \beta(\nu)$. The formula $\beta = s \circ \mathcal{S}$ therefore defines an injective map $s\colon \mathcal{S}(M(\mathbf{H}^+)) \to M(\mathbf{H}^+)$. Since s is injective, the equations

$$s \circ \mathcal{S} = \beta = \beta \circ \beta = s \circ \mathcal{S} \circ s \circ \mathcal{S}$$

imply that $\mathcal{S} \circ s$ is the identity in $\mathcal{S}(M(\mathbf{H}^+))$. In other words s is a section of $\mathcal{S}$.

To prove the analyticity, let $\phi = \mathcal{S}(\mu)$, μ in $M(\mathbf{H}^+)$, be a point in the domain of s. By Propostion 2 there exist a neighborhood V_μ of ϕ and a holomorphic map $s_\mu\colon V_\mu \to M(\mathbf{H}^+)$ such that $\mathcal{S}\circ s_\mu$ is the identity in V_μ. Then $s = s\circ\mathcal{S}\circ s_\mu = \beta\circ s_\mu$ in V_μ, so s is locally the composition of a holomorphic and a real-analytic map. □

Remark 8. Any Möbius transformation that maps $\mathbf{H}^+$ onto Δ has the form $A\circ g$, g in G, so Proposition 3(c) implies that the operator ex and the maps β and s do not depend on our particular choice of A in Definition 2.

5. Convergence of homeomorphisms

Let $\mathcal{H}(\mathbb{C})$ be the group of homeomorphisms of $\mathbb{C}$, with the (metrizable) topology of uniform convergence on compact sets, and let $\mathcal{H}(\overline{\mathbb{C}})$ be the group of homeomorphisms of $\overline{\mathbb{C}}$ with the (metric) topology of uniform convergence. We use the Euclidean metric for $\mathbb{C}$ and the spherical metric for $\overline{\mathbb{C}}$.

It is well known (see [1]) that these topologies make $\mathcal{H}(\mathbb{C})$ and $\mathcal{H}(\overline{\mathbb{C}})$ topological groups and that the natural identification of $\mathcal{H}(\mathbb{C})$ with the closed subgroup of $\mathcal{H}(\overline{\mathbb{C}})$ that fixes the point at infinity is a homeomorphism. For the reader's convenience we shall prove the last of these statements. In the following proposition and its proof $d_{\overline{\mathbb{C}}}$ will be the spherical metric.

Proposition 5. *Let $\{f_n\}$ be a sequence in $\mathcal{H}(\overline{\mathbb{C}})$ that converges pointwise to some f in $\mathcal{H}(\overline{\mathbb{C}})$. Suppose $f_n(\infty) = \infty$ for all n. Then $|f_n(z) - f(z)| \to 0$ uniformly on compact subsets of $\mathbb{C}$ if and only if $d_{\overline{\mathbb{C}}}(f_n(z), f(z)) \to 0$ uniformly on $\overline{\mathbb{C}}$.*

Proof. We shall use the notion of continuous convergence. Recall that by definition a sequence of maps f_n from $\mathbb{C}$ or $\overline{\mathbb{C}}$ to itself converges continuously to the map f if $f_n(z_n) \to f(z)$ whenever $z_n \to z$ in $\mathbb{C}$ or $\overline{\mathbb{C}}$. For sequences of continuous maps of $\mathbb{C}$ into itself continuous convergence is equivalent to uniform convergence on compact sets (with respect to the Euclidean metric). For sequences of continuous maps of $\overline{\mathbb{C}}$ into itself continuous convergence is equivalent to uniform convergence on compact sets (with respect to the spherical metric). See §§174–180 in [3].

Suppose $d_{\overline{\mathbb{C}}}(f_n(z), f(z)) \to 0$ uniformly on $\overline{\mathbb{C}}$. Then f_n converges continuously to f in $\overline{\mathbb{C}}$, hence in $\mathbb{C}$, so $|f_n(z) - f(z)| \to 0$ uniformly on compact subsets of $\mathbb{C}$.

Conversely, let f_n converge continuously to f in $\mathbb{C}$. We must show that f_n converges continuously to f in $\overline{\mathbb{C}}$. Let $z_n \to z$ in $\overline{\mathbb{C}}$ and let $f_n(z_n) \to w$ in $\overline{\mathbb{C}}$. We must show that $w = f(z)$. If $z \neq \infty$ then $w = f(z)$ by hypothesis. Suppose $z = \infty$. If $w \neq f(\infty)(= \infty)$, then (since f_n^{-1} converges continuously to f^{-1} in $\mathbb{C}$),

$$z_n = f_n^{-1}(f_n(z_n)) \to f^{-1}(w) \neq \infty,$$

which is impossible. □

6. Convergence of quasiconformal mappings

The following lemma states a useful continuity property of Bers's Schwarzian derivative map $\mathcal{S}$ and the barycentric map β. Its proof uses Proposition 5.

Lemma 6.1. *Let $\{\mu_n\}$ be a sequence in $M(\mathbf{H}^+)$. Suppose there is a number k such that $\|\mu_n\|_\infty \leq k < 1$ for all n. Suppose the sequence $\{f^{\mu_n}\}$ converges uniformly on compact subsets of $\mathbb{C}$ to a quasiconformal map f^μ, μ in $M(\mathbf{H}^+)$. Then $f_{\mu_n} \to f_\mu$, $\phi_{\mu_n} \to \phi_\mu$, and $\beta(\mu_n) \to \beta(\mu)$ as $n \to \infty$. The convergence is uniform on compact sets in $\mathbb{C}$, $\mathbf{H}^-$, and $\mathbf{H}^+$ respectively.*

Remark 9. We do not assume that $\mu_n \to \mu$ in any sense except that $f^{\mu_n} \to f^{\mu}$ uniformly on compact sets in $\mathbb{C}$ as $n \to \infty$.

Proof. Since the quasiconformal maps f_{μ_n} form a normal family (see [13]) we may assume that $\{f_{\mu_n}\}$ converges uniformly on compact sets in $\mathbb{C}$ as $n \to \infty$ to some quasiconformal mapping f of $\mathbb{C}$ onto itself. We must prove that $f = f_\mu$.

Let $g_n = f_{\mu_n} \circ (f^{\mu_n})^{-1}$ for each n. Since $\mathcal{H}(\mathbb{C})$ is a topological group, $\{g_n\}$ converges to $f \circ (f^{\mu})^{-1}$ uniformly on compact sets in $\mathbb{C}$. Each g_n is conformal in $\mathbf{H}^+$, so $f \circ (f^{\mu})^{-1}$ is conformal in $\mathbf{H}^+$. Therefore the Beltrami coefficient of f in $\mathbf{H}^+$ is μ. Since f fixes 0, 1, and ∞ and is conformal in $\mathbf{H}^-$, $f = f_\mu$ as required.

Since the convergence of f_{μ_n} to f_μ is uniform on compact subsets of $\mathbb{C}$, it is obvious that $\phi_{\mu_n} = S(f_{\mu_n}) \to S(f_\mu) = \phi_\mu$ as $n \to \infty$, uniformly on compact sets in $\mathbf{H}^-$.

Now Proposition 5 implies that the convergence of f^{μ_n} to f^{μ} is uniform in $\overline{\mathbb{C}}$ with respect to the spherical metric. Proposition 2 of [5] therefore implies that the barycentric extensions $f^{\beta(\mu_n)}$ converge to $f^{\beta(\mu)}$ as C^∞ functions on $\mathbf{H}^+$. In particular $\partial f^{\beta(\mu_n)}/\partial \bar z \to \partial f^{\beta(\mu)}/\partial \bar z$ and $\partial f^{\beta(\mu_n)}/\partial z \to \partial f^{\beta(\mu)}/\partial z$ as $n \to \infty$, uniformly on compact sets in $\mathbf{H}^+$. By Proposition 3(b), the Jacobian of $f^{\beta(\mu)}$ is positive at all points of $\mathbf{H}^+$, so $\beta(\mu_n) \to \beta(\mu)$ as required. □

7. Some remarks about vanishing at infinity

The space $L_0^\infty(\mathbf{H}^+, \Gamma)$ of Beltrami differentials for Γ that vanish at infinity on the hyperbolic Riemann surface X was defined in §2 (see Definition 1 and the paragraph that follows it). We shall need some additional properties of these Beltrami differentials.

Let d be the hyperbolic metric $d(z, w) = \tanh^{-1}\left|\frac{z-w}{z-\bar w}\right|$ on $\mathbf{H}^+$. Since Γ is a discrete group of hyperbolic isometries, d induces a quotient hyperbolic metric d_X on X that satisfies

$$(5) \qquad d_X(\varpi(a), q) = \min\{d(a, z) : \varpi(z) = q\} \qquad \text{for all } a \in \mathbf{H}^+ \text{ and } q \in X.$$

It is well known that d_X is a complete Riemannian metric of constant curvature -4.

For any a in $\mathbf{H}^+$ and $r > 0$ let $\bar D_{\mathbf{H}^+}(a; r)$ and $\bar D_X(\varpi(a); r)$ be the closed hyperbolic balls of radius r and centers a and $\varpi(a)$ respectively. One way to verify the completeness of d_X is to observe that $\bar D_X(\varpi(a); r)$ is compact because, by (5), it is the image of $\bar D_{\mathbf{H}^+}(a; r)$ under ϖ.

Let $\{p_n\}$ be a sequence in X. We say p_n approaches infinity in X and write $p_n \to \infty$ if and only if $\lim_{n\to\infty} d_X(p, p_n) = \infty$ for some, hence all, p in X. With these definitions the following lemma is obvious.

Lemma 7.1. *Let μ be a Beltrami differential for Γ. If μ is continuous on $\mathbf{H}^+$ then μ vanishes at infinity on X if and only if $\lim_{n\to\infty} \mu(z_n) = 0$ for every sequence $\{z_n\}$ in $\mathbf{H}^+$ such that $\varpi(z_n) \to \infty$.*

We shall need a similar result that applies to all μ in $L^\infty(\mathbf{H}^+, \Gamma)$, continuous or not. It will be a corollary of the following lemma.

Lemma 7.2. *Let $r > 0$ be given and let $\{z_n\}$ be a sequence in $\mathbf{H}^+$ such that $\varpi(z_n) \to \infty$. If E is a compact set in X, then $\varpi(\bar D_{\mathbf{H}^+}(z_n; r)) \subset X \setminus E$ for all but finitely many n.*

Proof. Suppose $p \in E$ and $p_n \in E \cap \varpi(\bar{D}_{\mathbf{H}^+}(z_n;r))$. If c is the (hyperbolic) diameter of E, then $d_X(p,\varpi(z_n)) \le d_X(p,p_n)+d_X(p_n,\varpi(z_n)) \le c+r$. That can happen for only finitely many n because $d_X(p,\varpi(z_n)) \to \infty$ as $n \to \infty$. □

Corollary 7.1. *Let μ be a Beltrami differential for Γ and let r be a positive real number. Then μ vanishes at infinity on X if and only if $\lim_{n\to\infty} \|\mu\|_{\bar{D}_{\mathbf{H}^+}(z_n;r)} = 0$ for every sequence $\{z_n\}$ in $\mathbf{H}^+$ such that $\varpi(z_n) \to \infty$.*

Proof. Suppose μ vanishes at infinity on X. If $\varepsilon > 0$ is given, there is a compact set $E \subset X$ such that $\|\mu\|_{\varpi^{-1}(X\setminus E)} < \varepsilon$. If $\varpi(z_n) \to \infty$, Lemma 7.2 gives us an N such that $\bar{D}_{\mathbf{H}^+}(z_n;r) \subset \varpi^{-1}(X \setminus E)$ for $n > N$. Therefore $\|\mu\|_{\bar{D}_{\mathbf{H}^+}(z_n;r)} < \varepsilon$ if $n > N$, so $\|\mu\|_{\bar{D}_{\mathbf{H}^+}(z_n;r)} < \varepsilon \to 0$.

If μ does not vanish at infinity on X choose an $\varepsilon > 0$ so that $\|\mu\|_{\varpi^{-1}(X\setminus E)} \ge 2\varepsilon$ for every compact set E in X. Choose p in X. Since the closed ball $\bar{D}_X(p,n)$ is compact for each $n \ge 1$, the set of points in $\varpi^{-1}(X \setminus \bar{D}_X(p,n))$ where $|\mu| \ge \varepsilon$ has positive measure. We can therefore choose a point z_n in $\varpi^{-1}(X \setminus \bar{D}_X(p,n))$ such that $\|\mu\|_{\bar{D}_{\mathbf{H}^+}(z_n;r)} \ge \varepsilon$. Then $\varpi(z_n) \to \infty$ but $\|\mu\|_{\bar{D}_{\mathbf{H}^+}(z_n;r)} \not\to 0$. □

Definition 3. A quadratic differential ϕ in $B(\mathbf{H}^-,\Gamma)$ vanishes at infinity on X^* if and only if the Beltrami differential μ_ϕ defined by (3) vanishes at infinity on X.

Remark 10. Obviously, ϕ vanishes at infinity on X^* if and only if μ_ϕ has the properties specified in Lemma 7.1 or Corollary 7.1.

8. G-equivariance of the Bers and barycentric mappings

It is well known that the group G acts on both $L^\infty(\mathbf{H}^+)$ and $B(\mathbf{H}^-)$ as a group of isometries via the formulas

$$
\begin{aligned}
g^*(\phi) &= (\phi \circ g)(g')^2 \qquad \text{if } \phi \in B(\mathbf{H}^-) \text{ and } g \in G,\\
g^*(\mu) &= (\mu \circ g)\overline{g'}/g' \qquad \text{if } \mu \in L^\infty(\mathbf{H}^+) \text{ and } g \in G.
\end{aligned}
\tag{6}
$$

Clearly, if Γ is the group of covering transformations for some holomorphic universal covering map $\varpi\colon \mathbf{H}^+ \to X$, then ϕ in $B(\mathbf{H}^-)$ belongs to $B(\mathbf{H}^-,\Gamma)$ if and only if $\gamma^*(\phi) = \phi$ for all γ in Γ. Similarly μ in $L^\infty(\mathbf{H}^+)$ belongs to $L^\infty(\mathbf{H}^+,\Gamma)$ if and only if $\gamma^*(\mu) = \mu$ for all γ in Γ.

Bers's Schwarzian derivative map $\mathcal{S}(\mu) = \phi_\mu$ (defined in §3), its barycentric section s, and the barycentric map $\beta(\mu)$ (both defined in §4) are all G-equivariant. Though the proofs of these facts are well known (see Lemma 10 in Bers's paper [2] and Lemmas 4 and 5 in [5]), we shall sketch them here.

Proposition 6. *For any g in G, μ in $M(\mathbf{H}^+)$, and ϕ in $\mathcal{S}(M(\mathbf{H}^+))$, we have*

$$\phi_{g^*(\mu)} = g^*(\phi_\mu), \qquad \beta(g^*(\mu)) = g^*(\beta(\mu)), \qquad \textit{and} \qquad s(g^*(\phi)) = g^*(s(\phi)).$$

Proof. Let g and μ be given. To prove the first formula choose a Möbius transformation g_μ so that $g_\mu^{-1} \circ f_\mu \circ g$ fixes 0, 1, and ∞. Then $g_\mu \circ f_{g^*(\mu)} = f_\mu \circ g$ in $\overline{\mathbb{C}}$. In $\mathbf{H}^-$, classical properties of the Schwarzian derivative give

$$\phi_{g^*(\mu)} = S(f_{g^*(\mu)}) = S(g_\mu \circ f_{g^*(\mu)}) = S(f_\mu \circ g) = (S(f_\mu)\circ g)(g')^2 = g^*(\phi_\mu).$$

For the second formula choose h in G so that $h \circ f^\mu \circ g$ fixes 0, 1, and ∞. Then $f^{g^*(\mu)} = h \circ f^\mu \circ g$ in $\overline{\mathbb{C}}$, hence on $\widehat{\mathbb{R}}$. Proposition 3(c) and the definition of β give

$$f^{\beta(g^*(\mu))} = h \circ f^{\beta(\mu)} \circ g$$

in $\mathbf{H}^+$. Computing the Beltrami coefficients of both sides of that identity we obtain the desired result.

Finally, given g in G and ϕ in $\mathcal{S}(M(\mathbf{H}^+))$, choose μ in $M(\mathbf{H}^+)$ so that $\phi = \phi_\mu$. Then

$$s(g^*(\phi)) = s(g^*(\phi_\mu)) = s(\phi_{g^*(\mu)}) = \beta(g^*(\mu)) = g^*(\beta(\mu)) = g^*(s(\phi_\mu)). \quad \square$$

Corollary 8.1. *Let the group Γ be as above. If μ belongs to $M(\mathbf{H}^+,\Gamma)$, then so does $\beta(\mu)$. If ϕ belongs to $\mathcal{S}(M(\mathbf{H}^+)) \cap B(\mathbf{H}^-,\Gamma)$, then $s(\phi)$ belongs to $M(\mathbf{H}^+,\Gamma)$. In particular, $\mathcal{S}(M(\mathbf{H}^+)) \cap B(\mathbf{H}^-,\Gamma) = \mathcal{S}(M(\mathbf{H}^+,\Gamma))$.*

Proof. Only the final sentence requires explanation. Proposition 2 (or the G-equivariance of $\mathcal{S}$) guarantees that $\mathcal{S}$ maps $M(\mathbf{H}^+,\Gamma)$ into $\mathcal{S}(M(\mathbf{H}^+)) \cap B(\mathbf{H}^-,\Gamma)$. To see that $\mathcal{S}$ maps $M(\mathbf{H}^+,\Gamma)$ onto $\mathcal{S}(M(\mathbf{H}^+)) \cap B(\mathbf{H}^-,\Gamma)$ we observe that if ϕ belongs to $\mathcal{S}(M(\mathbf{H}^+)) \cap B(\mathbf{H}^-,\Gamma)$ then $s(\phi) \in M(\mathbf{H}^+,\Gamma)$ and $\mathcal{S}(s(\phi)) = \phi$. $\square$

Remark 11. The statement that $\mathcal{S}(M(\mathbf{H}^+)) \cap B(\mathbf{H}^-,\Gamma) = \mathcal{S}(M(\mathbf{H}^+,\Gamma))$ was first proved by Kra in [11] for finitely generated groups Γ and by Tukia in [17] for arbitrary Fuchsian groups. The short proof given above was first described in [5].

9. The main theorem

Theorem 4. *Let μ and ν be Beltrami coefficients for Γ. The following statements are equivalent:*
(a) *$\beta(\mu) - \beta(\nu)$ vanishes at infinity on X,*
(b) *$[\mu]_{AT(X)} = [\nu]_{AT(X)}$,*
(c) *$\phi_\mu - \phi_\nu$ vanishes at infinity on X^*.*

Proof. (a) $\Longrightarrow$ (b): Propositions 1 and 4 imply that $[\beta(\mu)]_{T(X)} = [\mu]_{T(X)}$ and $[\beta(\nu)]_{T(X)} = [\nu]_{T(X)}$, so (b) follows from (a) and Corollary 2.1.

(b) $\Longrightarrow$ (c): Since $\phi_\mu = \phi_{\widetilde{\mu}}$ when $[\mu]_{T(X)} = [\widetilde{\mu}]_{T(X)}$, condition (b) and Corollary 2.1 allow us to assume that $\mu - \nu$ vanishes at infinity on X.

We shall use Lemma 7.1 and Corollary 7.1 to prove that $\phi_\mu - \phi_\nu$ vanishes at infinity on X^*. Let $\{z_n\}$ be a sequence in $\mathbf{H}^+$ such that $\varpi(z_n) \to \infty$. For each n let C_n be an automorphism of $\mathbf{H}^+$ that maps the point $i = \sqrt{-1}$ to z_n. Define μ_n and ν_n in $M(\mathbf{H}^+)$ by

$$\mu_n = C_n^*(\mu) \quad \text{and} \quad \nu_n = C_n^*(\nu). \tag{7}$$

Since $\mu - \nu$ vanishes at infinity on X, Corollary 7.1 and equations (6) and (7) imply that

$$\lim_{n\to\infty} \|\mu_n - \nu_n\|_{\bar{D}_{\mathbf{H}^+}(i;r)} = \lim_{n\to\infty} \|\mu - \nu\|_{\bar{D}_{\mathbf{H}^+}(z_n;r)} = 0 \qquad \text{for all } r > 0. \tag{8}$$

Equation (7) also implies that all the mappings f^{μ_n} and f^{ν_n} are K-quasiconformal, with K independent of n. Therefore, by passing to a subsequence if necessary, we can assume that $f^{\mu_n} \to f^{\mu_0}$ and $f^{\nu_n} \to f^{\nu_0}$ uniformly on $\overline{\mathbb{C}}$ as $n \to \infty$, for some μ_0 and ν_0 in $M(\mathbf{H}^+)$.

Let λ_n be the Beltrami coefficient of $f^{\mu_n} \circ (f^{\nu_n})^{-1}$. The chain rule gives

$$|\lambda_n \circ f^{\nu_n}| = \frac{|\mu_n - \nu_n|}{|1 - \overline{\nu_n}\,\mu_n|} \tag{9}$$

in $\mathbf{H}^+$.

Given a compact set E in $\mathbf{H}^+$, we choose $r > 0$ so large that $(f^{\nu_0})^{-1}(E)$ is contained in the open hyperbolic ball $D(i;r)$. Since $(f^{\nu_n})^{-1} \to (f^{\nu_0})^{-1}$ uniformly in $\overline{\mathbb{C}}$ as $n \to \infty$, we will have $(f^{\nu_n})^{-1}(E) \subset D(i;r)$ for all sufficiently large n. Equations (8) and (9) therefore imply that $\lim_{n\to\infty} \|\lambda_n\|_E = 0$.

Since E is arbitrary, the limit $f^{\mu_0} \circ (f^{\nu_0})^{-1}$ of the sequence of maps $f^{\mu_n} \circ (f^{\nu_n})^{-1}$ is conformal in $\mathbf{H}^+$. Since it maps $\mathbf{H}^+$ onto itself and fixes 0, 1, and ∞, it is the identity. Therefore $f^{\mu_0} = f^{\nu_0}$ in $\mathbf{H}^+$, and by symmetry the sequences $\{f^{\mu_n}\}$ and $\{f^{\nu_n}\}$ converge uniformly in $\overline{\mathbb{C}}$ to the same limit f^{μ_0}.

Therefore, by Lemma 6.1, the sequences $\{\phi_{\mu_n}\}$ and $\{\phi_{\nu_n}\}$ converge uniformly on compact sets in $\mathbf{H}^-$ to the same limit ϕ_{μ_0}, and $\phi_{\mu_n} - \phi_{\nu_n} \to 0$ uniformly on compact sets in $\mathbf{H}^-$ as $n \to \infty$. In particular

$$\lim_{n\to\infty} |\phi_{\mu_n}(-i) - \phi_{\nu_n}(-i)| = 0. \tag{10}$$

Now equation (7) and Proposition 6 give

$$\phi_{\mu_n} = C_n^*(\phi_\mu) \quad \text{and} \quad \phi_{\nu_n} = C_n^*(\phi_\nu) \tag{11}$$

in $\mathbf{H}^-$. Since $C_n(-i) = \overline{z_n}$ and $|C_n'(-i)| = |z_n - \overline{z_n}|/2$, equations (6), (10), and (11) imply the equation

$$\lim_{n\to\infty} \left[|\phi_\mu(\overline{z_n}) - \phi_\nu(\overline{z_n})| \times |z_n - \overline{z_n}|^2 \right] = 0.$$

Since this holds on a subsequence of any sequence in $\mathbf{H}^+$ whose image under ϖ tends to infinity on X, Lemma 7.1 implies that $\phi_\mu - \phi_\nu$ vanishes at infinity on X^*.

(c) $\Longrightarrow$ (a): We shall again use Lemma 7.1 and Corollary 7.1. Let $\{z_n\}$ be a sequence in $\mathbf{H}^+$ such that $\varpi(z_n) \to \infty$. For each n choose an automorphism C_n of $\mathbf{H}^+$ as before and define μ_n and ν_n in $M(\mathbf{H}^+)$ by the formulas (7). Since $\phi_\mu - \phi_\nu$ vanishes at infinity on X^*, Corollary 7.1 and equation (11) imply that

$$\sup\left\{ |\phi_{\mu_n}(\bar z) - \phi_{\nu_n}(\bar z)| \times |z - \bar z|^2 : z \in \bar D_{\mathbf{H}^+}(i;r) \right\} \to 0 \text{ as } n \to \infty$$

for all $r > 0$, so

$$\lim_{n\to\infty} (\phi_{\mu_n}(z) - \phi_{\nu_n}(z)) = 0 \qquad \text{for all } z \in \mathbf{H}^-. \tag{12}$$

Again we pass if necessary to a subsequence so that $f^{\mu_n} \to f^{\mu_0}$ and $f^{\nu_n} \to f^{\nu_0}$ uniformly on $\overline{\mathbb{C}}$ as $n \to \infty$, for some μ_0 and ν_0 in $M(\mathbf{H}^+)$.

Lemma 6.1 again implies that $\phi_{\mu_n} \to \phi_{\mu_0}$ and $\phi_{\nu_n} \to \phi_{\nu_0}$ uniformly on compact sets in $\mathbf{H}^-$ as $n \to \infty$. Equation (12) therefore implies that $\phi_{\mu_0} = \phi_{\nu_0}$ in $\mathbf{H}^-$. This implies that $[\mu_0]_{\mathbf{H}^+} = [\nu_0]_{\mathbf{H}^+}$, by Proposition 2 (with Γ equal to the trivial group). Therefore, by Propositions 1 and 4, $\beta(\mu_0) = \beta(\nu_0)$.

Since Lemma 6.1 also implies that $\beta(\mu_n) \to \beta(\mu_0)$ and $\beta(\nu_n) \to \beta(\nu_0)$ uniformly on compact subsets of $\mathbf{H}^+$ as $n \to \infty$, the equation $\beta(\mu_0) = \beta(\nu_0)$ implies that

$$\lim_{n\to\infty} |\beta(\mu_n)(i) - \beta(\nu_n)(i)| = 0.$$

Now equation (7) and Proposition 6 imply that $\beta(\mu_n) = \beta(C_n^*(\mu)) = C_n^*(\beta(\mu))$ for all n. Similarly $\beta(\nu_n) = C_n^*(\beta(\nu))$ for all n, so

$$\lim_{n\to\infty} |\beta(\mu)(z_n) - \beta(\nu)(z_n)| = \lim_{n\to\infty} |\beta(\mu_n)(i) - \beta(\nu_n)(i)| = 0. \quad \square$$

10. Proof of Theorem 1

Theorem 1 is an immediate consequence of our Theorem 4 and the results in §2.3 of [7], which we shall now summarize (see also [6] and Chapter 14 of [8]).

As usual, X is a hyperbolic Riemann surface and Γ is the group of covering transformations for a holomorphic universal covering map $\varpi\colon \mathbf{H}^+ \to X$. The set of quadratic differentials in $B(\mathbf{H}^-, \Gamma)$ that vanish at infinity on X^* is a closed subspace that we denote by $B_0(\mathbf{H}^-, \Gamma)$. We denote the quotient space $B(\mathbf{H}^-, \Gamma)/B_0(\mathbf{H}^-, \Gamma)$ by $\widehat{B}(\Gamma)$ and the quotient map from $B(\mathbf{H}^-, \Gamma)$ to $\widehat{B}(\Gamma)$ by P_B.

It is shown in [6] and [7] that the formula

$$\widehat{\mathfrak{B}}([\mu]_{AT(X)}) = P_B(\mathcal{S}(\mu)), \qquad \mu \in M(\mathbf{H}^+, \Gamma), \tag{13}$$

produces a well-defined locally injective map $\widehat{\mathfrak{B}}$ of $AT(X)$ onto a bounded open subset of $\widehat{B}(\Gamma)$ and that $AT(X)$ has a unique complex manifold structure such that the map $\widehat{\mathfrak{B}}$ is locally biholomorphic.

The map $\widehat{\mathfrak{B}}$ defined by (13) is the asymptotic Bers map referred to in the Introduction. Since $\widehat{\mathfrak{B}}\colon AT(X) \to \widehat{B}(\Gamma)$ is locally biholomorphic and its image is a bounded open set, the proof of Theorem 1 is reduced to showing that $\widehat{\mathfrak{B}}$ is an injective map. The required injectivity follows immediately from Theorem 4. Indeed, equation (13) implies that if μ and ν belong to $M(\mathbf{H}^+, \Gamma)$ and $\widehat{\mathfrak{B}}([\mu]_{AT(X)}) = \widehat{\mathfrak{B}}([\nu]_{AT(X)})$, then $\mathcal{S}(\mu) - \mathcal{S}(\nu)(= \phi_\mu - \phi_\nu)$ vanishes at infinity on X^*. By Theorem 4, this implies that $[\mu]_{AT(X)} = [\nu]_{AT(X)}$. ☐

Remark 12. The proof we have just presented used only one part (the implication (c) $\Longrightarrow$ (b)) of our main theorem. The implication (b) $\Longrightarrow$ (c) in that theorem provides a new proof that the map $\widehat{\mathfrak{B}}$ is well defined by formula (13).

11. The asymptotic Schwarzian derivative map and its asymptotic barycentric section

Recall that $L_0^\infty(\mathbf{H}^+, \Gamma)$ is the closed subspace of $L^\infty(\mathbf{H}^+, \Gamma)$ consisting of the Beltrami differentials μ that vanish at infinity on X. We denote the quotient space $L^\infty(\mathbf{H}^+, \Gamma)/L_0^\infty(\mathbf{H}^+, \Gamma)$ by $\widehat{L^\infty}(\Gamma)$, the quotient map from $L^\infty(\mathbf{H}^+, \Gamma)$ to $\widehat{L^\infty}(\Gamma)$ by P_L, and the open unit ball in $\widehat{L^\infty}(\Gamma)$ by $\widehat{M}(\Gamma)$. Observe that $\widehat{M}(\Gamma)$ is the image of $M(\mathbf{H}^+, \Gamma)$ under P_L.

Our next theorem concerns the barycentric section s of Bers's Schwarzian derivative map, which we defined in Lemma 4.1. According to Corollary 8.1, s maps $\mathcal{S}(M(\mathbf{H}^+, \Gamma))$ into $M(\mathbf{H}^+, \Gamma)$.

Theorem 5. *The map $s\colon \mathcal{S}(M(\mathbf{H}^+, \Gamma)) \to M(\mathbf{H}^+, \Gamma)$ induces a real-analytic map $\widehat{s}\colon P_B(\mathcal{S}(M(\mathbf{H}^+, \Gamma))) \to \widehat{M}(\Gamma)$ such that the diagram*

$$\begin{array}{ccc}
\mathcal{S}(M(\mathbf{H}^+, \Gamma)) & \xrightarrow{s} & M(\mathbf{H}^+, \Gamma) \\
{\scriptstyle P_B}\big\downarrow & & {\scriptstyle P_L}\big\downarrow \\
P_B(\mathcal{S}(M(\mathbf{H}^+, \Gamma))) & \xrightarrow{\widehat{s}} & \widehat{M}(\Gamma)
\end{array} \tag{14}$$

commutes.

Proof. Let μ and ν belong to $M(\mathbf{H}^+, \Gamma)$.

If $P_B(\phi_\mu) = P_B(\phi_\nu)$, then Theorem 4 and the definition of s give

$$P_L(s(\phi_\mu)) = P_L(\beta(\mu)) = P_L(\beta(\nu)) = P_L(s(\phi_\nu)),$$

so the formula $\widehat{s}(P_B(\phi_\mu)) = P_L(s(\phi_\mu))$ produces a well-defined map from the set $P_B(\mathcal{S}(M(\mathbf{H}^+,\Gamma)))$ to $\widehat{M}(\Gamma)$ that makes the diagram (14) commute.

The domain $P_B(\mathcal{S}(M(\mathbf{H}^+,\Gamma)))$ of $\widehat{s}$ is open in $\widehat{B}(\Gamma)$ because P_B is an open map and $\mathcal{S}(M(\mathbf{H}^+,\Gamma))$ is open in $B(\mathbf{H}^-,\Gamma)$. To prove that $\widehat{s}$ is real-analytic in its domain, consider the power series expansion

$$\widehat{s}(P_B(\phi_\mu) + P_B(\phi)) = \sum_{m=0}^{\infty} A_m(\phi,\phi,\ldots,\phi) \tag{15}$$

at ϕ_μ of the real-analytic map $\widehat{s}\circ P_B = P_L \circ s$.

The expansion (15) is valid for ϕ sufficiently close to zero. It provides a power series representation of $\widehat{s}$ at $P_B(\phi_\mu)$ if, for each $m > 0$, there is a map $\widehat{A}_m$ from $\widehat{B}(\Gamma)^m$ to $L^\infty(\mathbf{H}^+,\Gamma)/L_0^\infty(\mathbf{H}^+,\Gamma)$ that is continuous, symmetric, and m-linear over $\mathbb{R}$ and satisfies

$$\widehat{A}_m(P_B(\phi_1), P_B(\phi_2), \ldots, P_B(\phi_m)) = A_m(\phi_1,\phi_2,\cdots,\phi_m) \tag{16}$$

for all $(\phi_1,\phi_2,\cdots,\phi_m)$ in $B(\mathbf{H}^-,\Gamma)$. Equation (16) itself defines such a map $\widehat{A}_m$ if

$$A_m(\phi_1,\phi_2,\cdots,\phi_m) = 0 \quad \text{whenever some } \phi_j \text{ belongs to } B_0(\mathbf{H}^-,\Gamma). \tag{17}$$

Since $B_0(\mathbf{H}^-,\Gamma)$ is the kernel of P_B, the directional derivative of $\widehat{s}\circ P_B$ at ϕ_μ in the direction ϕ is zero for all ϕ in $B_0(\mathbf{H}^-,\Gamma)$. Therefore (17) holds, and $\widehat{s}$ is real-analytic. □

Theorems 4 and 5, together with some classical Teichmüller theory, make possible a new proof of the following basic result, first proved by Earle, Gardiner, and Lakic in [7]. We shall present it here, since it has some independent interest.

Proposition 7. *Bers's Schwarzian derivative map $\mathcal{S}$ induces a holomorphic split submersion $\widehat{\mathcal{S}}\colon \widehat{M}(\Gamma) \to \widehat{B}(\Gamma)$ such that the diagram*

$$\begin{array}{ccc} M(\mathbf{H}^+,\Gamma) & \xrightarrow{\mathcal{S}} & B(\mathbf{H}^-,\Gamma) \\ {\scriptstyle P_L}\downarrow & & {\scriptstyle P_B}\downarrow \\ \widehat{M}(\Gamma) & \xrightarrow{\widehat{\mathcal{S}}} & \widehat{B}(\Gamma) \end{array} \tag{18}$$

commutes.

Proof. Let μ and ν belong to $M(\mathbf{H}^+,\Gamma)$.

If $P_L(\mu) = P_L(\nu)$, then Corollary 2.1 gives $[\mu]_{AT} = [\nu]_{AT}$, so Theorem 4 implies that $P_B(\mathcal{S}(\mu)) = P_B(\mathcal{S}(\nu))$. The formula $\widehat{\mathcal{S}}\circ P_L = P_B\circ\mathcal{S}$ therefore produces a well-defined map $\widehat{\mathcal{S}}\colon \widehat{M}(\Gamma)\to\widehat{B}(\Gamma)$. Since $\mathcal{S}$ is holomorphic and both P_B and P_L are $\mathbb{C}$-linear, $\widehat{\mathcal{S}}$ clearly has a $\mathbb{C}$-linear Fréchet derivative at each point of $\widehat{M}(\Gamma)$. Therefore $\widehat{\mathcal{S}}$ is holomorphic in $\widehat{M}(\Gamma)$.

To prove that $\widehat{\mathcal{S}}$ is a holomorphic split submersion we shall study its real-analytic sections. We observe first that

$$\widehat{\mathcal{S}}(\widehat{M}(\Gamma)) = \widehat{\mathcal{S}}(P_L(M(\mathbf{H}^+,\Gamma))) = P_B(\mathcal{S}(M(\mathbf{H}^+,\Gamma))),$$

which is the domain of the real-analytic map $\widehat{s}$ defined in Theorem 5. We can therefore combine (14) and (18) into the commutative diagram

$$
\begin{array}{ccccc}
\mathcal{S}(M(\mathbf{H}^+,\Gamma)) & \xrightarrow{\ s\ } & M(\mathbf{H}^+,\Gamma) & \xrightarrow{\ \mathcal{S}\ } & B(\mathbf{H}^-,\Gamma) \\
{\scriptstyle P_B}\big\downarrow & & {\scriptstyle P_L}\big\downarrow & & {\scriptstyle P_B}\big\downarrow \\
\widehat{\mathcal{S}}(\widehat{M}(\Gamma)) & \xrightarrow{\ \widehat{s}\ } & \widehat{M}(\Gamma) & \xrightarrow{\ \widehat{\mathcal{S}}\ } & \widehat{B}(\Gamma).
\end{array}
\tag{19}
$$

Since $\mathcal{S}\circ s$ is the identity on $\mathcal{S}(M(\mathbf{H}^+,\Gamma))$, we see from (19) that $\widehat{\mathcal{S}}\circ\widehat{s}$ is the identity on $\widehat{\mathcal{S}}(\widehat{M}(\Gamma))$. Therefore $\widehat{s}$ is a real-analytic section of $\widehat{\mathcal{S}}$, and the chain rule implies that $\widehat{\mathcal{S}}'(\widehat{s}(\widehat{\phi}))\circ\widehat{s}'(\widehat{\phi})$ is the identity on $\widehat{B}(\Gamma)$ for any $\widehat{\phi}$ in $\widehat{\mathcal{S}}(\widehat{M}(\Gamma))$. In general the map $\widehat{s}'(\widehat{\phi})$ is not $\mathbb{C}$-linear, but the map $\widehat{\psi}\mapsto\frac{1}{2}(\widehat{s}'(\widehat{\phi})(\widehat{\psi})-i\,\widehat{s}'(\widehat{\phi})(i\,\widehat{\psi}))$, $\widehat{\psi}$ in $\widehat{B}(\Gamma)$, is easily seen to be a continuous $\mathbb{C}$-linear right inverse for $\widehat{\mathcal{S}}'(\widehat{s}(\widehat{\phi}))$.

Now let μ be any given point of $M(\mathbf{H}^+,\Gamma)$. The chain rule argument in the preceding paragraph shows that $\widehat{\mathcal{S}}'(P_L(\mu))$ has a continuous $\mathbb{C}$-linear right inverse if $\widehat{\mathcal{S}}$ has a real-analytic section whose image contains the point $P_L(\mu)$. We shall complete the proof by constructing such a section.

Let $f=(f^\mu)^{-1}\circ f^{\beta(\mu)}$. Then f equals the identity on $\widehat{\mathbb{R}}$ and $f\circ\gamma=\gamma\circ f$ for all γ in Γ, so the Beltrami coefficient σ of f in $\mathbf{H}^+$ belongs to $M(\mathbf{H}^+,\Gamma)$. Let ρ_f be the holomorphic map of $M(\mathbf{H}^+,\Gamma)$ onto itself such that

$$f^{\rho_f(\nu)}\circ f=f^\nu \qquad \text{for all } \nu \text{ in } M(\mathbf{H}^+,\Gamma).$$

Then $[\rho_f(\nu)]_{T(X)}=[\nu]_{T(X)}$ for all ν in $M(\mathbf{H}^+,\Gamma)$, and $\rho_f(\beta(\mu))=\mu$. Therefore we have $\mathcal{S}(\rho_f(s(\phi)))=\mathcal{S}(s(\phi))=\phi$ for all ϕ in $\mathcal{S}(M(\mathbf{H}^+,\Gamma))$, and in addition $\rho_f(s(\mathcal{S}(\mu)))=\rho_f(\beta(\mu))=\mu$, by the definition of s in Lemma 4.1. Therefore $\rho_f\circ s$ is a real-analytic section of $\mathcal{S}$, taking the value μ at $\mathcal{S}(\mu)$.

The explicit formula

$$(\rho_f(\nu)\circ f)\overline{f_z}/f_z=\frac{\nu-\sigma}{1-\overline{\sigma}\nu}, \qquad \nu\in M(\mathbf{H}^+,\Gamma),$$

implies that ρ_f induces a well-defined holomorphic map $\widehat{\rho}_f:\widehat{M}(\Gamma)\to\widehat{M}(\Gamma)$ such that $\widehat{\rho}_f\circ P_L=P_L\circ\rho_f$. It is easy to verify that $\widehat{\rho}_f\circ\widehat{s}$ is a real-analytic section of $\widehat{\mathcal{S}}$ and takes the value $P_L(\mu)$ at $\widehat{\mathcal{S}}(P_L(\mu))$. □

Remark 13. During our proof of Proposition 7 we showed that the map $\widehat{s}$ defined in Theorem 5 is a real-analytic section of the map $\widehat{\mathcal{S}}$ defined in Proposition 7. We call $\widehat{\mathcal{S}}$ and $\widehat{s}$ the asymptotic Schwarzian derivative map and its asymptotic barycentric section respectively. We shall use them in the proof of Theorem 2 (see formula (21) below).

Remark 14. We could give $AT(X)$ a topology and complex manifold structure simply by declaring the asymptotic Bers map $\widehat{\mathcal{B}}$ to be biholomorphic. However, it is more natural (see Theorem 2.11 of [7]) to think of $AT(X)$ as a quotient space of $\widehat{M}(\Gamma)$, from which it inherits a topology and complex structure via the quotient map $P_L(\mu)\mapsto[\mu]_{AT(X)}$, μ in $M(\mathbf{H}^+,\Gamma)$.

An important consequence of Proposition 7, already observed in [7], is that these two paths lead to the same result. The quotient map $P_L(\mu)\mapsto[\mu]_{AT(X)}$ is the map

$\widehat{\mathcal{B}}^{-1} \circ \widehat{\mathcal{S}}$. By declaring $\widehat{\mathcal{B}}$ to be biholomorphic we make that quotient map a holomorphic split submersion, and $AT(X)$ can have at most one topology and complex structure with that property.

12. Proof of Theorem 2

It will be useful to consider the commutative diagram

$$\begin{array}{ccc} T(X) & \xrightarrow{\mathcal{B}} & \mathcal{S}(M(\mathbf{H}^+,\Gamma)) \\ {\scriptstyle\pi}\big\downarrow & & {\scriptstyle P_B}\big\downarrow \\ AT(X) & \xrightarrow{\widehat{\mathcal{B}}} & \widehat{\mathcal{S}}(\widehat{M}(\Gamma)), \end{array} \tag{20}$$

in which $\pi\colon T(X) \to AT(X)$ is the quotient map $[\mu]_{T(X)} \mapsto [\mu]_{AT(X)}$, $\mathcal{B}$ is the classical Bers map $[\mu]_T \mapsto \mathcal{S}(\mu)$, and P_B and $\widehat{\mathcal{B}}$ are defined as in §10. Since $\mathcal{B}$ and $\widehat{\mathcal{B}}$ are biholomorphic, the fibers of π in $T(X)$ are homeomorphic to the fibers of P_B in $\mathcal{S}(M(\mathbf{H}^+,\Gamma))$. These fibers are readily analysed by using our main theorem and the barycentric section s of $\mathcal{S}$.

Let ϕ in $\mathcal{S}(M(\mathbf{H}^+,\Gamma)$ be given, and set $\mu = s(\phi)$ and $\widehat{\phi} = P_B(\phi)$. These will be fixed for the remainder of this section. First we prove

Lemma 12.1. *Let $V_\mu = \{\nu \in M(\mathbf{H}^+,\Gamma) : \nu - \mu \in L_0^\infty(\mathbf{H}^+,\Gamma)\}$, and let*

$$P_B^{-1}(\widehat{\phi}) = \{\psi \in \mathcal{S}(M(\mathbf{H}^+,\Gamma)) : \psi - \phi \in B_0(\mathbf{H}^-,\Gamma)\}$$

be the inverse image of $\widehat{\phi}$. Then $\mathcal{S}(V_\mu) \subset P_B^{-1}(\widehat{\phi})$, and $s(P_B^{-1}(\widehat{\phi})) \subset V_\mu$.

Proof. If $\nu \in V_\mu$ then, by Theorem 4, $\mathcal{S}(\nu) - \phi = \phi_\nu - \phi_\mu$ vanishes at infinity on X^*, so $\mathcal{S}(\nu) \in P_B^{-1}(\widehat{\phi})$.

If $\nu \in M(\mathbf{H}^+,\Gamma)$ and $\mathcal{S}(\nu) = \phi_\nu$ belongs to $P_B^{-1}(\widehat{\phi})$ then, again by Theorem 4, $\beta(\nu) - \beta(\mu)$ vanishes at infinity on X. Now $\beta(\nu) = s(\phi_\nu)$ and

$$\beta(\mu) = s(\mathcal{S}(\mu)) = s(\phi) = \mu,$$

so $s(\phi_\nu) - \mu$ vanishes at infinity on X, and $s(\phi_\nu) \in V_\mu$. □

Corollary 12.1. *For ψ in $P_B^{-1}(\widehat{\phi})$ and t in $[0,1]$ put*

$$\sigma(\psi,t) = \mathcal{S}(t\, s(\phi) + (1-t)\, s(\psi)).$$

Then σ is a real-analytic map of $P_B^{-1}(\widehat{\phi}) \times [0,1]$ into $P_B^{-1}(\widehat{\phi})$, so $P_B^{-1}(\widehat{\phi})$ is contractible.

Proof. Let (ψ,t) belong to $P_B^{-1}(\widehat{\phi}) \times [0,1]$. By Lemma 12.1, $s(\phi)$ and $s(\psi)$ belong to V_μ. Since V_μ is convex, Lemma 12.1 implies that $\sigma(\psi,t)$ belongs to $P_B^{-1}(\widehat{\phi})$. Therefore σ maps $P_B^{-1}(\widehat{\phi}) \times [0,1]$ into $P_B^{-1}(\widehat{\phi})$.

The remainder of the corollary follows readily. □

Since the map $P_B\colon \mathcal{S}(M(\mathbf{H}^+,\Gamma)) \to \widehat{\mathcal{S}}(\widehat{M}(\Gamma))$ has contractible fibers, so does the map $\pi\colon T(X) \to AT(X)$. In particular $T_0(X) = \pi^{-1}([0]_{AT(X)})$ is contractible.

To prove that $AT(X)$ is contractible we observe that it is homeomorphic (by the biholomorphic map $\widehat{\mathcal{B}}$) to $\widehat{\mathcal{S}}(\widehat{M}(\Gamma))$. The map

$$\widehat{\sigma}(\widehat{\psi},t) = \widehat{\mathcal{S}}((1-t)\,\widehat{s}(\widehat{\psi})), \qquad (\widehat{\psi},t) \in \widehat{\mathcal{S}}(\widehat{M}(\Gamma)) \times [0,1], \tag{21}$$

from $\widehat{\mathcal{S}}(\widehat{M}(\Gamma)) \times [0,1]$ to $\widehat{\mathcal{S}}(\widehat{M}(\Gamma))$ shows that $\widehat{\mathcal{S}}(\widehat{M}(\Gamma))$ is contractible. □

13. Proof of Theorem 3

If $[\mu]_T$ belongs to $T_0(X)$ then $[\mu]_{AT(X)} = [0]_{AT(X)}$ and, by Theorem 4, $\beta(\mu)$ vanishes at infinity on X. The quasiconformal map $w^{\beta(\mu)}\colon X \to X^{\beta(\mu)}$ represents the point $[\mu]_T$ (see §2 and §4), so it suffices to show that $w^{\beta(\mu)}$ is an asymptotic hyperbolic isometry. We shall prove the following precise form of that statement.

Lemma 13.1. *Let d_X and $d_{X^{\beta(\mu)}}$ be the hyperbolic metrics on X and $X^{\beta(\mu)}$ respectively. If $\mu \in M(\mathbf{H}^+, \Gamma)$ and $\beta(\mu)$ vanishes at infinity on X, then for any $\varepsilon > 0$ there is a compact set E in X such that*

$$(1+\varepsilon)^{-1} d_X(x,y) \le d_{X^{\beta(\mu)}}(w^{\beta(\mu)}(x), w^{\beta(\mu)}(y)) \le (1+\varepsilon) d_X(x,y) \tag{22}$$

for all (x,y) in $(X \times X) \setminus (E \times E)$.

First we shall prove an infinitesimal version of Lemma 13.1. Its statement requires a few definitions. The metrics d_X and $d_{X^{\beta(\mu)}}$ are Riemannian metrics induced by the Riemannian metric $ds = |dz|/|z - \bar{z}|$ on $\mathbf{H}^+$. In particular, each tangent vector to X or $X^{\beta(\mu)}$ has a hyperbolic length. We denote by $\|u\|_X$ and $\|v\|_{X^{\beta(\mu)}}$ the hyperbolic lengths of the tangent vectors u and v to X and $X^{\beta(\mu)}$ respectively. A unit tangent vector is a tangent vector of hyperbolic length one.

Let w_* be the map of tangent bundles induced by $w^{\beta(\mu)}$. If u is a tangent vector to X at x, then $w_*(u)$ is a tangent vector to $X^{\beta(\mu)}$ at $w^{\beta(\mu)}(x)$.

Lemma 13.2. *Let $\{p_n\}$ be a sequence of points in X such that $p_n \to \infty$, and let u_n be a unit tangent vector at p_n for each n. Under the assumptions stated above*

$$\lim_{n\to\infty} \|w_*(u_n)\|_{X^{\beta(\mu)}} = 1. \tag{23}$$

Proof. We choose a sequence $\{z_n\}$ in $\mathbf{H}^+$ such that $\varpi(z_n) = p_n$ for all n and imitate the proof of Theorem 4.

For each n let C_n be an automorphism of $\mathbf{H}^+$ that maps the point $i = \sqrt{-1}$ to z_n, and set $\mu_n = C_n^*(\beta(\mu))$. Since $\beta(\mu)$ vanishes at infinity on X, we conclude from Corollary 7.1 and the definition of μ_n that

$$\lim_{n\to\infty} \|\mu_n\|_{\bar{D}_{\mathbf{H}^+}(i;r)} = \lim_{n\to\infty} \|\beta(\mu)\|_{\bar{D}_{\mathbf{H}^+}(z_n;r)} = 0 \qquad \text{for all } r > 0.$$

Since all the f^{μ_n} are K-quasiconformal, with K independent of n, it follows readily that f^{μ_n} converges uniformly on $\overline{\mathbb{C}}$ to the identity map as $n \to \infty$.

Since $\mu_n = C_n^*(\beta(\mu))$, for each n there is an automorphism A_n of $\mathbf{H}^+$ such that

$$f^{\mu_n} = A_n \circ f^{\beta(\mu)} \circ C_n. \tag{24}$$

Let φ^μ be the restriction of $f^{\beta(\mu)}$ to $\widehat{\mathbb{R}}$. By definition of β, $f^{\beta(\mu)}$ is equal in $\mathbf{H}^+ \cup \widehat{\mathbb{R}}$ to the barycentric extension of φ^μ. Proposition 3 and equation (24) therefore imply that f^{μ_n} is the barycentric extension of $A_n \circ \varphi^\mu \circ C_n$ in $\mathbf{H}^+ \cup \widehat{\mathbb{R}}$ for all n.

Since the boundary values of f^{μ_n} converge to the identity on $\widehat{\mathbb{R}}$ uniformly in the spherical metric, Proposition 2 of [5] implies that f^{μ_n} converges to the identity in the C^∞ sense in $\mathbf{H}^+$. Therefore $\partial f^{\mu_n}/\partial\bar{z} \to 0$ and $\partial f^{\mu_n}/\partial z \to 1$ as $n \to \infty$, uniformly on compact sets in $\mathbf{H}^+$.

For any smooth map $f\colon \mathbf{H}^+ \to \mathbf{H}^+$ let f_* be the induced map of tangent bundles, and let $\|v\|_{\mathbf{H}^+}$ denote the hyperbolic length of any tangent vector v to $\mathbf{H}^+$. We have proved that for any given $\delta > 0$ there is an integer N such that

$$(1+\delta)^{-1} \le \|f_*^{\mu_n}(v)\|_{\mathbf{H}^+} \le (1+\delta)$$

whenever $n > N$ and v is a unit tangent vector to $\mathbf{H}^+$ at the point i.

Since all automorphisms of $\mathbf{H}^+$ are hyperbolic isometries, (24) implies that

$$(1+\delta)^{-1} \le \|f_*^{\beta(\mu)}(v_n)\|_{\mathbf{H}^+} \le (1+\delta) \tag{25}$$

whenever $n > N$ and v_n is a unit tangent vector to $\mathbf{H}^+$ at the point z_n. Let v_n be the lift to z_n of the unit tangent vector u_n to X at p_n. Since $\varpi\colon \mathbf{H}^+ \to X$ and $\varpi^{\beta(\mu)}\colon \mathbf{H}^+ \to X^{\beta(\mu)}$ are infinitesimal isometries, (23) follows from (25) and the equation $w^{\beta(\mu)} \circ \varpi = \varpi^{\beta(\mu)} \circ f^{\beta(\mu)}$ (see §2). □

Proof of Lemma 13.1. To make our notation less cumbersome we shall set $\beta(\mu) = \nu$ thoughout this proof. We denote the hyperbolic length of a smooth curve $c(t)$ in X or X^ν by $L_X(c)$ or $L_{X^\nu}(c)$.

Let $\varepsilon > 0$ be given. We shall derive the desired inequality (22) from Lemma 13.2. By that lemma we can choose a compact set E_0 in X so that

$$(1+\tfrac{\varepsilon}{2})^{-1}\|v\|_X \le \|w_*(v)\|_{X^\nu} \le (1+\tfrac{\varepsilon}{2})\|v\|_X \tag{26}$$

whenever v is a tangent vector to X at a point of $X \setminus E_0$.

Let C be the hyperbolic diameter of $w^\nu(E_0)$ and let x in X satisfy the condition $d_X(x, E_0) > 2C/\varepsilon$. We claim that

$$d_{X^\nu}(w^\nu(x), w^\nu(y)) \le (1+\varepsilon) d_X(x,y) \qquad \text{for all } y \text{ in } X. \tag{27}$$

To prove (27) consider a distance minimizing geodesic $c\colon [0,1] \to X$ that joins $x = c(0)$ to $y = c(1)$. If $c(t) \in X \setminus E_0$ for all t in $[0,1]$, then (26) gives

$$L_{X^\nu}(w^\nu \circ c) \le (1+\tfrac{\varepsilon}{2}) L_X(c) = (1+\tfrac{\varepsilon}{2}) d_X(x,y).$$

Since the curve $w^\nu \circ c$ joins $w^\nu(x)$ to $w^\nu(y)$ we obtain

$$d_{X^\nu}(w^\nu(x), w^\nu(y)) \le (1+\tfrac{\varepsilon}{2}) d_X(x,y). \tag{28}$$

If $c(t) \in E_0$ for some t in $[0,1]$ set

$$a = \min\{t \in [0,1] : c(t) \in E_0\} \quad \text{and} \quad b = \max\{t \in [0,1] : c(t) \in E_0\}.$$

The curves $c\colon [0,a] \to X$ and $c\colon [b,1] \to X$ are distance minimizing geodesics, and the sum of their lengths is at most $d_X(x,y)$. Since (apart from their endpoints) they avoid E_0, the reasoning that gave (28) yields

$$d_{X^\nu}(w^\nu(x), w^\nu(c(a))) + d_{X^\nu}(w^\nu(c(b)), w^\nu(y)) \le (1+\tfrac{\varepsilon}{2}) d_X(x,y). \tag{29}$$

Since $c(a)$ and $c(b)$ belong to E_0, the definition of C and our condition on x give

$$d_{X^\nu}(w^\nu(c(a)), w^\nu(c(b))) \le C < \tfrac{\varepsilon}{2} d_X(x, E_0) \le \tfrac{\varepsilon}{2} d_X(x, c(a)) \le \tfrac{\varepsilon}{2} d_X(x,y). \tag{30}$$

Adding inequalities (29) and (30) and applying the triangle inequality, we obtain (27), as claimed.

The same reasoning, with the roles of X and X^ν reversed, shows that

$$d_X(x,y) \le (1+\varepsilon) d_{X^\nu}(w^\nu(x), w^\nu(y)) \quad \text{for all } y \text{ in } X$$

if x in X satisfies the condition $d_{X^\nu}(w^\nu(x), w^\nu(E_0)) > 2D/\varepsilon$, where D is the hyperbolic diameter of E_0. We omit the details.

We conclude that (22) holds unless both x and y belong to the compact set

$$E = \{x \in X : d_X(x, E_0) \le 2C/\varepsilon \text{ or } d_{X^\nu}(w^\nu(x), w^\nu(E_0)) \le 2D/\varepsilon\}. \quad \square$$

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(Earle) Department of Mathematics, Cornell University, Ithaca, NY 14853
E-mail address: `cliff@math.cornell.edu`

(Markovic) Department of Mathematics, University of Warwick, Coventry, UK
E-mail address: `markovic@maths.warwick.ac.uk`

(Saric) CUNY Ph.D. Program in Mathematics, 365 Fifth Avenue, New York, NY 10016
Current address: Department of Mathematics, University of Southern California, 1042 W. 36th Place, Los Angeles, CA 90089
E-mail address: `saric@math.usc.edu`