Infinite dimensional Teichmüller spaces

Alastair Fletcher and Vladimir Markovic

School of Mathematical Sciences, University of Nottingham,
University Park, Nottingham, NG7 2RD, United Kingdom
email: alastair.fletcher@nottingham.ac.uk

Mathematics Department, Stony Brook University,
Stony Brook NY, 11794-3651, USA
email: markovic@math.sunysb.edu
and
Mathematics Institute, University of Warwick,
Coventry, CV4 7AL, UK
email: V.Markovic@warwick.ac.uk

Contents

1 Introduction

In this chapter, we consider some analytic properties of Teichmüller spaces, in particular those of infinite dimension. The Bers embedding maps the Teichmüller space $T(M)$ of a hyperbolic Riemann surface M biholomorphically onto a subset of a Banach space $Q(M)$ of holomorphic quadratic differentials. If M is of non-exceptional analytic type, then the dimension of $T(M)$ is finite if and only if the dimension of $Q(M)$ is finite if and only if M is of finite analytic type, that is, M is a compact Riemann surface of genus g with n punctures, where $2g + n \geq 5$. In this case $Q(M)$ is a finite dimensional vector space over $\mathbb{C}$ and is therefore reflexive.

Via the Bers embedding, it can be shown that the cotangent space of $T(M)$ at the base-point [0] can be identified with the Bergman space $A^1(M)$, the predual of $Q(M)$. In the finite dimensional case, $A^1(M) = Q(M)$. However, when M is of infinite analytic type, it is no longer true that $A^1(M)$ is reflexive, and so $A^1(M) \neq Q(M)$.

The problem of classifying biholomorphic maps between Teichmüller spaces can be reduced, via consideration of the Kobayashi and Teichmüller metrics, to a problem of classifying isometries between the cotangent spaces of the corresponding Riemann surfaces. The infinite dimensional case requires more machinery because the Bergman spaces of Riemann surfaces of infinite analytic type are not reflexive.

We will show that if there is a surjective $\mathbb{C}$-linear isometry between $A^1(M)$ and $A^1(N)$, then the Riemann surfaces M and N, assumed to be of non-exceptional type, are conformally equivalent. Note that we do not assume that the Riemann surfaces M and N are even homeomorphic. This result implies that every biholomorphic map between Teichmüller spaces $T(M)$ and $T(N)$ is induced by a quasiconformal mapping between M and N, and therefore the automorphism group of $T(M)$ is equal to the mapping class group of M.

We also prove a counterpoint to the above result on isometries of Bergman spaces. Namely, if M and N are any two Riemann surfaces of infinite analytic type, then the corresponding Bergman spaces will be isomorphic. This then implies that the Teichmüller spaces of any two Riemann surfaces of infinite analytic type are locally bi-Lipschitz equivalent.

The chapter ends with some open problems that have arisen as a result of work in this area.

2 Quasiconformal mappings and Teichmüller spaces

2.1 Quasiconformality

A map $g : (a, b) \to \mathbb{C}$ is absolutely continuous on the interval (a, b) if

$$g(x) = \int_a^x h(t)\, dt + g(a)$$

for $x \in (a, b)$ and $h \in L^1_{loc}(a, b)$, the space of locally integrable functions. If g is absolutely continuous, then it is differentiable almost everywhere and $g' = h$ almost everywhere.

Let Ω be a plane domain, $f : \Omega \to f(\Omega) \subset \mathbb{C}$ and let a rectangle $R \subset \Omega$ have sides parallel to the x and y axes. We say that f is absolutely continuous on lines (ACL) on R if f is absolutely continuous on almost every horizontal and vertical line in R. The map f is ACL on Ω if f is ACL on every rectangle $R \subset \Omega$.

Definition 2.1. *A homeomorphism $f : \Omega \to f(\Omega)$ is K-quasiconformal if and only if the following holds:*
(i) f is ACL on Ω,
(ii) $|f_{\bar{z}}| \leq k|f_z|$ almost everywhere in Ω, where $k = (K - 1)/(K + 1)$.

In particular, f is conformal if and only if f is 1-quasiconformal. There are equivalent definitions of quasiconformality, see [7].

Example. The ACL condition is certainly necessary, as we will show here. Let C be the Cantor set on $(0, 1)$ so that every $x \in C$ can be written as

$$x = \sum_{i=1}^{\infty} 2 \cdot 3^{-n_i}$$

for some subset $\{n_i\}$ of $\mathbb{Z}^+$. The Cantor function $F : (0, 1) \to (0, 1)$ is defined by setting

$$F(x) = \sum_{i=1}^{\infty} 2^{-n_i}$$

for $x \in C$, and extended to the whole of $(0, 1)$ be requiring that F be monotonically increasing. Since F is constant on connected components of the complement of C, F is differentiable almost everywhere, with derivative 0, but is not differentiable at points of C. Now define the function $f : (0, 1) \times (-\infty, \infty) \to \mathbb{C}$ given by

$$f(x + iy) = F(x) + x + iy.$$

Now, $F(x) + x$ is a homeomorphism of $(0, 1)$ onto $(0, 2)$, and so f is a homeomorphism, which is differentiable almost everywhere, and in fact $f_{\bar{z}} = 0$ almost everywhere. However f cannot be conformal on $(0,1) \times (-\infty, \infty)$ because it is not differentiable at any point of $C \times (-\infty, \infty)$. Moreover, f cannot be quasiconformal, because if it was, then it would have to be 1-quasiconformal and hence conformal. In conclusion, if $f : \Omega \to f(\Omega)$ is a homeomorphism, differentiable almost everywhere and $f_{\overline{z}} = 0$ almost everywhere, then this does not imply that f is conformal or quasiconformal.

Quasiconformality can also be defined for maps between Riemann surfaces, and note that in this chapter we are assuming that all our Riemann surfaces are hyperbolic, that is, they have the unit disk $\mathbb{D}$ as the universal cover. The map $f : M \to N$ is said to be K-quasiconformal at $p \in M$ if there are coordinate charts (U_p, π_p) on M and $(U_{f(p)}, \pi_{f(p)})$ on N such that $\pi_{f(p)} \circ f \circ \pi_p^{-1}$ is a K-quasiconformal mapping whose domain is the plane domain $\pi_p(U_p)$. The mapping f is then said to be K-quasiconformal if it is K-quasiconformal at all $p \in M$. Note that this definition is independent of the choice of coordinate chart, since the transition maps are conformal.

Let μ be a measurable $(-1,1)$-form on a Riemann surface M with

$$|\mu(p)| \leq k < 1$$

for almost all $p \in M$, so $\mu \in B(M)$, the open unit ball of $L^\infty(M)$. Such a μ is called a Beltrami differential. The Beltrami differential equation is

$$f_{\overline{z}} = \mu f_z.$$

The solution f (sometimes denoted f^μ) of the Beltrami differential equation is a quasiconformal mapping, and all quasiconformal mappings arise in this way, giving a correspondence between quasiconformal mappings and Beltrami differentials. The solution f^μ can be lifted to give a quasiconformal self-mapping of $\mathbb{D}$. If the lifted solution is normalized to fix three points of $\partial\mathbb{D}$, then the correspondence between quasiconformal mappings and Beltrami differentials is one-to-one. See [7] for the proof of these statements.

2.2 Teichmüller space

Let $\mathcal{F}$ be the family of all quasiconformal self-mappings of $\mathbb{D}$ which are normalised so that their extensions to $\partial\mathbb{D}$ fix $1, -1, i$. From the solution of the Beltrami differential equation, there is a one to one correspondence between $\mathcal{F}$ and the open unit ball B of $L^\infty(\mathbb{D})$. Now, we can put an equivalence relation $\sim_{\mathcal{F}}$ on elements of $\mathcal{F}$ by declaring that $f_1 \sim_{\mathcal{F}} f_2$ if and only if the extensions of f_1 and f_2 to $\partial\mathbb{D}$ agree on $\partial\mathbb{D}$. Equivalently, two elements μ_1 and μ_2 of B are related by $\sim_{\mathcal{F}}$ if and only if the extensions of the normal solutions f^{μ_1} and f^{μ_2} to $\partial\mathbb{D}$ agree on $\partial\mathbb{D}$.

Definition 2.2. *The set of equivalence classes of $\mathcal{F}$ under $\sim_{\mathcal{F}}$ is called the universal Teichmüller space $T(\mathbb{D})$.*

The deformation space $\mathrm{Def}(M)$ of a Riemann surface M is the set of pairs (N, f) where N is a Riemann surface, and $f : M \to N$ is a quasiconformal map. For any plane domain Ω with its appropriate Riemann map f, we have $(\Omega, f) \in \mathrm{Def}(\mathbb{D})$. An equivalence relation $\sim$ can be defined on $\mathrm{Def}(M)$ by requiring that $(N_1, f_1) \sim (N_2, f_2)$ if and only if

$$f_1 \circ f_2^{-1} : N_2 \to N_1$$

is homotopic to a conformal map $g : N_2 \to N_1$. Two maps f and g between hyperbolic Riemann surfaces are homotopic if they can be lifted to mappings of $\mathbb{D}$ which agree on $\partial\mathbb{D}$.

Definition 2.3. *The Teichmüller space $T(M)$ of a Riemann surface M is given by*

$$\mathrm{Def}(M)/\sim .$$

The base-point of $T(M)$ is the Teichmüller class of the identity mapping. This definition agrees with the definition of universal Teichmüller space. To see this, first note that since all the quasiconformal images of $\mathbb{D}$ are conformally equivalent, only the normalized quasiconformal self-mappings f^μ of $\mathbb{D}$ need be considered. Then $f^{\mu_2} \circ (f^{\mu_1})^{-1}$ is homotopic to a conformal map if and only if $f^{\mu_2} \circ (f^{\mu_1})^{-1}$ is the identity mapping on $\partial\mathbb{D}$. Therefore f^{μ_1} agrees with f^{μ_2} on $\partial\mathbb{D}$, which is precisely the definition for f^{μ_1} and f^{μ_2} to determine the same point of universal Teichmüller space.

As in the considerations for the universal Teichmüller space, $T(M)$ can be considered as a space of Beltrami differentials under the corresponding equivalence relation. The base-point of $T(M)$ is the Teichmüller class of $0 \in B(M)$. Given a Beltrami differential μ on a Riemann surface M, μ lifts to a Beltrami differential $\widetilde{\mu}$ on $\mathbb{D}$ which satisfies

$$\widetilde{\mu} = (\widetilde{\mu} \circ g)\, \frac{\overline{g'}}{g'}$$

for every g in the covering group of M. We write f_μ for the quasiconformal mapping of the plane which has the complex dilatation $\widetilde{\mu}$ on $\mathbb{D}$, and 0 on $\mathbb{D}^*$, where $\mathbb{D}^* = \mathbb{C} \setminus \overline{\mathbb{D}}$. Note that $\mathbb{D}^*$ is the image of $\mathbb{D}$ under the reflection in $\partial\mathbb{D}$ given by $z \mapsto 1/\overline{z}$ for $z \in \mathbb{D}$.

Theorem 2.4. *The following are equivalent:*

(i) The Beltrami differentials μ and ν on the Riemann surface M are equivalent under $\sim$,

(ii) $f^\mu|_{\partial\mathbb{D}} = f^\nu|_{\partial\mathbb{D}}$,

(iii) $f_\mu|_{\mathbb{D}^*} = f_\nu|_{\mathbb{D}^*}$.

2.3 Teichmüller metric

For $[f_0], [g_0] \in T(M)$, we can define the Teichmüller distance

$$d_T([f_0],[g_0]) = \frac{1}{2} \inf \log K_{g\circ f^{-1}} \tag{2.1}$$

where the infimum is taken over all maps f and g in the Teichmüller classes of f_0 and g_0 respectively, and where K_f is the maximal dilatation of f. We write d_T instead of $d_{T(M)}$ for brevity where it is clear which Teichmüller space $[f]$ and $[g]$ are in. We can replace inf by min in (2.1) and d_T is in fact a metric. For the details, we refer to [7].

The Teichmüller metric on $B(M)$ is given by

$$d_B(\mu,\nu) = \frac{1}{2}\log \frac{1+||(\mu-\nu)/(1-\overline{\mu}\nu)||_\infty}{1-||(\mu-\nu)/(1-\overline{\mu}\nu)||_\infty},$$

for $\mu,\nu \in B(M)$. The Teichmüller metric d_T can be expressed as the quotient of d_B,

$$d_T([\mu_0],[\nu_0]) = \frac{1}{2} \inf_{\mu\in[\mu_0],\nu\in[\nu_0]} \log \frac{1+||(\mu-\nu)/(1-\overline{\mu}\nu)||_\infty}{1-||(\mu-\nu)/(1-\overline{\mu}\nu)||_\infty},$$

for $[\mu_0],[\nu_0] \in T(M)$.

2.4 Schwarzian derivatives and quadratic differentials

If f is holomorphic in a domain Ω and $f'(z) \neq 0$ in Ω, then the Schwarzian derivative of f is

$$S_f = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2. \tag{2.2}$$

If also $f(z) \neq 0$ in Ω, then a direct computation shows

$$S_f(z) = S_{1/f}(z)$$

and this formula shows how to define the Schwarzian derivative of a meromorphic function at simple poles. Thus the Schwarzian derivative can be defined for locally injective meromorphic functions, and S_f is itself holomorphic. Let

$$A(z) = \frac{az+b}{cz+d}$$

be a Möbius transformation, then differentiating gives

$$\frac{A''(z)}{A'(z)} = -\frac{2c}{cz+d}$$

and

$$\left(\frac{A''(z)}{A'(z)}\right)' = \frac{2c^2}{(cz+d)^2}$$

from which we see that

$$S_A = 0.$$

Conversely, starting with the equation $S_f = 0$ and setting $g = f''/f'$, then (2.2) gives $g' = g^2/2$. Solving this differential equation shows that every solution of $S_f = 0$ is a Möbius transformation. Schwarzian derivatives satisfy the composition rule

$$S_{f\circ g} = (S_f \circ g)g'^2 + S_g \tag{2.3}$$

and so if g is a Möbius transformation and $S_g = 0$ then

$$S_{f\circ g} = (S_f \circ g)g'^2. \tag{2.4}$$

On the other hand, if f is a Möbius transformation, then

$$S_{f\circ g} = S_g. \tag{2.5}$$

To define the Schwarzian derivative at ∞, assume that f is locally injective and meromorphic in a neighbourhood of ∞, then $h(z) = f(1/z)$ is defined in a neighbourhood of 0. Using (2.4),

$$z^4 S_h(z) = S_f(1/z)$$

and so we can define

$$S_f(\infty) = \lim_{z\to 0} z^4 S_h(z)$$

and S_f is holomorphic at ∞. Thus the Schwarzian derivative can be defined for a locally injective meromorphic function f on any domain Ω. The following theorem shows that the Schwarzian derivative can be prescribed.

Theorem 2.5. *Let g be a holomorphic function in a simply connected domain Ω. Then there is a meromorphic function f in Ω such that*

$$S_f = g$$

which is unique up to an arbitrary Möbius transformation.

See [7] for the proof. Let $Q(M)$ be the space of holomorphic quadratic differentials on a Riemann surface M equipped with the norm

$$||\widetilde{\varphi}||_{Q(M)} = \sup_{p\in M} \rho_M^{-2}(p)|\widetilde{\varphi}(p)|,$$

where ρ_M is the hyperbolic density on M, and noting that $\rho_M^{-2}|\widetilde{\varphi}|$ is a function on M, whereas $|\widetilde{\varphi}|$ is usually not. We will call $Q(M)$ the Bers space, and

$||.||_{Q(M)}$ the Bers norm. It is straightforward to see that $Q(M)$ is a Banach space.

Now let f_1 and f_2 be meromorphic functions on a domain Ω and also let $h : \Omega' \to \Omega$ be conformal. Using the invariance property (2.3),

$$S_{f_1 \circ h} - S_{f_2 \circ h} = (S_{f_1} \circ h - S_g \circ h) h'^2.$$

Since the hyperbolic density is conformally invariant, $\rho_{\Omega'} = (\rho_\Omega \circ h)|h'|$, then by writing $w = h(z)$, we have the invariance formula

$$\frac{|S_{f_1}(w) - S_{f_2}(w)|}{\rho^2_\Omega(w)} = \frac{|S_{f_1 \circ h}(z) - S_{f_2 \circ h}(z)|}{\rho^2_{\Omega'}(z)}.$$

In terms of the Bers norm, this is

$$||S_{f_1} - S_{f_2}||_{Q(\Omega)} = ||S_{f_1 \circ h} - S_{f_2 \circ h}||_{Q(\Omega')}.$$

If $f_2 = h^{-1}$ is a conformal mapping of Ω, then

$$||S_{f_1} - S_{f_2}||_{Q(\Omega)} = ||S_{f_1 \circ f_2^{-1}}||_{Q(f_2(\Omega))}.$$

In the special case of f_1 being the identity,

$$||S_{f_2}||_{Q(\Omega)} = ||S_{f_2^{-1}}||_{Q(f_2(\Omega))}.$$

Lastly, if $f_2 = h^{-1}$ is a Möbius transformation, then

$$||S_{f_1}||_{Q(\Omega)} = ||S_{f_1 \circ f_2^{-1}}||_{Q(f_2(\Omega))},$$

which shows that $||S_f||_{Q(\Omega)}$ is completely invariant with respect to Möbius transformations.

Recalling the quasiconformal mappings f^μ and f_μ, we have that $f_\mu|_{\mathbb{D}^*}$ is a conformal map, and the set of such maps characterizes $T(M)$. If $M \simeq \mathbb{D}/G$, then for every $g \in G$, $f_\mu \circ g \circ f_\mu^{-1}$ is a Möbius transformation. Therefore, using the transformation rules (2.4) and (2.5), we have

$$S_{f_\mu|_{\mathbb{D}^*}} = S_{(f_\mu \circ g \circ f_\mu^{-1}) \circ f_\mu|_{\mathbb{D}^*}} = S_{f_\mu \circ g|_{\mathbb{D}^*}} = (S_{f_\mu|_{\mathbb{D}^*}} \circ g) g'^2.$$

This shows that the Schwarzian derivative is a quadratic differential for the group G acting on $\mathbb{D}^*$, and its projection is a holomorphic quadratic differential on $\mathbb{D}^*/G$, which is the mirror image of the Riemann surface M, denoted by M^*.

2.5 Bers embedding and complex structure on Teichmüller space

We have the mapping

$$\mu \mapsto S_{f_\mu|_{\mathbb{D}^*}}$$

which maps the open unit ball $B(M)$ of $L^\infty(M)$ into the space of quadratic differentials $Q(M^*)$. This induces a mapping

$$\lambda_M : T(M) \to Q(M^*), \tag{2.6}$$

which is called the Bers embedding. The first thing to note is that if two holomorphic functions f_1 and f_2 on $\mathbb{D}^*$ have the same Schwarzian derivative, then by Theorem 2.5, one is equal to the other post-composed by a Möbius transformation. Now since both functions are normalized at 3 points, then f_1 and f_2 must be identical on $\mathbb{D}^*$, and therefore determine the same Teichmüller class. This shows that the mapping λ_M is one-to-one onto its image. Further, the image of $T(M)$ under λ_M is open in $Q(M^*)$, the proof of which can be found in [7].

In fact, $\mu \mapsto S_{f_\mu|_{\mathbb{D}^*}}$ is a holomorphic map from $B(M)$ into $Q(M^*)$. For a given $\mu \in B(M)$, write

$$G^\mu = \{f^\mu \circ g \circ (f^\mu)^{-1} : g \in G\},$$

where $M \simeq \mathbb{D}/G$, and $M^\mu \simeq \mathbb{D}/G^\mu$ is a Riemann surface which is quasiconformally equivalent to M. Let $\widetilde{\alpha_\mu} : B(M) \to B(M^\mu)$ be the mapping given by

$$f^{\widetilde{\alpha_\mu}(\nu)} = f^\nu \circ (f^\mu)^{-1}$$

or, by writing out in full,

$$\widetilde{\alpha_\mu}(\nu) = \left(\frac{\nu - \mu}{1 - \overline{\mu}\nu} \left(\frac{f_z^\mu}{|f_z^\mu|} \right)^2 \right) \circ (f^\mu)^{-1}.$$

The function $\widetilde{\alpha_\mu}$ maps $B(M)$ bijectively onto $B(M^\mu)$, and it follows that $\widetilde{\alpha_\mu}$ is holomorphic. Note that the induced mapping α_μ, given by $\alpha_\mu([\nu]) = [\widetilde{\alpha_\mu}(\nu)]$ is a bijective isometry of $T(M)$ onto $T(M^\mu)$. For $\mu \in B(M)$ and $\nu \in B(M^\mu)$ we write

$$\Lambda_\mu(\nu) = S_{f_\nu|_{\mathbb{D}^*}}. \tag{2.7}$$

This mapping of $B(M^\mu)$ into $Q((M^\mu)^*)$ is holomorphic. Now, in the ball

$$B_\mu(0, 1/2) = \{\varphi \in Q((M^\mu)^*) : ||\varphi||_Q < 2\},$$

the mapping (2.7) has a section $\sigma_\mu : B_\mu(0, 2) \to B(M^\mu)$. We see that σ_μ is holomorphic. Let π be the canonical projection of $B(M)$ onto $T(M)$, and let λ and λ_μ be the Bers embeddings of $T(M)$ and $T(M^\mu)$, respectively, into $Q(M^*)$ and $Q((M^\mu)^*)$. The collection

$$\{V_\mu = (\pi \circ (\widetilde{\alpha_\mu})^{-1} \circ \sigma_\mu)(B_\mu(0, 1/2)) : \mu \in B(M)\}$$

is an open covering of $T(M)$. Indeed, V_μ is the pre-image of $B_\mu(0, 1/2)$ under the homeomorphism $h_\mu = \lambda_\mu \circ \alpha_\mu$ of $T(M)$ onto $Q((M^\mu)^*)$.

Theorem 2.6. *The atlas*

$$\{(V_\mu, h_\mu) : \mu \in B(M)\} \tag{2.8}$$

defines a complex structure on Teichmüller space $T(M)$. The Bers embedding $[\mu] \to S_{f_\mu|_{\mathbb{D}^*}}$ of $T(M)$ into $Q(M^*)$ is holomorphic with respect to this structure.

Proof. Choose two elements $\mu_1, \mu_2 \in B(M)$ such that $V_{\mu_1} \cap V_{\mu_2}$ is non-empty. In $h_{\mu_1}(V_{\mu_1} \cap V_{\mu_2})$, we have

$$h_{\mu_2} \circ h_{\mu_1}^{-1} = \Lambda_{\mu_2} \circ \widetilde{\alpha_{\mu_2}} \circ (\widetilde{\alpha_{\mu_1}})^{-1} \circ \sigma_{\mu_1}.$$

We know that all the mappings on the right hand side of this equation are holomorphic, and so $h_{\mu_2} \circ h_{\mu_1}^{-1}$ is holomorphic. Switching μ_1 and μ_2 in this calculation shows that $h_{\mu_2} \circ h_{\mu_1}^{-1}$ is biholomorphic, and so (2.8) defines a complex structure for $T(M)$. To show that the Bers embedding is holomorphic, we have to show that $\lambda \circ h_\mu^{-1}$ is holomorphic in $B_\mu(0, 1/2)$. Now,

$$\lambda \circ h_\mu^{-1} = \Lambda \circ (\widetilde{\alpha_\mu})^{-1} \circ \sigma_\mu$$

and since all the mappings on the right hand side are holomorphic, then $\lambda \circ h_\mu^{-1}$ must also be holomorphic. □

A Riemann surface M is said to be of finite analytic type if it is a compact Riemann surface of genus g with a finite number n of punctures. It has exceptional type if $2g + n < 5$. All non-hyperbolic Riemann surfaces have exceptional type.

If M is of finite analytic type, then $Q(M)$ can be identified with its predual space $A^1(M)$, the subset of $L^1(M)$ consisting of holomorphic quadratic differentials on M with finite norm

$$||\varphi||_1 = \int_M |\varphi|,$$

for $\varphi \in A^1(M)$. The Banach space $A^1(M)$ is called the Bergman space. In fact, every linear functional on $A^1(M)$, $L : A^1(M) \to \mathbb{C}$ has the form

$$L(\varphi) = \int_M \rho_M^{-2} \overline{\psi} \varphi$$

for some $\psi \in Q(M)$, and $L \equiv 0$ if and only if $\psi \equiv 0$. et $\widetilde{M}$ be a compact Riemann surface of genus $g \geq 0$ and let E be a finite, possibly empty, subset of $\widetilde{M}$ which contains exactly $n \geq 0$ points. We assume that $2g+n \geq 5$, so that the Riemann surface $M = \widetilde{M} \backslash E$ has non-exceptional finite type. Each $\varphi \in A^1(M)$ can be regarded as a quadratic differential on $\widetilde{M}$ which is holomorphic except for isolated singularities at the points of M. The integrability of φ implies that the singularities of φ are either removable or simple poles, so $A^1(M)$ is the space of meromorphic quadratic differentials on $\widetilde{M}$ whose poles, if any, are simple and belong to E.

Proposition 2.7. *If $M = \widetilde{M} \setminus E$ as above, then the dimension of $A^1(M)$ is $3g - 3 + n \geq 2$. If $x \in E$, then some $\varphi \in A^1(M)$ has a pole at x.*

The Riemann surface M is of infinite analytic type if M has infinite genus or an infinite number of punctures. If M is of infinite analytic type, then the dimension of $A^1(M)$ is infinite. For example, if M has an infinite number of punctures at the points (z_n) for $n = 1, 2, ...$, then there exist functions f_n, each of which have a simple pole at z_n, and which are linearly independent. This points us in the direction of the following result, proved in [9].

Theorem 2.8. *A Riemann surface M is of finite analytic type if and only if the dimension of $A^1(M)$ is finite.*

If the dimension of $A^1(M)$ is infinite, then $A^1(M)$ is a proper subset of $Q(M)$. Therefore, via the Bers embedding, the dimension of the Teichmüller space $T(M)$ is infinite if and only if M is not of finite analytic type.

3 Biholomorphic maps between Teichmüller spaces

In this section, we will classify biholomorphic maps between Teichmüller spaces by reducing the problem to the cotangent space. That is, all surjective linear isometries between Bergman spaces of Riemann surfaces of non-exceptional type are geometric, which in particular implies that the two Riemann surfaces are conformally related.

3.1 Kobayashi metric

Let X be any connected complex Banach manifold, and let $H(\mathbb{D}, X)$ be the set of all holomorphic maps from $\mathbb{D}$ into X. The Kobayashi function $\delta_X : X \times X \to [0, +\infty]$ is

$$\delta_X(x, y) = \inf \left\{ \rho_{\mathbb{D}}(0, t) : f(0) = x, f(t) = y \text{ for some } f \in H(\mathbb{D}, X) \right\},$$

provided the set of such maps is non-empty, and $+\infty$ otherwise. If X and Y are connected complex Banach manifolds and $f : X \to Y$ is holomorphic, then

$$\delta_Y(f(x_1), f(x_2)) \leq \delta_X(x_1, x_2),$$

for all $x_1, x_2 \in X$, and with equality if f is biholomorphic.

Definition 3.1. *The Kobayashi pseudo-metric σ_X on X is the largest pseudo-metric on X such that*

$$\sigma_X(x,y) \leq \delta_X(x,y), \tag{3.1}$$

for all $x, y \in X$. If δ_X is a metric, as it will be in our cases, then σ_X is a metric and σ_X and δ_X are equal.

Note that when we say σ_X is the largest pseudo-metric, we mean that if d_X is any other pseudo-metric which satisfies (3.1), then

$$d_X(x,y) \leq \sigma_X(x,y)$$

for all $x, y \in X$. The Kobayashi pseudo-metric is the largest metric under which holomorphic mappings are distance decreasing.

Example 1. If X is $\mathbb{C}$ or $\widehat{\mathbb{C}}$, then the linear maps $f_n(z) = nz$, together with their inverses, from $\mathbb{D}$ into X show that $\delta_X \equiv 0$. Hence $\sigma_X \equiv 0$. Furthermore, if X is a torus or the punctured plane $\mathbb{C} \setminus \{0\}$, then there is a covering map π from $\mathbb{C}$ onto X. Since π is a contraction in the respective pseudo-metrics, $\sigma_X \equiv 0$.

Example 2. Let X be a hyperbolic Riemann surface, that is, it has $\mathbb{D}$ as its universal cover. If π is the covering map from $\mathbb{D}$ onto X, then $\pi \in H(\mathbb{D}, X)$ and so is a contraction in the corresponding Kobayashi pseudo-metrics. However, every $f \in H(\mathbb{D}, X)$ lifts to a map $\widetilde{f} \in H(\mathbb{D}, \mathbb{D})$ such that $f = \pi \circ g$. Thus δ_X is equal to the quotient pseudo-metric on X with respect to the covering map π and the hyperbolic metric on $\mathbb{D}$. That is, δ_X coincides with the hyperbolic metric on X. Since δ_X is a pseudo-metric, $\sigma_X = \delta_X$ and so the Kobayashi pseudo-metric is equal to the hyperbolic metric on X.

Example 3. Let B be the unit ball in a complex Banach manifold X and pick $x \in B$. The linear function $f(t) = tx/||x||$ maps the unit disk $\mathbb{D}$ into the unit ball B and maps $||x||$ to x and 0 to 0. Therefore

$$\sigma_B(0,x) \leq \sigma_{\mathbb{D}}(0, ||x||).$$

However, via the Hahn-Banach Theorem, there exists a continuous linear functional L on X such that $L(x) = ||x||$ and $||L|| = 1$. Thus L maps B into the unit disk $\mathbb{D}$ and so

$$\sigma_{\mathbb{D}}(0, ||x||) \leq \sigma_B(0,x).$$

The definition of the hyperbolic metric on $\mathbb{D}$ leads to

$$\sigma_B(0,x) = \frac{1}{2} \tanh^{-1} ||x||.$$

Proposition 3.2. *For all $\mu, \nu \in B(M)$, we have*

$$\delta_{B(M)}(\mu,\nu) = d_{B(M)}(\mu,\nu).$$

Proof. We can assume that $\mu \neq \nu$. First we also assume that $\mu = 0$. Suppose that $f \in H(\mathbb{D}, B(M))$, $f(0) = 0$ and $f(t) = \nu$ for some $t \in \mathbb{D}$. By the Schwarz Lemma,

$$|t| \geq ||f(t)||_\infty = ||\nu||_\infty.$$

Taking the infimum over all such f,

$$\delta_{B(M)}(0, \nu) \geq \rho_{\mathbb{D}}(0, ||\nu||_\infty = d_{B(M)}(0, \nu).$$

Choose the function $f(t) = t\nu/||\nu||_\infty$ and observe that

$$\delta_{B(M)}(0, \nu) \leq \rho_{\mathbb{D}}(0, ||\nu||_\infty) = d_{B(M)}(0, \nu)$$

since $f(||\nu||_\infty) = \nu$. Now if $\mu \neq 0$, observe that the function $f : B(M) \to L^\infty(M)$ defined by

$$f(\lambda) = \frac{\mu - \lambda}{1 - \overline{\mu}\lambda}$$

is a holomorphic map of $B(M)$ onto itself. Therefore

$$\delta_{B(M)}(\mu, \nu) = \delta_{B(M)}(f(\mu), f(\nu)) = d_{B(M)}(0, f(\nu)) = d_{B(M)}(\mu, \nu).$$

□

As we have seen, the Teichmüller metric on $B(M)$ induces a quotient metric on $T(M)$:

$$d_T([\mu], [\nu]) = \inf \left\{ d_{B(M)}(\mu_0, \nu_0) : \mu_0, \nu_0 \in B(M), \mu_0 \in [\mu], \nu_0 \in [\nu] \right\},$$

for all $[\mu], [\nu] \in T(M)$. We can now prove Royden's theorem on the equality of Teichmüller and Kobayashi metrics on $T(M)$, proved in [16].

Theorem 3.3. *The Teichmüller and Kobayashi metrics on $T(M)$ coincide, that is,*

$$d_T([\mu], [\nu]) = \delta_{T(M)}([\mu], [\nu]).$$

Proof. Fix $[\mu], [\nu] \in T(M)$. We have

$$\delta_{T(M)}([\mu], [\nu]) \leq \inf \left\{ \delta_{B(M)}(\mu_0, \nu_0) : \mu_0 \in [\mu], \nu_0 \in [\nu] \right\}.$$

Hence by Proposition 3.2,

$$\delta_{T(M)}([\mu], [\nu]) \leq d_T([\mu], [\nu]).$$

For the opposite inequality, choose $f \in H(\mathbb{D}, T(M))$ such that $f(0) = [\mu]$ and $f(t) = [\nu]$ for some $t \in \mathbb{D}$. Using a theorem of Slodkowski (see [4]), we can write $\pi \circ g = f$ with

$$g \in H(\mathbb{D}, B(M)).$$

Using Proposition 3.2,

$$\rho_{\mathbb{D}}(0,t) \geq d_{B(M)}(g(0),g(t)) \geq d_T(\pi(g(0)),\pi(g(t))) = d_T([\mu],[\nu]).$$

Taking the infimum over all such f, we obtain

$$\delta_{T(M)}([\mu],[\nu]) \geq d_T([\mu],[\nu]).$$

□

Corollary 3.4. *Let M and N be two hyperbolic Riemann surfaces. Then every biholomorphic map between $T(M)$ and $T(N)$ preserves Teichmüller distances.*

3.2 The infinitesimal Teichmüller metric

Due to Corollary 3.4, investigating biholomorphic self-mappings of $T(M)$ reduces to the study of biholomorphic Teichmüller isometries. Now, biholomorphic Teichmüller isometries preserve the infinitesimal Teichmüller metric, as we will see below.

Recall the Bers embedding (2.6) of $T(M)$ into $Q(M^*)$, where M^* is the mirror image of M. For the rest of this chapter, we will for brevity (and to avoid confusion with our notation for Banach duals) write $Q(M)$ instead of $Q(M^*)$ and $A^1(M)$ for $A^1(M^*)$, but bear in mind that the Bers embedding maps $T(M)$ onto a subset of the Bers space of the mirror image of M. Via the Bers embedding, we can regard $Q(M)$ as the tangent space to $T(M)$ at its base-point $[0]$ (where $0 \in B(M)$). Further, we can define an isomorphism θ of $Q(M)$ onto the Banach dual $(A^1(M))^*$ of $A^1(M)$ via

$$\theta(\varphi)(f) = \int_M \rho_M^{-2}\varphi(\overline{z})f(z)\,dx\,dy,$$

for $\varphi \in Q(M)$ and $f \in A^1(M)$ and where ρ_M is the hyperbolic density on M. This was proved by Bers in [1]. We can therefore identify $(A^1(M))^*$ with the tangent space to $T(M)$ at its base-point. This identifies the cotangent space with $A^1(M)$ in the finite dimensional case, since then $A^1(M)$ is reflexive. Further, the standard norm

$$||L|| = \sup\{|L(f)| : f \in A^1(M), ||f||_1 \leq 1\},$$

for $L \in (A^1(M))^*$, on $(A^1(M))^*$ is exactly the infinitesimal Teichmüller metric for tangent vectors at the base-point of $T(M)$.

3.3 Isometries of Bergman spaces

Let M and N be two hyperbolic Riemann surfaces and let $f : T(M) \to T(N)$ be a biholomorphic map that sends base-point to base-point. We have seen that the derivative of f at the base-point of $T(M)$ is a $\mathbb{C}$-linear isometry of $(A^1(M))^*$ onto $(A^1(N))^*$. In the finite dimensional case, we immediately have that the adjoint of that derivative is a $\mathbb{C}$-linear isometry of $A^1(N)$ onto $A^1(M)$. However, in the infinite dimensional case, we need to use a theorem of Earle and Gardiner's, see [2], which says that if there is an invertible $\mathbb{C}$-linear isometry $F : (A^1(M))^* \to (A^1(N))^*$, then there is always an invertible $\mathbb{C}$-linear isometry $L : A^1(N) \to A^1(M)$ which is the adjoint of F. In this way, we pass from biholomorphic maps between Teichmüller spaces to linear isometries between Bergman spaces.

There are two obvious types of isometries between $A^1(N)$ and $A^1(M)$. The map $\varphi \mapsto \theta\varphi$ is an isometry of $A^1(M)$ onto itself whenever $\theta \in \mathbb{C}$ has $|\theta| = 1$. Also, if α is a conformal map of M onto N, each $\varphi \in A^1(N)$ can be pulled back to a quadratic differential $\alpha^*(\varphi)$ on $A^1(M)$, and the map $\varphi \mapsto \alpha^*(\varphi)$ is an isometry.

Definition 3.5. *If M and N are Riemann surfaces, then a surjective linear isometry $T : A^1(M) \to A^1(N)$ is called geometric if there exists a conformal map $\alpha : M \to N$ and a complex number θ with $|\theta| = 1$ such that*

$$T^{-1}(\varphi) = \theta(\varphi \circ \alpha)\alpha'^2,$$

for every $\varphi \in A^1(N)$.

Theorem 3.6. *Suppose that M and N are Riemann surfaces which are of non-exceptional finite type and that $T : A^1(M) \to A^1(N)$ is a surjective complex-linear isometry. Then T is geometric.*

Royden proved Theorem 3.6 in [16] in the case where M and N are compact and hyperbolic, and his method was extended to Riemann surfaces of non-exceptional finite type, even though M and N are not assumed to be homeomorphic, by Earle and Kra in [3] and Lakic in [12]. Some further special cases of Theorem 3.6 were proved by Matsuzaki in [13]. Markovic proved 3.6 in full generality, that is, for the infinite analytic type case, in [14]. As in [5], we will use the methods of [14] to prove Theorem 3.6 in the finite analytic case, which gives a good indication of the methods used, without going into the technical detail required for the general case.

Let $\widetilde{M}$ be a compact Riemann surface of genus $g \geq 0$ and let E be a finite, possibly empty, subset of $\widetilde{M}$ which contains exactly $n \geq 0$ points. We assume that $2g + n \geq 5$, so that the Riemann surface $M = \widetilde{M} \setminus E$ has non-exceptional finite type.

We will consider projective embeddings of $\widetilde{M}$ associated with $A^1(M)$. Let k be a positive integer, and let $\mathbb{P}^k$ be the k-dimensional complex projective space. Each point $(z_0, ..., z_k) \in \mathbb{C}^{k+1} \setminus \{0\}$ determines a point $[(z_0, ..., z_k)] \in \mathbb{P}^k$.

The formula

$$\pi_0(z_1, ..., z_k) = [(1, z_1, ..., z_k)]$$

defines a holomorphic map of $\mathbb{C}^k$ onto a dense open subset of $\mathbb{P}^k$.

Let $\mathcal{M}(\widetilde{M})$ be the field of meromorphic functions on $\widetilde{M}$. For any divisor D on $\widetilde{M}$, we define $\mathcal{O}_D(\widetilde{M})$ to be the complex vector space of all functions in $\mathcal{M}(\widetilde{M})$, including the zero function, that are multiples of the divisor $-D$, that is

$$\mathcal{O}_D(\widetilde{M}) = \{f \in \mathcal{M}(\widetilde{M}) : ord_x(f) \geq -D(x), \forall x \in \widetilde{M}\}.$$

Proposition 3.7. *Let $M = \widetilde{M} \setminus E$ as above, and let $\varphi_0, ..., \varphi_k$ be a basis for $A^1(M)$. Set $f_j = \varphi_j/\varphi_0$, for $j = 1, ..., k$ and set*

$$M_0 = \widetilde{M} \setminus \{x \in \widetilde{M} : \text{ some } f_j \text{ has a pole at } x\}.$$

Let $F : M_0 \to \mathbb{C}^k$ be the holomorphic map $F = (f_1, ..., f_k)$. There is a unique holomorphic embedding $\Phi : \widetilde{M} \to \mathbb{P}^k$ such that

$$\Phi(x) = \pi_0(F(x)),$$

for all $x \in M_0$.

Proof. Consider the divisor $D = (\varphi_0) + \chi_E$ on $\widetilde{M}$. Clearly, $A^1(M)$ is the set of meromorphic quadratic differentials $\varphi = f\varphi_0$ such that $f \in \mathcal{O}_D(\widetilde{M})$.

Again, $\deg(D) \geq 2g + 1$. Since the functions $1, f_1, ..., f_k$ are a basis for $\mathcal{O}_D(\widetilde{M})$, we can conclude the map

$$x \mapsto [(1, f_1(x), ..., f_k(x))],$$

for $x \in \widetilde{M}$, when interpreted appropriately at the poles of the f_j, defines a holomorphic embedding of $\widetilde{M}$ into $\mathbb{P}^k$. □

Corollary 3.8. *The map F defined above is a homeomorphism of M_0 onto a closed subset of $\mathbb{C}^k$.*

Proof. Since F is holomorphic on M_0, it is continuous. Since $\Phi^{-1} \circ \pi_0 \circ F$ is the identity map of M_0 to itself, F is a homeomorphism. To see that $F(M_0)$ is a closed set, consider a sequence (x_n) in M_0 such that $F(x_n)$ converges to $z = (z_1, ..., z_k)$ in $\mathbb{C}^k$. We may assume that x_n converges to some point $\widetilde{x} \in \widetilde{M}$. Then $f_j(\widetilde{x}) = z_j \neq \infty$ for $j = 1, ..., k$ and so $\widetilde{x} \in M_0$ and $z = F(\widetilde{x})$. □

Definition 3.9. *Suppose that $f_1, ..., f_n$ are μ-measurable functions on X and that $g_1, ..., g_n$ are ν-measurable functions on Y. Writing $F = (f_1, ..., f_n)$ and $G = (g_1, ..., g_n)$, which we consider as $\mathbb{C}^n$ valued functions, then F and G are* *equimeasurable if*

$$\mu(F^{-1}(E)) = \nu(G^{-1}(E))$$

for every Borel set $E \subset \mathbb{C}$*.*

The following theorem on a condition for equimeasurability and the previous lemmas are due to Rudin in [17].

Theorem 3.10. *Let* $0 < p < \infty$*,* $p \neq 2, 4, 6, ...$*,* $n \in \mathbb{N}$*, and let* μ *and* ν *be measures on measurable spaces* X *and* Y *respectively. If for* $1 \leq i \leq n$*,* $f_i \in L^p(\mu)$ *and* $g_i \in L^p(\nu)$*, and*

$$\int_X |1 + z_1 f_1 + ... + z_n f_n|^p \, d\mu = \int_Y |1 + z_1 g_1 + ... z_n g_n|^p \, d\nu$$

for all $(z_1, ..., z_n) \in \mathbb{C}^n$*, then* $(f_1, ..., f_n)$ *and* $(g_1, ..., g_n)$ *are equimeasurable.*

We are now in a position to prove Theorem 3.6 for the finite analytic case.

Proof of Theorem 3.6. We will prove this theorem in the case where the given Riemann surfaces M and N are the complements of finite sets in compact Riemann surfaces $\widetilde{M}$ and $\widetilde{N}$. Again, for the proof in full generality, see [14]. Note that we do not assume $\widetilde{M}$ and $\widetilde{N}$ have the same genus.

Let $\varphi_0, ..., \varphi_k$ be a basis for $A^1(M)$, and define M_0 and the map

$$F = (f_1, ..., f_k)$$

as in the previous proposition. Set $\psi_j = T(\varphi_j)$ for $j = 0, ..., k$. Since

$$T : A^1(M) \to A^1(N)$$

is a surjective $\mathbb{C}$-linear isometry, $\psi_0, ..., \psi_k$ is a basis for $A^1(N)$. Set $g_j = \psi_j/\psi_0$, for $j = 1, ..., k$, and set

$$N_0 = \widetilde{N} \setminus \{y \in \widetilde{N} : \text{ some } g_j \text{ has a pole at } y\}.$$

Let $G : N_0 \to \mathbb{C}^k$ be the holomorphic embedding map $G = (g_1, ..., g_k)$. By Proposition 3.10, there is a holomorphic embedding $\Psi : \widetilde{N} \to \mathbb{P}^k$ such that $\Psi = \pi_0 \circ G$ on N_0.

Let μ and ν be the finite positive Borel measures on M_0 and N_0 defined by

$$\mu(A) = \int_A |\varphi_0|$$

for all Borel sets $A \subset M_0$, and

$$\nu(B) = \int_B |\psi_0|$$

for all Borel sets $B \subset N_0$. Since T is a $\mathbb{C}$-linear isometry, we have

$$\int_{M_0} \left| 1 + \sum_{j=1}^{k} \lambda_j f_j \right| d\mu = \int_M \left| \varphi_0 + \sum_{j=1}^{k} \lambda_j \varphi_j \right| =$$

$$\int_N \left| \psi_0 + \sum_{j=1}^{k} \lambda_j \psi_j \right| = \int_{N_0} \left| 1 + \sum_{j=1}^{k} \lambda_j g_j \right| d\nu,$$

for all $(\lambda_1, ..., \lambda_k) \in \mathbb{C}^k$. Therefore, the maps F and G, and the measures μ and ν satisfy Rudin's equimeasurability condition, Theorem 3.10. Applying this to the closed subset $F(M_0)$ of $\mathbb{C}^k$, we obtain

$$||\varphi_0|| = \int_{M_0} |\varphi_0| = \mu(M_0) = \nu(G^{-1}(F(M_0)))$$

$$= \int_{G^{-1}(F(M_0))} |\psi_0| \leq \int_{N_0} |\psi_0| = ||\psi_0||.$$

Since $||\psi_0|| = ||\varphi_0||$, the weak inequality here is actually an equality, and then $G^{-1}(F(M_0))$ has full measure in N_0. Since it is a closed subset of N_0, $G^{-1}(F(M_0))$ equals N_0, and $G(N_0)$ is contained in $F(M_0)$.

Similarly, applying the equimeasurability condition of Theorem 3.7 to the set $G(N_0)$, we find that $F(M_0)$ is a subset of $G(N_0)$. Therefore the sets $F(M_0)$ and $G(N_0)$ are equal, and so are their images under the map π_0 from $\mathbb{C}^k$ to $\mathbb{P}^k$. Now $\pi_0(F(M_0)) = \Phi(M_0)$ is dense in the compact set $\Phi(\widetilde{M})$, and $\pi_0(G(N_0))$ is dense in the compact set $\Psi(\widetilde{N})$, and so the sets $\Phi(\widetilde{M})$ and $\Psi(\widetilde{N})$ are equal.

Let $h : \widetilde{N} \to \widetilde{M}$ be the bijective holomorphic map $\Phi^{-1} \circ \Psi$. The restriction of h to N_0 satisfies $F \circ h = G$ and $h(N_0) = M_0$. From the definitions of F and G, we obtain

$$\frac{T(\varphi_j)}{T(\varphi_0)} = \frac{\psi_j}{\psi_0} = g_j = f_j \circ h = \frac{\varphi_j}{\varphi_0} \circ h = \frac{h^*(\varphi_j)}{h^*(\varphi_0)}$$

for $j = 1, ..., k$, and so

$$\frac{T(\varphi)}{T(\varphi_0)} = \frac{h^*(\varphi)}{h^*(\varphi_0)}$$

for all $\varphi \in A^1(M)$, where we write $h^*(\varphi)$ for the pullback of φ by h. Let K be any compact set in N_0. Applying the equimeasurability condition to the compact set $G(K)$ in $\mathbb{C}^k$, we obtain

$$\int_K |T(\varphi_0)| = \int_K |\psi_0| = \nu(K) = \mu(h(K)) = \int_{h(K)} |\varphi_0| = \int_K |h^*(\varphi_0)|.$$

Since K is arbitrary, we must have $|T(\varphi_0)| = |h^*(\varphi_0)|$ in N_0, and hence in all of $\widetilde{N}$. Therefore $T(\varphi_0) = e^{it}h^*(\varphi_0)$ for some $t \in \mathbb{R}$, and we see that

$$T(\varphi) = e^{it}h^*(\varphi), \tag{3.2}$$

for all $\varphi \in A^1(M)$. To complete the proof, we need to show that $h(N) = M$. By Proposition 3.7, N is the set of points in $\widetilde{N}$ where every $T(\varphi)$ is finite, and $h^{-1}(M)$ is the set of points in $\widetilde{N}$ where every $h^*(\varphi)$ is finite. These sets coincide by (3.2). □

We have that every biholomorphic map between two Teichmüller spaces $T(M)$ and $T(N)$ is induced by a quasiconformal map between M and N, unless one of them has exceptional type. The automorphism group of $T(M)$, denoted $\mathrm{Aut}(T(M))$ is the group of all biholomorphic self-mappings of $T(M)$.

Every quasiconformal mapping $g : M \to N$ induces a mapping $\rho_g : T(M) \to T(N)$ given by

$$\rho_g([f]) = [f \circ g^{-1}].$$

The mapping class group $MC(M)$ is the group of all Teichmüller classes of quasiconformal maps from the Riemann surface M onto itself. Further, every $g \in MC(M)$ induces an automorphism ρ_g of $T(M)$. Theorem 3.6 immediately gives us the following result.

Theorem 3.11. *If M is a Riemann surface of non-exceptional type, then*

$$\mathrm{Aut}(T(M)) = MC(M).$$

4 Local rigidity of Teichmüller spaces

In the previous section, we saw that a surjective linear isometry between the Bergman spaces of two Riemann surfaces implies that the two Riemann surfaces are conformally equivalent. In this section, we will use a classical Banach space result of Pelczynski [15] to prove a result of Fletcher [6] which shows that when two Riemann surfaces M and N are of infinite analytic type, their Bergman spaces will always be isomorphic. This then implies, via the Bers embedding, that their Teichmüller spaces will be locally bi-Lipschitz equivalent.

4.1 Projections on Banach spaces

Let $\mathbb{A}$ be two dimensional Lebesgue measure on the domain $\Omega \subseteq \mathbb{C}$. Then $L^1(\Omega)$ is the Banach space of measurable functions on Ω which have finite

L^1-norm given by

$$||f||_1 = \int_\Omega |f(z)|\, d\mathbb{A}(z) < \infty,$$

for $f \in L^1(\Omega)$. The Banach space $A^1(\Omega)$ is the subset of $L^1(\Omega)$ consisting of holomorphic functions.

Lemma 4.1. *Let Ω be a simply connected precompact subset of a Riemann surface M. Then given $\epsilon > 0$, there exists a projection $P : L^1(\Omega) \to L^1(\Omega)$ such that $||P|| < 1$,*

$$||P(f) - f|| < \epsilon,$$

for all $f \in A^1(\Omega)$ satisfying $||f||_1 \le 1$, and $P(L^1(\Omega))$ is isometric to $(l^1)_n$, where $(l^1)_n$ is the n-dimensional subspace of l^1 with all terms except possibly the first n being equal to 0.

Proof. We can for simplicity assume that Ω is a bounded simply connected plane domain. Subdivide Ω into a finite number of subsets, $\Omega_1, ..., \Omega_n$. For a given $f \in L^1(\Omega)$, define λ_i to be $\int_{\Omega_i} f$. We have

$$\sum_{i=1}^n |\lambda_i| = \sum_{i=1}^n \left| \int_{\Omega_i} f \right| \le \int_\Omega |f| < \infty.$$

Define the map $P : L^1(\Omega) \to L^1(\Omega)$ by

$$P(f) = \sum_{i=1}^n \frac{\lambda_i}{m(\Omega_i)} \mathbf{1}_{\Omega_i},$$

where $\mathbf{1}_{\Omega_i}$ denotes the indicator function of Ω_i, and m is the usual two dimensional Lebesgue measure of Ω_i. The map P is clearly linear and bounded ($||P|| \le 1$ in fact), and also a projection, since $P^2 = P$.

We can define a map $\mu : P(L^1(\Omega)) \to (l^1)_n$ given by

$$\mu(P(f)) = (\lambda_1, ..., \lambda_n, 0, ...).$$

Now, $||\mu(P(f))||_{l^1} = \sum_{i=1}^n |\lambda_i|$. Also,

$$||P(f)||_1 = \int_\Omega |P(f)| = \int_\Omega \left| \sum_{i=1}^n \frac{\lambda_i}{m(\Omega_i)} \mathbf{1}_{\Omega_i} \right| = \sum_{i=1}^n \int_{\Omega_i} \left| \frac{\lambda_i}{m(\Omega_i)} \right| = \sum_{i=1}^n |\lambda_i|$$

since the supports of $\mathbf{1}_{\Omega_i}$ are disjoint. Hence μ is isometric, and so $P(L^1(\Omega))$ is isometric to $(l^1)_n$.

We now have to show that we can find a fine enough subdivision of Ω so that for the corresponding projection P, $||P(f) - f|| < \epsilon$ for $f \in A^1(\Omega)$ with

$||f||_1 \leq 1$. Since Ω is precompact in M,

$$\sup\{|f(z)|\}$$

is bounded, where the supremum is taken over all $f \in A^1(M)$ with $||f||_1 \leq 1$ and over all $z \in \Omega$. This means that

$$\Theta = \{f|_\Omega : f \in A^1(M), ||f||_1 \leq 1\}$$

is a normal family, and hence is equicontinuous, ie. for all $f \in \Theta$ and for all $\epsilon > 0$, there exists a $\delta > 0$ such that if $|z - z_0| < \delta$, for $z, z_0 \in \Omega$, then $|f(z) - f(z_0)| < \epsilon$.

If $B(z_i, \delta)$ is a ball centred at z_i of Euclidean radius δ, then for any holomorphic function f,

$$\frac{1}{m(B(z_i, \delta))} \int_{B(z_i,\delta)} f = f(z_i).$$

If now Ω is subdivided into $\Omega_1, ..., \Omega_n$, with each $\Omega_i \subset B(z_i, \delta)$ for some z_i, and P is the projection corresponding to this subdivision, then

$$\int_{\Omega_i} |f - P(f)| \leq \int_{B(z_i,\delta)} |f(z) - f(z_i)| < \epsilon m(B(z_0, \delta))$$

recalling that $m(B(z_i, \delta))$ is the area of $B(z_i, \delta)$, and noting that the last inequality follows from the equicontinuity of Θ. Hence

$$\int_\Omega |f - P(f)| < \epsilon m(\Omega)$$

and since we are assuming that $m(\Omega)$ is finite, and ϵ can be made as small as wished, then we have the desired conclusion that $||P - I||$ can be as small as desired for P corresponding to a suitably fine subdivision of Ω. □

Lemma 4.2. *Let Y be a complemented subspace of a Banach space X, and let $T : Y \to X$ be a linear operator satisfying*

$$||T - I|_Y|| < \epsilon. \tag{4.1}$$

Then if ϵ is sufficiently small, $T(Y)$ is closed and complemented in X.

Proof. Let $S = I|_Y - T$. Then $||S|| < \epsilon$. Let also $P_Y : X \to Y$ be a projection, which is guaranteed to exist since Y is a complemented subspace of X. Then we have

$$X = \operatorname{Im}(P_Y) \oplus ker(P_Y). \tag{4.2}$$

Define $\widetilde{S} : X \to X$ by

$$\widetilde{S} = S \circ P_Y.$$

Then $\widetilde{S}$ is an extension of S and

$$||\widetilde{S}|| \leq ||P_Y||.||S||.$$

Thus if $\epsilon < ||P_Y||^{-1}$, we have $||\widetilde{S}|| < 1$. Now set

$$\widetilde{T} = I - \widetilde{S} : X \to X,$$

so that

$$\widetilde{T}(x) = x - P_Y(x) + T(P_Y(x)) \tag{4.3}$$

for $x \in X$. Then $\widetilde{T}$ is an extension of T, and $||\widetilde{S}|| < 1$ implies that $\widetilde{T}$ is invertible and therefore a homeomorphism. Since $\widetilde{T}$ is a homeomorphism, the image of $\widetilde{T}$ is closed and, furthermore, since Y is a closed subspace of X, $T(Y) = \widetilde{T}(Y)$ is closed. The fact that $T(Y) = \widetilde{T}(Y)$ follows from (4.3). From (4.2) and (4.3), it follows that $T(Y)$ is complementary to $\widetilde{T}(ker(P_Y))$ in $\widetilde{T}(X)$. Furthermore, we can rewrite (4.3) as

$$\widetilde{T}(x) = (I - P_Y)(x - T(P_Y(x))) + P_Y(T(P_Y(x))) \tag{4.4}$$

for $x \in X$, where the first term on the right hand side of (4.4) is an element of $ker(P_Y)$ and the second term on the right hand side of (4.4) is an element of Y. From the hypothesis (4.1) and the fact that T is invertible, it follows that $T : Y \to T(Y)$ is invertible and $P_Y : T(Y) \to Y$ is invertible so that the image of $P_Y \circ T$ is the whole of Y. It then follows from (4.4) that the image of $\widetilde{T}$ is the whole of X and therefore $T(Y)$ is complemented in X. □

4.2 Bergman kernels and projections on $L^1(M)$

The Bergman kernel on $\mathbb{D} \times \mathbb{D}$ is given by

$$K(z, \zeta) = \frac{1}{(1 - z\overline{\zeta})^4},$$

Every hyperbolic Riemann surface M has the disk $\mathbb{D}$ as its universal cover, that is, there is a Fuchsian covering group G such that $M \simeq \mathbb{D}/G$. Let $\pi : \omega \to M$ be the covering map from a fundamental region ω of $\mathbb{D}/G$ to M, chosen so that π is injective. Now, given such a covering group G, form the Poincaré theta series given by

$$F(z, \zeta) = \sum_{\gamma \in G} K(\gamma(z), \zeta)\gamma'(z)^2.$$

Definition 4.3. *Let M be a hyperbolic Riemann surface with covering group G of $\mathbb{D}$ over M. The Bergman kernel function for $M \times M$ is given by the* *projection of F to M. That is,*

$$K_M(\pi(z),\pi(\zeta))\pi'(z)^2\overline{\pi'(\zeta)^2} = F(z,\zeta).$$

Lemma 4.4. *The kernel function $K_M : M \times M \to \mathbb{C}$ defined above is holomorphic in the first argument, antiholomorphic in the second argument, and satisfies the following properties, where $p, q \in M$,*

(i) $K_M(p,q) = \overline{K_M(q,p)}$;

(ii) *for every conformal $f : M \to M$, $K_M(f(p), f(q))f'(p)^2\overline{f'(q)^2} = K_M(p,q)$;*

(iii) $\int_M |K_M(p,q)|\, d\mathbb{A}(p) \le \frac{\pi}{4}\rho_M^2(q)$, *where $\mathbb{A}(p)$ is the area measure on M in the p-coordinate;*

(iv) *for every $\varphi \in A^1(M)$,*

$$\varphi(p) = \frac{12}{\pi}\int_M \rho_M^{-2}(q)K_M(p,q)\varphi(q)\, d\mathbb{A}(q),$$

where $\mathbb{A}(q)$ is the area measure on M in the q-coordinate;

(v) *for each fixed $q \in M$,*

$$\sup_{p\in M} |K_M(p,q)|\rho_M^{-2}(p) < \infty.$$

Proof. We will just prove the third property since it will be used shortly. See [6, 8] for more details. Let $p = \pi(z)$ and $q = \pi(\zeta)$ for $p, q \in M$ and $z, \zeta \in \mathbb{D}$. We have

$$\int_M |K_M(p,q)|\, d\mathbb{A}(p) = \int_\omega |F(z,\zeta)|\, d\mathbb{A}(z)$$

where ω is a fundamental region for M in $\mathbb{D}$,

$$= \int_\omega \left| \sum_{\gamma\in G} K_{\mathbb{D}}(\gamma(z),\zeta)\gamma'(z)^2 \right| d\mathbb{A}(z) \le \sum_{\gamma\in G}\int_{\gamma(\omega)} |K_{\mathbb{D}}(z,\zeta)|\, d\mathbb{A}(z)$$

$$= \int_{\mathbb{D}} |K_{\mathbb{D}}(z,\zeta)|\, d\mathbb{A}(z) \le \frac{\pi}{4}\rho(z)^2 = \frac{\pi}{4}\rho_M(p)^2,$$

which completes the proof. □

Define the linear map $P : L^1(M) \to A^1(M)$ by

$$(P(\varphi))(\mu) = \frac{12}{\pi}\int_M \rho_M^{-2}(q)K_M(p,q)\varphi(q)\, d\mathbb{A}(q) \tag{4.5}$$

for $p, q \in M$. For any $\varphi \in L^1(M)$, it is clear that the integral formula for $P(\varphi)$ means that $P(\varphi)$ will be holomorphic, so the image of P is indeed $A^1(M)$.

Theorem 4.5. *There exists a bounded linear projection $\theta : L^1(M) \to A^1(M)$, given by $\theta : \varphi \mapsto P(\varphi)$ for $\varphi \in L^1(M)$.*

Proof. The map θ is clearly linear, and bounded, since

$$||P(\varphi)||_1 = \int_M |P(\varphi(p))| d\mathbb{A}(p) = \frac{12}{\pi} \int_M \left| \int_M \rho_M^{-2}(q) K_M(p,q) \varphi(q) \, d\mathbb{A}(q) \right| d\mathbb{A}(p)$$

$$\leq \frac{12}{\pi} \int_M \left(\int_M |K_M(p,q)| \, d\mathbb{A}(p) \right) \rho_M^{-2}(q) |\varphi(q)| \, d\mathbb{A}(q)$$

by Fubini's theorem, which we can apply by the fifth property in Lemma 4.4, and then using the third property of Lemma 4.4 gives

$$||P(\varphi)||_1 \leq 3 \int_M |\varphi(q)| \, d\mathbb{A}(q)$$

Hence $||\theta|| \leq 3$. The integral reproducing formula given in (4.5) shows that $\theta|_{A^1(M)}$ is the identity, $\theta^2 = \theta$, and so θ is a projection. □

4.3 Isomorphisms of Bergman spaces

Let $X_1, X_2, \ldots$ be Banach spaces with norms $||x_i||_i$ for $i = 1, 2, \ldots$ and $x_i \in X_i$. Also let $p > 0$. Then the Banach space $(X_1 \oplus X_2 \oplus \ldots)_p$ has elements of the form $(x_1, x_2, \ldots)$, for $x_i \in X_i$, and norm given by

$$||(x_1, x_2, \ldots)||_p = \left(\sum_{i=1}^{\infty} ||x_i||_i^p \right)^{1/p}. \tag{4.6}$$

Theorem 4.6. *If M is a hyperbolic Riemann surface of infinite analytic type, then $A^1(M)$ is isomorphic to the sequence space l^1.*

Proof. As discussed previously, we know that this theorem applies to all Riemann surfaces where the dimension of $A^1(M)$ is infinite, for example, the plane punctured at the integer lattice points, or an infinite genus surface.

We first subdivide M in an appropriate way. For every $p \in M$, there exists an open subset $U_p \subset M$ containing p, and a chart π_p such that $\pi_p(U_p)$ is a disk in $\mathbb{C}$ and $\pi_p(p) = 0$. Let V_p be an open simply connected set in M whose closure is contained in U_p, so that in particular $\pi_p(V_p)$ is a precompact subset of $\pi_p(U_p)$.

As p varies through M, $(V_p)_{p \in M}$ forms an open cover of M, and it is possible to find a countable subset $p_1, p_2, \ldots$ such that

$$M = \bigcup_{i=1}^{\infty} V_{p_i}$$

Now modify the subsets V_{p_i} to give a disjoint partition of M in the following way: define $M_1 = V_{p_1}$, and then inductively,

$$M_n = V_{p_n} \setminus \left(\bigcup_{i=1}^{n-1} V_{p_i} \right).$$

$$\begin{array}{ccccccc} L^1(M) & \xrightarrow{R=\oplus_i R_i} & \oplus_i L^1(M_i) & \xrightarrow{P=\oplus_i P_i} & \oplus_i P_i(L^1(M_i)) = \Lambda & \xrightarrow{isometry} & l^1 \\ \Big\downarrow \theta & & \Big\downarrow \widetilde{\theta} & & \Big\downarrow \widehat{\theta} & & \\ A^1(M) & \xrightarrow{R} & R(A^1(M)) & \xrightarrow{T} & \oplus_i P_i(R_i(A^1(M))) & & \end{array}$$

Refer to the diagram above for the following definitions. Let $R_i : L^1(M) \to L^1(M_i)$ be the restriction map given by $R_i(f) = f|_{M_i}$, for $f \in L^1(M)$. Define the operator $R : L^1(M) \to (L^1(M_1) \oplus L^1(M_2) \oplus ...)_1$ by

$$R(f) = (R_1(f), R_2(f), ...),$$

for $f \in L^1(M)$. The operator R is isometric, since

$$||R(f)||_1 = \sum_{i=1}^{\infty} ||R_i(f)||_1 = \sum_{i=1}^{\infty} \int_{M_i} |f| = \int_M |f| = ||f||_1,$$

using (4.6), and R is also clearly surjective. Now, given $\epsilon_i > 0$, by Lemma 4.1, we can find a projection P_i of $L^1(M_i)$ into itself such that $||P_i|| \leq 1$, $P_i(L^1(M_i))$ is isometric to $(l^1)_{\alpha_i}$ for some $\alpha_i \in \mathbb{Z}^+$, and $||P_i(R_i(f)) - R_i(f)||_1 \leq \epsilon_i$ for all $f \in A^1(M)$ with $||f|| < 1$.

Let

$$\Lambda = (P_1(L^1(M_1)) \oplus P_2(L^1(M_2)) \oplus ...)_1,$$

a subspace of $(L^1(M_1) \oplus L^1(M_2) \oplus ...)_1$. Since each $P_i(L^1(M_i))$ is isometric to $(l^1)_{\alpha_i}$ for some $\alpha_i \in \mathbb{Z}^+$, Λ is isometric to l^1. Now we define the operator $T : R(A^1(M)) \to \Lambda$ by

$$T(R_1(f), R_2(f), ...) = (P_1(R_1(f)), P_2(R_2(f)), ...).$$

Since the dimension of $A^1(M)$ is infinite, $R(A^1(M))$ must also be infinite dimensional. We also have

$$||T(\xi) - \xi||_1 \leq \left(\sum_{i=1}^{\infty} \epsilon_i \right) ||\xi||_1$$

for $\xi \in R(A^1(M))$, and so given $\epsilon > 0$, it is possible to choose the $(\epsilon_i)_i$ so that $||T(\xi) - \xi||_1 < \epsilon ||\xi||_1$, for $\xi \in R(A^1(M))$.

There exists a bounded linear projection $\theta : L^1(M) \to A^1(M)$ by Theorem 4.5. Therefore, there is a bounded linear projection $\widetilde{\theta} : R(L^1(M)) \to R(A^1(M))$, given by

$$\widetilde{\theta}(R_1(f), R_2(f), ...) = (R_1(\theta(f)), R_2(\theta(f)), ...)$$

which is clearly linear, bounded and satisfies $\widetilde{\theta}^2 = \widetilde{\theta}$. Therefore $R(A^1(M))$ is complemented in $R(L^1(M))$. Thus, by Lemma 4.2, if ϵ is small enough, $T(R(A^1(M)))$ is complemented in $R(L^1(M))$ and, in particular, Λ. This follows since if $W \subset Y$ is complemented in X, then there exists a projection $S : X \to W$, $(Im(S)) \cap Y$ is complemented in Y and so W is complemented in Y. The projection from Λ onto $T(R(A^1(M)))$ is denoted in the diagram above by $\widehat{\theta}$.

If $\epsilon < 1$, then $||T - I|| < 1$, and Lemma 4.2 gives that T is thus invertible and an isomorphism. By a classical result due to Pelczynski [15], every infinite dimensional complemented subspace of l^1 is isomorphic to l^1, and so $A^1(M)$ is isomorphic to l^1. □

By taking the Banach duals of the Banach spaces in the statement of Theorem 4.6, we immediately get the following results.

Corollary 4.7. *If M is a hyperbolic Riemann surface of infinite analytic type, then $Q(M)$ is isomorphic to the sequence space l^∞, and we will denote this isomrphism by α_M^*.*

Corollary 4.8. *If M and N are two hyperbolic Riemann surfaces of infinite analytic type, then $A^1(M)$ and $A^1(N)$ are isomorphic, and $Q(M)$ and $Q(N)$ are isomorphic.*

4.4 Local bi-Lipschitz equivalence of Teichmüller spaces

We have the following situation,

$$\lambda_M : T(M) \hookrightarrow Q(M), \qquad \alpha_M^* : Q(M) \to l^\infty$$

where the image of the Bers embedding λ_M is contained in $Q(M)$. Since λ_M is a locally bi-Lipschitz mapping, there exists a neighbourhood, X_M, of the identity class in $T(M)$ such that $\lambda_M|_{X_M}$ is bi-Lipschitz. Since α_M^* is an isomorphism, X_M is mapped onto a neighbourhood of the origin of l^∞ by $\alpha_M^* \circ \lambda_M$. If

$$Y_M = (\alpha_M^* \circ \lambda_M)(X_M),$$

then X_M and Y_M are bi-Lipschitz equivalent.

Lemma 4.9. *If M and N are two hyperbolic Riemann surfaces with infinite dimensional Bergman spaces, then a neighbourhood of the identity class in $T(M)$ is bi-Lipschitz equivalent to a neighbourhood of the identity class in $T(N)$.*

Proof. Consider the neighbourhoods of the identity class in the respective Teichmüller spaces given by X_M and X_N, and consider their images in l^∞ under the respective maps $\alpha_M^* \circ \lambda_M$ and $\alpha_N^* \circ \lambda_N$, given by Y_M and Y_N.

$$T(M) \xrightarrow{\lambda_M} Q(M) \xrightarrow{\alpha_M^*} l^\infty \xleftarrow{\alpha_N^*} Q(N) \xleftarrow{\lambda_N} T(N).$$

The sets Y_M and Y_N are both open neighbourhoods of the origin in l^∞, and so $Y := Y_M \cap Y_N$ is also an open neighbourhood of the origin. Since $\alpha_M^* \circ \lambda_M$ is a bi-Lipschitz mapping of X_M, it has an inverse on Y, and

$$((\alpha_M^* \circ \lambda_M)^{-1})(Y) \subseteq X_M$$

is an open neighbourhood of the origin in $T(M)$.

Thus $(\alpha_N^* \circ \lambda_N) \circ (\alpha_M^* \circ \lambda_M)^{-1}$ is a bi-Lipschitz mapping from a neighbourhood of the identity class in $T(M)$, namely $((\alpha_M^* \circ \lambda_M)^{-1})(Y)$, to a neighbourhood of the identity class in $T(N)$, namely $(\alpha_N^* \circ \lambda_N)(Y)$. □

Theorem 4.10. *If M and N are two hyperbolic Riemann surfaces with infinite dimensional Bergman spaces, then their Teichmüller spaces are locally bi-Lipschitz equivalent.*

Proof. Let $M \simeq \mathbb{D}/G$, and identify $T(M)$ with $T(G)$. Recall that a chart for the neighbourhood of the identity class in $T(f_\mu \circ G \circ (f_\mu)^{-1})$ is a chart for the neighbourhood of $[\mu]$ in $T(M)$. Thus charts for any $[\mu] \in T(M)$ and $[\nu] \in T(N)$ correspond to charts for the respective identity classes in $T(f_\mu \circ \Gamma \circ (f_\mu)^{-1})$ and $T(f_\nu \circ \Gamma_1 \circ (f_\nu)^{-1})$.

Lemma 4.9 gives a bi-Lipschitz mapping between neighbourhoods of these two identity classes, and hence we have a bi-Lipschitz mapping between neighbourhoods of $[\mu] \in T(M)$ and $[\nu] \in T(N)$. □

5 Open problems

If X and Y are connected complex Banach manifolds, then the Kobayashi metrics on the respective spaces are the largest metrics for which holomorphic maps between X and Y are distance decreasing. Conversely, the smallest metric under which holomorphic mappings are distance decreasing is called the Carathéodory metric. The Carathéodory distance on a connected complex

Banach manifold X is

$$C(x,y) = \sup_{f \in H(\mathbb{D},X)} \{\rho_{\mathbb{D}}(0,t) : f(0) = x, f(t) = y\},$$

for $x, y \in X$ and where $\rho_{\mathbb{D}}$ is the hyperbolic metric on $\mathbb{D}$.

Problem 5.1. *We can define the Carathéodory metric on Teichmüller space just as we did for the Kobayashi metric. The problem is, is the Carathéodory metric equal to the Teichmüller metric (or, equivalently, the Kobayashi metric) on Teichmüller space? Results in this direction can be found in [10, 11], where it is shown that the Carathéodory and Teichmüller metrics coincide on abelian Teichmüller disks.*

Problem 5.2. *Markovic's proof of Theorem 3.6 in the general case, see [14], involves how $A^1(M)$ separates points. That is, we say $A^1(M)$ separates $p, q \in M$ if there exists $\varphi \in A^1(M)$ such that $\varphi(p) = 0$ and $\varphi(q) \neq 0$. If M is of non-exceptional type, Markovic proves that the set of points E of M which are not separated by $A^1(M)$ is discrete, which is enough to prove Theorem3.6. The problem is, can E be shown to be empty?*

Problem 5.3. *Let $I : \mathbb{D} \to T(M)$ be an isometry. If I is holomorphic and the dimension of $T(M)$ is finite, then the image of I is a Teichmüller disk. The problem is, do all isometries from $\mathbb{D}$ into $T(M)$, which are not necessarily holomorphic, have a Teichmüller disk as their image?*

Problem 5.4. *Theorem 4.6 shows that there exists a constant C_M depending on M such that*

$$\frac{||\alpha_M(\varphi)||_{l^1}}{C_M} \leq ||\varphi||_1 \leq C_M ||\alpha_M(\varphi)||_{l^1}, \tag{5.1}$$

for all $\varphi \in A^1(M)$. Is there a universal constant C such that (5.1) holds, with C replacing C_M, independently of M?

References

[1] L. Bers, Automorphic forms and Poincaré series for finitely generated Fuchsian groups, *Amer. Math. J.*, **87** (1965), 196–214.

[2] C. Earle and F. Gardiner, Geometric isomorphisms between infinite dimensional Teichmüller spaces, *Trans. Amer. Math. Soc.*, **348**, no. 3 (1996), 1163–1190.

[3] C. Earle and I. Kra, On holomorphic mappings between Teichmüller spaces, *Contributions to Analysis*, Academic Press, New York (1974), 107–124.

[4] C. Earle, I. Kra and S. Krushkal, Holomorphic motions and Teichmüller spaces, *Trans. Amer. Math. Soc.*, **343**, no.2 (1994), 927–948.

[5] C. Earle and V. Markovic, Isometries between the spaces of L^1 holomorphic quadratic differentials on Riemann surfaces of finite type, *Duke Math. J.*, **120**, no.2 (2003), 433–440.

[6] A. Fletcher, Local Rigidity of Infinite Dimensional Teichmüller Spaces, *J. London Math. Soc.*, **74** (1) (2006), 26–40.

[7] A. Fletcher and V. Markovic, *Quasiconformal mappings and Teichmüller theory*, OUP (2007).

[8] F. Gardiner, *Teichmüller Theory and Quadratic Differentials*, John Wiley and Sons (1987).

[9] I. Kra, *Automorphic forms and Kleinian groups*, Benjamin, Reading, Massachusetts (1972).

[10] I. Kra, The carathéodory metric on abelian Teichmüller disks, *J. Analyse Math.*, **40** (1981), 129–143

[11] S. Krushkal, Grunsky coefficient inequalities, Carathéodory metric and extremal quasiconformal mappings, *Comment. Math. Helv.*, **64**, no. 4 (1989), 650–660.

[12] N. Lakic, An isometry theorem for quadratic differentials on Riemann surfaces of finite genus, *Trans. Amer. Math. Soc.*, **349**, no.7 (1997), 2951-2967.

[13] K. Matsuzaki, Inclusion relations between the Bers embeddings of Teichmüller spaces, *Israel J. Math.*, **140** (2004), 113–123.

[14] V. Markovic, Biholomorphic maps between Teichmüller spaces, *Duke Math. J.*, **120**, no. 2 (2003), 405–431.

[15] A. Pelczynski, Projections in certain Banach spaces, *Studia. Math.*, **19** (1960), 209–228.

[16] H. Royden, Automorphisms and isometries of Teichmüller space in 'Advances of the Theory of Riemann Surfaces,' *Ann. Math. Stud.*, **66** (1971), 369–384.

[17] W. Rudin, L^p-isometries and equimeasurability, *Indiana Univ. Math. J.*, **25**, no. 3 (1976) 215–228.