

# DECOMPOSING DIFFEOMORPHISMS OF THE SPHERE

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ABSTRACT. A central problem in the theory of quasiconformal and bi-Lipschitz mappings is whether they can be written as a composition of such mappings with small distortion. In this paper we prove a decomposition result for  $C^1$  diffeomorphisms of the sphere. Namely we show that given  $\epsilon > 0$ , every  $C^1$  diffeomorphism of the sphere  $S^n$  can be written as a composition of bi-Lipschitz mappings with isometric distortion at most  $1 + \epsilon$ .

## 1. INTRODUCTION

1.1. **Background.** A bi-Lipschitz homeomorphism  $f : X \rightarrow Y$  between metric spaces is a mapping  $f$  such that  $f$  and  $f^{-1}$  satisfy a uniform Lipschitz condition, that is, there exists  $L \geq 1$  such that

$$\frac{d_X(x, y)}{L} \leq d_Y(f(x), f(y)) \leq L d_X(x, y)$$

for all  $x, y \in X$ . The smallest such constant  $L$  is called the *isometric distortion* of  $f$ . In the metric space setting, a homeomorphism  $f : X \rightarrow Y$  is called quasiconformal if there exists a constant  $H \geq 1$  such that

$$H_f(x) := \limsup_{r \rightarrow 0} \frac{\sup\{d_Y(f(x), f(y)) : d_X(x, y) \leq r\}}{\inf\{d_Y(f(x), f(y)) : d_X(x, y) \geq r\}} \leq H$$

for all  $x \in X$ . The constant  $H$  is called the *conformal distortion* of  $f$ . This definition coincides with the perhaps more familiar analytic definition of quasiconformal mappings in  $\mathbb{R}^n$ .

Let  $S^n$  be the sphere of dimension  $n$  and denote by  $QC(S^n)$  and  $LIP(S^n)$  the orientation preserving quasiconformal and bi-Lipschitz homeomorphisms, respectively, of  $S^n$ . An old central problem in this area is the following.

**Conjecture 1.1.** *Let  $f$  be in either  $QC(S^n)$  or  $LIP(S^n)$ . Then for every  $\epsilon > 0$  we can find homeomorphisms  $f_k \in QC(S^n)$  or  $f_k \in LIP(S^n)$ ,  $k = 1, \dots, m$ , such that  $f$  can be written as a composition  $f = f_m \circ \dots \circ f_1$ , where each  $f_k$  has conformal distortion at most  $1 + \epsilon$  or each  $f_k$  has isometric distortion at most  $1 + \epsilon$  respectively.*

The conjecture is known for the class  $QC(S^2)$  and is essentially a consequence of solving the Beltrami equation in the plane, see for example [2]. The quasisymmetric case  $QC(S^1)$  also follows from the dimension 2 case.

It is well-known that every  $L$ -bi-Lipschitz homeomorphism between two intervals can be factored into bi-Lipschitz mappings with smaller isometric distortion  $\alpha$ . Such a factorisation can be written explicitly in the following way. Let  $f : I \rightarrow I'$  be an  $L$ -bi-Lipschitz mapping. Then  $f$  can be written as  $f = f_2 \circ f_1$ , where

$$f_1(x) = \int_{x_0}^x |f'(t)|^\lambda dt,$$

$x_0 \in I$  is fixed,  $\lambda = \log_L \alpha$ ,  $f_1$  is  $\alpha$ -bi-Lipschitz and  $f_2 = f \circ f_1^{-1}$  is  $L/\alpha$ -bi-Lipschitz. It follows that to factorise an  $L$ -bi-Lipschitz mapping into  $\alpha$ -bi-Lipschitz mappings requires  $N < \log_\alpha L + 1$  factors.

In dimension 2, Freedman and He [4] studied the logarithmic spiral map  $s_k(z) = ze^{ik \log |z|}$ , which is an  $L$ -bi-Lipschitz mapping of the plane where  $|k| = L - 1/L$ . They showed that  $s_k$  requires  $N \geq |k|(\alpha^2 - 1)^{-1/2}$  factors to be represented as a composition of  $\alpha$ -bi-Lipschitz mappings. Gutlyanskii and Martio [6] studied a related class of mappings in dimension 2, and generalized this to a class of volume preserving bi-Lipschitz automorphisms of the unit ball  $\mathbb{B}^3$  in 3 dimensions. Beyond these particular examples, however, very little is known about factorizing bi-Lipschitz mappings in dimension 2 and higher, and factorizing quasiconformal mappings in dimension 3 and higher.

A natural question to ask is whether diffeomorphisms of the sphere  $S^n$  can be decomposed into diffeomorphisms that are  $C^1$  close to the identity. The answer in general is negative as the exotic spheres of Milnor [9] provide an obstruction. This is seen as follows.

In [9], it is shown that there exist topological 7-spheres which are not diffeomorphic to the standard 7-sphere  $S^7$ . This implies that there exists a  $C^1$  diffeomorphism  $f : S^6 \rightarrow S^6$  such that there is no  $C^1$  path of diffeomorphisms of  $S^6$  that connects the identity and  $f$ . Next, we show that there exists  $\delta_0 > 0$  such that  $f$  cannot be written as  $f = f_m \circ \dots \circ f_1$ , where  $f_k : S^6 \rightarrow S^6$  are diffeomorphisms that are  $\delta_0$  close to the identity in the  $C^1$  metric. Recall the following standard result [1]:

- For each  $n > 0$ , there exists a constant  $\epsilon_n > 0$  such that every  $C^1$  diffeomorphism  $g : S^n \rightarrow S^n$  that is  $\epsilon_n$  close to the identity in the  $C^1$  metric, there exists a  $C^1$  path of diffeomorphisms of  $S^n$  that connects the identity and  $g$ .

We may take  $\delta_0 = \epsilon_6$ . Then if we suppose that  $f$  can be written as  $f = f_m \circ \dots \circ f_1$ , where  $f_k : S^6 \rightarrow S^6$  are diffeomorphisms that are  $\delta_0$  close to the identity in the  $C^1$  metric, we would have that each  $f_k$  can be connected to the identity by a  $C^1$  path of diffeomorphisms and thus  $f$  can be as well which is a contradiction.

There are two facts that might be obstructions to the factorization theorem. One is the Milnor example. The second fact is that not all topological manifolds of dimension at least 5 admit differentiable structures [7]. The same result is true (but much harder to prove) in dimension 4 and it follows from the works of Donaldson and Freedman [3]. On the other hand, a deep

result of Sullivan [11] states that they always admit a bi-Lipschitz structure. The recent results of Bonk, Heinonen and Wu [14] which state that closed bi-Lipschitz manifolds where the transition maps have small enough distortion admit a  $C^1$  structure, raises the question of whether a factorization theorem in this case would contradict Sullivan's theorem.

**1.2. Main results.** Since some  $C^1$  diffeomorphisms of  $S^n$  cannot be decomposed into  $C^1$  diffeomorphisms with derivative close to the identity, that suggests the question of trying to factor them into bi-Lipschitz mappings of small isometric distortion.

The main result of this paper states that one can find a path connecting the identity and any  $C^1$  diffeomorphism of  $S^n$  which is a composition of bi-Lipschitz paths, a notion that will be made more precise in §2.

**Theorem 1.2.** *Let  $f : S^n \rightarrow S^n$  be a  $C^1$  diffeomorphism. Then there exist bi-Lipschitz paths  $A_t, p_t^1, p_t^2 : S^n \rightarrow S^n$  for  $t \in [0, 1]$  such that  $A_0, p_0^1$  and  $p_0^2$  are all the identity, and  $A_1 \circ p_1^2 \circ p_1^1 = f$ .*

**Remark 1.3.** *It is not a priori true that a composition of bi-Lipschitz paths is another bi-Lipschitz path since issues arise at points of non-differentiability.*

As a corollary to this theorem, we find that  $C^1$  diffeomorphisms of the sphere  $S^n$  can be decomposed into bi-Lipschitz mappings of arbitrarily small isometric distortion.

**Theorem 1.4.** *Let  $f : S^n \rightarrow S^n$  be a  $C^1$  diffeomorphism. Given  $\epsilon > 0$ , there exists  $m \in \mathbb{N}$ , depending on  $f$  and  $\epsilon$ , such that  $f$  decomposes as  $f = f_m \circ \dots \circ f_1$ , where  $f_k$  is  $(1 + \epsilon)$ -bi-Lipschitz with respect to the spherical metric  $\chi$ , and  $\chi(f_k(x), x) < \epsilon$  for all  $x \in S^n$  and for  $k = 1, \dots, m$ .*

**Remark 1.5.** *The bi-Lipschitz factors  $f_k : S^n \rightarrow S^n$  in the above theorem might not always be diffeomorphisms according to the discussion at the end of Section 1.1. It turns out that each  $f_k$  is differentiable at every point of  $S^n$  but it is not  $C^1$ , that is the derivative is not continuous. See the remark after Lemma 2.7.*

In §2, we will state several intermediate lemmas and prove Theorem 1.2 and Corollary 1.4 assuming these lemmas hold. The proofs of the lemmas are postponed to §3.

## 2. OUTLINE OF PROOF

**2.1. Some notation.** We will first fix some notation. Let  $S^n = \mathbb{R}^n \cup \{\infty\}$  be the sphere of dimension  $n$ . Denote by  $d$  the Euclidean metric on  $\mathbb{R}^n$  and by  $\chi$  the spherical metric on  $S^n$ , so that

$$d(x, y) = |x - y|,$$

for  $x, y \in \mathbb{R}^n$  and

$$\chi(x, y) = \frac{|x - y|}{\sqrt{1 + |x|^2} \sqrt{1 + |y|^2}}$$

for  $x, y \in S^n \setminus \{\infty\}$ . If  $y$  is the point at infinity,

$$\chi(x, \infty) = \frac{1}{\sqrt{1 + |x|^2}}.$$

Let  $B_d(x, r) = \{y \in \mathbb{R}^n : d(x, y) \leq r\}$  and  $B_\chi(x, r) = \{y \in S^n : \chi(x, y) \leq r\}$  be the closed balls centred at  $x$  of respectively Euclidean and spherical radius  $r$ . We say that a diffeomorphism  $f$  is supported on a set  $U \subset S^n$  if  $f$  is the identity on the complement  $S^n \setminus U$ .

**2.2. Diffeomorphisms supported on balls.** We first need to show that a  $C^1$  diffeomorphism with a fixed point can be written as a composition of  $C^1$  diffeomorphisms supported on spherical balls.

**Lemma 2.1.** *Let  $f : S^n \rightarrow S^n$  be a  $C^1$  diffeomorphism with at least one fixed point. Then there exist  $x_1, x_2 \in S^n$  and  $r_1, r_2 > 0$  such that  $f$  decomposes as  $f = f^2 \circ f^1$  where  $f^1, f^2$  are  $C^1$  diffeomorphisms supported on spherical balls  $B_1 = B_\chi(x_1, r_1), B_2 = B_\chi(x_2, r_2)$  in  $S^n$ , and so that neither  $B_1$  nor  $B_2$  are  $S^n$ .*

To prove the lemma, we will need to make use of the following result of Munkres [10, Lemma 8.1] as formulated in [13].

**Theorem 2.2** ([10]). *Let  $h : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be an orientation preserving  $C^k$  diffeomorphism for  $1 \leq k \leq \infty$ . Then there exists a  $C^k$  diffeomorphism  $\tilde{h} : \mathbb{R}^n \rightarrow \mathbb{R}^n$  which coincides with the identity near  $0 \in \mathbb{R}^n$  and  $h$  near infinity.*

*Proof of Lemma 2.1.* Suppose that  $f : S^n \rightarrow S^n$  is a  $C^1$  diffeomorphism with a fixed point in  $S^n$ . Identifying  $S^n$  with  $\overline{\mathbb{R}^n}$ , without loss of generality we can assume  $f$  fixes the point at infinity. Then by Theorem 2.2, there exists a  $C^1$  diffeomorphism  $\tilde{f}$  and real numbers  $r_1, r_2 > 0$  such that  $\tilde{f}|_{B_\chi(0, r_1)}$  is the identity and  $\tilde{f}|_{B_\chi(\infty, r_2)}$  is equal to  $f$ . We can then write

$$f = \left( f \circ \tilde{f}^{-1} \right) \circ \tilde{f}$$

where  $f^2 := f \circ \tilde{f}^{-1}$  is supported on the ball  $S^n \setminus B_\chi(\infty, r_2)$  and  $f^1 := \tilde{f}$  is supported on the ball  $S^n \setminus B_\chi(0, r_1)$ .  $\square$

**2.3. Bi-Lipschitz paths.** We shall postpone the proofs of the lemmas in this section until §3. Let us now define the notion of a bi-Lipschitz path.

**Definition 2.3.** *Let  $(X, d_X)$  be a metric space. A path  $h : [0, 1] \rightarrow LIP(X)$  is called a bi-Lipschitz path if for every  $\epsilon > 0$ , there exists  $\delta > 0$  such that if  $s, t, \in [0, 1]$  with  $|s - t| < \delta$ , the following two conditions hold:*

- (i) for all  $x \in X$ ,  $d_X(h_s \circ h_t^{-1}(x), x) < \epsilon$ ;
- (ii) we have that  $h_s \circ h_t^{-1}$  is  $(1 + \epsilon)$ -bi-Lipschitz with respect to  $d_X$ .

We need the following lemmas on bi-Lipschitz paths.

**Lemma 2.4.** *Let  $h_t : [0, 1] \rightarrow LIP(\mathbb{R}^n)$  be a bi-Lipschitz path with respect to  $d$ . Then  $h_t : [0, 1] \rightarrow LIP(S^n)$  is a bi-Lipschitz path with respect to  $\chi$ .*

**Lemma 2.5.** *Let  $h_t : [0, 1] \rightarrow LIP(\mathbb{R}^n)$  be a bi-Lipschitz path with respect to  $d$  and let  $g : S^n \rightarrow S^n$  be a Möbius transformation. Then the path  $g \circ h_t \circ g^{-1}$  is bi-Lipschitz with respect to  $\chi$  on  $S^n$ .*

**Remark 2.6.** *It can be shown that a bi-Lipschitz path  $h_t : [0, 1] \rightarrow LIP(M)$  on a closed manifold  $M$  remains bi-Lipschitz after conjugation by a conformal map  $g : M \rightarrow M$ . The condition that  $g$  is conformal cannot be weakened to  $g$  being a diffeomorphism.*

The following lemma is the main step in the proof of Theorem 1.2.

**Lemma 2.7.** *Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a  $C^1$  diffeomorphism supported in  $B_d(0, 1/3)$ . Then there exists a path  $h_t : [0, 1] \rightarrow LIP(\mathbb{R}^n)$  which is bi-Lipschitz with respect to  $d$ , connecting the identity  $h_0$  and  $h_1 = f$ .*

**Remark 2.8.** *It follows from the construction in the proof of Lemma 2.7 below that each  $h_t$  is differentiable at every point of  $\mathbb{R}^n = S^n$ , but  $h_t$  is not  $C^1$ . The derivative of  $h_t$  is not continuous at  $\infty$  (but it is continuous at every point of  $\mathbb{R}^n$ ).*

**2.4. Proofs of the main results.** Assuming the intermediate results above, the proof of Theorem 1.2 proceeds as follows.

*Proof of Theorem 1.2.* Let  $f : S^n \rightarrow S^n$  be a  $C^1$  diffeomorphism. There exists  $A \in SO(n)$  such that  $A \circ f$  has a fixed point in  $S^n$ . Note that if  $n$  is even, then  $f$  automatically has a fixed point and we can take  $A$  to be the identity.

By Lemma 2.1, we can write  $A \circ f = f^2 \circ f^1$  where  $f^i$  is supported on the spherical ball  $B_i$  for  $i = 1, 2$ . By standard spherical geometry, see e.g. [12], for  $i = 1, 2$ , there exist Möbius transformations  $g_i$  such that  $g_i^{-1} \circ f^i \circ g_i$  is supported on  $B_d(0, 1/3)$ .

Now, applying Lemma 2.7 to  $g_i^{-1} \circ f^i \circ g_i$ , we obtain two bi-Lipschitz paths  $h_t^i$ , for  $i = 1, 2$ , with respect to  $d$  on  $\mathbb{R}^n$ . Consider the paths

$$p_t^i = g_i \circ h_t^i \circ g_i^{-1}$$

for  $i = 1, 2$ , where  $p_0^i$  is the identity and  $p_1^i = f^i$ .

It follows by Lemma 2.5 that  $p_t^i$  is bi-Lipschitz with respect to  $\chi$  on  $S^n$ . Then  $p_t^2 \circ p_t^1$  is a composition of bi-Lipschitz paths, with respect to  $\chi$ , connecting the identity and  $A \circ f$ . Since  $A^{-1} \in SO(n)$ , there is a bi-Lipschitz path  $A_t$  connecting the identity  $A_0$  and  $A_1 = A^{-1}$ . We conclude that  $A_t \circ p_t^2 \circ p_t^1$  is a composition of three bi-Lipschitz paths, which connects the identity and  $f$ . This completes the proof.  $\square$

*Proof of Theorem 1.4.* Let  $\epsilon > 0$ . By Theorem 1.2,  $A_t, p_t^1$  and  $p_t^2$  are all bi-Lipschitz paths with respect to  $\chi$  on  $S^n$ ,  $A_0 \circ p_0^2 \circ p_0^1$  is the identity and  $A_1 \circ p_1^2 \circ p_1^1 = f$ .

Given a bi-Lipschitz path  $h_t$ , we can choose  $0 = t_1 < t_2 < \dots < t_{j+1} = 1$  such that  $g_k = h_{t_{k+1}} \circ h_{t_k}^{-1}$  is  $(1 + \epsilon)$ -bi-Lipschitz for  $k = 1, \dots, j$  and  $h_1 = h_{t_{j+1}} = g_j \circ \dots \circ g_1$ . Applying this observation to the bi-Lipschitz paths  $A_t, p_t^1$  and  $p_t^2$ , there exists  $j(1), j(2), j(3) \in \mathbb{N}$  such that

$$\begin{aligned} A_1 &= A_{1,j(1)} \circ A_{1,j(1)-1} \circ \dots \circ A_{1,1}, \\ p_1^1 &= p_{1,j(2)}^1 \circ p_{1,j(2)-1}^1 \circ \dots \circ p_{1,1}^1, \\ p_1^2 &= p_{1,j(3)}^2 \circ p_{1,j(3)-1}^2 \circ \dots \circ p_{1,1}^2, \end{aligned}$$

and each map in these three decompositions is  $(1 + \epsilon)$ -bi-Lipschitz with respect to  $\chi$ , and also only moves points in  $S^n$  by at most spherical distance  $\epsilon$ . In view of  $A_1 \circ p_1^2 \circ p_1^1 = f$ , this proves the theorem with  $m = j(1) + j(2) + j(3)$ .  $\square$

### 3. PROOFS OF THE LEMMAS

We will prove Lemma 2.4 and Lemma 2.5 first, before proving the main Lemma 2.7. The main ideas are contained in Section 3.3. In Section 3.1 we compute a routine change of metrics.

**3.1. Proof of Lemma 2.4.** Let  $h_t : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a bi-Lipschitz path with respect to  $d$ . Then each  $h_t$  extends to a mapping  $S^n \rightarrow S^n$  which fixes the point at infinity. Let  $s, t \in [0, 1]$  and consider the mapping  $g = h_s \circ h_t^{-1}$ . Since  $h_t$  is a bi-Lipschitz path, choose  $\delta > 0$  small enough so that if  $|s - t| < \delta$  then  $d(g(x), x) < \epsilon$  for all  $x \in \mathbb{R}^n$  and  $g$  is  $(1 + \epsilon)$ -bi-Lipschitz with respect to  $d$ .

Property (i) of Definition 2.3 is satisfied for  $\chi$  since  $\chi(g(x), x) \leq d(g(x), x)$ , for  $x \in \mathbb{R}^n$ , and  $g$  fixes the point at infinity.

We now show that  $h_t$  satisfies property (ii) of Definition 2.3. The fact that  $h_t$  is a bi-Lipschitz path with respect to  $d$  and the formula for the spherical distance give

$$\begin{aligned} \chi(g(x), g(y)) &= \frac{|g(x) - g(y)|}{\sqrt{1 + |g(x)|^2} \sqrt{1 + |g(y)|^2}} \\ &\leq \frac{(1 + \epsilon)|x - y|}{\sqrt{1 + |g(x)|^2} \sqrt{1 + |g(y)|^2}} \\ (1) \quad &= (1 + \epsilon)\chi(x, y) \left( \frac{1 + |x|^2}{1 + |g(x)|^2} \right)^{1/2} \left( \frac{1 + |y|^2}{1 + |g(y)|^2} \right)^{1/2}, \end{aligned}$$

for  $x, y \in \mathbb{R}^n$ . Since  $d(g(x), x) < \epsilon$ , it follows that

$$\frac{1 + |x|^2}{1 + (|x| + \epsilon)^2} \leq \frac{1 + |x|^2}{1 + |g(x)|^2} \leq \frac{1 + |x|^2}{1 + (|x| - \epsilon)^2}.$$

Therefore,

$$\left( 1 + \frac{\epsilon(\epsilon + 2|x|)}{1 + |x|^2} \right)^{-1} \leq \frac{1 + |x|^2}{1 + |g(x)|^2} \leq \left( 1 + \frac{\epsilon(\epsilon - 2|x|)}{1 + |x|^2} \right)^{-1}$$

and so it follows that given  $\epsilon > 0$ , we can choose  $\epsilon'$  small enough so that

$$(2) \quad \frac{1}{1 + \epsilon'} \leq \frac{1 + |x|^2}{1 + |g(x)|^2} \leq 1 + \epsilon'$$

for all  $x \in \mathbb{R}^n$ . By (1) and (2), it follows that

$$(3) \quad \chi(g(x), g(y)) \leq (1 + \epsilon)(1 + \epsilon')\chi(x, y),$$

for all  $x, y \in \mathbb{R}^n$ . We can conclude that given  $\epsilon > 0$ , we can choose  $\xi > 0$  small enough so that

$$(4) \quad \chi(g(x), g(y)) \leq (1 + \xi)\chi(x, y)$$

for all  $x, y \in \mathbb{R}^n$ . The reverse inequality follows by applying (4) to  $g^{-1}$ . Therefore condition (ii) of Definition 2.3 holds for  $x, y \in \mathbb{R}^n$  with  $\delta$ , and  $\xi$  playing the role of  $\epsilon$ .

Finally, if  $x \in \mathbb{R}^n$  and  $y = \infty$ , then

$$\chi(g(x), \infty) = \frac{1}{\sqrt{1 + |g(x)|^2}} = \chi(x, \infty) \left( \frac{1 + |x|^2}{1 + |g(x)|^2} \right)^{1/2}$$

and we then apply (2) as above. This completes the proof of Lemma 2.4.

**3.2. Proof of Lemma 2.5.** Recall that  $h_t$  is a bi-Lipschitz path with respect to  $d$  on  $\mathbb{R}^n$  and that  $g : S^n \rightarrow S^n$  is a Möbius transformation. We can write

$$g = C \circ B,$$

where  $B : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is an affine map and  $C$  is a spherical isometry. To see this, let  $x \in S^n$  be the point such that  $g(\infty) = x$ . Then there exists a (non-unique) spherical isometry  $C$  such that  $C(\infty) = x$  and then the map  $B = C^{-1} \circ g$  is affine.

We first show that  $B \circ h_t \circ B^{-1}$  is a bi-Lipschitz path with respect to  $d$  on  $\mathbb{R}^n$ . Since  $B : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is an affine map, there is a real number  $\alpha > 0$  such that

$$d(B(x), B(y)) = \alpha d(x, y),$$

for all  $x, y \in \mathbb{R}^n$ . Since  $h_t$  is a bi-Lipschitz path with respect to  $d$ , write  $f = h_s \circ h_t^{-1}$ , with  $|s - t| < \delta$  small enough so that  $d(f(x), x) < \epsilon$  and  $f$  is  $(1 + \epsilon)$ -bi-Lipschitz with respect to  $d$ . Then

$$\begin{aligned} d(B(f(B^{-1}(x))), x) &= d(B(f(B^{-1}(x))), B(B^{-1}(x))) \\ &\leq \alpha d(f(B^{-1}(x)), B^{-1}(x)) \\ &< \alpha \epsilon, \end{aligned}$$

for all  $x \in \mathbb{R}^n$ . Therefore  $B \circ h_t \circ B^{-1}$  satisfies condition (i) of Definition 2.3 with  $\delta$  and  $\alpha \epsilon$ . Next,

$$\begin{aligned} d(B(f(B^{-1}(x))), B(f(B^{-1}(y)))) &= \alpha d(f(B^{-1}(x)), f(B^{-1}(y))) \\ &\leq \alpha(1 + \epsilon)d(B^{-1}(x), B^{-1}(y)) \\ &= (1 + \epsilon)d(x, y) \end{aligned}$$

and so  $B \circ h_t \circ B^{-1}$  satisfies condition (ii) of Definition 2.3 with  $\delta$  and  $\epsilon$ .

By Lemma 2.4,  $B \circ h_t \circ B^{-1}$  is also bi-Lipschitz with respect to  $\chi$  on  $S^n$ . It remains to show that  $C \circ B \circ h_t \circ B^{-1} \circ C^{-1} = g \circ h_t \circ g^{-1}$  is a bi-Lipschitz path with respect to  $\chi$  on  $S^n$ .

Since  $B \circ h_t \circ B^{-1}$  is a bi-Lipschitz path with respect to  $\chi$ , write  $f = B \circ h_s \circ h_t^{-1} \circ B^{-1}$ , with  $|s - t| < \delta$  small enough so that  $\chi(f(x), x) \leq \epsilon$  and  $f$  is  $(1 + \epsilon)$ -bi-Lipschitz with respect to  $\chi$ . Then

$$\begin{aligned} \chi(C(f(C^{-1}(x))), x) &= \chi(C(f(C^{-1}(x))), C(C^{-1}(x))) \\ &= \chi(f(C^{-1}(x)), C^{-1}(x)) \\ &< \epsilon, \end{aligned}$$

for all  $x \in S^n$ . Therefore  $C \circ B \circ h_t \circ B^{-1} \circ C^{-1}$  satisfies condition (i) of Definition 2.3 with  $\delta$  and  $\epsilon$ . Next,

$$\begin{aligned} \chi(C(f(C^{-1}(x))), C(f(C^{-1}(y)))) &= \chi(f(C^{-1}(x)), f(C^{-1}(y))) \\ &\leq (1 + \epsilon)\chi(C^{-1}(x), C^{-1}(y)) \\ &= (1 + \epsilon)\chi(x, y), \end{aligned}$$

and so  $C \circ B \circ h_t \circ B^{-1} \circ C^{-1}$  satisfies condition (ii) of Definition 2.3 with  $\delta$  and  $\epsilon$ . This completes the proof.

**3.3. Proof of Lemma 2.7.** We first set some notation. If  $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is differentiable at  $x \in \mathbb{R}^n$ , write  $D_x g$  for the derivative of  $g$  at  $x$  and let

$$\|D_x g\| = \max_{y \in \mathbb{R}^n \setminus \{0\}} \frac{|(D_x g)(y)|}{|y|}$$

be the operator norm of the linear map  $D_x g$ . Note that we are regarding the derivative here as a mapping from  $\mathbb{R}^n$  to  $\mathbb{R}^n$  given by the matrix of partial derivatives  $\partial g_i / \partial x_j$ , and not as a mapping between tangent spaces.

Recall that  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a  $C^1$  diffeomorphism supported on the ball  $B_0 := B_d(0, 1/3)$ . Write  $A_t : \mathbb{R}^n \rightarrow \mathbb{R}^n$  for the translation  $A_t(x_1, x_2, \dots, x_n) = (x_1 + t, x_2, \dots, x_n)$  and define  $B_t = A_t(B_0)$ . Write  $e_1 = (1, 0, \dots, 0)$ .

Define  $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$  by

$$g(x) = \begin{cases} (A_m \circ f \circ A_m^{-1})(x) & \text{if } x \in B_m, \quad m \in \mathbb{N}, \\ x & \text{otherwise.} \end{cases}$$

Then  $g$  is a propagated version of  $f$ , supported in  $\cup_{m=1}^{\infty} B_m$ . We can extend  $g$  to a mapping on  $S^n$  by defining  $g$  to fix the point at infinity.

**Remark 3.1.** *The origins of definition of the propagated version of  $f$  can be traced to [8] and [5].*

**Lemma 3.2.** *The map  $g$  is  $C^1$  on  $\mathbb{R}^n$  and, further, satisfies the following properties:*

- (i)  $g$  is uniformly continuous on  $\mathbb{R}^n$ , that is, for all  $\epsilon > 0$ , there exists  $\delta > 0$  such that for all  $x, y \in \mathbb{R}^n$  satisfying  $|x - y| < \delta$ , we have  $|g(x) - g(y)| < \epsilon$ ;



(ii) there exists  $T > 0$  such that

$$(5) \quad \|D_x g\| \leq T$$

for all  $x \in \mathbb{R}^n$ ;

(iii) there exists a function  $\eta : [0, \infty] \rightarrow [0, \infty]$  for which  $\eta(0) = 0$ ,  $\eta$  is continuous at 0 and

$$(6) \quad \|D_x g - D_y g\| \leq \eta(|x - y|)$$

for all  $x, y \in \mathbb{R}^n$ . The function  $\eta$  is the modulus of continuity of  $Dg$ .

Further, we may assume that  $g^{-1}$  also satisfies these three conditions, by changing the constants and modulus of continuity if necessary.

*Proof.* First note that  $f$  is  $C^1$  by hypothesis, and satisfies the three claims of the lemma because it is supported in a compact subset of  $\mathbb{R}^n$ . Since  $g$  is a propagated version of  $f$ , it satisfies the three claims of the lemma with the same constants as  $f$ . The last claim follows since  $f^{-1}$  is also  $C^1$ , and  $g^{-1}$  is a propagated version of  $f^{-1}$ .  $\square$

**Definition 3.3.** For  $t \in [0, 1]$ , let

$$h_t = g^{-1} \circ A_t^{-1} \circ g \circ A_t.$$

By Lemma 3.2 and [12, Lemma 1.54], which says that Euclidean translations in  $\mathbb{R}^n$  are bi-Lipschitz with respect to  $\chi$ ,  $h_t$  is bi-Lipschitz with respect to both  $d$  and  $\chi$ . The following lemma follows from the definition of  $h_t$ .

**Lemma 3.4.** The path  $h_t$ ,  $t \in [0, 1]$ , connects the identity and the diffeomorphism  $f$  through bi-Lipschitz mappings, that is

$$h_0 = id, \text{ and } h_1 = f.$$

We now want to show that this is a bi-Lipschitz path.

**Lemma 3.5.** Given  $\epsilon > 0$ , there exists  $\delta > 0$  such that if  $s, t \in [0, 1]$  satisfy  $|s - t| < \delta$ , then

$$d(h_s \circ h_t^{-1}(x), x) \leq \epsilon,$$

for all  $x \in \mathbb{R}^n$ .

*Proof.* Writing  $h_s \circ h_t^{-1}$  out in full gives

$$(7) \quad h_s \circ h_t^{-1} = g^{-1} \circ A_s^{-1} \circ g \circ A_s \circ A_t^{-1} \circ g^{-1} \circ A_t \circ g.$$

Considering first the middle four functions in this expression, write

$$(8) \quad P_{s,t}(x) = g \circ A_s \circ A_t^{-1} \circ g^{-1}(x).$$

Then the fact that

$$d(g(x), g(y)) \leq \sup_x \|D_x g\| \cdot d(x, y),$$

and (5) gives

$$\begin{aligned} d(P_{s,t}(x), x) &= d(g(g^{-1}(x) + (s-t)e_1), g(g^{-1}(x))) \\ &\leq Td(g^{-1}(x) + (s-t)e_1, g^{-1}(x)) \\ &= T|s-t|, \end{aligned}$$

for all  $x \in \mathbb{R}^n$ . Next, by using the fact that translations are isometries of  $\mathbb{R}^n$ , the triangle inequality and the previous inequality applied to  $x + te_1$ , we obtain

$$\begin{aligned} d(A_s^{-1} \circ P_{s,t} \circ A_t(x), x) &= d(P_{s,t}(x + te_1) - se_1, x) \\ &= d(P_{s,t}(x + te_1), (x + te_1) + (s-t)e_1) \\ &\leq d(P_{s,t}(x + te_1), (x + te_1)) + d(x + te_1, x + te_1 + (s-t)e_1) \\ (9) \qquad \qquad \qquad &\leq (T+1)|s-t|, \end{aligned}$$

for all  $x \in \mathbb{R}^n$ . Finally, we use (5) with  $g^{-1}$  and (9) applied to  $g(x)$  to obtain

$$\begin{aligned} d(h_s \circ h_t^{-1}(x), x) &= d(g^{-1} \circ A_s^{-1} \circ P_{s,t} \circ A_t \circ g(x), g^{-1}(g(x))) \\ &\leq Td(A_s^{-1} \circ P_{s,t} \circ A_t \circ g(x), g(x)) \\ &\leq T(T+1)|s-t|, \end{aligned}$$

for all  $x \in \mathbb{R}^n$ . We can therefore take  $\delta = \epsilon/T(T+1)$ .  $\square$

**Lemma 3.6.** *Given  $\epsilon > 0$ , there exists  $\delta > 0$  such that if  $s, t \in [0, 1]$  satisfy  $|s-t| < \delta$ , then*

$$\|D_x(h_s \circ h_t^{-1}) - I\| < \epsilon$$

for all  $x \in \mathbb{R}^n$ , where  $I$  is the identity mapping.

*Proof.* Recalling the strategy of the proof of the previous lemma, we will consider the middle six terms of (7) and work outwards. Recall the definition of  $P_{s,t}$  from (8) and write  $Q_{s,t} = A_s^{-1} \circ P_{s,t} \circ A_t$ . Observe that

$$D_x Q_{s,t} = D_{A_t(x)} P_{s,t}$$

and

$$D_x P_{s,t} = D_{A_s \circ A_t^{-1} \circ g^{-1}(x)} g \circ D_x g^{-1}$$

since the derivative of  $A_t$  is the identity. By this observation, the chain rule gives

$$(10) \qquad \|D_x(Q_{s,t}) - I\| = \|(D_{A_s \circ A_t^{-1} \circ g^{-1} \circ A_t(x)} g) \circ (D_{A_t(x)} g^{-1}) - I\|.$$

We can write the right hand side of (10) as

$$\| \left[ (D_{A_s \circ A_t^{-1} \circ g^{-1} \circ A_t(x)} g) - ((D_{A_t(x)} g^{-1})^{-1}) \right] \circ (D_{A_t(x)} g^{-1}) \|.$$

Using this, and applying the formula for the derivative of an inverse  $(D_{A_t(x)} g^{-1})^{-1} = D_{g^{-1}(A_t(x))} g$  and (5) applied to  $g^{-1}$ , yields from (10) that

$$(11) \qquad \|D_x(Q_{s,t}) - I\| \leq T \| (D_{A_s \circ A_t^{-1} \circ g^{-1} \circ A_t(x)} g) - (D_{g^{-1} \circ A_t(x)} g) \|.$$

We then apply (6) to the right hand side of (11) to give

$$(12) \quad \begin{aligned} \|D_x(Q_{s,t}) - I\| &\leq T\eta(|A_s \circ A_t^{-1} \circ g^{-1} \circ A_t(x) - g^{-1} \circ A_t(x)|) \\ &= T\eta(|s - t|), \end{aligned}$$

for all  $x \in \mathbb{R}^n$ .

Now, consider the derivative of  $h_s \circ h_t^{-1} = g^{-1} \circ Q_{s,t} \circ g$ . By the chain rule, we have

$$(13) \quad \|D_x(g^{-1} \circ Q_{s,t} \circ g) - I\| = \|(D_{Q_{s,t}(g(x))}g^{-1}) \circ (D_{g(x)}Q_{s,t}) \circ (D_xg) - I\|.$$

We can write the right hand side of (13) as

$$\|(D_{Q_{s,t}(g(x))}g^{-1}) \circ [D_{g(x)}Q_{s,t} - I] \circ (D_xg) + (D_{Q_{s,t}(g(x))}g^{-1}) \circ (D_xg) - I\|.$$

Applying the triangle inequality and (5) for  $g$  and  $g^{-1}$  to this expression yields

$$(14) \quad \|D_x(g^{-1} \circ Q_{s,t} \circ g) - I\| \leq T^2\|D_{g(x)}Q_{s,t} - I\| + \|(D_{Q_{s,t}(g(x))}g^{-1}) \circ (D_xg) - I\|$$

We next apply (12) to the first term on the right hand side of (14), and re-write the second term to give

$$(15) \quad \|D_x(g^{-1} \circ Q_{s,t} \circ g) - I\| \leq T^3\eta(|s - t|) + \|[D_{Q_{s,t}(g(x))}g^{-1} - (D_xg)^{-1}] \circ (D_xg)\|$$

We use the formula  $(D_xg)^{-1} = D_{g(x)}g^{-1}$  and (5) applied to  $g$  on the second term on the right hand side of (15) to yield

$$\|D_x(g^{-1} \circ Q_{s,t} \circ g) - I\| \leq T^3\eta(|s - t|) + T\|D_{Q_{s,t}(g(x))}g^{-1} - D_{g(x)}g^{-1}\|$$

Finally, (6) and (9) give

$$\begin{aligned} \|D_x(g^{-1} \circ Q_{s,t} \circ g) - I\| &\leq T^3\eta(|s - t|) + T\eta(|Q_{s,t}(g(x)) - g(x)|) \\ &\leq T^3\eta(|s - t|) + T\eta((T + 1)|s - t|). \end{aligned}$$

Since  $\lim_{x \rightarrow 0} \eta(x) = 0$ , the lemma follows.  $\square$

Lemmas 3.4, 3.5 and 3.6 together show that  $h_t$  is a bi-Lipschitz path with respect to  $d$  connecting the identity and  $f$ . This completes the proof.

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