# Classification of continuously transitive circle groups

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**Abstract**

Let $G$ be a closed transitive subgroup of $\mathrm{Homeo}(\mathbb{S}^1)$ which contains a non-constant continuous path $f\colon [0,1] \to G$. We show that up to conjugation $G$ is one of the following groups: $\mathrm{SO}(2,\mathbb{R})$, $\mathrm{PSL}(2,\mathbb{R})$, $\mathrm{PSL}_k(2,\mathbb{R})$, $\mathrm{Homeo}_k(\mathbb{S}^1)$, $\mathrm{Homeo}(\mathbb{S}^1)$ . This verifies the classification suggested by Ghys in [5]. As a corollary we show that the group $\mathrm{PSL}(2,\mathbb{R})$ is a maximal closed subgroup of $\mathrm{Homeo}(\mathbb{S}^1)$ (we understand this is a conjecture of de la Harpe). We also show that if such a group $G < \mathrm{Homeo}(\mathbb{S}^1)$ acts continuously transitively on $k$–tuples of points, $k > 3$, then the closure of $G$ is $\mathrm{Homeo}(\mathbb{S}^1)$ (c.f. [1]).

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## 1 Introduction

Let $\mathrm{Homeo}(\mathbb{S}^1)$ denote the group of orientation preserving homeomorphisms of $\mathbb{S}^1$ which we endow with the uniform topology. Let $G$ be a subgroup of $\mathrm{Homeo}(\mathbb{S}^1)$ with the topology induced from $\mathrm{Homeo}(\mathbb{S}^1)$ . We say that $G$ is transitive if for every two points $x, y \in \mathbb{S}^1$, there exists a map $f \in G$, such that $f(x) = y$. We say that a group $G$ is closed if it is closed in the topology of $\mathrm{Homeo}(\mathbb{S}^1)$ . A continuous path in $G$ is a continuous map $f\colon [0,1] \to G$.

Let $\mathrm{SO}(2,\mathbb{R})$ denote the group of rotations of $\mathbb{S}^1$ and $\mathrm{PSL}(2,\mathbb{R})$ the group of Möbius transformations. The first main result we prove describes transitive subgroups of $\mathrm{Homeo}(\mathbb{S}^1)$ that contain a non constant continuous path.

**Theorem 1.1** *Let $G$ be a transitive subgroup of $\mathrm{Homeo}(\mathbb{S}^1)$ which contains a non constant continuous path. Then one of the following mutually exclusive possibilities holds,*

(1) *$G$ is conjugate to $\mathrm{SO}(2,\mathbb{R})$ in $\mathrm{Homeo}(\mathbb{S}^1)$*

(2) *$G$ is conjugate to $\mathrm{PSL}(2,\mathbb{R})$ in $\mathrm{Homeo}(\mathbb{S}^1)$*

(3) *For every $f \in \mathrm{Homeo}(\mathbb{S}^1)$ and each finite set of points $x_1, \ldots, x_n \in \mathbb{S}^1$ there exists $g \in G$ such that $g(x_i) = f(x_i)$ for each $i$*

(4) *$G$ is a cyclic cover of a conjugate of $\mathrm{PSL}(2,\mathbb{R})$ in $\mathrm{Homeo}(\mathbb{S}^1)$ and hence conjugate to $\mathrm{PSL}_k(2,\mathbb{R})$ for some $k > 1$*

(5) *$G$ is a cyclic cover of a group satisfying condition 3 above*

Here we write $\mathrm{PSL}_k(2,\mathbb{R})$ and $\mathrm{Homeo}_k(\mathbb{S}^1)$ to denote the cyclic covers of the groups $\mathrm{PSL}(2,\mathbb{R})$ and $\mathrm{Homeo}(\mathbb{S}^1)$ respectively, for some $k \in \mathbb{N}$.

The proof begins by showing that the assumptions of the theorem imply that $G$ is continuously 1–transitive. This means that if we vary points $x, y \in \mathbb{S}^1$ in a continuous fashion, then we can choose corresponding elements of $G$ which map $x$ to $y$ that also vary in a continuous fashion. In Theorems 3.8 and 3.10 we show that this leads us to two possibilities, either $G$ is conjugate to $\mathrm{SO}(2,\mathbb{R})$, or $G$ is a cyclic cover of a group which is continuously 2–transitive.

We then analyse groups which are continuously 2–transitive and show that they are infact all continuously 3–transitive. Furthermore, if such a group is not continuously 4–transitive, we show that it is a convergence group and hence conjugate to $\mathrm{PSL}(2,\mathbb{R})$. On the other hand if it is continuously 4–transitive, then we use an induction argument to show that it is continuously $n$–transitive for all $n \geq 4$. This implies that for every $f \in \mathrm{Homeo}(\mathbb{S}^1)$ and each finite set of points $x_1, \ldots, x_n \in \mathbb{S}^1$ there exists a group element $g$ such that $g(x_i) = f(x_i)$ for each $i$.

The remaining possibilities, namely cases 2 and 3, arise when the aforementioned cyclic cover is trivial.

In the case where the group $G$ is also closed we can use Theorem 1.1 to make the following classification.

**Theorem 1.2** *Let $G$ be a closed transitive subgroup of* $\mathrm{Homeo}(\mathbb{S}^1)$ *which contains a non constant continuous path. Then one of the following mutually exclusive possibilities holds,*

(1) *$G$ is conjugate to* $\mathrm{SO}(2, \mathbb{R})$ *in* $\mathrm{Homeo}(\mathbb{S}^1)$

(2) *$G$ is conjugate to* $\mathrm{PSL}_k(2, \mathbb{R})$ *in* $\mathrm{Homeo}(\mathbb{S}^1)$ *for some $k \geq 1$*

(3) *$G$ is conjugate to* $\mathrm{Homeo}_k(\mathbb{S}^1)$ *in* $\mathrm{Homeo}(\mathbb{S}^1)$ *for some $k \geq 1$*

The above theorem provides the classification of closed, transitive subgroups of $\mathrm{Homeo}(\mathbb{S}^1)$ that contain a non-trivial continuous path. This classification was suggested by Ghys for all transitive and closed subgroups of $\mathrm{Homeo}(\mathbb{S}^1)$ (See [5]).

One well known problem in the theory of circle groups is to prove that the group of Möbius transformations is a maximal closed subgroup of $\mathrm{Homeo}(\mathbb{S}^1)$ . We understand that this is a conjecture of de la Harpe (see [1]). The following theorem follows directly from our work and answers this question.

**Theorem 1.3** $\mathrm{PSL}(2, \mathbb{R})$ *is a maximal closed subgroup of* $\mathrm{Homeo}(\mathbb{S}^1)$ .

In the following five sections we develop the techniques needed to prove our results. Here we prove the results about the transitivity on $k$–tuples of points. In Section 7. we give the proofs of all the main results stated above.

## 2 Continuous Transitivity

Let $G < \mathrm{Homeo}(\mathbb{S}^1)$ be a transitive group of orientation preserving homeomorphisms of $\mathbb{S}^1$. We begin with some definitions which generalize the notion of transitivity.

Set,

$$P_n = \{(x_1, \ldots, x_n) : x_i \in \mathbb{S}^1, x_i = x_j \iff i = j\}$$

to be the set of distinct $n$–tuples of points in $\mathbb{S}^1$. Two $n$–tuples

$$(x_1, \dots, x_n), (y_1, \dots, y_n) \in P_n$$

have matching orientations if there exists $f \in \mathrm{Homeo}(\mathbb{S}^1)$ such that $f(x_i) = y_i$ for each $i$.

**Definition 2.1** *$G$ is $n$–transitive if for every pair $(x_1, \dots, x_n), (y_1, \dots, y_n) \in P_n$ with matching orientations there exists $g \in G$ such that $g(x_i) = y_i$ for each $i$.*

**Definition 2.2** *$G$ is uniquely $n$–transitive if it is $n$–transitive and for each pair $(x_1, \dots, x_n), (y_1, \dots, y_n) \in P_n$ with matching orientations there is exactly one element $g \in G$ such that $g(x_i) = y_i$. Equivalently, the only element of $G$ fixing $n$ distinct points is the identity.*

Endow $\mathbb{S}^1$ with the standard topology and $P_n$ with the topology it inherits as a subspace of the $n$–fold Cartesian product $\mathbb{S}^1 \times \cdots \times \mathbb{S}^1$. These are metric topologies. With the topology on $P_n$ being induced by the distance function

$$d_{P_n}((x_1, \dots, x_n), (y_1, \dots, y_n)) = \max\{d_{\mathbb{S}^1}(x_i, y_i) : i = 1, \dots, n\},$$

where $d_{\mathbb{S}^1}$ is the standard Euclidean distance function on $\mathbb{S}^1$.

Endow $G$ with the uniform topology. This is also a metric topology, induced by the distance function,

$$d_G(g_1, g_2) = \sup\{\max\{d_{\mathbb{S}^1}(g_1(x), g_2(x)), d_{\mathbb{S}^1}(g_1^{-1}(x), g_2^{-1}(x))\} : x \in \mathbb{S}^1\}$$

A path in a topological space $X$ is a continuous map $\gamma\colon [0,1] \to X$. If $\mathcal{X}\colon [0,1] \to P_n$ is a path in $P_n$ we will write $x_i(t) = \pi_i \circ \mathcal{X}(t)$, where $\pi_i$ is projection onto the $i$–th component of $\mathbb{S}^1 \times \cdots \times \mathbb{S}^1$, so that we can write $\mathcal{X}(t) = (x_1(t), \cdots, x_n(t))$. We will call a pair of paths $\mathcal{X}, \mathcal{Y}\colon [0,1] \to P_n$ *compatible* if there exists a path $h\colon [0,1] \to \mathrm{Homeo}(\mathbb{S}^1)$ with $h(t)(x_i(t)) = y_i(t)$ for each $i$ and $t$.

**Definition 2.3** *$G$ is continuously $n$–transitive if for every compatible pair of paths $\mathcal{X}, \mathcal{Y}\colon [0,1] \to P_n$ there exists a path $g\colon [0,1] \to G$ with the property that $g(t)(x_i(t)) = y_i(t)$ for each $i$ and $t$.*

**Definition 2.4** *A continuous deformation of the identity in $G$ is a non constant path of homeomorphisms $f_t \in G$ for $t \in [0,1]$ with $f_0 = id$.*

We have the following lemma.

**Lemma 2.5** *For $n \geq 2$ the following are equivalent:*

(1) *$G$ is continuously $n$–transitive*

(2) *$G$ is continuously $n-1$–transitive and the following holds. For every $(a_1, \ldots, a_{n-1}) \in P_{n-1}$ and $x \in \mathbb{S}^1 \setminus \{a_1, \ldots, a_{n-1}\}$ there exists a continuous map $F_x \colon I_x \to G$ satisfying the following conditions,*

   (a) *$F_x(y)$ fixes $a_1, \ldots, a_{n-1}$ for all $y \in I_x$*

   (b) *$(F_x(y))(x) = y$ for all $y \in I_x$*

   (c) *$F_x(x) = id$*

   *where $I_x$ is the component of $\mathbb{S}^1 \setminus \{a_1, \ldots, a_{n-1}\}$ containing $x$.*

(3) *$G$ is continuously $n-1$–transitive and there exists $(a_1, \ldots, a_{n-1}) \in P_{n-1}$ with the following property. There is a component $I$ of $\mathbb{S}^1 \setminus \{a_1, \ldots, a_{n-1}\}$, a point $\tilde{x} \in I$ and a continuous map $F_{\tilde{x}} \colon I \to G$ satisfying the following conditions,*

   (a) *$F_{\tilde{x}}(y)$ fixes $a_1, \ldots, a_{n-1}$ for all $y \in I$*

   (b) *$(F_{\tilde{x}}(y))(\tilde{x}) = y$ for all $y \in I$*

   (c) *$F_{\tilde{x}}(\tilde{x}) = id$*

(4) *$G$ is continuously $n-1$–transitive and there exists $(a_1, \ldots, a_{n-1}) \in P_{n-1}$ with the following property. There is a component $I$ of $\mathbb{S}^1 \setminus \{a_1, \ldots, a_{n-1}\}$, such that for each $x \in I$ there exists a continuous deformation of the identity $f_t$, satisfying $f_t(a_i) = a_i$ for each $t$ and $i$ and $f_t(x) \neq x$ for some $t$.*

**Proof** We start by showing $[1 \Rightarrow 4]$. As $G$ is continuously $n$–transitive, it will automatically be continuously $n-1$ transitive. Take $(a_1, \ldots, a_{n-1}) \in P_{n-1}$ and $x \in \mathbb{S}^1 \setminus \{a_1, \ldots, a_{n-1}\}$. Let $I_x$ be the component of $\mathbb{S}^1 \setminus \{a_1, \ldots, a_{n-1}\}$ which contains $x$. Take $y \in I_x \setminus \{x\}$ and let $x_t$ be an injective path in $I_x$ with $x_0 = x$ and $x_1 = y$.

Let $\mathcal{X} \colon [0,1] \to P_n$ be the constant path defined by $\mathcal{X}(t) = (a_1, \ldots, a_{n-1}, x_0)$ and let $\mathcal{Y} \colon [0,1] \to P_n$ be the path defined by $\mathcal{Y}(t) = (a_1, \ldots, a_{n-1}, x_t)$. Then since $x_t \in I_x$ for every time $t$ these form an compatible pair of paths. Consequently, there exists a path $g_t \in G$ which fixes each $a_i$ and such that $g_t(x) = (x_t)$. Defining $f_t = g_t \circ (g_0^{-1})$ gives us the required continuous deformation of the identity.

We now show that $[4 \Rightarrow 3]$. For $\tilde{x} \in I$ set $K_{\tilde{x}}$ to be the set of points $x \in I$ for which there is a path of homeomorphisms $f_t \in G$ satisfying,

(1) $f_0 = id$

(2) $f_t(a_i) = a_i$ for each $i$ and $t$

(3) $f_1(\tilde{x}) = x$.

Obviously, $K_{\tilde{x}}$ will be a connected subset of $I$ and hence an interval for each $\tilde{x} \in I$.

Choose $\tilde{x} \in I$ and take $x \in K_{\tilde{x}}$. Let $f_t$ and $g_t$ be continuous deformations of the identity which fix the $a_i$ for all $t$ and such that $f_{t_0}(x) \neq x$ for some $t_0 \in (0, 1]$ and $g_1(\tilde{x}) = x$. $f_t$ exists by the assumptions of condition 4. and $g_t$ exists because $x \in K_{\tilde{x}}$. The following paths show that the interval between $f_{t_0}(x)$ and $(f_{t_0})^{-1}(x)$ is contained in $K_{\tilde{x}}$

$$h_1(t) = \begin{cases} g_{2t} & t \in [0, 1/2] \\ f_{t_0(2t-1)} \circ g_1 & t \in [1/2, 1] \end{cases}$$

$$h_2(t) = \begin{cases} g_{2t} & t \in [0, 1/2] \\ (f_{t_0(2t-1)})^{-1} \circ g_1 & t \in [1/2, 1] \end{cases} .$$

As $x$ is contained in this interval and cannot be equal to either of its endpoints we see that $K_{\tilde{x}}$ is open for every $\tilde{x} \in I$. On the other hand, $\tilde{x} \in K_{\tilde{x}}$ for each $\tilde{x} \in I$ and if $x_1 \in K_{x_2}$ then $K_{x_1} = K_{x_2}$. Consequently, the sets $\{K_{\tilde{x}} : \tilde{x} \in I\}$ form a partition of $I$ and hence $K_{\tilde{x}} = I$ for every $\tilde{x} \in I$.

We now construct the map $F_{\tilde{x}}$. To do this, take a nested sequence of intervals $[x_n, y_n]$ containing $\tilde{x}$ for each $n$ and such that $x_n, y_n$ converge to the endpoints of $I$ as $n \to \infty$. We define $F_{\tilde{x}}$ inductively on these intervals. Since $K_{\tilde{x}} = I$ we can find a path of homeomorphisms $f_t \in G$ satisfying,

(1) $f_0 = id$

(2) $f_t(a_i) = a_i$ for each $i$ and $t$

(3) $f_1(\tilde{x}) = x_1$.

We now show that there exists a path $\bar{f}_t \in G$, which also satisfies the above, but with the additional condition that the path $\bar{f}_t(\tilde{x})$ is simple.

To see this, let $[x^*, \tilde{x}]$ be the largest subinterval of $[x_1, \tilde{x}]$ for which there exists a path $\bar{f}_t \in G$ which satisfies,

(1) $\bar{f}_0 = id$

(2) $\bar{f}_t(a_i) = a_i$ for each $i$ and $t$

(3) $\bar{f}_1(\tilde{x}) = x^*$

(4) $\bar{f}_t(\tilde{x})$ is simple.

We want to show that $x^* = x_1$. Assume for contradiction that $x^* \neq x_1$. Then since $x^* \in [x_1, \tilde{x}]$ there exists $s \in [0,1]$ such that $f_s(\tilde{x}) = x^*$ and for small $\epsilon > 0$, we have that $f_{s+\epsilon}(\tilde{x}) \notin [x^*, \tilde{x}]$. Then if we concatenate the path $\bar{f}_t$ with $f_{s+\epsilon} \circ f_s^{-1} \circ \bar{f}_1$ for small $\epsilon$ we can construct a simple path satisfying the same conditions as $\bar{f}_t$ but on a interval strictly bigger that $[x^*, \tilde{x}]$, this contradicts the maximality of $x^*$ and we deduce that $x^* = x_1$.

We can use the path $\bar{f}_t$ to define a map $F^1_{\tilde{x}} \colon [x_1, y_1] \to G$ satisfying,

(1) $F^1_{\tilde{x}}(y)$ fixes each $a_i$ for each $y \in I$

(2) $(F^1_{\tilde{x}}(y))(\tilde{x}) = y$ for all $y \in I$

(3) $F^1_{\tilde{x}}(\tilde{x}) = id$.

by taking paths of homeomorphisms that move $\tilde{x}$ to $x_1$ and $y_1$ along simple paths in $\mathbb{S}^1$.

Now assume we have defined a map $F^k_{\tilde{x}} \colon [x_k, y_k] \to G$ satisfying,

(1) $F^k_{\tilde{x}}(y)$ fixes each $a_i$ for each $y \in I$

(2) $(F^k_{\tilde{x}}(y))(\tilde{x}) = y$ for all $y \in I$

(3) $F^k_{\tilde{x}}(\tilde{x}) = id$.

We can use the same argument used to produce $F^1_{\tilde{x}}$ to show that there exists a map $\mathfrak{F}_{x_k} \colon [x_{k+1}, x_k] \to G$ such that $\mathfrak{F}_{x_k}(x)$ fixes the $a_i$ for each $x$, $\mathfrak{F}_{x_k}(x_k) = id$ and $(\mathfrak{F}_{x_k}(x))(x_k) = x$. Similarly there exists a map $\mathfrak{F}_{y_k} \colon [y_k, y_{k+1}] \to G$ such that $\mathfrak{F}_{y_k}(x)$ fixes the $a_i$ for each $x$, $\mathfrak{F}_{y_k}(y_k) = id$ and $(\mathfrak{F}_{y_k}(x))(y_k) = x$.

This allows us to define, $F^{k+1}_{\tilde{x}} \colon [x_{k+1}, y_{k+1}] \to G$ by,

$$F^{k+1}_{\tilde{x}}(x) = \begin{cases} F^k_{\tilde{x}}(x) & x \in [x_k, y_k] \\ (\mathfrak{F}_{x_k}(x)) \circ F^k_{\tilde{x}}(x_k) & x \in [x_{k+1}, x_k] \\ (\mathfrak{F}_{y_k}(x)) \circ F^k_{\tilde{x}}(y_k) & x \in [y_k, y_{k+1}] \end{cases}$$

Inductively, we can now define the full map $F_{\tilde{x}} \colon I \to G$.

We now show that $[3 \Rightarrow 2]$. So take $x' \in I$ with $x' \neq \tilde{x}$ and define $F_{x'} \colon I \to G$ by

$$F_{x'}(y) = F_{\tilde{x}}(y) \circ (F_{\tilde{x}}(x'))^{-1} \tag{1}$$

Then $F_{x'}$ satisfies,

(1) $F_{x'}(y)$ fixes $a_1, \ldots, a_{n-1}$ for all $y \in I$

(2) $(F_{x'}(y))(x') = y$ for all $y \in I$

(3) $F_{x'}(x') = id$

Moreover, we can use (1) to define a map $F\colon I \times I \to G$ which is continuous in each variable and satisfies,

(1) $F(x, y)$ fixes $a_1, \ldots, a_{n-1}$ for all $x, y \in I$

(2) $(F(x, y))(x) = y$ for all $x, y \in I$

(3) $F(x, x) = id$ for all $x \in I$.

Now take $x'$ to be a point in $\mathbb{S}^1 \setminus I \cup \{a_1, \ldots, a_{n-1}\}$ and let $I'$ be the component of $\mathbb{S}^1 \setminus \{a_1, \ldots, a_{n-1}\}$ which contains $x'$. Then since $G$ is continuously $n-1$–transitive there exists $g \in G$ which permutes the $a_i$ so that $g(I) = I'$. Define $F_{x'}\colon I' \to G$ by,

$$F_{x'}(y) = g \circ F_{g^{-1}(x')}(g^{-1}(y)) \circ g^{-1}$$

for $y \in I'$. Then $F_{x'}$ satisfies,

(1) $F_{x'}(y)$ fixes $a_1, \ldots, a_{n-1}$ for all $y \in I$

(2) $(F_{x'}(y))(x') = y$ for all $y \in I'$

(3) $F_{x'}(x') = id$.

Now let $(b_1, \ldots, b_{n-1}) \in P_{n-1}$ have the same orientation as $(a_1, \ldots, a_{n-1})$ then since $G$ is continuously $n-1$–transitive there exists $g \in G$ so that $g(a_i) = b_i$ for each $i$. Let $x' \in \mathbb{S}^1 \setminus \{b_1, \ldots, b_{n-1}\}$ and let $I'$ be the component of $\mathbb{S}^1 \setminus \{b_1, \ldots, b_{n-1}\}$ in which it lies. Define $F_{x'}\colon I' \to G$ by,

$$F_{x'}(y) = g \circ F_{g^{-1}(x')}(g^{-1}(y)) \circ g^{-1}$$

for $y \in I'$. Then $F_{x'}$ satisfies,

(1) $F_{x'}(y)$ fixes $b_1, \ldots, b_{n-1}$ for all $y \in I$

(2) $(F_{x'}(y))(x') = y$ for all $y \in I'$

(3) $F_{x'}(x') = id$

and we have that $[3 \Rightarrow 2]$

Finally we have to show that $[2 \Rightarrow 1]$. Let $\mathcal{X}, \mathcal{Y}\colon [0,1] \to P_n$ be an compatible pair of paths. We define $\mathcal{X}'\colon [0,1] \to P_{n-1}$ by,

$$\mathcal{X}'(t) = (x_1(t), \ldots, x_{n-1}(t))$$

and $\mathcal{Y}'\colon [0,1] \to P_{n-1}$ by,

$$\mathcal{Y}'(t) = (y_1(t), \ldots, y_{n-1}(t)).$$

Notice that $\mathcal{X}'$ and $\mathcal{Y}'$ will also be a compatible pair of paths. Furthermore, as $G$ is continuously $n-1$–transitive there will exist a path $g'\colon [0,1] \to G$ such that $g'(t)(x_i(t)) = y_i(t)$ for $1 \leq i \leq n-1$.

The paths $\mathcal{X}', \mathcal{Y}'\colon [0,1] \to P_{n-1}$ will also be compatible with the constant paths,

$$\mathcal{X}_0'\colon [0,1] \to P_{n-1}$$
$$\mathcal{X}_0'(t) = \mathcal{X}'(0)$$

and,

$$\mathcal{Y}_0'\colon [0,1] \to P_{n-1}$$
$$\mathcal{Y}_0'(t) = \mathcal{Y}'(0)$$

respectively. So that there exist paths $g_x', g_y'\colon [0,1] \to G$ with $g_x'(x_i(0)) = x_i(t)$ and $g_y'(y_i(0)) = y_i(t)$ for $1 \leq i \leq n-1$. Furthermore, by pre composing with $(g_x'(0))^{-1}$ and $(g_y'(0))^{-1}$ if necessary, we can assume that $g_x'(0) = g_y'(0) = id$.

We now construct a path $g_x\colon [0,1] \to G$ which satisfies,

$$g_x(t)(x_i(0)) = x_i(t)$$

for $1 \leq i \leq n$. To do this let $I$ be the component of $\mathbb{S}^1 \setminus \{x_1(0), \ldots, x_{n-1}(0)\}$ containing $x_n(0)$. By assumption we have a continuous map $F_{x_n(0)}\colon I \to G$ satisfying

(1) $F_{x_n(0)}(y)$ fixes $x_1(0), \ldots, x_{n-1}(0)$ for all $y \in I$
(2) $(F_{x_n(0)}(y))(x) = y$ for all $y \in I$
(3) $F_{x_n(0)}(x) = id$

Define $g_x\colon [0,1] \to G$ by

$$g_x(t) = g_x'(t) \circ (F_{x_n(0)}((g_x'(t))^{-1}(x_n(t))))^{-1}.$$

Then $g_x(t)(x_i(0)) = x_i(t)$ for $1 \leq i \leq n$. We can repeat this process with $g_y'$ to construct a path $g_y\colon [0,1] \to G$ satisfying $g_y(t)(y_i(0)) = y_i(t)$ for $1 \leq i \leq n$.

The map $g'(0)$ which we defined earlier will map $x_i(0)$ to $y_i(0)$ for $1 \leq i \leq n-1$. Moreover, $g'(0)(x_n(0))$ will lie in the same component of $\mathbb{S}^1 \setminus \{y_1(0), \ldots y_{n-1}(0)\}$ as $y_n(0)$. So we have a map $F_{g'(0)(x_n(0))}(y_n(0))$ which maps $g'(0)(x_n(0))$ to $y_n(0)$ and fixes the other $y_i(0)$. Putting all of this together allows us to define $g\colon [0,1] \to G$ by,

$$g(t) = g_y(t) \circ F_{g'(0)(x_n(0))}(y_n(0)) \circ g'(0) \circ (g_x(t))^{-1}.$$

This is a path in $G$ which satisfies $g_t(x_i(t)) = y_i(t)$ for each $i$ and $t$. Since we can do this for any two compatible paths, $G$ is continuously $n$–transitive and we have shown that $[2 \Rightarrow 1]$.

□

**Proposition 2.6** *If $G$ is $1$–transitive and there exists a continuous deformation of the identity $f_t\colon [0,1] \to G$ in $G$, then $G$ is continuously $1$–transitive.*

**Proof** Let $x_0 \in \mathbb{S}^1$ be such that $f_{t_0}(x_0) \neq x_0$ for some $t_0 \in [0,1]$. Take $x \in \mathbb{S}^1$ then there exists $g \in G$ such that $g(x) = x_0$. Consequently, $g^{-1} \circ f_t \circ g$ is a continuous deformation of the identity which doesn't fix $x$ for some $t$. Since these deformations exist for each $x \in \mathbb{S}^1$ the proof follows in exactly the same way as $[4 \Rightarrow 1]$ from the proof of Lemma 2.5. $\square$

¿From now on we will assume that $G$ contains a continuous deformation of the identity, and hence is continuously 1–transitive.

# 3 The set $J_x$

**Definition 3.1** *For $x \in \mathbb{S}^1$ we define $J_x$ to be the set of points $y \in \mathbb{S}^1$ which satisfy the following condition. There exists a continuous deformation of the identity $f_t \in G$ which fixes $x$ for all $t$ and such that $f_{t_0}(y) \neq y$ for some $t_0 \in [0,1]$.*

It follows directly from this definition that $x \notin J_x$.

**Lemma 3.2** *$J_{f(x)} = f(J_x)$ for every $f \in G$ and $x \in \mathbb{S}^1$.*

**Proof** Let $y \in J_{f(x)}$ and let $f_t$ be the corresponding continuous deformation of the identity with $f_{t_0}(y) \neq y$. Then $f^{-1} \circ f_t \circ f$ is also a continuous deformation of the identity which now fixes $x$, and for which $f_{t_0}(f^{-1}(y)) \neq f^{-1}(y)$. This means that $f^{-1}(y) \in J_x$ and hence $y \in f(J_x)$ so that $J_{f(x)} \subset f(J_x)$. The other inclusion is an identical argument. $\square$

**Lemma 3.3** *$J_x$ is open for every $x \in \mathbb{S}^1$.*

**Proof** Let $y \in J_x$ and take $f_t$ to be the corresponding continuous deformation of the identity with $f_{t_0}(y) \neq y$ for some $t_0 \in [0,1]$. Then since $f_{t_0}$ is continuous there exists a neighborhood $U$ of $y$ such that $f_{t_0}(z) \neq z$ for all $z \in U$. This implies that $U \subset J_x$ and hence that $J_x$ is open. $\square$

**Lemma 3.4** *$J_x = \emptyset$ for every $x \in \mathbb{S}^1$ or $J_x$ has a finite complement for every $x \in \mathbb{S}^1$.*

To prove this lemma we will use the Hausdorff maximality Theorem which we now recall.

**Definition 3.5** *A set $\mathcal{P}$ is partially ordered by a binary relation $\leq$ if,*

(1) $a \leq b$ *and* $b \leq c$ *implies* $a \leq c$

(2) $a \leq a$ *for every* $a \in \mathcal{P}$

(3) $a \leq b$ *and* $b \leq a$ *implies that* $a = b$

**Definition 3.6** *A subset $\mathcal{Q}$ of a partially ordered set $\mathcal{P}$ is totally ordered if for every pair $a, b \in \mathcal{Q}$ either $a \leq b$ or $b \leq a$. A totally ordered subset $\mathcal{Q} \subset \mathcal{P}$ is maximal if for any member $a \in \mathcal{P} \setminus \mathcal{Q}$, $\mathcal{Q} \cup \{a\}$ is not totally ordered.*

**Theorem 3.7** (**Hausdorff Maximality Theorem**) *Every nonempty partially ordered set contains a maximal totally ordered subset.*

We now prove Lemma 3.4.

**Proof** Assume that there exists $x \in \mathbb{S}^1$ for which $J_x = \emptyset$. Then for every $y \in \mathbb{S}^1$ there exists a map $g \in G$ such that $g(x) = y$. Consequently,

$$J_y = J_{g(x)} = g(J_x) = g(\emptyset) = \emptyset$$

for every $y \in \mathbb{S}^1$.

Assume that $J_x \neq \emptyset$ for every $x \in \mathbb{S}^1$ and let $S_x = \mathbb{S}^1 \setminus J_x$ denote the complement of $J_x$. This means that $S_x$ consists of the points $y \in \mathbb{S}^1$ such every continuous deformation of the identity which fixes $x$ also fixes $y$. The set $\mathcal{P} = \{S_x : x \in \mathbb{S}^1\}$ is partially ordered by inclusion so that by Theorem 3.7 there exists a maximal totally ordered subset, $\mathcal{Q} = \{S_x : x \in A\}$, where $A$ is the appropriate subset of $\mathbb{S}^1$.

If we set $\mathcal{S} = \bigcap_{x \in A} S_x$ then we have the following,

(1) $\mathcal{S} \neq \emptyset$

(2) If $x \in \mathcal{S}$ then $S_x = \mathcal{S}$

1 follows from the fact that $\mathcal{S}$ is the intersection of a descending family of compact sets, and hence is nonempty.

To see that 2 is also true, fix $x \in \mathcal{S}$. Then from the definition of $\mathcal{S}$, we will have $x \in S_a$ for each $a \in A$. In other words, if we take $a \in A$, then every continuous deformation of the identity which fixes $a$ will also fix $x$. Furthermore, if $y \in S_x$

then every continuous deformation of the identity which fixes $a$ not only fixes $x$ but $y$ too, so that $S_x \subset S_a$. This is true for every $a \in A$ so that $S_x \subset \mathcal{S}$. On the other hand, by the maximality of $\mathcal{Q}$, $\mathcal{S}$ must contain $S_x$. Consequently, if $x \in \mathcal{S}$ then $S_x = \mathcal{S}$.

Fix $x_0 \in \mathcal{S}$ and assume for contradiction that $S_{x_0}$ is infinite. Take a sequence $x_n \in S_{x_0}$ and let $x_{n_k}$ be a convergent subsequence with limit $x'$. This limit will also be in $S_{x_0}$ as it is closed. As $J_{x_0}$ is a nonempty open subset of $\mathbb{S}^1$ it will contain an interval $(a,b)$ with $a,b \in S_{x_0}$. Take maps $g_a, g_b \in G$ so that $g_a(x') = a$ and $g_b(x') = b$. Since $x', a \in S_{x_0}$ we have that,

$$g_a(S_{x_0}) = g_a(S_{x'}) = S_{g_a(x')} = S_a = S_{x_0}$$

and similarly for $g_b$. As a result $g_a(x_n), g_b(x_n) \in S_{x_0}$ for each $n$, but $g_a, g_b$ are orientation preserving homeomorphisms so that at least one of these points will lie in $(a,b)$, a contradiction.

We have shown that $S_{x_0}$ is finite. If we now take any other point $x \in \mathbb{S}^1$ then there exists a map $g \in G$ such that $g(x_0) = x$. This means that the set $S_x = S_{g(x_0)} = g(S_{x_0})$ will also be finite and we are done.

$\square$

**Theorem 3.8** *If $J_x = \emptyset$ for all $x \in \mathbb{S}^1$ then $G$ is conjugate in* $\mathrm{Homeo}(\mathbb{S}^1)$ *to the group of rotations* $\mathrm{SO}(2,\mathbb{R})$.

We require the following lemma for the proof of this Theorem.

**Lemma 3.9** *If $f\colon \mathbb{R} \to \mathbb{R}$ is a homeomorphism which conjugates translations to translations, then it is an affine map.*

**Proof** Let $f$ be a homeomorphism which conjugates translations to translations and set $f_1 = T \circ f$ where $T$ is the translation that sends $f(0)$ to 0. Then $f_1$ fixes 0 and also conjugates translations to translations. In particular there exists $\alpha$ such that $f_1$ conjugates $x \mapsto x+1$ to the map $x \mapsto x+\alpha$. Notice that $\alpha \neq 0$ since the identity is only conjugate to itself.

Now define $f_2 = f_1 \circ M_\alpha$ where $M_\alpha(x) = \alpha x$. A simple calculation shows that $f_2$ conjugates $x \mapsto x+1$ to itself and conjugates translations to translations. Since $f_2$ fixes 0 and conjugates $x \mapsto x+1$ to itself, we deduce that it must fix all the integer points.

Now, for $n \in \mathbb{N}$ let $\gamma \in \mathbb{R}$ be such that $(f_2)^{-1} \circ T_{1/n} \circ f_2 = T_\gamma$ where $T_\alpha(x) = x+\alpha$. It follows that,

$$T_1 = (f_2)^{-1} \circ (T_{1/n})^n \circ f_2 = ((f_2)^{-1} \circ T_{1/n} \circ f_2)^n = (T_\gamma)^n$$

so that $\gamma = 1/n$ and $(f_2)^{-1} \circ T_{1/n} \circ f_2 = T_{1/n}$ for every $n \in \mathbb{N}$. Combining this with the fact that $f_2$ fixes 0, we see that $f_2$ must fix all the rational points and hence is the identity. This implies that $f_1$ and hence $f$ are affine.

□

We can now prove Theorem 3.8.

**Proof** Let $\widehat{G} < G$ denote the path component of the identity in $G$. We are going to show that $\widehat{G}$ is a compact group. Proposition 4.1 in [5] will then imply that it is conjugate in $\mathrm{Homeo}(\mathbb{S}^1)$ to a subgroup of $\mathrm{SO}(2,\mathbb{R})$. Moreover, as $\widehat{G}$ is 1–transitive it will be equal to the whole of $\mathrm{SO}(2,\mathbb{R})$.

For $x \in \mathbb{S}^1$ let $\pi_x \colon \mathbb{R} \to \mathbb{S}^1$ be the usual projection map which sends each integer to $x$ and for each integer translation $T \colon \mathbb{R} \to \mathbb{R}$ satisfies $\pi_x \circ T = \pi_x$.

If we fix $x \in \mathbb{S}^1$ then since $G$ is continuously 1–transitive we can choose a continuous path $g\colon [0,1] \to G$ such that $g(t)(x) = \pi_x(t)$ and $g(0) = id$. Notice that this path is contained in $\widehat{G}$ and $g(1)$ is not necessarily the identity even though it fixes $x$.

For $x \in \mathbb{S}^1$ we define a continuous map $F_x \colon \mathbb{R} \to \widehat{G}$ by,

$$F_x(t) = g(t - [t]) \circ g(1)^{[t]} \tag{$*$}$$

where $[t]$ is the greatest integer less than or equal to $t$. Set $f = F_x(1)$. Note that $F_x(n) = f^n$ for every $n \in \mathbb{Z}$.

We claim that $F_x$ has the following properties,

(1) $F_x(t)(x) = \pi_x(t)$ for every $t \in \mathbb{R}$

(2) $F_x(0) = id$

(3) The map $F_x$ is a surjection, that is $F_x(\mathbb{R}) = \widehat{G}$

(4) If the map $f = F_x(1)$ is not equal to the identity map then $F_x$ is a bijection

The first two properties follow directly from the definition. To see that the third property holds, let $h_s$ be a path in $\widehat{G}$, $s \geq 0$, $h_0 = id$. Let $\alpha(s) = h_s(x)$. We have that $\alpha$ is a continuous map from the non-negative reals $\mathbb{R}^+$ into the circle. Since the set $\mathbb{R}^+$ is contractible we can lift the map $\alpha$ into the universal cover of the circle. That is, there is a map $\beta\colon \mathbb{R}^+ \to \mathbb{R}$ such that $\pi_x \circ \beta = \alpha$. We have $F_x(\beta(s))(x) = h_s(x)$. Then $({h_s}^{-1} \circ F_x(\beta(s)))(x) = x$. It follows from the assumption of the theorem that $F_x(\beta(s)) = h_s$ and $F_x$ is surjective. The map $F_x$ is injective for $0 \leq t < 1$, because $F_x(t)(x) = \pi_x(t)$. If $F_x(1)$ is not the identity, and since $F_x(1)(x) = x$ we have that $F_x(m) = F_x(n)$ if and only if $m = n$, for every two integers $m, n$. This implies the fourth property.

It follows from $(*)$, and the surjectivity of $F_x$, that $\widehat{G}$ is a compact group if and only if the cyclic group generated by $F_x(1) = f$ is a compact group. We will prove that $f = id$.

Assume that $f$ is not the identity map. Since $F_x$ is a bijection for each $t \in \mathbb{R}$ there exists a unique $s_n(t) \in \mathbb{R}$ such that,

$$f^n \circ F_x(t) \circ f^{-n} = F_x(s_n(t)). \qquad (**)$$

This defines a function $s_n \colon \mathbb{R} \to \mathbb{R}$ which we claim is continuous for each $n$. To see this, fix $n$ and let $t_m \in \mathbb{R}$ be a convergent sequence with limit $t'$. Since $F_x$ is continuous,

$$f^n \circ F_x(t_m) \circ f^{-n} \longrightarrow f^n \circ F_x(t') \circ f^{-n}$$

and so $F_x(s_n(t_m)) \to F_x(s_n(t'))$ as $m \to \infty$.

Now, if $s_n(t_{m_k})$ is a convergent subsequence, with limit $t_0$, then by continuity $F_x(s_n(t_{m_k}))$ will converge to $F_x(t_0)$. Since $F_x$ is a bijection this gives us that $t_0 = s_n(t')$. Consequently, if the sequence $s_n(t_m)$ were bounded, then it would converge to $t'$.

Assume now that the sequence $s_n(t_m)$ is unbounded and take a divergent subsequence $s_n(t_{m_k})$. Consider the corresponding sequence,

$$F_x(s_n(t_{m_k})) = g(s_n(t_{m_k}) - [s_n(t_{m_k})]) \circ f^{[s_n(t_{m_k})]}.$$

Since $s_n(t_{m_k}) - [s_n(t_{m_k})] \in [0,1)$ for each $m$, there exists a subsequence $t_{m_{k_l}}$ of $t_{m_k}$ such that $s_n(t_{m_{k_l}}) - [s_n(t_{m_{k_l}})]$ converges to some $t_0 \in [0,1]$. Now since $g$ is continuous and the sequence $F_x(s_n(t_m))$ converges to a homeomorphism $F_x(s_n(t'))$ we have that $f^{[s_n(t_{m_{k_l}})]}$ converges to a homeomorphism as $l \to \infty$. However, as $s_n(t_{m_k})$ is divergent $[s_n(t_{m_{k_l}})]$ will be divergent too.

Let $S_f$ denote the set of fixed points of $f$. Note that $x \in S_f$. Since we assume that $f$ is not the identity we have that $\mathbb{S}^1 \setminus S_f$ is non-empty. Let $J$ be a component of $\mathbb{S}^1 \setminus S_f$ and let $a, b \in \mathbb{S}^1$ be its endpoints. Since $f$ fixes $J$, and has no fixed points inside $J$ we deduce that on compact subsets of $J$ the sequence $f^{[s_n(t_{m_{k_l}})]}$ converges to one of the endpoints and consequently, can not converge to a homeomorphism. This is a contradiction, so $s_n(t_m)$ can not be unbounded and $s_n$ is continuous.

Notice that $s_n(0) = 0$ and if $t \in \mathbb{Z}$ then $F_x(t)$ will commute with the $f^n$ so we have $s_n(m) = m$ for all $m \in \mathbb{Z}$. This yields that $s_n([0,1]) = [0,1]$ for every $n \in \mathbb{Z}$.

Let $U_f \subset \mathbb{S}^1$ be the set defined as follows. We say that $y \in U_f$ if there exists an open interval $I$, $y \in I$, such that $|f^n(I)| \to 0$, $n \to \infty$. Here $|f^n(I)|$ denotes the length of the corresponding interval. The set $U_f$ is open. We show that $U_f$ is non-empty and not equal to $\mathbb{S}^1$. As before, let $J$ be a component of $\mathbb{S}^1 \setminus S_f$ and let $a, b \in \mathbb{S}^1$ be its endpoints. Since $f$ fixes $J$, and has no fixed points inside $J$ we deduce that on compact subsets of $J$ the sequence $f^n$ converges to one of the endpoints, say $a$. This shows that $J \subset U_f$. Also, this shows that the point $b$ does not belong to $U_f$.

Let $y \in U_f$, and let $I$ be the corresponding open interval so that $y \in I$ and $|f^n(I)| \to 0$, $n \to \infty$. Set $f^n(I) = I_n$. Consider the interval $F_x(s_n(t))(I_n)$, $t \in [0,1]$. Since $s_n([0,1]) = [0,1]$ we have that $F_x(s_n([0,1]))$ is a compact family of homeomorphisms. This allows us to conclude that $|F_x(s_n(t))(I_n)| \to 0$, $n \to \infty$, uniformly in $n$ and $t \in [0,1]$. Set $J_t = F_x(t)(I)$. From $(**)$ we have that $|f^n(J_t)| \to 0$, $n \to \infty$, for a fixed $t \in [0,1]$. This implies that the point $F_x(t)(y)$ belongs to the set $U_f$ for every $t \in [0,1]$.

Let $J$ be a component of $U_f$, and let $a, b$ be its endpoints. Note that the points $a, b$ do not belong to $U_f$. Since $F_x(t)$ is a continuous path and $F_x(0) = id$, for small enough $t$ we have that $F_x(t)(J) \cap J \neq \emptyset$. Since $F_x(t)(J) \subset U_f$, and since $a, b$ are not in $U_f$ we have that $F_x(t)(J) = J$. By continuity this extends to hold for every $t \in [0,1]$. But this means that $F_x(t)(a) = a$ for every $t \in [0,1]$. However, for appropriately chosen inverse $t_0 = \pi_x^{-1}(a)$, we have that $F_x(t_0)(x) = a$, which contradicts the fact that $F_x(t_0)$ is a homeomorphism. This shows that $f = id$, and therefore we have proved that $\widehat{G}$ is a compact group.

To finish the argument, it remains to show that $G = \widehat{G}$. Let $\Phi \in \mathrm{Homeo}(\mathbb{S}^1)$ be a map which conjugates $\widehat{G}$ to $\mathrm{SO}(2,\mathbb{R})$ and take $g \in G \setminus \widehat{G}$. Since $\widehat{G}$ is a normal subgroup of $G$, $\Phi \circ g \circ \Phi^{-1}$ conjugates rotations to rotations. Lifting to the universal cover we get that every lift of $\Phi \circ g \circ \Phi^{-1}$ conjugates translations to translations. If we choose one then by Lemma 3.9 it will be affine. On the other hand, it must be periodic, and hence is a translation. So that $\Phi \circ g \circ \Phi^{-1}$ is itself a rotation and we are done.

□

**Theorem 3.10** *If $J_x \neq \emptyset$ then one of the following is true,*

(1) *$J_x = \mathbb{S}^1 \setminus \{x\}$ in which case $G$ is continuously 2–transitive*

(2) *There exists $R \in \mathrm{Homeo}(\mathbb{S}^1)$ which is conjugate to a finite order rotation and satisfies $R \circ g = g \circ R$ for every $g \in G$. Moreover, $G$ is a cyclic cover of a group $G_\Gamma$ which is continuously 2–transitive, where the covering transformations are the cyclic group generated by $R$*

**Proof** If $J_x = \mathbb{S}^1 \setminus \{x\}$ then we are in case 4 of Lemma 2.5 with $n = 2$. In this situation we know that $G$ will be continuously 2–transitive.

We already know that $S_x = \mathbb{S}^1 \setminus J_x$ must contain $x$ and by Lemma 3.4 must be finite. Moreover, as $f(J_x) = J_{f(x)}$ the sets $S_x$ contain the same number of points for each $x \in \mathbb{S}^1$. Define $R\colon \mathbb{S}^1 \to \mathbb{S}^1$ by taking $R(x)$ to be the first point of $S_x$ you come to as you travel anticlockwise around $\mathbb{S}^1$. Now take $g \in G$ and $x \in \mathbb{S}^1$, then since $J_{g(x)} = g(J_x)$ and $g$ is orientation preserving $R \circ g(x) = g \circ R(x)$ for all $x \in \mathbb{S}^1$.

We now show that $R$ is a homeomorphism. To see this take any continuous path $x_t \in \mathbb{S}^1$, we will show that $R(x_t) \to R(x_0)$ as $t \to 0$. Since $G$ is continuously 1–transitive, there exists a continuous path $g_t \in G$ satisfying $g_t(x_t) = x_0$, so that,

$$\lim_{t\to 0} R(x_t) = \lim_{t\to 0} (g_t)^{-1}(R(g_t(x_t))) = \lim_{t\to 0} (g_t)^{-1}(R(x_0)) = R(x_0).$$

where the first equality follows from the fact that $R \circ g(x) = g \circ R(x)$ for all $x \in \mathbb{S}^1$. This shows that $R$ is continuous. If we take $y \notin J_x$ then $J_x \subset J_y$, and hence $S_x \supset S_y$ but in this case since $S_x$ and $S_y$ contain the same number of points they will be equal. Consequently, $R$ has an inverse defined by taking $R^{-1}(x)$ to be the first point of $S_x$ you come to by traveling clockwise around $\mathbb{S}^1$ and this inverse is continuous by the same argument as for $R$. Consequently, $R \in \mathrm{Homeo}(\mathbb{S}^1)$. Furthermore, $R$ is of finite order equal to the number of points in $S_x$ and hence conjugate to a rotation.

Let $\Gamma$ denote the cyclic subgroup of $\mathrm{Homeo}(\mathbb{S}^1)$ generated by $R$. Define $\pi\colon \mathbb{S}^1 \to \mathbb{S}^1/\Gamma \cong \mathbb{S}^1$, in the usual way with $\pi(x)$ being the orbit of $x$ under $\Gamma$. Since $R \circ g(x) = g \circ R(x)$ for all $x \in \mathbb{S}^1$, each $g \in G$ defines a well defined homeomorphism of the quotient space $\mathbb{S}^1/\Gamma$ which we call $g_\Gamma$. This gives us a homomorphism $\pi_\Gamma\colon G \to \mathrm{Homeo}(\mathbb{S}^1)$, defined by $\pi_\Gamma(g) = g_\Gamma$. Let $G_\Gamma$ denote the image of $G$ under $\pi_\Gamma$, then $G$ is a cyclic cover of $G_\Gamma$.

It remains to see that $G_\Gamma$ is continuously 2–transitive. This follows from the fact that if we take $x_0 \in \mathbb{S}^1$ then $J_{\pi(x_0)} = \pi(J_{x_0})$, where $J_{\pi(x_0)}$ is the set of points that can be moved by continuous deformations of the identity in $G_\Gamma$ which fix $\pi(x_0)$. Consequently, $J_{\pi(x_0)} = \mathbb{S}^1 \setminus \{x_0\}$ so that $G_\Gamma$ is continuously 2–transitive by the first part of this proposition. □

## 4 Implications of continuous 2–transitivity

We now know that if $G$ is transitive and contains a continuous deformation of the identity then it is either conjugate to the group of rotations $\mathrm{SO}(2,\mathbb{R})$, is continuously 2–transitive, or is a cyclic cover of a group which is continuously 2–transitive. For the rest of the paper we assume that $G$ is continuously 2–transitive and examine which possibilities arise.

For $n \geq 2$ and $(x_1 \dots x_n) \in P_n$ we define $J_{x_1 \dots x_n}$ to be the subset of $\mathbb{S}^1$ containing the points $x \in \mathbb{S}^1$ which satisfy the following condition. There exists a continuous deformation of the identity $f_t \in G$, with $f_t(x_i) = x_i$ for each $i$ and $t$ and such that there exists $t_0 \in [0,1]$ with $f_{t_0}(x) \neq x$. This generalizes the earlier definition of $J_x$ and we get the following analogous results.

**Lemma 4.1** *$J_{f(x_1)\dots f(x_n)} = f(J_{x_1 \dots x_n})$ for every $f \in G$.*

**Lemma 4.2** *$J_{x_1 \dots x_n}$ is open.*

We also have the following.

**Lemma 4.3** *If $J_{x_1 \dots x_n}$ is nonempty and $G$ is continuously $n$–transitive, then it is equal to $\mathbb{S}^1 \setminus \{x_1 \dots x_n\}$.*

**Proof** Assume that $J_{x_1 \dots x_n} \subset \mathbb{S}^1 \setminus \{x_1, \dots, x_n\}$ is nonempty. By Lemma 4.2 it is also open and hence is a countable union of open intervals. Pick one of these, and call its endpoints $b_1$ and $b_2$. Assume for contradiction that at least one of $b_1$ and $b_2$ is not one of the $x_i$. Interchanging $b_1$ and $b_2$ if necessary we can assume that this point is $b_1$. Since $G$ is continuously $n$–transitive there exist elements of $G$ which cyclically permute the $x_i$. Using these elements and the fact that $J_{f(x_1)\dots f(x_n)} = f(J_{x_1 \dots x_n})$ for every $f \in G$, we can assume without loss of generality that $b_1$ and hence the whole interval lies in the component of $\mathbb{S}^1 \setminus \{x_1, \dots, x_n\}$ whose endpoints are $x_1$ and $x_2$.

We now claim that $J_{b_1, b_2, x_3, \dots, x_n} \supset J_{x_1 \dots x_n}$. To see this, take $x \in J_{x_1 \dots x_n}$, then there exists a continuous deformation of the identity $f_t$ which fixes $x_1, \dots, x_n$ and for which there exists $t_0$ such that $f_{t_0}(x) \neq x$. Now since $b_1, b_2 \notin J_{x_1 \dots x_n}$, $f_t$ must also fix $b_1$ and $b_2$ for all $t$, consequently we can use $f_t$ to show that $x \in J_{b_1, b_2, x_3, \dots, x_n}$. In particular, this means that $J_{b_1, b_2, x_3, \dots, x_n}$ contains the whole interval between $b_1$ and $b_2$.

Take $g \in G$ which maps $\{b_1, b_2\}$ to $\{x_1, x_2\}$ and fixes the other $x_i$, such an element exists as $G$ is continuously $n$–transitive. Then,

$$J_{x_1,x_2,x_3,\ldots,x_n} = J_{g(b_1),g(b_2),g(x_3),\ldots,g(x_n)} = g(J_{b_1,b_2,x_3,\ldots,x_n})$$

so that $J_{x_1\ldots x_n}$ must contain the whole interval between $x_1$ and $x_2$. This is a contradiction, since $b_1$ lies between $x_1$ and $x_2$ but is not in $J_{x_1\ldots x_n}$. □

**Proposition 4.4** *Let $G$ be continuously $n$–transitive for some $n \geq 2$ and suppose there exist $n$ distinct points $a_1, \ldots, a_n \in \mathbb{S}^1$ and a continuous deformation of the identity $g_t \in G$, which fixes each $a_i$ for all $t$. Then $G$ is continuously $n+1$ transitive.*

**Proof** $J_{a_1\ldots a_n} \neq \emptyset$ so by Lemma 4.3 $J_{a_1\ldots a_n} = \mathbb{S}^1 \setminus \{a_1, \ldots, a_n\}$. We can now apply Lemma 2.5 to see that $G$ is continuously $n+1$–transitive. □

**Corollary 4.5** *If $G$ is continuously 2–transitive and there exists $g \in G \setminus \{id\}$ with an open interval $I \subset \mathbb{S}^1$ such that the restriction of $g$ to $I$ is the identity, then $G$ is continuously $n$–transitive for every $n \geq 2$.*

**Proof** Let $I \subset \mathbb{S}^1$ be a maximal interval on which $g$ acts as the identity, so that if $I' \supset I$ is another interval containing $I$ then $g$ doesn't act as the identity on $I'$. Let $a$ and $b$ be the endpoints of $I$ and let $a_t$ and $b_t$ be continuous injective paths with $a_0 = a, b_0 = b$ and $a_t, b_t \notin I$ for each $t \neq 0$. This is possible because $g \neq id$ so that $\mathbb{S}^1 \setminus I$ will be a closed interval containing more than one point. Let $g_t$ be a continuous path in $G$ so that $g_0 = id$, $g_t(a) = a_t$ and $g_t(b) = b_t$, such a path exists as $G$ is continuously 2–transitive.

Consider the path $h_t = g^{-1} \circ g_t \circ g \circ g_t^{-1}$ since $g_0 = id$ we get $h_0 = id$. Now $g_t \circ g \circ g_t^{-1}$ acts as the identity on the interval between $a_t$ and $b_t$ and by maximality of $I$, $g^{-1}$ will not act as the identity for $t \neq 0$. Consequently, $h_t$ is a continuous deformation of the identity which acts as the identity on $I$. So if $G$ is continuously $k$–transitive for $k \geq 2$, by taking $k$–points in $I$ and using proposition 4.4 we get that $G$ is $k+1$–transitive. As a result, since $G$ is continuously 2–transitive it will be $n$–transitive for every $n \geq 2$. □

$\mathrm{SO}(2, \mathbb{R})$ is an example of a subgroup of $\mathrm{Homeo}(\mathbb{S}^1)$ which is continuously 1–transitive but not continuously 2–transitive. However, as the next result shows, there are no subgroups of $\mathrm{Homeo}(\mathbb{S}^1)$ which are continuously 2–transitive but not continuously 3–transitive.

**Proposition 4.6** *If $G$ is continuously 2–transitive, then it is continuously 3–transitive.*

**Proof** Let $a, b \in \mathbb{S}^1$ be distinct points. Construct two injective paths $a(t), b(t)$ in $\mathbb{S}^1$ with disjoint images, such that $a(0) = a$, $b(0) = b$ and such that $a(t)$ and $b(t)$ lie in the same component of $\mathbb{S}^1 \setminus \{a, b\}$ for $t \in (0, 1]$. We label this component $I$ and the other $I'$.

Since $G$ is continuously 2–transitive, there exists a path $g(t) \in G$ such that $g(0) = id$, $g(t)(a) = a(t)$ and $g(t)(b) = b(t)$ for every $t$. Now for every $t$ the restriction of $g(t)$ to the closure of $I$, is a continuous map of a closed interval into itself, and hence must have a fixed point, $c(t)$. This point will normally not be unique, but since $g(t)$ is continuous, for a small enough time interval we can choose it to depend continuously on $t$. Likewise for the restriction of $g(t)^{-1}$ to the closure of $I'$, for a small enough time interval we can choose a path of fixed points $d(t)$, which must therefore also be fixed points for $g(t)$.

Now pick points $c \in I$ and $d \in I'$. Using continuous 2–transitivity of $G$ construct a path $h(t) \in G$ such that $h(t)(c) = c(t)$ and $h(t)(d) = d(t)$. Then $h_t^{-1} \circ g(t) \circ h_t$ is only the identity when $t = 0$ because the same is true of $g(t)$ and we have constructed a continuous deformation of the identity which fixes $c$ and $d$ for all $t$. Consequently we can use Proposition 4.4 to show that $G$ is continuously 3–transitive. $\square$

# 5 Convergence Groups

**Definition 5.1** *A subgroup $G$ of* $\mathrm{Homeo}(\mathbb{S}^1)$ *is a convergence group if for every sequence of distinct elements $g_n \in G$, there exists a subsequence $g_{n_k}$ satisfying one of the following two properties,*

(1) *There exists $g \in G$ such that,*

$$\lim_{k\to\infty} g_{n_k} = g \quad \textit{and} \quad \lim_{k\to\infty} g_{n_k}^{-1} = g^{-1}$$

*uniformly in $\mathbb{S}^1$.*

(2) *There exist points $x_0, y_0 \in \mathbb{S}^1$ such that,*

$$\lim_{k\to\infty} g_{n_k} = x_0 \quad \textit{and} \quad \lim_{k\to\infty} g_{n_k}^{-1} = y_0$$

*uniformly on compact subsets of $\mathbb{S}^1 \setminus \{y_0\}$ and $\mathbb{S}^1 \setminus \{x_0\}$ respectively.*

The notion of convergence groups was introduced by Gehring and Martin [4] and they have proceeded to play a central role in geometric group theory. The following theorem has been one of the most important and we shall make frequent use of it.

**Theorem 5.2** *$G$ is a convergence group if and only if it is conjugate in* $\mathrm{Homeo}(\mathbb{S}^1)$ *to a subgroup of* $\mathrm{PSL}(2,\mathbb{R})$.

This Theorem was proved by Gabai in [3]. Prior to that, Tukia [7] proved this result in many cases and Hinkkanen [6] proved it for non discrete groups. Casson and Jungreis proved it independently using different methods [2]. See [2], [3], [7] for references to other papers in this subject.

For the rest of this section we shall assume that $G$ is continuously $n$–transitive, but not continuously $n+1$–transitive for some $n \geq 3$.

Take $(x_1, \ldots, x_{n-1}) \in P_{n-1}$ and define,

$$G_0 = \{g \in G : g(x_i) = x_i \quad i = 1, \ldots, n-1\}$$

Choose a component $I$ of $\mathbb{S}^1 \setminus \{x_1, \ldots, x_{n-1}\}$ and denote its closure by $\bar{I}$. We construct a homomorphism $\Phi\colon G_0 \to \mathrm{Homeo}(\mathbb{S}^1)$ as follows. Take $g \in G_0$, then since $g$ fixes the endpoints of $I$ and is orientation preserving, we can restrict it to a homeomorphism $g'$ of $\bar{I}$. By identifying the endpoints of $\bar{I}$ we get a copy of $\mathbb{S}^1$ and we define $\Phi(g)$ to be the homeomorphism of $\mathbb{S}^1$ that $g'$ descends to under this identification. We label the identification point $\bar{x}$ and set $\mathcal{G}_0 = \Phi(G_0)$ to be the image of $G_0$ under $\Phi$.

In this situation Lemma 2.5 implies the following. For every $x \in I$, there exists a continuous map $F_x\colon \mathbb{S}^1 \setminus \bar{x} \to \mathcal{G}_0$ satisfying the properties,

(1) $(F_x(y))(x) = y \quad \forall\ y \in \mathbb{S}^1 \setminus \bar{x}$

(2) $F_x(x) = id$

**Proposition 5.3** $\Phi\colon G_0 \to \mathcal{G}_0$ *is an isomorphism.*

**Proof** Surjectivity is trivial. If we assume that $\Phi$ is not injective then there will exist $g \in G_0$ which is non-trivial and acts as the identity on $I$. Then by Corollary 4.5 $G$ will be $n+1$ transitive, a contradiction. □

Let $\widehat{G}_0$ denote the path component of the identity in $G_0$, we now analyze the group $\widehat{\mathcal{G}}_0 = \Phi(\widehat{G}_0)$.

**Proposition 5.4** *$\widehat{\mathcal{G}}_0$ is a convergence group.*

**Proof** Choose $x \in I$ then we know there exists a continuous map $F_x\colon \mathbb{S}^1\backslash\bar{x} \to \mathcal{G}_0$ satisfying the properties,

(1) $(F_x(y))(x) = y \quad \forall\ y \in \mathbb{S}^1 \setminus \bar{x}$

(2) $F_x(x) = id$.

Now since $F_x(x) = id$ and $F_x$ is continuous, the image of $F_x$ will lie entirely in $\widehat{\mathcal{G}}_0$.

In fact, $F_x$ gives a bijection between $\mathbb{S}^1 \setminus \bar{x}$ and $\widehat{\mathcal{G}}_0$. To see this we first observe that injectivity follows directly from condition 1. To see that it is also surjective, take $g \in \widehat{\mathcal{G}}_0$. Then there exists a path $g_t \in \widehat{\mathcal{G}}_0$ for $t \in [0,1]$ with $g_0 = id$ and $g_1 = g$. So that $g_t(x)$ is a path in $\mathbb{S}^1 \setminus \bar{x}$ from $x$ to $g(x)$. Consider the path $(F_x(g_t(x)))^{-1} \circ g_t$ in $\widehat{\mathcal{G}}_0$, it fixes $x$ for every $t$, and so must be the identity for each $t$. Otherwise, by proposition 4.4, $G$ would be continuously $n+1-$transitive, which would contradict our assumptions. As a result $g = F_x(g(x))$ so $F_x$ is a bijection, with inverse given by evaluation at $x$.

Fix $x_0 \in \mathbb{S}^1\backslash\bar{x}$, let $g_n$ be a sequence of elements of $\widehat{\mathcal{G}}_0$ and consider the sequence of points $g_n(x_0)$, since $\mathbb{S}^1$ is compact $g_n(x_0)$ has a convergent subsequence $g_{n_k}(x_0)$ converging to some point $x'$. If $x' \neq \bar{x}$ then by continuity of $F_{x_0}$, $g_{n_k}$ will converge to $F_{x_0}(x')$. Now if there does not exist a subsequence of $g_n(x_0)$ converging to some $x' \neq \bar{x}$, then take a subsequence $g_{n_k}$ such that $g_{n_k}(x_0)$ converges to $\bar{x}$. If we can show that $g_{n_k}(x)$ converges to $\bar{x}$ for every $x \in \mathbb{S}^1 \setminus \bar{x}$ then we shall be done.

Suppose for contradiction that there exists $x \in \mathbb{S}^1\backslash\bar{x}$ such that $g_{n_k}(x)$ does not converge to $\bar{x}$. Then there exists a subsequence of $g_{n_k}(x)$ which converges to $x' \neq \bar{x}$, but then by the previous argument the corresponding subsequence of $g_{n_k}$ will converge to the homeomorphism $F_x(x')$. This is a contradiction since $F_x(x')(x_0)$ would have to equal $\bar{x}$.

$\square$

**Corollary 5.5** *Let $g$ be an element of $\widehat{\mathcal{G}}_0$. If $g$ fixes a point in $\mathbb{S}^1 \setminus \bar{x}$ then it is the identity.*

**Proof** Let $x \in \mathbb{S}^1 \setminus \bar{x}$ be a fixed point of $g$. From the previous proof we know that $F_x\colon I \to \widehat{\mathcal{G}}_0$ is a bijection. So that $F_x(g(x)) = g$, but $g$ fixes $x$ so that $g = F_x(x) = id$. $\square$

**Corollary 5.6** *The restriction of the action of $\widehat{\mathcal{G}}_0$ to $\mathbb{S}^1 \setminus \bar{x}$ is conjugate to the action of $\mathbb{R}$ on itself by translation.*

**Proof** By Theorem 5.2 and Proposition 5.4 $\widehat{\mathcal{G}}_0$ is conjugate in $\mathrm{Homeo}(\mathbb{S}^1)$ to a subgroup of $\mathrm{PSL}(2,\mathbb{R})$ which fixes the point $\bar{x}$. Moreover, from Corollary 5.5 this is the only point fixed by a non trivial element. By identifying $\mathbb{S}^1$ with $\mathbb{R} \cup \{\infty\}$ so that $\bar{x}$ is identified with $\{\infty\}$ in the usual way, we see that $\widehat{\mathcal{G}}_0$ is conjugate to a subgroup of the Möbius group acting on $\mathbb{R} \cup \{\infty\}$. Since every element will fix $\{\infty\}$, their restriction to $\mathbb{R}$ will be an element of $\mathrm{Aff}(\mathbb{R})$ acting without fixed points, so can only be a translation. On the other hand the group must act transitively on $\mathbb{R}$ and so must be the full group of translations. This gives the result. $\square$

**Proposition 5.7** *The restriction of the action of $G_0$ to $I$ is conjugate to the action of a subgroup of the affine group $\mathrm{Aff}(\mathbb{R})$ on $\mathbb{R}$. In particular, each non trivial element of $G_0$ can act on $I$ with at most one fixed point.*

**Proof** The restriction of $\widehat{\mathcal{G}}_0$ to $\mathbb{S}^1 \setminus \bar{x}$ is isomorphic to the restriction of $\widehat{G}_0$ to $I$. So that by Corollary 5.6 there exists a homeomorphism $\phi\colon I \to \mathbb{R}$ which conjugates the restriction of $\widehat{G}_0$ to $I$, to the action of $\mathbb{R}$ on itself by translation. Take $h \in G_0 \setminus \widehat{G}_0$ then $h' = \phi \circ h \circ \phi^{-1}$ is a self-homeomorphism of $\mathbb{R}$. Since $\widehat{G}_0$ is a normal subgroup of $G_0$, $h'$ conjugates every translation to another one and so by Lemma 3.9 is itself an affine map and the proof is complete. $\square$

Let $g$ be a nontrivial element of $G_0$, then $g \in \widehat{G}_0$ if and only if it acts on each component of $\mathbb{S}^1 \setminus \{x_1, \ldots, x_{n-1}\}$ as a conjugate of a non trivial translation. Furthermore, if $g \notin \widehat{G}_0$ then it acts on each component of $\mathbb{S}^1 \setminus \{x_1, \ldots, x_{n-1}\}$ as a conjugate of a affine map which is not a translation, each of which must have a fixed point. This situation cannot actually arise as the next proposition will show.

**Proposition 5.8** $G_0 = \widehat{G}_0$

**Proof** Let $g \in G_0 \setminus \widehat{G}_0$, then $g$ acts on each component of $\mathbb{S}^1 \setminus \{x_1, \ldots, x_{n-1}\}$ as a conjugate of a affine map which is not a translation. Consequently, $g$ will have a fixed point in each component of $\mathbb{S}^1 \setminus \{x_1, \ldots, x_{n-1}\}$. Label the fixed points of $g$ in the components of $\mathbb{S}^1 \setminus \{x_1, \ldots, x_{n-1}\}$ whose boundaries both contain $x_1$ as $y_1$ and $y_2$. Since $G$ is $n$–transitive, there exists a map $g'$ which sends $y_1$ to $x_1$ and fixes all the other $x_i$. Then $g' \circ g \circ (g')^{-1}$ fixes all the $x_i$

and hence is an element of $G_0$. On the other hand, $g' \circ g \circ (g')^{-1}$ also fixes $g'(x_1)$ and $g'(y_2)$ which lie in the same component of $\mathbb{S}^1 \setminus \{x_1, \ldots, x_{n-1}\}$, this is impossible since every non-trivial element of $G_0$ can only have one fixed point in each component of $\mathbb{S}^1 \setminus \{x_1, \ldots, x_{n-1}\}$. □

**Corollary 5.9** *The restriction of the action of $G_0$ to $I$ is conjugate to the action of $\mathbb{R}$ on itself by translation. In particular the action is free.*

We finish this section by comparing the directions that a non-trivial element of $G_0$ moves points in different components of $\mathbb{S}^1 \setminus \{x_1, \ldots, x_{n-1}\}$. So endow $\mathbb{S}^1$ with the anti-clockwise orientation, this gives us an ordering on any interval $I \subset \mathbb{S}^1$, where for distinct points $x, y \in I$, $x \prec y$ if one travels in an anti-clockwise direction to get from $x$ to $y$ in $I$. Let $g \in G_0 \setminus \{id\}$ if $I$ is a component of $\mathbb{S}^1 \setminus \{x_1, \ldots, x_{n-1}\}$ then we shall say that $g$ acts *positively* on $I$ if $x \prec g(x)$ and *negatively* if $x \succ g(x)$ for one and hence every $x \in I$.

Let $I$ and $I'$ be the two components of $\mathbb{S}^1 \setminus \{x_1, \ldots, x_{n-1}\}$ whose boundaries contain $x_i$. Labeled so that in the order on the closure of $I$, $x \prec x_i$ for each $x \in I$, whereas in the order on the closure of $I'$, $x_i \prec x$ for each $x \in I'$. Then we have the following,

**Proposition 5.10** *Let $g$ be a non trivial element of $G_0$, if $g$ acts positively on $I$ then it acts negatively on $I'$ and if $g$ acts negatively on $I$ then it acts positively on $I'$.*

**Proof** Let $x, x' \in I$ and $y, y' \in I'$ be points such that $x \prec x'$ and $y \succ y'$. There exists $g \in G$ fixing $x_1, \ldots, x_{i-1}$ and $x_{i+1}, \ldots, x_{n-1}$ and sending $x$ to $x'$ and $y$ to $y'$. This map will have a fixed point $\tilde{x}$ between $x'$ and $y'$, since it maps the interval between them into itself.

Let $g' \in G$ fix $x_1, \ldots, x_{i-1}$ and $x_{i+1}, \ldots, x_{n-1}$ and send $\tilde{x}$ to $x_i$. Then $g_0 = g' \circ g \circ (g')^{-1}$ will fix $x_1, \ldots, x_{n-1}$ and hence lie in $G_0$. Moreover, $g_0$ acts positively on $I$ and negatively on $I'$.

Now let $g_1 \in G_0$ be any non-trivial element which acts positively on $I$. Then there exists a path $g_t$ in $G_0$ from $g_0 = g' \circ g \circ (g')^{-1}$ to $g_1$, so that $g_t \neq id$ for any $t$. Since $g_t$ is never the identity and $g_0$ acts negatively on $I'$, $g_1$ must also act negatively on $I'$.

If $h \in G_0$ is a non-trivial element which acts negatively on $I$, then $h^{-1}$ will act positively on $I$. So that, by the above argument, $h^{-1}$ will act negatively on $I'$. This means that $h$ will act positively on $I'$ as required.

□

**Corollary 5.11** *If $G$ is $n$–transitive but not $n+1$–transitive for $n \geq 3$ then $n$ is odd.*

**Proof** Let $g$ be a non-trivial element of $G_0$ which acts positively on some component $I$ of $\mathbb{S}^1 \setminus \{x_1, \ldots, x_{n-1}\}$. Then by Proposition 5.10 as we travel around $\mathbb{S}^1$ in an anti-clockwise direction the manner in which it acts on each component will alternate between negative and positive. Consequently, if $n$ was even, when we return to $I$ we would require that $g$ acted negatively on $I$, a contradiction, so $n$ is odd. □

# 6 Continuous 3–transitivity and beyond

We begin this section by analyzing the case where $G$ is continuously 3–transitive but not continuously 4–transitive. We shall show that such a group is a convergence group and consequently conjugate to a subgroup of $\mathrm{PSL}(2, \mathbb{R})$.

Fix distinct points $x_0, y_0 \in \mathbb{S}^1$ and define

$$G_0 = \{g \in G : g(x_0) = x_0, g(y_0) = y_0\}$$

$$\bar{G} = \{g \in G : g(x_0) = x_0\}$$

then we have the following propositions.

**Proposition 6.1** *$G_0$ is a convergence group.*

**Proof** ¿From Corollary 5.9, we know that the restriction of $G_0$ to each of the components of $\mathbb{S}^1 \setminus \{x_0, y_0\}$ is conjugate to the action of $\mathbb{R}$ on itself by translation. Let $g_n$ be a sequence of distinct elements of $G_0$ and take a point $x \in \mathbb{S}^1 \setminus \{x_0, y_0\}$. Then the sequence of points $g_n(x)$ will have a convergent subsequence $g_{n_k}(x)$. If this sequence converges to $x_0$ or $y_0$, then from Proposition 5.10 so will the sequences $g_{n_k}(y)$ for all $y \in \mathbb{S}^1 \setminus \{y_0\}$ or $\mathbb{S}^1 \setminus \{x_0\}$ respectively.

Let $I_x$ be the component of $\mathbb{S}^1 \setminus \{x_0, y_0\}$ containing $x$. Assume that the sequence of points $g_{n_k}(x)$ converges to a point $x' \in I_x$. Now let $y$ be a point in the other component, $I_y$ of $\mathbb{S}^1 \setminus \{x_0, y_0\}$, and consider the sequence of points $g_{n_k}(y)$ in $I_y$. If it had a subsequence which converged to $x_0$ or $y_0$ then the sequence $g_{n_k}(x)$ would have to as well. This is impossible so $g_{n_k}(y)$ must stay within a compact subset of $I_y$ and hence $g_{n_k}$ has a subsequence, $g_{n_{k_l}}$ for which $g_{n_{k_l}}(y)$ converges to some point $y' \in I_y$.

By Corollary 5.9 there exist self homeomorphisms of $I_x$ and $I_y$ to which the sequence $g_{n_{k_l}}$ converges uniformly on $I_x$ and $I_y$ respectively. Gluing these together at $x_0$ and $y_0$ gives us an element of $\mathrm{Homeo}(\mathbb{S}^1)$ which $g_{n_k}$ converges to uniformly. Consequently, $G_0$ is a convergence group. □

**Proposition 6.2** *$\bar{G}$ is a convergence group.*

**Proof** Let $f_n$ be a sequence of elements of $\bar{G}$. If for every $y \in \mathbb{S}^1 \setminus \{x_0\}$ every convergent subsequence of $f_n(y)$ converges to $x_0$ then we would be done. So assume that this is not the case, take $y \in \mathbb{S}^1 \setminus \{x_0\}$ such that the sequence of points $f_n(y)$ has a convergent subsequence $f_{n_k}(y)$ converging to some point $\tilde{y} \neq x_0$. Let $I$ be a small open interval around $\tilde{y}$, not containing $x_0$ then since $G$ is continuously 3–transitive, there exists a map $F_{\tilde{y}} \colon I \to \bar{G}$ satisfying the following,

(1) $F_{\tilde{y}}(x)(\tilde{y}) = x$ for all $x \in I$

(2) $F_{\tilde{y}}(\tilde{y})$ is the identity

Let $g_1, g_2 \in \bar{G}$ satisfy $g_1(\tilde{y}) = y_0$ and $g_2(y_0) = y$ consider the sequence,

$$h_k = g_1 \circ F_{\tilde{y}}(f_{n_k}(y))^{-1} \circ f_{n_k} \circ g_2$$

of elements of $\bar{G}$. They all fix $y_0$, and since $g_1 \circ F_{\tilde{y}}(f_{n_k}(y))^{-1}$ converges to $g_1$ as $k \to \infty$ we have the following.

(1) If $h_k$ contains a subsequence $h_{k_l}$ such that there exists a homeomorphism $h$ with,

$$\lim_{l\to\infty} h_{k_l} = h \quad \textit{and} \quad \lim_{l\to\infty} (h_{k_l})^{-1} = h^{-1}$$

then so does $f_{n_k}$.

(2) Furthermore, if there exist points $x', y' \in \mathbb{S}^1$ and a subsequence $h_{k_l}$ of $h_k$ such that,

$$\lim_{l\to\infty} h_{k_l} = x' \quad \textit{and} \quad \lim_{l\to\infty} (h_{k_l})^{-1} = y'$$

uniformly on compact subsets of $\mathbb{S}^1 \setminus \{y'\}$ and $\mathbb{S}^1 \setminus \{x'\}$ respectively, then so does $f_{n_k}$ ($x'$ and $y'$ will be replaced by $g_1^{-1}(x')$ and $g_1^{-1}(y')$).

Now, since $G_0$ is a convergence group, one of the above situations must occur. Consequently, $\bar{G} = \{g \in G : g(x_0) = x_0\}$ is a convergence group. □

**Proposition 6.3** *If $G$ is a subgroup of* $\mathrm{Homeo}(\mathbb{S}^1)$ *which is continuously 3–transitive but not continuously 4–transitive then $G$ is a convergence group.*

**Proof** This proof is almost identical to the previous one but we write it out in full for clarity.

Choose $x_0 \in \mathbb{S}^1$ and let $f_n$ be a sequence of elements of $G$. Then since $\mathbb{S}^1$ is compact, the sequence of points $f_n(x_0)$ will have a convergent subsequence, $f_{n_k}(x_0)$, converging to some point $\tilde{x}$. Let $I$ be a small open interval around $\tilde{x}$, then since $G$ is continuously 3–transitive, there exists a map $F_{\tilde{x}}\colon I \to G$ satisfying the following,

(1) $F_{\tilde{x}}(x)(\tilde{x}) = x$ for all $x \in I$

(2) $F_{\tilde{x}}(\tilde{x})$ is the identity

Let $g \in G$ send $\tilde{x}$ to $x_0$ and consider the sequence,

$$h_k = g \circ F_{\tilde{x}}(f_{n_k}(x_0))^{-1} \circ f_{n_k}$$

of elements of $G$. They all fix $x_0$, and since $g \circ F_{\tilde{x}}(f_{n_k}(x_0))^{-1}$ converges to $g$ as $k \to \infty$ we have the following.

(1) If $h_k$ contains a subsequence $h_{k_l}$ such that there exists a homeomorphism $h$ with,
$$\lim_{l\to\infty} h_{k_l} = h \quad \textit{and} \quad \lim_{l\to\infty} (h_{k_l})^{-1} = h^{-1}$$
then so does $f_{n_k}$.

(2) Furthermore, if there exist points $x', y' \in \mathbb{S}^1$ and a subsequence $h_{k_l}$ of $h_k$ such that,
$$\lim_{l\to\infty} h_{k_l} = x' \quad \textit{and} \quad \lim_{l\to\infty} (h_{k_l})^{-1} = y'$$
uniformly on compact subsets of $\mathbb{S}^1 \setminus \{y'\}$ and $\mathbb{S}^1 \setminus \{x'\}$ respectively, then so does $f_{n_k}$ ($x'$ and $y'$ will be replaced by $g^{-1}(x')$ and $g^{-1}(y')$).

Now, since $\bar{G} = \{g \in G : g(x_0) = x_0\}$ is a convergence group $G$ is too. $\square$

We now look at the case where $G$ is continuously 4–transitive. In this case, we show that $G$ must be $n$–transitive for every $n \in \mathbb{N}$.

**Theorem 6.4** *If $G$ is continuously $n$–transitive for $n \geq 4$, then it is continuously $n+1$–transitive.*

**Proof** Fix $n \geq 4$ and assume for contradiction that $G$ is continuously $n$–transitive but not continuously $n+1$–transitive. Take $(a_1, \ldots, a_{n-2}) \in P_{n-2}$ and define,

$$\bar{G} = \{g \in G : g(a_i) = a_i \ \ \forall i\}$$

Let $I$ be a component of $\mathbb{S}^1 \setminus \{a_1, \ldots, a_{n-2}\}$. Construct a homomorphism $\Psi\colon \bar{G} \to \mathrm{Homeo}(\mathbb{S}^1)$ in the same way as $\Phi\colon G_0 \to \mathrm{Homeo}(\mathbb{S}^1)$ was constructed in Section 5. Explicitly, take $g \in \bar{G}$, restrict it to a self homeomorphism of $\bar{I}$ and identify the endpoints to get an element of $\mathrm{Homeo}(\mathbb{S}^1)$.

Let $\bar{\mathcal{G}}$ denote the image of $\bar{G}$ under $\Psi$. Then as in Proposition 5.3 $\bar{\mathcal{G}}$ is isomorphic to $\bar{G}$. Using the arguments from the earlier Propositions in this section we can show that $\bar{\mathcal{G}}$ is a convergence group and hence conjugate to a subgroup of $\mathrm{PSL}(2,\mathbb{R})$. On the other hand, $\bar{\mathcal{G}}$ is 2–transitive on $I$ and every element fixes the identification point. This means that the action of $\bar{G}$ on $I$ must be conjugate to the action of $\mathrm{Aff}(\mathbb{R})$ on $\mathbb{R}$.

Let $I$ and $I'$ be two components of $\mathbb{S}^1 \setminus \{a_1, \ldots, a_{n-2}\}$ and let $\phi\colon I \to \mathbb{R}$ be a homeomorphism which conjugates the action of $\bar{G}$ on $I$ to the action of $\mathrm{Aff}(\mathbb{R})$ on $\mathbb{R}$. Let $a_{n-1}, a'_{n-1}$ be two distinct points in $I'$. Consider the groups

$$G_0 = \{g \in \bar{G} : g(a_{n-1}) = a_{n-1}\}$$

and

$$G'_0 = \{g \in \bar{G} : g(a'_{n-1}) = a'_{n-1}\}$$

They each act transitively on $I$ and by Corollary 5.5 and Proposition 5.8 without fixed points. Consequently, $\phi$ conjugates both of these actions to the action of $\mathbb{R}$ on itself by translation. Let $g \in G_0$ and $g' \in G'_0$ be elements which are conjugated to $x \mapsto x+1$ by $\phi$. Then $g^{-1} \circ g'$ acts on $I$ as the identity. However, if it is equal to the identity, then $g' = g$ fixes $a_{n-1}$ and $a'_{n-1}$, this is impossible as non-trivial elements of $\bar{G}$ can have at most one fixed point in $I'$. So $g^{-1} \circ g$ is a non-trivial element of $G$ which acts as the identity on $I$ and so by Corollary 4.5 we have that $G$ is continuously $n+1$–transitive. □

# 7 Summary of Results

**Theorem 7.1** *Let $G$ be a transitive subgroup of* $\mathrm{Homeo}(\mathbb{S}^1)$ *which contains a non constant continuous path. Then one of the following mutually exclusive possibilities holds,*

(1) *$G$ is conjugate to* $\mathrm{SO}(2,\mathbb{R})$ *in* $\mathrm{Homeo}(\mathbb{S}^1)$

(2) *$G$ is conjugate to* $\mathrm{PSL}(2,\mathbb{R})$ *in* $\mathrm{Homeo}(\mathbb{S}^1)$

(3) *For every $f \in \mathrm{Homeo}(\mathbb{S}^1)$ and each finite set of points $x_1, \ldots, x_n \in \mathbb{S}^1$ there exists $g \in G$ such that $g(x_i) = f(x_i)$ for each $i$.*

(4) *$G$ is a cyclic cover of a conjugate of* $\mathrm{PSL}(2,\mathbb{R})$ *in* $\mathrm{Homeo}(\mathbb{S}^1)$ *and hence conjugate to* $\mathrm{PSL}_k(2,\mathbb{R})$ *for some* $k>1$

(5) *$G$ is a cyclic cover of a group satisfying condition 3 above*

**Proof** Let $f\colon\ [0,1]\to G$ be a non constant continuous path. Then

$$f(0)^{-1}\circ f\colon\ [0,1]\to G$$

is a continuous deformation of the identity in $G$. Consequently, Proposition 2.6 tells us that $G$ is continuously 1–transitive.

If $J_x=\emptyset$ for every $x\in\mathbb{S}^1$ then by Theorem 3.8 $G$ is conjugate to $\mathrm{SO}(2,\mathbb{R})$ in $\mathrm{Homeo}(\mathbb{S}^1)$ . If $J_x\neq\emptyset$ for some and hence all $x\in\mathbb{S}^1$ then by Theorem 3.10 $G$ is either continuously 2–transitive or is a cyclic cover of a group $G'$ which is continuously 2–transitive.

So assume that $G$ is continuously 2–transitive, then by Proposition 4.6 it is continuously 3–transitive. If moreover $G$ is not continuously 4–transitive, then by Proposition 6.3 it is a convergence group and hence conjugate to a subgroup of $\mathrm{PSL}(2,\mathbb{R})$. On the other hand, since $G$ is continuously 3–transitive, it is 3–transitive, and hence must be conjugate to the whole of $\mathrm{PSL}(2,\mathbb{R})$.

If we now assume that $G$ is continuously 4–transitive then by Theorem 6.4 it is continuously $n$–transitive and hence $n$–transitive for every $n\in\mathbb{N}$. So if we take $f\in\mathrm{Homeo}(\mathbb{S}^1)$ and a finite set of points $x_1,\ldots,x_n\in\mathbb{S}^1$ there exists $g\in G$ such that $g(x_i)=f(x_i)$ and we are done.

□

**Theorem 7.2** *Let $G$ be a closed transitive subgroup of* $\mathrm{Homeo}(\mathbb{S}^1)$ *which contains a non constant continuous path. Then one of the following mutually exclusive possibilities holds,*

(1) *$G$ is conjugate to* $\mathrm{SO}(2,\mathbb{R})$ *in* $\mathrm{Homeo}(\mathbb{S}^1)$

(2) *$G$ is conjugate to* $\mathrm{PSL}_k(2,\mathbb{R})$ *in* $\mathrm{Homeo}(\mathbb{S}^1)$ *for some* $k\geq 1$

(3) *$G$ is conjugate to* $\mathrm{Homeo}_k(\mathbb{S}^1)$ *in* $\mathrm{Homeo}(\mathbb{S}^1)$ *for some* $k\geq 1$

**Proof** Since $G$ is a transitive subgroup of $\mathrm{Homeo}(\mathbb{S}^1)$ which contains a non constant continuous path, Theorem 7.1 applies. It remains to show that if $G$ satisfies condition 3 in Theorem 7.1 then its closure is $\mathrm{Homeo}(\mathbb{S}^1)$ .

To see this, let $f$ be an arbitrary element of $\mathrm{Homeo}(\mathbb{S}^1)$. If we can find a sequence of elements of $G$ which converges uniformly to $f$ then we shall be done. So let $\{a_n : n\in\mathbb{N}\}$ be a countable and dense set of points in $\mathbb{S}^1$. Choose a sequence of maps $g_n\in G$ so that $g_n(a_k)=f(a_k)$ for $1\leq k\leq n$. Then $g_n$ will converge uniformly to $f$ so that the closure of $G$ will equal $\mathrm{Homeo}(\mathbb{S}^1)$ . □

**Theorem 7.3** $\mathrm{PSL}(2,\mathbb{R})$ *is a maximal closed subgroup of* $\mathrm{Homeo}(\mathbb{S}^1)$ .

**Proof** Let $G$ be a closed subgroup of $\mathrm{Homeo}(\mathbb{S}^1)$ containing $\mathrm{PSL}(2,\mathbb{R})$. Then $G$ is 3–transitive and by applying Theorem 7.2 we can see that $\mathrm{Homeo}(\mathbb{S}^1)$ and $\mathrm{PSL}(2,\mathbb{R})$ are the only possibilities for $G$. □

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