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THE SURFACE SUBGROUP AND THE EHRENEPREIS CONJECTURES

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ABSTRACT. We survey our recent results including the Surface Subgroup Theorem and the Ehrenpreis Conjecture. Applications and future direction are discussed.

1. INTRODUCTION

1.1. **The Surface Subgroup Theorem.** One of the corollaries of the Geometrization Theorem is that most 3-manifolds admit hyperbolic structure. Therefore when studying topology of a 3-manifold one can often assume that the manifold is endowed with a hyperbolic structure. This greatly expands the tool-kit that is available bringing hyperbolic geometry, analysis and dynamics into play.

An essential step in the eventual proof of the Virtual Haken and the Virtual Fibering Conjectures is the Surface Subgroup Theorem:

Theorem 1.1 (Kahn-Markovic). *Every closed hyperbolic 3-manifold contains a quasifuchsian surface subgroup.*

Recall that every hyperbolic manifold $\mathbf{M}^3$ can be represented as the quotient $\mathbf{M}^3 = \mathbb{H}^3/\mathcal{G}$, where $\mathbb{H}^3$ is the hyperbolic 3-ball and $\mathcal{G}$ a Kleinian group. Using geometry and relying on fine statistical properties of the frame flow on hyperbolic manifolds we proved in [14] the following result which implies the Surface Subgroup Theorem:

Theorem 1.2 (Kahn-Markovic). *Let $\mathbf{M}^3 = \mathbb{H}^3/\mathcal{G}$ denote a closed hyperbolic 3-manifold. Given any $\epsilon > 0$, there exists a $(1+\epsilon)$-quasifuchsian group $G < \mathcal{G}$.*

(Recall that a group is K-quasifuchsian if it is K-quasiconformal deformation of a Fuchsian group.)

The nearly geodesic surfaces we constructed in [14] have large genus (it can be shown that the genus of S grows polynomially with $\frac{1}{\epsilon}$). Moreover, each such surface $f(S) \subset \mathbf{M}^3$ represents the trivial homology class in $H_2(\mathbf{M}^3, \mathbb{Z})$.

A three holed sphere with a hyperbolic metric and geodesic boundary is called a pair of pants (after Thurston). Given $R > 0$, the pair of pants whose all 3 cuffs have the same length $2R$ is called the R-perfect pair of pants. Let $\mathbf{S}(R)$ denote a genus 2 Riemann surface that is obtained by gluing two R-perfect pairs of pants along their cuffs with the twist of $+1$. The induced orbifold is denoted by $\mathrm{Orb}(R)$.

Theorem 1.2 is proved by the showing that for a given closed hyperbolic 3-manifold $\mathbf{M}^3$ and for every $\epsilon > 0$ and every large enough R, there exists a Riemann surface $S = S(\epsilon, R)$ and a continuous map $f : S \to \mathbf{M}^3$ such that the induced map between universal covers $\partial f : \partial \mathbb{H}^2 \to \partial \mathbb{H}^3$ is $(1 + \epsilon)$-quasisymmetric. Moreover, the surface S admits a decomposition into R-perfect pants that are glued to each other with the twist of $+1$. In particular, such a Riemann surface S is a regular holomorphic cover of the Model Orbifold $\mathrm{Orb}(R)$.

For fixed $\epsilon, R > 0$, a good pair of pants (in a given hyperbolic 3-manifold) is a pair of pants whose cuffs have complex half-length ϵ close to R (see Section 3). In order to find the map $f : S \to \mathbf{M}^3$ one is guided by the following principles:

(1) Do not start by trying to specify the surface S.
(2) Instead, consider the good pants $\mathbf{\Pi}$ immersed in $\mathbf{M}^3$ as the building blocks and eventually construct the surface $f(S) \subset \mathbf{M}^3$ by appropriately assembling together all the good pants from $\mathbf{\Pi}$.

Typically there are many ways in which one can assemble the pants and get an immersed surface.

Consider any finite formal sum $W \in \mathbb{N}\mathbf{\Pi}$. Taking two copies of each pair of pants (with opposite orientations) one obtains the new formal sum $2W \in \mathbb{N}\mathbf{\Pi}$. Then along every geodesic in $\mathbf{M}^3$ that appears as a boundary curve of some of the pants from $2W$ one can pair off the pairs of pants that contain that geodesic as a boundary component (there may be many ways in which one can pair off these pants and we choose one way of doing it for each such geodesic). One can now assemble these pairs of pants according to the instructions to construct a closed surface in $\mathbf{M}^3$.

So, we have constructed a closed surface S and a map $f : S \to \mathbf{M}^3$, but the induced map between fundamental groups is not necessarily injective. For example, if W denotes a single pair of good pants then the surface S is a genus two surface obtained by gluing together two pairs of pants. However, the corresponding map $f : S \to \mathbf{M}^3$ collapses one pair of pants onto the other and therefore the induced map between fundamental groups is not injective.

Observe that every such surface $f(S) \subset \mathbf{M}^3$ represents the trivial homology class in $H_2(\mathbf{M}^3, \mathbb{Z})$. This is because each pair of pants from W is used twice and with different orientations.

It is clear from the previous discussion that if one wants to glue pairs of good pants in $\mathbf{M}^3$ to get a nearly geodesic surface (and thus a quasifuchsian surface) then any two pairs of pants that are glued along a geodesic should meet at an angle that is close to π. It turns out that in order to assemble good pants and construct a nearly geodesic surface in $\mathbf{M}^3$, what is needed is that the good pants are equidistributed in $\mathbf{M}^3$, which follows from the exponential mixing of the frame flow. We will explain this in more details in the next section, but here we state the mixing principle:

Lemma 1.1 (Exponential Mixing). *Let $\mathbf{M}^3$ denote a closed hyperbolic manifold (in particular a hyperbolic surface). There exists $\mathbf{q} > 0$ that depends only on $\mathbf{M}^3$ such that the following holds. Let $\psi, \phi : \mathcal{F}(\mathbf{M}^3) \to \mathbb{R}$ be two C^1 functions (here $\mathcal{F}(\mathbf{M}^3)$ denotes the frame bundle, if $\mathbf{M}^3$ is a Riemann surface this is just the tangent bundle). Then assuming that the volume of the frame bundle $\mathcal{F}(\mathbf{M}^3)$ is equal to 1, for every $r \in \mathbb{R}$ the inequality*

$$\left| \int\limits_{\mathcal{F}(\mathbf{M}^3)} (\mathbf{g}_r^* \psi)(x)\phi(x) \, d\Lambda(x) - \int\limits_{\mathcal{F}(\mathbf{M}^3)} \psi(x) \, d\Lambda(x) \int\limits_{\mathcal{F}(\mathbf{M}^3)} \phi(x) \, d\Lambda(x) \right| \leq Ce^{-\mathbf{q}|r|},$$

holds, where $C > 0$ only depends on the C^∞ norm of ψ and ϕ.

2. The Ehrenpreis Conjecture

The Ehrenpreis conjecture was an old conjecture in the theory of Riemann surfaces. The idea is that although two Riemann surfaces S and T do not have a common finite cover (all covers in this proposal are regular and unbranched) one should still be able to interpolate between certain finite covers of S and T respectively (according to Gromov this statement goes back to Riemann). The precise formulation of the conjecture is as follows:

Conjecture 2.1 (Ehrenpreis Conjecture). *Let S and T denote two closed Riemann surfaces of genus at least 2 and let $\epsilon > 0$. Then there exists finite covers S_1 and T_1 of S and T respectively, such that S_1 and T_1 are $(1 + \epsilon)$ quasiconformal to each other (that is there exists $(1 + \epsilon)$-quasiconformal map $f : S_1 \to T_1$).*

In [16] J. Kahn and I have announced a proof of this conjecture.

Remark. *The Enhrenpreis Conjecture is harder to prove because there may be more pants on one side of a closed geodesic than the other. So we need to add in a signed sum of pants so that there are an equal number on both sides of every good geodesic. Computing this correction requires the "good pants homology", which we develop in* [16].

In fact, we prove this conjecture by proving the statement that every closed hyperbolic Riemann surface has a virtual decomposition into good pairs of pants that are glued by a twist that is nearly equal to $+1$. Recall that (ϵ, R)-good pair of pants is a pair of pants whose cuffs have the length ϵ-close to R. We prove the following virtual decomposition type theorem:

Theorem 2.1 (Kahn-Markovic). *Let S be a hyperbolic surface and let $\epsilon > 0$. Then for every large enough $R > 0$, the surface S has a finite cover S_1 that can be decomposed into (ϵ, R)-good pants such that every two adjacent pairs of good pants are glued with the twist that is $\frac{\epsilon}{R}$ close to $+1$.*

We then show that this surface S_1 is quasiconformally close to a finite cover of the model orbifold $\mathrm{Orb}(R)$ (see above for the definition of the Model orbifold $\mathrm{Orb}(R)$):

Theorem 2.2 (Kahn-Markovic). *Let S be a closed hyperbolic Riemann surface. Then for every $K > 1$, and every large enough $R > 0$ there are finite covers S_1 and O_1 of the surface S and the model orbifold $\mathrm{Orb}(R)$ respectively, and a K-quasiconformal map $f : S_1 \to O_1$.*

The Ehrenpreis conjecture is an immediate corollary of this theorem.

Theorem 2.3 (Kahn-Markovic). *Let S and M denote two closed Riemann surfaces. For any $K > 1$, one can find finite degree covers S_1 and M_1 of S and M respectively, such that there exists a K-quasiconformal map $f : S_1 \to M_1$.*

The proof of the above Virtual Decomposition theorem follows from the equidistribution of the good pants (in much the same way as in the proof of the Surface Subgroup Theorem) and from the Correction theory that we will outline below.

3. The Setup and Main Ideas

3.1. **The feet of a pair of pants.** A pair of pants is a compact hyperbolic Riemann surface with geodesic boundary that is homeomorphic to the sphere minus three disjoint round open disks. Any such pair of pants is determined by the lengths of the three boundary components,

which are called cuffs. For reasons which will become clear in the next section, we will prefer to work with the half-lengths, which of course are half the lengths of the three cuffs. In particular, an R-perfect pair of pants is a pair of pants whose three half-lengths are equal to R (for a given $R > 0$).

An orthogeodesic for a compact hyperbolic surface S with geodesic boundary is a proper geodesic arc which is orthogonal to the boundary of S at both endpoints. The long orthogeodesics for a pair of pants are the three embedded orthogeodesics which divide S into two components—these are the embedded orthogeodesics from a cuff to itself. The short orthogeodesics are the three other embedded orthogeodesics (from one cuff to another); the three short orthogeodesics together divide S into two right-angled hexagons. Because a right-angled hexagon is determined by the lengths of three alternating sides, these two right-angled hexagons must be isometric. It follows that the six endpoints of the three short orthogeodesics divide the three cuffs into six segments such that each cuff is divided into two equal segments. At each endpoint of an orthogeodesic η, there is a unique normal vector to the boundary that generates η (via the geodesic flow); we call this normal vector the foot of η at that endpoint. We say that the feet of a pair of pants at a given cuff are the feet of the two short orthogeodesics from that cuff to the two other cuffs. Thus there are two feet of a pair of pants in the normal bundle of each cuff of the pants.

3.2. **Good and Perfect Panted Surfaces.** Now suppose that we are given a closed hyperbolic Riemann surface S of genus $g > 1$, and a maximal collection $\mathcal{C}$ of disjoint curves on S. (By a *curve* we mean an (smooth) isotopy class of smoothly embedded closed curves). Each of these curves can be uniquely realized as a closed geodesic on S; together they divide S into $2g - 2$ pairs of pants. For each closed geodesic γ, there are two of these pairs of pants with γ as boundary (or γ appears as two boundaries of the same pair of pants). We can then find two pairs of feet, and holding up the (universal cover of the) cuff vertically, we see that the two feet on the right are a certain distance above the two feet on the left—except that this distance is only defined up to the half-length of the cuff. Therefore, to each cuff C, we have two invariants: the positive real half-length of the geodesic representative γ of C, and the shear, which is defined up to the half-length of γ. There is a natural topology on panted surfaces (of a given genus), for which these $6g - 6$ invariants provide local coordinates.

An R-perfect panted surface is one for which all of the cuffs of the pants have half-length R, and all of the shears are equal to 1. An R, ϵ-good pair of pants is one for which all three cuffs have half-length within ϵ of R, and an R, ϵ-good panted surface is one made of out of good pants, for which all of the shears are within ϵ/R of 1. (Sometimes we will write perfect for R-perfect, and good for R, ϵ-good). For any good panted surface $S, \mathcal{C}$, there is a path through good panted surfaces to a perfect panted panted surface $S', \mathcal{C}'$ and a homeomorphism $h: S \to S'$ determined (up to isotopy) by that path. (The path is determined up to homotopy rel endpoints, and hence the homeomorphism is determined up to isotopy). We say that $S', \mathcal{C}'$ is the *perfect version* of $S, \mathcal{C}$, and that h is the *perfecting homeomorphism.*

We prove the following theorem which provides a criterion for two large genus surfaces to be close to each other in the corresponding Moduli space with respect to the Teichmüller metric.

Theorem 3.1. *There exists R_0, K_0, and $\epsilon_0 > 0$ such that the following holds. Suppose that $S, \mathcal{C}$ is an R, ϵ good panted surface, and $R > R_0$, $\epsilon < \epsilon_0$. Let $S', \mathcal{C}'$ be the perfect version of $S, \mathcal{C}$. Then there is a $K_0 \epsilon$-quasiconformal diffeomorphism $h: S \to S'$ that is homotopic to the perfecting homeomorphism.*

The proof of this theorem is very delicate and we omit it here (the reader can see Section 2 in [14]). It should be stressed that the requirement that pants are glued with the twist by $+1$ plays a vital and subtle role and the criterion would not hold without it.

Theorem 2.3 follows if we can prove the following:

Theorem 3.2. *For every closed hyperbolic Riemann surface S we can find a finite cover $\hat{S}$ and a maximal set $\mathcal{C}$ of disjoint curves on $\hat{S}$ such that $\hat{S}, \mathcal{C}$ is a good panted surface.*

or, more precisely, if we can prove the following:

Theorem 3.3. *For every closed hyperbolic Riemann surface S, $\epsilon > 0$, and $R > R_0(S, \epsilon)$, we can find a finite cover $\hat{S}$ and a maximal set $\mathcal{C}$ of disjoint curves on $\hat{S}$ such that $\hat{S}, \mathcal{C}$ is an R, ϵ good panted surface.*

Let us briefly explain this implication. We glue two R-perfect pairs of pants together with a shear of 1 at each cuff to obtain an R-perfect surface S_R with an orientation-preserving isometry group of size 12. The model orbifold O_R is the quotient of S_R by this group of isometries; the three cuffs of half-length R on S_R map to a single segment η_R of length $R/2$ on O_R connecting two of the the order 2 points on O_R. *Any R-perfect panted surface S is a finite cover of O_R in such a way*

that the R-cuffs of S are the components of the pre-image of η_R by the cover. It follows that any two R-perfect panted surfaces have a common finite cover. Then given two surfaces S and T, and $\epsilon > 0$, we find $R, \epsilon/2K_0$ good panted covers $\hat{S}$ and $\hat{T}$ (for any R sufficiently large). By Theorem 3.1, these are each $\epsilon/2$ close to perfect surfaces, which then have a common cover. Therefore $\hat{S}$ and $\hat{T}$ have common covers within ϵ of each other in the Teichmüller metric, and we are finished.

Recall that the Teichmüller metric on the moduli space of compact Riemann surfaces of genus g is defined so that the distance between S and S' is $\log K$, where K is the infimum of K for which there exists a K-quasiconformal diffeomorphism $h \colon S \to S'$. We will often write $1+\epsilon$-quasiconfomal when we should really be writing e^ϵ-quasiconformal, and so forth—the reader can make the necessary modifications.

3.3. **Building a good cover.** We can now begin to describe how we prove Theorem 3.3. Recall that S is our given closed hyperbolic Riemann surface. A good curve for S (really an R, ϵ good curve) will be a closed geodesic γ (or the associated free homotopy class) whose half-length $\mathbf{hl}(\gamma)$ is within ϵ of R. We will denote the set of R, ϵ good curves by $\Gamma_{\epsilon, R}$; it is a finite set, with size asymptotic to $4\epsilon e^{2R}/2R$ when R is large.

Now let Π be a topological pair of pants, and let $f \colon \Pi \to S$ be a π_1 injective immersion. Then there is a unique hyperbolic metric on Π (up to pullback by a diffeomorphism isotopic to the identity) such that Π becomes a geometric pair of pants (with geodesic boundary) and f is isotopic to an isometric immersion. If Π is then a good pair of pants (for some R, ϵ), then we say that f represents an immersed good pair of pants in S. For any R and ϵ, there is a finite set $\Pi_{\epsilon, R} \equiv \Pi_{\epsilon, R}(S)$ of good pairs of pants in S. Using the exponential mixing of the geodesic flow on S, and the consequent estimates on the number of long orthogeodesic segments connecting a pair of geodesic segments on S, we prove that the feet of good pants are evenly distributed in the normal bundle of every good geodesic:

Theorem 3.4. *Suppose that $\gamma \in \Gamma_{\epsilon, R}$, and let I be an interval in the (square root of the) normal bundle for γ. The number $n(\gamma, I)$ of feet of pants in $\Pi_{\epsilon, R}$ that lie in I is estimated by*

$$n(\gamma, I) = \frac{n(\gamma)|I|}{2\mathbf{l}(\gamma)} + O(e^{(1-\alpha)R}),$$

where $n(\gamma)$ is the total number of pairs of feet on both sides on γ, and $\alpha \equiv \alpha(S)$. Moreover,

$$n(\gamma) \sim 2\epsilon^2 R e^R / \operatorname{Area}(S),$$

for R large given S and ϵ.

What is important in this statement is that the error term for $n(\gamma, I)$ is exponentially small (in R) compared to $n(\gamma)$. Up to this error term, the feet of the pants with γ as a boundary are evenly distributed on the normal bundle of γ. Let us suppose, by some miracle, that the distribution of feet is also *balanced*: that there are exactly as many feet on one side of γ as the other. (By the two sides of γ we mean the two components of the unit normal bundle for γ). Then it is a simple and elementary exercise to show that there is a bijection $\sigma \colon \Pi_\gamma^+ \to \Pi_\gamma^-$ (where Π_γ^+ and Π_γ^- are the pants with feet on the left and right sides of the unit normal bundle of the oriented geodesic γ) such that for any pair of pants $\pi \in \Pi_\gamma^+$, the feet of $\sigma(\pi)$ on γ are $1 + O(e^{-\alpha R})$ above the feet of π on γ. We then use this bijection to glue the pants in Π_γ^- to the pants of Π_γ^+ (along the cuffs that map to γ), and doing this with every $\gamma \in \Gamma_{\epsilon,R}$, we obtain a closed surface, made of the pants in $\Pi_{\epsilon,R}$, that is a finite cover of S. Because the shears are exponentially close to 1, and an exponentially small number is less than ϵ/R when R is large, we have obtained a good panted cover of S, and have thereby proven Theorem 3.3.

Of course, we have no reason to believe that there are exactly the same number of pants on the two sides of γ. We will describe in a few paragraphs how to correct this imbalance, but first we will describe the analogous construction in a closed hyperbolic three-manifold M, and we will see that in three dimensions, the work is a bit easier, because there is no imbalance to correct.

3.4. **Working in three dimensions.** Suppose that $f \colon \Pi \to M$ is a π_1-injective map from a topological pair of pants Π to M. We are interested in describing f up to homotopy. We can assume that f maps the boundaries of Π to closed geodesics $\gamma_0, \gamma_1, \gamma_2$ in M. We can also assume that f maps three disjoint arcs in Π (connecting the three boundary components) into three geodesic segments η_0, η_1, η_2 such that η_0 connects γ_1 and γ_2 and meets both geodesics orthogonally (and similarly for η_1 and η_2). These three arcs will divide Π into two (filled) hexagons, and f will map the boundary of each of these hexagons into skew right-angled hexagons.

Skew right-angled hexagons in $\mathbb{H}^3$ are very much like right-angled hexagons in $\mathbb{H}^2$, with $\mathbb{R}$ replaced by $\mathbb{C}$. That is, a skew right-angled

hexagon is determined by the complex length of three alternating sides. The real part of the complex length is the real length, and the imaginary part, which is defined up to multiples of $2\pi i$, is the amount of rotation from one adjacent side to the other adjacent side, after one adjacent side has been translated along the given side to meet the other adjacent side. Because the complex lengths of the η_i are the same in both skew right-angled hexagons, the two hexagons are isometric, and hence each γ_i is divided into two segments by the endpoints of $\eta_{i\pm1}$, and these two segments have equal complex length (with respect to the $\eta_{i\pm1}$). We call this complex length the complex half-length $\mathbf{hl}(\gamma)$ of γ. The *feet* or initial vectors of the orthogeodesics $\eta_{i\pm1}$ are elements of the unit normal bundle $N^1(\gamma)$, which is a torsor for $\mathbb{C}/(2\pi i\mathbb{Z}+2\mathbf{hl}(\gamma)\mathbb{Z})$, and the difference of between the two feet is exactly $\mathbf{hl}(\gamma)$. Thus we can think of the unordered pair of feet as living in $N^1(\sqrt{\gamma})$, the set of unordered pairs that differ by $\mathbf{hl}(\gamma)$; it is a torsor for $\mathbb{C}/(2\pi\mathbb{Z} + \mathbf{hl}(\gamma)\mathbb{Z})$.

We let $\Gamma_{\epsilon,R}$ be the good closed geodesics in M (so $\gamma \in \Gamma_{\epsilon,R}$ if the complex length $\mathbf{l}(\gamma)$ satisfies $|\mathbf{l}(\gamma) - 2R| < 2\epsilon$), and we let $\mathbf{\Pi}_{\epsilon,R}$ be the good pants in M (so $f\colon \Pi \to M$ is in $\mathbf{\Pi}_{\epsilon,R}$ if for each $\gamma \in f(\partial\Pi)$ we have $|\mathbf{hl}(\gamma) - R| < \epsilon$).

We can then prove the analogue of Theorem 3.4 for distribution of the feet of good pants:

Theorem 3.5. *Suppose that $\gamma \in \Gamma_{\epsilon,R}$, and let $I \times J$ be a rectangle in the (square root of the) normal bundle for γ. The number $n(\gamma, I)$ of feet of pants in $\mathbf{\Pi}_{\epsilon,R}$ that lie in I is estimated by*

$$n(\gamma, I) = \frac{n(\gamma)|I \times J|}{2\pi\Re(\mathbf{hl}(\gamma))} + O(e^{(1-\alpha)R}),$$

where $n(\gamma)$ is the total number of pairs of feet in $N^1(\sqrt{\gamma})$, and $\alpha \equiv \alpha(M)$. Moreover,

$$n(\gamma) \sim 8\epsilon^4 Re^{2R}/\mathrm{Vol}(M)$$

for R large given M and ϵ.

It then follows that if $A_\gamma \subset N^1(\sqrt{\gamma})$ is the set of pairs of feet of good pants on γ, then we can find a permutation $\sigma\colon A_\gamma \to A_\gamma$ such that

$$|\sigma(x) - x - \pi i - 1| < \epsilon/R$$

for every $x \in A$.

Then we can assemble the pants of $\mathbf{\Pi}_{\epsilon,R}$ into a "good panted surface group representation" using the "doubling trick". We take two copies of every pair of pants in $\mathbf{\Pi}_{\epsilon,R}$, and give them the two possible orientations. Then for any $\gamma \in \Gamma_{\epsilon,R}$, we have two sets, Π_γ^+ and Π_γ^-, of oriented pants with γ as boundary, where each pair of pants in Π_γ^+ induces a "positive"

orientation on γ (arbitrarily chosen), and the opposite holds for Π_γ^-. We then find $\hat{\sigma}\colon \Pi_\gamma^+ \to \Pi_\gamma^-$ such that the pair of feet of $\hat{\sigma}(\Pi)$ on γ is σ applied to the pair of feet of Π on γ. In this way we pair off all of the boundary components of the two copies of the good pants.

We then obtain an immersed panted surface $f\colon S \to M$ (with a maximal set $\mathcal{C}$ of curves on S). It is an ϵ, R good panted surface group representation in the following sense: the restriction of f to every component of $S - \bigcup \mathcal{C}$ is a pair of good pants, and for every $C \in \mathcal{C}$, the complex shear coordinates—the difference (in $N^1(\sqrt{\gamma})$) between the feet of the pants on one side of C and the other—is within ϵ of $i\pi + 1$.

It follows that f is essential by the following theorem (closely analogous to Theorem 3.1), which gives a way to certify the injectivity of the induced homomorphism $\rho\colon \pi_1(S) \to \mathrm{Isom}(\mathbb{H}^3)$.

Theorem 3.6. *There exists R_0, K_0, and $\epsilon_0 > 0$ such that the following holds. Suppose that $\rho\colon \pi_1(S) \to \mathrm{Isom}(\mathbb{H}^3)$ is an R, ϵ good panted surface group representation, and $R > R_0$, $\epsilon < \epsilon_0$. Then we can find an R-perfect panted Fuchsian group (which we then think of as acting on $\mathbb{H}^3$), and an equivariant map $h\colon \mathbb{H}^3 \to \mathbb{H}^3$ that extends to be $K_0\epsilon$-quasiconformal on the boundary. In particular, ρ is a faithful, discrete, and quasifuchsian representation.*

3.5. **The good pants homology and the Ehrenpreis conjecture.** We now return the the problem of proving Theorem 3.3, which implies the Ehrenpreis conjecture. We will let $\Gamma_{\epsilon,R}$ denote the set of *oriented* geodesics, and we will let $\mathbb{Z}\Gamma_{\epsilon,R}$ denote the set of integral formal sums of elements of $\Gamma_{\epsilon,R}$, where we will think of opposite orientations of the same geodesic as summing to zero. Let $\partial\colon \Pi_{\epsilon,R} \to \mathbb{Z}\Gamma_{\epsilon,R}$ be the obvious boundary map. We prove that when R is large given S and ϵ, there is a map $q\colon \mathbb{Q}\Gamma_{\epsilon,R} \to \mathbb{Q}\Pi_{300\epsilon,R}$ such that, for any $\alpha \in \mathbb{Q}\Pi_{\epsilon,R}$,

(1) $\partial q(\partial \alpha) = \partial \alpha$, and
(2) $||q(\alpha)||_\infty \leq e^{-R} P(R) ||\alpha||_\infty$ for any weighted sum α of good curves.

(Where $P(R)$ is a polynomial in R that depends only on S and ϵ). Letting $\alpha \equiv \Sigma\Pi_{\epsilon,R}$ be the formal sum of the good pants, we replace α with $\alpha' = \alpha - q(\partial \alpha)$ to obtain a "balanced" sum of pants ($\partial \alpha' = 0$) with the same equidistribution properties[1] as in Theorem 3.4 (because $q(\partial \alpha)$ is small compared to α). We can then pair these pants across every good geodesic to obtain an immersed (or covering) panted surface which, by Theorem 3.1, is $1 + \epsilon$ quasiconformally equivalent to the corresponding perfect surface, thus proving the Ehrenpreis conjecture.

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[1]We should observe as well that α' is positive!

We will briefly outline the construction of the map q and the demonstration of the estimate (2). We define the "good pants homology" as $\mathbb{Q}\Gamma_{\epsilon,R}/\partial\mathbb{Q}\mathbf{\Pi}_{\epsilon,R}$; if two sums of good curves differ by an element of $\partial\mathbb{Q}\mathbf{\Pi}_{\epsilon,R}$, we will say that they are $\mathbf{\Pi}_{\epsilon,R}$ homologous. We prove that, if A_i, B_j, U, V $(i, j = 0, 1)$ are elements of $\pi_1(S, *)$ such that the broken closed geodesic $[\cdot A_i \cdot U \cdot B_j \cdot V \cdot]$ has "bounded inefficiency" and the geodesic segments $\cdot U \cdot$ and $\cdot V \cdot$ are sufficiently long, then

$$\sum_{i,j=0,1} (-1)^{i+j}[A_i U B_j V] \equiv 0$$

in $\mathbf{\Pi}_{\epsilon,R}$ (really $\Pi_{300\epsilon,R}$) homology, provided, of course, that the $[A_i U B_j V]$ are the free homotopy classes (or, if you like, conjugacy classes in $\pi_1(S, *)$) of good curves.

This then permits us to define, for $A, T \in \pi_1(S, *)$,

$$A_T \equiv \frac{1}{2}\left([T A T^{-1} U] - [T A^{-1} T^{-1} U]\right),$$

where U is fairly arbitrary. Then A_T in good pants homology is independent of the choice of U. We can show through a series of lemmas (see [16]) that $(XY)_T \equiv X_T + Y_T$ in good pants homology; this then implies that any element of $\mathbb{Q}\mathbf{\Pi}_{\epsilon,R}$ that is zero in $H_1(S)$ is zero in $\mathbf{\Pi}_{\epsilon,R}$ homology.

We have not yet said anything about the function q. The idea is that whenever we prove that two formal sums of curves are equal in good pants homology, we produce a sum of good pants (the "witness" to the homology) whose boundary is equal to the difference of the two formal sums. When we make an arbitrary choice in determining the sum of good pants, we take the average of the results of our choices as our witness. When one identity in good pants homology is proved using another one, the witness for the latter is used to build the witness for the former. In this way, when we prove that

$$(XY)_T \equiv X_T + Y_T$$

in good pants homology, we can explicitly produce a function $g\colon \pi_1(S) \times \pi_1(S) \to \mathbb{Q}\mathbf{\Pi}_{\epsilon,R}$ such that $(XY)_T - X_T - Y_T = \partial g(X, Y)$.

We then let $g_1, \ldots g_{2n}$ be a standard set of generators for $\pi_1(S, *)$; then $[g_1], \ldots, [g_{2n}]$ also form a basis for $H_1(S)$, and so does $\mathfrak{g} = \{(g_1)_T, \ldots, (g_{2n})_T\}$, because $X \equiv X_T$ in H_1. For any $\gamma \in \Gamma_{\epsilon,R}$, we can find a unique $\hat{\gamma} \in \mathbb{Z}\mathfrak{g}$ that is equal to γ in $H_1(S)$. Then in the course of proving that $\gamma \equiv \hat{\gamma}$ in $\mathbf{\Pi}_{\epsilon,R}$ homology, we produce $q\colon \mathbb{Q}\Gamma_{\epsilon,R} \to \mathbb{Q}\mathbf{\Pi}_{\epsilon,R}$ such that

$$\partial q(\gamma) = \gamma - \hat{\gamma}.$$

The identity (1) then follows for q, because $\widehat{\partial \alpha} = 0$ (because $\partial \alpha \equiv 0$ in H_1) for any $\alpha \in \mathbb{Q}\Pi_{\epsilon,R}$ (where we have extended $\gamma \mapsto \hat{\gamma}$ to $\mathbb{Q}\Pi_{\epsilon,R}$).

It remains only to show the estimate (2) for this q. Again, for each identity that we prove in [16], and each resulting implicit definition of a witness, we produce a corresponding estimate for the "witness function", using the previous estimates. Is this manner we produce the desired inequality.

4. Applications

4.1. **Virtual Classification of 3-Manifolds.** A subsurface $S \subset \mathbf{M}^3$ (here S is a compact surface, possibly with boundary) is **essential** if the induced map between fundamental groups is an injection. The surface is **incompressible** in $\mathbf{M}^3$ if it is embedded in $\mathbf{M}^3$ and if every homotopically non-trivial simple loop on S is mapped onto a homotopically non-trivial closed curve in $\mathbf{M}^3$. Every essential embedded surface is incompressible, and the converse is a well-known theorem.

Machinery has been developed to study hyperbolic 3-manifolds that are Haken. A manifold is Haken if it contains an incompressible surface. If $\mathbf{M}^3$ is Haken, one can cut $\mathbf{M}^3$ along its incompressible surface to obtain a 3-manifold with boundary (which may be disconnected). Furthermore, hyperbolic 3-manifolds with boundary are known to be Haken so one can continue to cut until arriving at indecomposable pieces. This is known as the Haken hierarchy and it is a cornerstone of 3-dimensional topology. Although many 3-manifolds are not Haken it was conjectured by Thurston that every such manifold has a finite degree cover that is. This was known as the Virtual Haken Conjecture.

Thurston made an even stronger conjecture called the Virtual Fibering Conjecture. Let $\mathbb{S}_g$ denote a closed surface of genus $g \geq 2$. Given a homeomorphism $f : \mathbb{S}_g \to \mathbb{S}_g$, let $\mathbf{M}_f^3$ denote the corresponding mapping torus. Then $\mathbf{M}_f^3$ is a closed 3-manifold that fibers over the circle. Thurston proved that $\mathbf{M}_f^3$ is hyperbolic if and only if f is homotopic to a pseudo-Anosov homeomorphism of $\mathbb{S}_g$. The Virtual Fibering Conjecture Thurston stated that every hyperbolic 3-manifold has a finite degree cover that fibers over the circle.

These two conjectures were major driving forces behind the research in three dimensional topology in recent decades. Building on the Surface Subgroup Theorem of Kahn-Markovic and the work by Wise [23] and Haglund-Wise [12], Agol [2] completed the proofs of both conjectures. Below we state the main steps in the proof.

A group is cubulated if it is acting properly and co-compactly on a CAT(0) cube complex. It turns out that each cubulated hyperbolic

group has a rich (hidden) underlying stucture. Wise developed this theory [23], although in his work he used an additional assumption that cubulated hyperbolic groups have a certain Haken hierarchy. Under this assumption he, and in collaboration with Haglund, Hruska, Hsu and others, showed that such groups can be embedded in Right Angled Artin groups which implies that such cubulated hyperbolic groups have many deep properties like being linear, LERF (that is, finitely generated subroups are separable), etc. In particular, if the fundamental group of a hyperbolic 3-manifold satisfies these assumption, it follows from Wise's theory that this 3-manifold is Virtually Haken. Moreover, using the Agol's criterion for virtual fibering [1], it also follows that such a 3-manifold virtually fibers over the circle.

In order for Wise's theory to be applied to hyperbolic 3-manifolds it has to be shown that 3-manifold groups are cubulated and that Wise's assumption on the Haken hierarchy can be dropped.

In the course of proving the Surface Subgroup Theorem we proved that given any two points on the 2-sphere there is a surface subgroup whose limit set separates these two points. Combining this fact and the Sageev construction [22], Bergeron-Wise showed:

Theorem 4.1 (Bergeron-Wise). *The fundamental group of a closed hyperbolic 3-manifold is cubulated.*

Finally, Agol [1] proved Wise's conjecture that cubulated hyperbolic groups are virtually special (which in particular means that Wise's assumption on the Haken hierarchy is not needed for his theory to work), and thus he was able to prove the Virtual Haken Theorem and the Virtual Fibering Theorem:

Theorem 4.2 (Agol). *Every closed hyperbolic 3-manifold has a finite cover that fibers over the circle. In particular, every hyperbolic 3-manifold has a finite cover that is Haken.*

The reader may want to consult the comprehensive survey article [3] by Aschenbrenner-Friedl-Wilton for a complete overview of these theories.

4.2. **Counting Problems for Essential Surfaces and Moduli spaces.** Counting closed geodesics in negatively curved manifold is an old and profound subject. Standard results (that are essentially corollaries of the mixing properties of geodesic flows on negatively curved manifolds) state that the number of closed geodesics of length at most L grows exponentially with L. For hyperbolic manifolds (and in particular for Riemann surfaces) this asymptotic is precisely known (Margulis [19]) with excellent bounds on error terms.

Analogously, in a given hyperbolic 3-manifold $\mathbf{M}^3$ one can count the number of essential surfaces (up to homotopy) live inside $\mathbf{M}^3$. Let $s(\mathbf{M}^3, g)$ denote the number (up to homotopy) of genus g incompressible surfaces of $\mathbf{M}^3$. The following counting result was proved in [15]:

Theorem 4.3 (Kahn-Markovic). *Let $\mathbf{M}^3$ be a closed hyperbolic 3-manifold. There exist constants $0 < c_1 \leq c_2$, such that the inequality $(c_1 g)^{2g} \leq s(\mathbf{M}^3, g) \leq (c_2 g)^{2g}$, holds for every large g.*

- A difficult (and perhaps deep) conjecture is to prove that for some constant $c = c(\mathbf{M}^3) > 0$ the formula

$$\lim_{g \to \infty} \frac{\sqrt[2g]{s(\mathbf{M}^3, g)}}{g} = c$$

  holds. A positive answer to this conjecture would represent a kind of the Prime Number Theorem for counting essential surfaces in 3-manifolds analogous to the Margulis' Prime Number Theorem for counting closed geodesics [19].
- Another important question is: What does a random essential surface of genus g (for some large g) inside $\mathbf{M}^3$ look like? Is this a quasifuchsian surface or is it a geometrically infinite surface (according to Thurston, Bonahon and Canary a geometrically infinite closed surface in $\mathbf{M}^3$ is a virtual fiber)?

4.3. **Homology Of Curves And Surfaces In Hyperbolic 3-Manifolds.** In manifolds of negative curvature each homotopy class of closed curves can be realized by a unique geodesic. Given that closed curves have such nice geometric representatives, Thurston recently asked if one can represent each homology class in $H_2(\mathbf{M}^3, \mathbb{Z})$ by a nearly geodesic representative. The following theorem is proved using the methods from [14].

Theorem 4.4 (Liu-Markovic). *Every rational second homology class of a closed hyperbolic 3-manifold has a positive integral multiple represented by an oriented connected closed quasi-Fuchsian subsurface.*

It is well known that every homology class in $H_2(\mathbf{M}^3, \mathbb{Z})$ can be represented as a sum (with integer coefficients) of connected incompressible surfaces in $\mathbf{M}^3$. Such an incompressible surface may be quasifuchsian but it also can be a non-geometrically finite (and thus non quasifuchsian) incompressible surface. At any rate, this result shows that we can replace any such sum of incompressible surfaces with a connected quasifuchsian surface without changing the homology class.

Let γ_1 and γ_2 denote two oriented closed curves inside a closed hyperbolic 3-manifold $\mathbf{M}^3$. Moving into a general position one can show

that if γ_1 and γ_2 are homologous in $\mathbf{M}^3$ then γ_1 and $-\gamma_2$ bound an immersed surface in $\mathbf{M}^3$. Topologically it is much more significant when two homologous closed curves γ_1 and $-\gamma_2$ bound an essential surface inside $\mathbf{M}^3$ (a surface, possibly with boundary, is essential in $\mathbf{M}^3$ if its fundamental group injects into the fundamental group of $\mathbf{M}^3$). The following claim asserts that this property is always true in the rational homology $H_1(\mathbf{M}^3, \mathbb{Q})$. In particular, every closed homologically trivial curve in a closed hyperbolic 3-manifold $\mathbf{M}^3$ rationally bounds an essential surface in $\mathbf{M}^3$. This answers a question of D. Calegari in the case of hyperbolic 3-manifolds (this problem is wide open for hyperbolic groups for example).

Theorem 4.5 (Liu-Markovic). *Every rationally null-homologous, π_1 injectively immersed oriented closed 1-submanifold in a closed hyperbolic 3-manifold has an equidegree finite cover which bounds an oriented connected compact immersed quasi-Fuchsian subsurface.*

The following two very recent results by Hongbin Sun are heavily dependent on the Virtual Haken Theorem and Theorem 4.5.

Theorem 4.6 (Sun). *Let A be a finite Abelian group. Then every closed hyperbolic 3-manifold $\mathbf{M}^3$ has a finite degree cover $\mathbf{M}_1^3$ such that A is a direct summand in $Tor\big(H_1(\mathbf{M}^3, \mathbb{Z})\big)$.*

Theorem 4.7 (Sun). *For any closed oriented hyperbolic 3-manifold $\mathbf{M}^3$, and any closed oriented 3-manifold $\mathbf{N}^3$, there exists a finite cover $\mathbf{M}_1^3$ of $\mathbf{M}^3$, and a degree-2 map $f : \mathbf{M}_1^3 \to \mathbf{N}^3$, i.e. $\mathbf{M}^3$ virtually 2-dominates $\mathbf{N}^3$.*

Very recently, Ursula Hamenstead (see [13]) showed that most closed, rank-one locally symmetric spaces contain surface subgroups. In particular, she proves that every closed complex hyperbolic contains a surface subgroup.

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