Homogenization of random quasiconformal mappings and random Delauney triangulations

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Abstract

In this paper, we solve two problems dealing with the homogenization of random media. We show that a random quasiconformal mapping is close to an affine mapping, while a circle packing of a random Delauney triangulation is close to a conformal map, confirming a conjecture of K. Stephenson. We also show that on a Riemann surface equipped with a conformal metric, a random Delauney triangulation is close to being circle packed.

1 Introduction

1.1 Random quasiconformal mappings

Our model of a random quasiconformal mapping depends on a probability measure λ on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. For each cell in a square grid in the complex plane, randomly assign a complex number in the unit disk according to the measure λ. In other words, the values of the cells are independent and identically distributed (i.i.d.) random variables with

$$\mathbb{P}(\# \in E) = \lambda(E), \qquad E \subset \mathbb{D} \text{ measurable.}$$

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The collection of these numbers defines a Beltrami coefficient $\mu(z)$ on $\mathbb{C}$ which is constant on the cells of the grid.

We write w^μ for the unique injective solution of the Beltrami equation $\overline{\partial} w(z) = \mu(z)\partial w(z)$ which fixes $0, 1, \infty$. If λ is supported on a compact subset of the unit disk, the existence of w^μ is guaranteed by the measurable Riemann mapping theorem and $w^\mu : \mathbb{C} \to \mathbb{C}$ is a homeomorphism. In general, $w^\mu(z)$ exists and is unique by virtue of $K_\mu = \frac{1+|\mu|}{1-|\mu|}$ being locally bounded, but may not be surjective. We refer to w^μ as a random quasiconformal mapping even though it may not be a genuine quasiconformal mapping.

Our first main theorem says that if the mesh size $\delta > 0$ is small, then with high probability, w^μ is close to an affine transformation $A_\lambda = w^{\mu_\lambda}$ determined by the measure λ. (Since an affine map has constant dilatation, μ_λ is a constant with absolute value less than 1.)

Theorem 1.1. *Suppose λ is a probability measure on the unit disk. For each cell of a square grid in the plane of mesh size δ, randomly assign a complex number in the unit disk according to the measure λ. There exists an affine transformation A_λ such that for any compact set $K \subset \mathbb{C}$ and $\varepsilon > 0$, when $\delta \le \delta_0(K, \varepsilon)$ is sufficiently small, $\|w^\mu - A_\lambda\|_{C(K)} < \varepsilon$ holds with probability at least $1 - \varepsilon$.*

We do not know how to explicitly determine the affine map A_λ for general λ. However, if λ respects the $90°$ symmetry of the model, then A_λ is the identity mapping:

Corollary 1.2. *If $d\lambda(z) = d\lambda(-z)$, then $A_\lambda(z) = z$ is the identity map.*

Proof. Observe that $w^\mu(iz)/w^\mu(i)$ is the normalized quasiconformal mapping with dilatation $-\mu(iz)$. Since the square grid is invariant under multiplication by i, the random quasiconformal maps $w^\mu(iz)/w^\mu(i)$ and $w^\mu(z)$ are equally likely. Therefore, the affine transformation $A_\lambda(z)$ satisfies the relation $A_\lambda(iz)/A_\lambda(i) = A_\lambda(z)$ which forces $A_\lambda(z) = z$. □

In an unpublished manuscript [2], K. Astala, S. Rohde, E. Saksman and T. Tao gave a different proof of Theorem 1.1 for random quasiconformal mappings with uniformly bounded distortion. A recent blog post [12] by T. Tao gives a brief summary and a discussion of the results. The method of [2] is based on the homogenization of iterated singular integrals. By contrast, our proof is more elementary: it is based on the geometric definition of quasiconformal mappings.

We first show that with high probability, a random quasiconformal map is roughly quasiconformal, that is, it stretches the moduli of all rectangles whose sides have length $\geq \varepsilon$ by a bounded amount. This uses a simple lemma about percolation on the square grid that we have learned from a paper of P. Mathieu [8] on random walk in random environments.

We then show that there exists an extremal direction such that with positive probability, the random quasiconformal map stretches moduli of squares in that direction by approximately the maximal amount. We promote *positive probability* to *high probability* by subdividing a square that is stretched by approximately the maximal amount into a large number of small squares. On one hand, the moduli of the images of the small squares are independent random variables since they are disjoint, while one the other hand, the extremality of the big square forces all small squares to be extremal.

The above argument shows that there is a sequence of good scales $\delta_k \to 0$, such that with high probability, the random quasiconformal mapping w^μ constructed using the square grid of mesh size δ_k is close to an affine mapping on any fixed compact subset of the plane. To show that all sufficiently small scales are good, we use the following principle: if an orientation-preserving homeomorphism is conformal off a random set of small measure, then it is close to a conformal map.

1.2 Random Delauney triangulations

A *circle packing* $\mathcal{P} = \{C_i\}$ is a collection of circles in the plane with disjoint interiors. The *tangency pattern* of $\mathcal{P}$ is an embedded graph in the plane whose vertices are centers of circles in $\mathcal{P}$ and edges are line segments which connect centers of tangent circles. The Koebe-Andreev-Thurston Circle Packing Theorem [7, 13] says that any finite triangulation $\mathcal{T}$ of a topological disk admits a *maximal circle packing* $\mathcal{P} = \bigcup C_i \subset \mathbb{D}$ whose boundary circles are horocycles. Furthermore, this maximal packing is unique up to Möbius transformations.

For a discrete set of points V in the plane, the *Voronoi tessellation* is a decomposition of the complex plane $\mathbb{C} = \bigcup F_x$ where F_x consists of all points $z \in \mathbb{C}$ for which $\min_{y \in V} |y-z| = |x-z|$. Each F_x is a polygon, although it could be unbounded. If the points in V are in general position, that is, if no three points lie on a line and no four points lie on a circle, then one can define the *Delauney triangulation* as the dual graph to the Voronoi tessellation. Namely, its vertex set is V, and there is an edge between x and y if the intersection $F_x \cap F_y$ is a line segment. Since the union of all triangles in a Delauney triangulation is the convex hull of V, it is a topological disk.

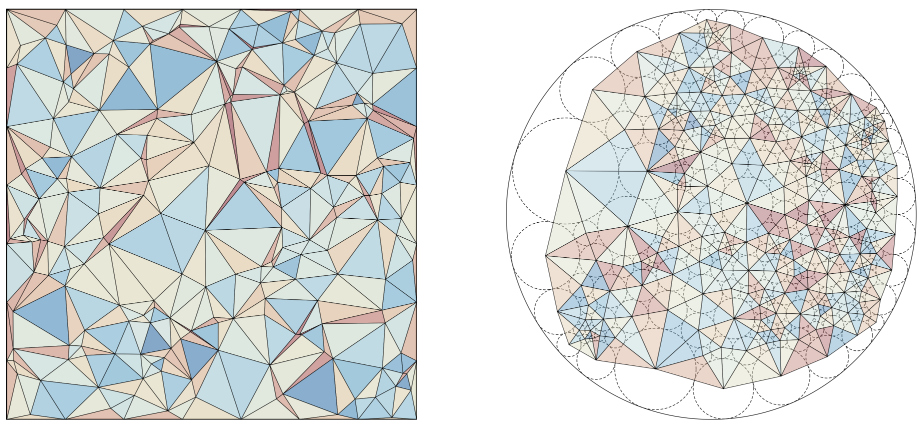

Figure 1: A random Delauney triangulation and its circle packing.

Let $\Omega \subset \mathbb{C}$ be a simply-connected domain bounded by C^1 curve. Randomly choose $N \geq 1$ points in Ω with respect to Lebesgue measure. Due to technical reasons, we also need to throw in $\asymp \sqrt{N}$ equally-spaced points on $\partial\Omega$. The *random Delauney triangulation* is defined as the union of the Delauney triangles contained in Ω. Based on numerical experiments, Kenneth Stephenson suggested that when $N \geq 1$ is large, then with high probability, the maximal circle packing of a random Delauney triangulation approximates a conformal map $\varphi : \Omega \to \mathbb{D}$. In this paper, we prove this conjecture.

To be precise, fix two points $z_1, z_2 \in \Omega$. For each $i = 1, 2$, let $v_i \in \mathcal{T}$ be the closest point to z_i (in case of a tie, choose v_i arbitrarily). Let $\mathcal{P}$ be the maximal circle packing of $\mathcal{T}$ normalized so that C_{v_1} is centered at the origin while the center of C_{v_2} lies on $(0, 1)$. The *circle packing map* $\varphi_{\mathcal{P}} : \operatorname{carr} \mathcal{T} \to \operatorname{carr} \mathcal{P}$ is the piecewise linear map that takes points of $\mathcal{T}$ to centers of circles and is linear on triangles. Here, the *carrier* of a triangulation is simply the union of all the triangles in the triangulation.

Theorem 1.3. *Let Ω be a bounded simply-connected domain in the plane with C^1 boundary and $\varphi : \Omega \to \mathbb{D}$ be the conformal map with $\varphi(z_1) = 0$ and $0 < \varphi(z_2) < 1$. Consider the random Delauney triangulation with N points. For any compact set $K \subset \Omega$ and $\varepsilon > 0$, when $N \geq N_0(K, \varepsilon)$ is sufficiently large, $\|\varphi_{\mathcal{P}} - \varphi\|_{C(K)} < \varepsilon$ holds with probability at least $1 - \varepsilon$.*

The above theorem remains true if the random set of points is generated using a Poisson point process of high intensity.

The proof of Theorem 1.3 is similar to that of Theorem 1.1 in that most of the work goes into showing that with high probability, the circle packing map $\varphi_{\mathcal{P}}$ is roughly quasiconformal. One first shows that with high probability, the discrete modulus of every rectangle $\mathbf{R} \subset \Omega$ whose sides have length $\geq \varepsilon$ is bounded from above and below. The discrete modulus is simple to analyze since it only depends on the combinatorics of the Delauney triangulation in $\mathbf{R}$. In general, the discrete and continuous moduli of $\varphi_{\mathcal{P}}(\mathbf{R})$ are unrelated, however, if the triangulation in question has bounded valence, the two notions of modulus agree up to a multiplicative constant. While a random Delauney triangulation may have vertices of arbitrarily large valence, they are quite rare and can be "avoided" using a percolation argument.

At this point, we are presented with a second difficulty. In the setting of random quasiconformal mappings, the modulus of $w^\mu(\mathbf{R})$ only depends on the Beltrami coefficient on R, however, in a circle packing, the modulus of $\varphi_{\mathcal{P}}(\mathbf{R})$ also depends on the behaviour of the triangulation outside of $\mathbf{R}$. However, if all circles in the packing have small radii, then by a fundamental result of He and Schramm [5], $\operatorname{Mod} \varphi_{\mathcal{P}}(\mathbf{R})$ is determined by the combinatorics of the triangulation in $\mathbf{R}$ up to small error which tends to 0 as the radii of the circles in the packing shrink.

Remark. Let Σ be a compact Riemann surface of genus $g \geq 2$. Consider the random Delauney triangulation on Σ with respect to the hyperbolic metric. According to [11, Proposition 9.1], the maximal circle packing will live on a Riemann surface $\Sigma_{\mathcal{P}}$ homeomorphic to Σ, however, the complex structure may be different. Using the methods of this paper, one can show that when the number of points N is large, then with high probability, the Riemann surface $\Sigma_{\mathcal{P}}$ is close to Σ in the Teichmüller space $\mathcal{T}_g$ of Riemann surfaces of genus g, and furthermore, the mapping $\varphi_{\mathcal{P}}$ is uniformly close to the Teichmüller map from $\Sigma \to \Sigma_{\mathcal{P}}$.

Remark. One may alternatively uniformize a random Delauney triangulation $\mathcal{T}$ by thinking of each triangle in $\mathcal{T}$ as an equilateral one. More precisely, one builds a Riemann surface $\mathcal{S}$ out of equilateral triangles which has the same adjacency relations as $\mathcal{T}$. Our arguments show that when the number of Delauney points is large, the piecewise conformal map from $\mathcal{S}$ to $\mathcal{T}$ is close to a conformal map. We leave the details to the interested reader.

1.3 Random walk in random environments

For comparison, we mention some results about random walk in random environments. Let λ be a probability measure on $(0, \infty)$. For each edge in the square grid $\mathbb{Z}^2$, randomly choose its conductance according to λ. Let S_n be the random walk in $\mathbb{Z}^2$ which starts at the origin, and at each step, the walker moves from a vertex x to an adjacent vertex $y \sim x$ with probability

$$\frac{c(x,y)}{\sum_{z \sim x} c(x,z)}.$$

In 2004, Sidoravicius and Sznitman [10] showed that if the conductances are uniformly bounded away from zero and infinity, then $S_n/\sqrt{n}$ converges to Brownian motion, as in the unweighted case. Several years later, P. Mathieu [8] and M. Biskup and T. Prescott [4] independently showed the convergence of simple random walk to Brownian motion when the conductances are allowed to get arbitrarily close to zero. For a survey on the random conductance model, see [3].

The model of random quasiconformal maps can be interpreted as a continuous analogue of simple random walk in random media where one simulates Brownian motion in a random environment: in each cell of the square grid, Brownian motion is to be stretched in some direction depending on the dilatation. Essentially, this process simulates the image of Brownian motion under the quasiconformal map. This has been studied by Osada [9] under the name of *homogenization of diffusing processes*, although he only discussed the case of bounded distortion.

2 Moduli of curve families

By a *conformal rectangle* $\mathbf{R}$, we mean a Jordan domain in the plane with four marked boundary points. In this paper, all conformal rectangles will be *marked*, i.e. equipped with a distinguished pair of opposite sides. The Schwarz-Cristoffel formula provides a conformal map from $\mathbf{R}$ onto a geometric rectangle $[0, m] \times [0, 1]$. If one insists that the marked sides of $\mathbf{R}$ are mapped onto the vertical sides of $[0, m] \times [0, 1]$, then the number $m \in (0, \infty)$ is determined uniquely. The number $m := \operatorname{Mod} \mathbf{R}$ is known as the *modulus* of $\mathbf{R}$.

Given a geometric rectangle $\mathbf{R}$, we denote the length of its marked sides by $\ell_1(\mathbf{R})$ and the length of the unmarked sides by $\ell_2(\mathbf{R})$. Then, $\operatorname{Mod} \mathbf{R} = \ell_2(\mathbf{R})/\ell_1(\mathbf{R})$. We denote the side length of a square $\mathbf{S}$ by $\ell(\mathbf{S})$. Any square has modulus 1.

Similarly, any doubly-connected domain $\mathbf{A} \subset \mathbb{C}$ can be mapped onto a round annulus $\{z : r < |z| < R\}$. The modulus of the doubly-connected domain $\mathbf{A}$ is defined as $\operatorname{Mod} \mathbf{A} := \frac{1}{2\pi} \log \frac{R}{r}$. It is well known that two doubly-connected domains are conformally equivalent if and only if their moduli coincide.

To estimate moduli of conformal rectangles and doubly-connected domains, one uses moduli of curve families. We will work with two notions of moduli of curves: a discrete one and a continuous one.

In the continuous setting, a *metric* $\rho(z)$ is a non-negative measurable function defined on a domain $\Omega \subset \mathbb{C}$. One can use $\rho(z)$ to measure lengths of rectifiable curves:

$$\ell_\rho(\gamma) = \int_\gamma \rho(z)|dz|.$$

The total area of ρ is defined as

$$A(\rho) = \int_\Omega \rho(z)^2 |dz|^2.$$

As usual, $|dz|$ denotes 1-dimensional Lebesgue measure while $|dz|^2$ denotes 2-dimensional Lebesgue measure.

A metric is said to be *admissible* for a family of rectifiable curves Γ contained in Ω if the length of every curve $\gamma \in \Gamma$ is at least 1. The *modulus* of the curve family Γ is defined as

$$\operatorname{Mod}\Gamma := \inf_{\rho} A(\rho),$$

where the infimum is taken over all admissible metrics ρ. If one finds a conformal metric ρ such that $\ell_\rho(\gamma) \geq L$ for any $\gamma \in \Gamma$, then $\operatorname{Mod}\Gamma \leq A(\rho)/L^2$.

The notions of modulus of a conformal rectangle and the modulus of a doubly-connected domain are special cases of the above construction: for a conformal rectangle $\mathbf{R}$, let $\Gamma_\leftrightarrow$ be the family of curves connecting the distinguished pair of opposite sides of $\mathbf{R}$ and $\Gamma_\updownarrow$ denote the conjugate family which connects the other pair of opposite sides. Then, $\operatorname{Mod}\mathbf{R} = \operatorname{Mod}\Gamma_\updownarrow$ and

$$\operatorname{Mod}\Gamma_\leftrightarrow \cdot \operatorname{Mod}\Gamma_\updownarrow = 1. \tag{2.1}$$

The modulus of a doubly-connected domain $\mathbf{A}$ is equal to the modulus of the family of curves that separate the two boundary components.

For compact sets $E, F \subset \mathbb{C}$, the *Hausdorff distance* $d(E,F)$ is defined as the minimal number $t \geq 0$ such that any point of E is within t of some point of F and vice versa. To define the Hausdorff distance between two conformal rectangles, one also needs to make sure that the marked sides line up.

It is easy to see that modulus of a conformal rectangle varies continuously in the Hausdorff topology. The following lemma says that the modulus of the image of a conformal rectangle under a quasiconformal map does not change much under small perturbations:

Lemma 2.1. *Suppose $f : \mathbb{C} \to \mathbb{C}$ is a K-quasiconformal mapping and $\mathbf{S}, \mathbf{S}'$ are two squares in the plane. For any $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon, K) > 0$, so that if the relative Hausdorff distance $d(\mathbf{S}, \mathbf{S}')/\ell(\mathbf{S}) < \delta$, then $|\operatorname{Mod} f(\mathbf{S}') - \operatorname{Mod} f(\mathbf{S})| < \varepsilon$.*

In the discrete setting, a *metric* $\rho(v)$ is a function on the vertices of a planar graph G. The area of ρ is defined as

$$A(\rho) = \sum_{v \in G} \rho(v)^2.$$

A path $\gamma = \langle x_0, x_1, x_2, \ldots, x_n \rangle$ is a collection of vertices such that $x_i \sim x_{i+1}$ are connected by an edge. One can use ρ to measure lengths of paths:

$$\ell_\rho(\gamma) = \sum_{v \in \gamma} \rho(v).$$

The notions of admissibility and discrete modulus are defined as in the continuous case. By a *combinatorial rectangle* $\mathbf{R}$, we mean a topological rectangle enclosed by a finite collection of edges of G, and four vertices of G have been marked on $\partial \mathbf{R}$.

3 Roughly quasiconformal maps

According to the geometric definition of quasiconformality, an orientation-preserving homeomorphism $f : U \to \mathbb{C}$ is quasiconformal if it distorts moduli of rectangles in U by a bounded amount. It is well known that one can test quasiconformality by looking at round annuli of modulus $\frac{1}{2\pi} \log 2$ or at rectangles of modulus 10. For convenience of the reader, we recall the proofs.

Lemma 3.1. *Suppose an orientation-preserving homeomorphism $f : U \to \mathbb{C}$ distorts moduli of all annuli $\mathbf{A} = A(z, r, 2r) \subset U$ by a bounded amount:*

$$(1/K) \cdot \operatorname{Mod} \mathbf{A} \;\leq\; \operatorname{Mod} f(\mathbf{A}) \;\leq\; K \cdot \operatorname{Mod} \mathbf{A}\,. \tag{3.1}$$

Then, f is K' quasiconformal, where K' depends on K.

In the proof below, we will use the following standard estimate: suppose F is a compact connected set (e.g. an interval) contained in a simply-connected domain Ω. If $\operatorname{Mod}(\Omega \setminus F) \geq m$ is bounded from below, then

$$\operatorname{dist}(\partial\Omega, F) \geq c \operatorname{diam} F,$$

for some $c > 0$ which depends only on $m > 0$.

Proof. We will show that f satisfies the following quasisymmetry condition: there exists a constant $C = C(K) > 0$ such that

$$\frac{\sup_{|z-x|=2r} |f(z) - f(x)|}{\inf_{|z-x|=2r} |f(z) - f(x)|} \leq C, \qquad \text{whenever } B(x, 4r) \subset U. \tag{3.2}$$

Once we show (3.2), the lemma follows from the equivalence of quasiconformality and quasisymmetry, e.g. see [1, Theorerm 3.4.1].

Suppose $|y-x| = r$ and $|z-x| = 2r$. Consider the annulus $\mathbf{A} = A(x, r, 2r)$. Since $f(\mathbf{A})$ separates $f(x), f(y)$ from $f(z)$ and $\operatorname{Mod} f(\mathbf{A})$ is bounded from below, we have

$$|f(z) - f(y)| \geq c\,|f(y) - f(x)|,$$
$$|f(z) - f(x)| \geq c\,|f(y) - f(x)|,$$

for some constant $c > 0$ which depends only on K. It follows that

$$\inf_{|z-x|=2r} |f(z) - f(x)| \geq c \sup_{|y-x|=r} |f(y) - f(x)|. \tag{3.3}$$

For the reverse inequality, note that if y is the midpoint of x and z then we also have

$$|f(x) - f(y)| \geq c\,|f(y) - f(z)|,$$

in which case,

$$\sup_{|z-x|=2r} |f(z) - f(x)| \leq \sup_{|z-x|=2r} \big\{|f(z) - f(y)| + |f(y) - f(x)|\big\} \tag{3.4}$$
$$\leq (1 + c^{-1}) \sup_{|y-x|=r} |f(y) - f(x)|. \tag{3.5}$$

Putting (3.3) and (3.5) together completes the proof. □

Corollary 3.2. *Under the assumptions of Lemma 3.1, there exist constants* $C_1, C_2 > 0$ *and* $0 < \alpha < \beta$*, depending only on* K*, such that*

$$C_1 \left|\frac{z-x}{y-x}\right|^{\alpha} \leq \left|\frac{f(z) - f(x)}{f(y) - f(x)}\right| \leq C_2 \left|\frac{z-x}{y-x}\right|^{\beta}, \tag{3.6}$$

whenever $|y - x| < |z - x|$ *and* $B(x, 2|z - x|) \subset U$.

Sketch of proof. Let $r = |y - x|, R = |z - x|$ and $m = \lfloor \log(R/r) \rfloor$. The upper bound in (3.6) follows after applying (3.5) $m+1$ times, while the lower bound follows from

$$\operatorname{Mod} f\big(A(x, r, R)\big) \geq \sum_{j=0}^{m-1} \operatorname{Mod} f\big(A(x, 2^j r, 2^{j+1} r)\big) \geq \frac{m}{K} \cdot \frac{1}{2\pi} \log 2$$

and the fact that $f\big(B(x, 2r)\big)$ is essentially round, cf. (3.2). □

Lemma 3.3. *Suppose an orientation-preserving homeomorphism $f : U \to \mathbb{C}$ distorts moduli of all rectangles with aspect ratio* 10 *contained in U by a bounded amount:*

$$(1/K) \cdot \operatorname{Mod} \mathbf{R} \le \operatorname{Mod} f(\mathbf{R}) \le K \cdot \operatorname{Mod} \mathbf{R}. \tag{3.7}$$

Then, f is K' quasiconformal, where K' depends on K.

Proof. We follow the argument from Hinkkanen's paper [6]. Let $\mathbf{A}$ denote the standard annulus $\{z : 1 < |z| < 2\}$ of modulus $\frac{1}{2\pi} \log 2$. Consider the following collection of 8 rectangles of modulus 10:

$$\mathbf{P}_0 = e^{2\pi i/8}\big([1, 1.3] \times [-1.5, 1, 5]\big), \qquad \mathbf{P}_j = e^{2\pi i(j/4)} \cdot \mathbf{P}_0,$$
$$\mathbf{Q}_0 = [0.5, 2.5] \times [-0.1, 0.1], \qquad \mathbf{Q}_j = e^{2\pi i(j/4)} \cdot \mathbf{Q}_0,$$

with $j = 0, 1, 2, 3$. A brief inspection of Figure 9.4 shows that if γ is a curve that connects the boundary components of $\mathbf{A}$, then γ contains a crossing which joins a pair of opposite sides of some $\mathbf{P}_j$ or $\mathbf{Q}_j$.

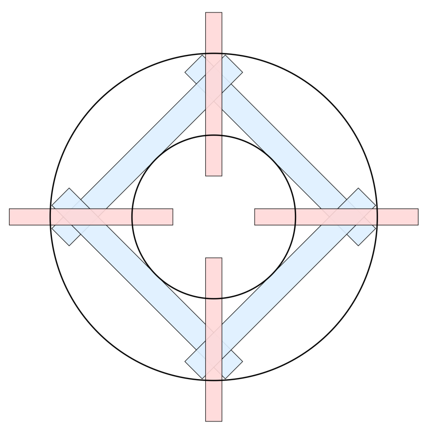

Figure 2: The rectangles $\mathbf{P}_j$ and $\mathbf{Q}_j$.

An arbitrary round annulus of modulus $\frac{1}{2\pi} \log 2$ in U can be expressed as the image of $\mathbf{A}$ under a complex-linear map $L(z) = az + b$. By assumption, for each $j = 0, 1, 2, 3$, we can find a metric $\rho^*_{L(\mathbf{P}_j)}$ of area $\le 10K$ which is admissible for $f\big(\Gamma_{\updownarrow}(L(\mathbf{P}_j))\big)$ and a metric $\rho^*_{L(\mathbf{Q}_j)}$ of area $\leq 10K$ which is admissible for $f\big(\Gamma_{\updownarrow}(L(\mathbf{Q}_j))\big)$. Since the metric

$$\rho^*_{L(\mathbf{A})} = \sum_{j=0}^{3} \rho^*_{L(\mathbf{P}_j)} + \sum_{j=0}^{3} \rho^*_{L(\mathbf{Q}_j)}$$

has area at most $8^2 \cdot 10\,K = 640\,K$ and is admissible for the family of curves that connect the boundary components of $f(L(\mathbf{A}))$, $\operatorname{Mod} f(L(\mathbf{A})) \geq \frac{1}{640\,K}$ is bounded from below by a definite constant (depending on K).

Finding an upper bound for $\operatorname{Mod} f(L(\mathbf{A}))$ amounts to constructing a metric which is admissible for the family of curves which separate the boundary components of $f(L(\mathbf{A}))$. Since this only requires one rectangle, e.g. $L(\mathbf{Q}_0)$, $\operatorname{Mod} f(L(\mathbf{A})) \leq 10\,K$. □

Remark. Strangely enough, it is not known whether an orientation-preserving homeomorphism $f : U \to V$ which distorts moduli of squares by a bounded amount must be quasiconformal. However, if f is known to be differentiable almost everywhere, then by examining the behaviour of f near points of differentiability, it is easy to see that f is K quasiconformal if and only if it distorts moduli of squares in U by at most K.

Let $\mathcal{R}(\varepsilon)$ denote the set of rectangles in the plane whose sides have length at least ε. For a bounded domain $U \subset \mathbb{C}$, let $\mathcal{R}_U(\varepsilon)$ denote the collection of rectangles in $\mathcal{R}(\varepsilon)$ that are compactly contained in U. We say that an orientation-preserving homeomorphism $f : U \to V$ is (K, ε) *roughly quasiconformal* if

$$(1/K) \cdot \operatorname{Mod} \mathbf{R} \;\leq\; \operatorname{Mod} f(\mathbf{R}) \;\leq\; K \cdot \operatorname{Mod} \mathbf{R}, \qquad \mathbf{R} \in \mathcal{R}_U(\varepsilon),$$

for some $K \geq 1$. We have the following compactness criterion for families of roughly quasiconformal maps:

Lemma 3.4. *Let $U \subset \mathbb{C}$ be a domain in the complex plane containing $0, 1$. Suppose $f_n : U \to \mathbb{C}$ is a sequence of (K, ε_n) roughly quasiconformal maps with $\varepsilon_n \to 0$ as $n \to \infty$. Assume that $f_n(0) = 0$, $f_n(1) = 1$ for each n. Then the sequence $\{f_n\}$ is uniformly equicontinuous on compact subsets of U and any subsequential limit is an $L(K)$ quasiconformal homeomorphism.*

Proof. The proof of Lemma 3.3 shows that f_n distorts the moduli of annuli $A(z, r, 2r) \subset U$ by a bounded amount, provided that $r \geq 5\varepsilon_n$ and $B(z, 4r) \subset U$. The proof of Corollary 3.2 shows that f_n satisfies (3.6) whenever

$$5\varepsilon_n \leq |y - x| \leq |z - x| \leq \frac{\text{dist}(x, \partial U)}{2}.$$

In view of the normalization $f_n(0) = 0$, $f_n(1) = 1$, we see that if two points $x, y \in U$ are close to each other, then their images under f_n are close to each other, and conversely, if two points are far apart, their images under f_n remain far apart. In other words, any subsequential limit f is a homeomorphism. Since $\text{Mod}\, f_n\big(A(z, r, 2r)\big) \to \text{Mod}\, f\big(A(z, r, 2r)\big)$, f distorts the moduli of annuli $A(z, r, 2r) \subset U$ by a bounded amount, and is therefore quasiconformal by Lemma 3.1. □

For future reference, we define the following collections of squares:

- Let $\mathcal{S}(\varepsilon)$ denote the set of squares in the plane with side length between ε and 1. For a bounded domain $U \subset \mathbb{C}$, let $\mathcal{S}_U(\varepsilon)$ denote the collection of squares in $\mathcal{S}(\varepsilon)$ that are compactly contained in U.
- For $0 < \varepsilon \leq 1$, let $\mathcal{S}'(\varepsilon)$ denote the set of squares in the plane that belong to one of the grids $e^{2\pi i(k/n)} \cdot j\varepsilon\mathbb{Z}^2$, $1 \leq j, k \leq n$ where $n = \lceil 1/\varepsilon \rceil$. If U is a bounded domain, the collection $\mathcal{S}'_U(\varepsilon)$ of squares compactly contained in U is finite. By construction, $\bigcup_{\varepsilon > 0} \mathcal{S}'_U(\varepsilon)$ is dense in the set of all squares contained in U of diameter at most 1.

4 A lemma on percolation

In this section, we present a lemma on percolation which will allow us to control the moduli of images of rectangles under an orientation-preserving homeomorphism that is quasiconformal off a random set of small measure.

Consider the infinite square grid $\mathbb{Z}^2$. Fix the percolation parameter $0 < r < 1$. In the discrete setting, we colour vertices of $\mathbb{Z}^2$ in two colours: we colour a vertex *yellow* with probability r and *blue* with probability $1 - r$. For two points $x, y \in \mathbb{Z}^2$, we define their *combinatorial distance* $d_{\mathbb{Z}^2}(x, y)$ as the minimal length of a path

$$x_0, x_1, x_2, \ldots, x_n, \qquad x_0 = x, \quad x_n = y,$$

where $x_i \sim x_{i+1}$ are adjacent vertices. By the Pythagorean theorem, the combinatorial distance is within a factor of $\sqrt{2}$ of the Euclidean distance. We are more interested in the *chemical distance* $d_{\text{chem}}(x, y)$ which minimizes the number of blue vertices along paths that connect x to y. The following lemma says that if the points x, y are at macroscopic distance from one another, then the chemical distance is also equivalent to the Euclidean distance:

Lemma 4.1. *There exists a universal constant $0 < r_0 < 1/2$ so that if the percolation parameter $0 < r < r_0$, then with probability $\geq 1 - 1/N^2$, for two points $x, y \in [-N, N] \times [-N, N]$ with $|x - y| \geq \log N$,*

$$\frac{9}{10} \cdot d_{\mathbb{Z}^2}(x, y) \;\leq\; d_{\text{chem}}(x, y) \;\leq\; d_{\mathbb{Z}^2}(x, y).$$

Proof. The number of non-self intersecting paths in $[-N, N] \times [-N, N]$ of length L is bounded above by $(2N+1)^2 \cdot 4^L$ since there are $(2N+1)^2$ choices for the initial vertex and at most four choices for each following vertex. Since the probability that a fixed path of length L contains at least $L/10$ yellow vertices is at most

$$\sum_{j=\lceil L/10 \rceil}^{L} \binom{L}{j} r^j (1-r)^{L-j} \;\leq\; 2^L \sum_{j=\lceil L/10 \rceil}^{L} r^j \;\leq\; 2^{L+1} \cdot r^{L/10},$$

the probability that some path of length $L \geq \log N$ contains at least $L/10$ yellow vertices is bounded above by

$$\sum_{L \geq \log N} (2N+1)^2 4^L \cdot 2^{L+1} r^{L/10} \;\leq\; 2(2N+1)^2 \sum_{L \geq \log N} (8r^{1/10})^L.$$

A simple computation shows that the last quantity is at most $1/N^2$ provided r is small enough: if $\alpha := 8r^{1/10} < 1/2$, then

$$\sum_{L \geq \log N} (8r^{1/10})^L \;=\; \sum_{L \geq \log N} \alpha^L \;\leq\; 2\alpha^{\log N} \;=\; 2N^{\log \alpha},$$

so the lemma holds if $\alpha < e^{-4}$. □

In the continuous setting, one independently colours each cell of $\mathbb{Z}^2$ either blue or yellow: *yellow* with probability r and *blue* with probability $1-r$. The Euclidean distance minimizes the length of a rectifiable curve that connects two given points in the plane. The continuous analogue of the chemical distance is defined by instead minimizing the part of the length that is contained in the blue squares.

Lemma 4.2. *There exists a universal constant $0 < r_0 < 1/2$ so that if the percolation parameter $0 < r < r_0$ is sufficiently small, then with probability $\geq 1 - 1/N^2$, for two points $x, y \in [-N, N] \times [-N, N]$ with $|x-y| \geq \sqrt{2}\log N$, their continuous chemical distance is comparable to the Euclidean distance:*

$$\frac{1}{2} \cdot |x-y| \;\leq\; |x-y|_{\text{chem}} \;\leq\; |x-y|.$$

Proof. When computing the continuous chemical distance, it suffices to take the infimum over paths which enter yellow squares at most once and the path within any yellow square is a line segment. We deduce the continuous case from the discrete case. If a continuous path γ has length $L \geq \sqrt{2}\log N$, then the length of its discrete itinerary (with loops removed) is at least $\log N$. With high probability, at most $L/10$ of these squares will be yellow, and therefore, the total length of γ in the yellow squares is at most $(L/10)\sqrt{2} < L/2$. In this case, the total length of γ in the blue squares is at least $L/2$. $\square$

Modifications. (i) By making $r_0 > 0$ sufficiently small, one can replace $9/10$ and $1/2$ in the lemmas above with arbitrary constants less than 1, while the factor $\sqrt{2}$ in $\sqrt{2}\log N$ can be removed.

(ii) The proof of Lemma 4.1 applies to any graph with $\lesssim N^2$ vertices of bounded valence.

(iii) Fix an odd integer $m \geq 1$. As before, colour a cell *yellow* with probability r and *blue* with probability $1-r$. We call a cell $\square$ *deep blue* if every cell $\hat{\square} \subset m\square$ is blue, where $m\square$ denotes the square with the same center as $\square$ and side length $m \cdot \ell(\square)$. We claim that there exists a universal constant $r_0(m)$ such that if the percolation parameter $r < r_0(m)$, then the total length that a continuous path spends in the deep blue squares is at least $|x-y|/2$, provided that the distance between the endpoints $|x-y| \geq \sqrt{2}\log N$.

To prove the discrete version of the claim, note that if there is a path $x = x_0, x_1, x_2, \dots, x_n = y$ which contains more than $L/10$ vertices that are not deep blue, then there exists a chain $x = x'_0, x'_1, x'_2, \dots, x'_n = y$ with $d_{\mathbb{Z}^2}(x'_j, x'_{j+1}) = O(m)$, $0 \le j \le n-1$ which contains more than $L/10$ yellow vertices. More precisely, the chain $x'_0, x'_1, x'_2, \dots, x'_n$ is formed by replacing each non-deep blue vertex x_j by a yellow vertex x'_j with $d_{\mathbb{Z}^2}(x_j, x'_j) = O(m)$. The discrete case now follows from modification (ii). As before, the continuous case may be obtained from the discrete case.

(iv) We can slightly weaken the independence assumption when deciding to colour a cell blue or yellow: it is enough to require that the colours of any finite collection of cells $\square_1, \square_2, \dots, \square_n$ with $d_{\mathbb{Z}^2}(\square_i, \square_j) \ge m$, $i \ne j$ are chosen independently. To see the discrete version of the statement, observe that if a path of length L contains $\gtrsim L$ yellow vertices, then it contains an m-separated set of $\gtrsim L/m^2$ yellow vertices. Again, the continuous case follows from the discrete case.

5 Approximate conformality

In this section, we run percolation with parameter r on $\mathbf{S}_1 = [-1,1] \times [-1,1]$ with mesh size $\delta = 1/N$. The following lemma says that if an orientation-preserving homeomorphism is conformal outside a random set of small measure, then it is close to a conformal map:

Lemma 5.1. *For any $\varepsilon > 0$, there exist r_0, N_0 so that if $r < r_0$ and $N > N_0$, then with probability at least $1-\varepsilon$, any orientation-preserving homeomorphism $f : \mathbf{S}_1 \to \mathbb{C}$ that is conformal on the blue squares satisfies*

$$1 - \varepsilon \,<\, \operatorname{Mod} f(\mathbf{S}) \,<\, 1 + \varepsilon, \qquad \forall\, \mathbf{S} \in \mathcal{S}_{\mathbf{S}_1}(\varepsilon).$$

Proof. Recall that $\mathcal{S}_{\mathbf{S}_1}(\varepsilon)$ denotes the collection of squares compactly contained in $\mathbf{S}_1$ with $\ell(\mathbf{S}) \ge \varepsilon$. To a square $\mathbf{S} \in \mathcal{S}_{\mathbf{S}_1}(\varepsilon)$, we associate the metric $\rho_{\mathbf{S}} = \chi_{\mathscr{B} \cap \mathbf{S}}$ where $\mathscr{B}$ is the union of the blue squares. To estimate $\operatorname{Mod} f(\mathbf{S})$, we use the conformal metric

$$\rho^*_{\mathbf{S}}(w) = \frac{1}{|(f^{-1})'(w)|} \cdot \chi_{f(\mathscr{B} \cap \mathbf{S})}(w), \qquad w \in f(\mathbf{S}).$$

By construction, $A(\rho^*_{\mathbf{S}}) = A(\rho_{\mathbf{S}}) \le \ell(\mathbf{S})^2$. By modification (i) of Lemma 4.2, for any $\eta > 0$, with probability at least $1-\varepsilon$, the inequality $\ell_{\rho^*_{\mathbf{S}}}(f(\gamma)) = \ell_{\rho_{\mathbf{S}}}(\gamma) \ge (1-\eta)\ell(\mathbf{S})$ holds for any curve $\gamma \in \Gamma_{\updownarrow}(\mathbf{S})$ and any square $\mathbf{S} \in \mathcal{S}_{\mathbf{S}_1}(\varepsilon)$, as long as $N > N_0$ is sufficiently large and $r < r_0$ is sufficiently small. Since the metric $\frac{1}{(1-\eta)\ell(\mathbf{S})} \cdot \rho^*_{\mathbf{S}}$ is admissible for $\Gamma_{\updownarrow}(f(\mathbf{S}))$,

$$\operatorname{Mod} f(\mathbf{S}) \le \frac{1}{(1-\eta)^2}, \qquad \forall\, \mathbf{S} \in \mathcal{S}_{\mathbf{S}_1}(\varepsilon).$$

By working with the horizontal path family $\Gamma_{\leftrightarrow}(\mathbf{S})$, we get the complementary estimate

$$\operatorname{Mod} f(\mathbf{S}) \ge (1-\eta)^2, \qquad \forall\, \mathbf{S} \in \mathcal{S}_{\mathbf{S}_1}(\varepsilon).$$

It remains to choose $\eta = \varepsilon/3$. □

We say that an orientation-preserving homeomorphism $f : \mathbf{S}_1 \to \mathbb{C}$ is ε-*close to linear* on $\square$ if

$$\|f - L_\square\|_{C(3\square)} \le \varepsilon \cdot a_\square \ell(\square) \tag{5.1}$$

for some complex-linear map $L_\square(z) = a_\square z + b_\square$.

In practice, f will be close to conformal on $\eta^{-1}\square$, where $\eta > 0$ is small, from which the condition (5.1) follows from Koebe's distortion theorem.

Lemma 5.2. *For any $\varepsilon > 0$, there exist r_0, N_0 so that if $r < r_0$ and $N > N_0$, then with probability at least $1-\varepsilon$, any orientation-preserving homeomorphism $f : \mathbf{S}_1 \to \mathbb{C}$ that is ε-close to linear on the blue squares satisfies*

$$1 - \varepsilon \,<\, \operatorname{Mod} f(\mathbf{S}) \,<\, 1 + \varepsilon, \qquad \forall\, \mathbf{S} \in \mathcal{S}_{\mathbf{S}_1}(\varepsilon).$$

Proof. Let $\mathscr{B}$ denote the union of the blue cells. By assumption, f takes a blue cell $\square$ approximately to $\square^* := a_\square \cdot \square + b_\square$:

$$(1-2\varepsilon)\square^* \subset f(\square) \subset (1+2\varepsilon)\square^*.$$

Given a square $\mathbf{S} \in \mathcal{S}_{\mathbf{S}_1}(\varepsilon)$, define $\rho_{\mathbf{S}} = \chi_{\mathscr{B}\cap\mathbf{S}}$ and

$$\rho^*_{\mathbf{S}} \,=\, \chi_{f(\mathbf{S})} \sum_{\square \subset \mathscr{B}} \rho^*_\square \,=\, \chi_{f(\mathbf{S})} \sum_{\square \subset \mathscr{B}} \frac{1}{a_\square} \cdot \chi_{(1+4\varepsilon)\square^*}.$$

By design, $\rho_{\mathbf{S}}$ is a metric on $\mathbf{S}$ while $\rho^*_{\mathbf{S}}$ is a metric on $f(\mathbf{S})$. Since the set where different $(1+4\varepsilon)\square^*$ overlap has area $O(\varepsilon)$ and only four $(1+4\varepsilon)\square^*$ can overlap at a time,

$$A(\rho^*_{\mathbf{S}}) \le 1 + C_1\varepsilon, \tag{5.2}$$

where C_1 is a universal constant.

By modification (i) of Lemma 4.2, when $N > N_0$ is large and $r < r_0$ is small, with probability $\ge 1-\varepsilon$,

$$\ell_{\rho_{\mathbf{S}}}(\gamma) \ge (1-\varepsilon)\ell(\mathbf{S}), \qquad \gamma \in \Gamma_{\updownarrow}(\mathbf{S}).$$

We claim that

$$\ell_{\rho^*_{\mathbf{S}}}(f(\gamma)) \ge (1-\varepsilon)\ell(\mathbf{S}), \qquad \gamma \in \Gamma_{\updownarrow}(\mathbf{S}). \tag{5.3}$$

We can straighten a curve $\gamma : [a,b] \to \mathbf{S}$ with respect to the grid $\delta\mathbb{Z}^2$ by replacing each connected component of $\gamma \cap \square$ with a line segment. In general, the straightened curve $\overline{\gamma}$ may be composed of infinitely many segments. To prove the claim, it suffices to show that

$$\ell_{\rho^*_{\mathbf{S}}}(f(\gamma)) \ge \ell_{\rho_{\mathbf{S}}}(\overline{\gamma}). \tag{5.4}$$

Examine a connected component γ_j of $\gamma \cap \square$. Since the $\rho^*_\square$ length of $f(\gamma_j)$ is at least $1/a_\square$ times the distance between the endpoints, $\ell_{\rho^*_\square}(f(\gamma_j)) \ge \ell_{\rho_\square}(\overline{\gamma}_j) - 2\varepsilon\,\ell(\square)$. This is close to what we want, however, the errors $2\varepsilon\,\ell(\square)$ can accumulate.

We now slightly modify the definition of $\overline{\gamma}$ to resolve this issue. Given a cell $\square \subset \delta\mathbb{Z}^2$, consider the set

$$\Theta(\square) := \big\{ t \in [a,b] : f(\gamma(t)) \in (1+4\varepsilon)\square^* \big\}.$$

For a connected component $[t_1, t_2] \subset \Theta(\square)$, record the first entry and last exit times τ_1, τ_2 of $f(\gamma(t))$ in $f(\square)$. We refer to $\gamma([\tau_1,\tau_2])$ as an *excursion* of γ in $\square$. Each excursion can slightly protrude outside of $\square$ and there could be several excursions through a single cell.

Let $\overline{\gamma}$ be the curve obtained from γ by straightening all the excursions (in order of parametrization). With the updated definition, $\overline{\gamma}$ is composed of finitely many segments and (5.4) holds since the defect $2\varepsilon\,\ell(\square)$ is covered by $\ell_{\rho^*_\square}\big(f(\gamma([t_1,\tau_1]))\big) + \ell_{\rho^*_\square}\big(f(\gamma([\tau_2,t_2]))\big)$.

The estimates (5.2) and (5.3) show that $\operatorname{Mod} f(\mathbf{S}) \le 1 + C\varepsilon$ where $C > 0$ is a universal constant. Working with the horizontal path family $\Gamma_{\leftrightarrow}(\mathbf{S})$ in place of $\Gamma_{\updownarrow}(\mathbf{S})$ gives the lower bound. $\square$

6 Homogenization of quasiconformal maps

In this section, we prove a variant of Theorem 1.1 where the dilatation is randomized on a bounded open set $\Omega \subset \mathbb{C}$.

Theorem 6.1. *Let $\Omega \subset B(0,R)$ be a bounded open set in the plane whose boundary has zero measure. Consider the square grid in the plane of mesh size $\delta > 0$. For each square of side length δ compactly contained in Ω, select μ according to the measure λ. Outside of the δ-approximation of Ω, set $\mu = \mu_0$, where μ_0 is a fixed Beltrami coefficient on the plane with $\|\mu_0\|_\infty < 1$. For any $\varepsilon > 0$,*

$$\mathbb{P}\left[\sup_{z \in B(0,R)} \left| w^\mu(z) - w^{\mu_\lambda \cdot \chi_{\Omega^\delta} + \mu_0 \cdot \chi_{\mathbb{C}\setminus\Omega^\delta}}(z) \right| < \varepsilon \right] \to 1, \qquad \text{as } \delta \to 0,$$

where μ_λ is a constant that depends only on λ.

6.1 Rough quasiconformality

By the results of Section 4, random quasiconformal mappings are roughly quasiconformal:

Lemma 6.2. *For any $\varepsilon > 0$, when the mesh size $\delta < \delta_0(\varepsilon)$ is sufficiently small, the probability that w^μ is (K, ε) roughly quasiconformal on $B(0, 2R)$ is at least $1 - \varepsilon$.*

Proof. Choose $0 < k_1 < 1$ so that

$$\|\mu_0\|_\infty \le k_1, \qquad \lambda\big(\{z : k_1 < |z| < 1\}\big) < r_0,$$

where r_0 is the constant from Lemma 4.2. Colour a cell $\square$ in $\delta\mathbb{Z}^2$ *blue* if $|\mu(\square)| \le k_1$ and *yellow* otherwise. Let $\mathscr{B}$ denote the union of the blue squares. We will show the lemma holds with $K = 4K_1$ where $K_1 = \frac{1+k_1}{1-k_1}$.

Recall that $\mathcal{R}_{B(0,2R)}(\varepsilon)$ denotes the collection of rectangles compactly contained in $B(0,2R)$ whose sides have length at least ε. To a rectangle $\mathbf{R} \in \mathcal{R}_{B(0,2R)}(\varepsilon)$, associate the metric $\rho_{\mathbf{R}} = \chi_{\mathscr{B}\cap\mathbf{R}}$.

Claim. *When the mesh size $\delta < \delta_0(\varepsilon)$ is sufficiently small, the inequality*

$$\ell_{\rho_{\mathbf{R}}}(\gamma) \geq \ell_1(\mathbf{R})/2 \tag{6.1}$$

holds with probability at least $1-\varepsilon$, for any curve $\gamma \in \Gamma_{\updownarrow}(\mathbf{R})$ and any rectangle $\mathbf{R} \in \mathcal{R}_{B(0,2R)}(\varepsilon)$.

Proof. Let x and y be the endpoints of the curve γ. Since x and y lie on opposite horizontal sides of $\mathbf{R}$, $|x-y| \geq \ell_1(\mathbf{R})$.

Since the $\rho_{\mathbf{R}}$-length of a curve measures the amount of time it spends in the blue squares, $\ell_{\rho_{\mathbf{R}}}(\gamma) \geq |x-y|_{\text{chem}}$. According to Lemma 4.2, we have $(1/2)\cdot|x-y| \leq |x-y|_{\text{chem}}$. Putting these estimates together gives

$$\frac{1}{2}\cdot \ell_1(\mathbf{R}) \leq \frac{1}{2}\cdot|x-y| \;\leq\; |x-y|_{\text{chem}} \leq \ell_{\rho_{\mathbf{R}}}(\gamma),$$

which proves the claim. □

We now finish the proof of Lemma 6.2. To estimate $\operatorname{Mod} w^{\mu}(\mathbf{R})$, we define the metric $\rho^*_{\mathbf{R}}$ on $w^{\mu}(\mathbf{R})$ by setting

$$\rho^*_{\mathbf{R}}(w) = \left[\operatorname{Jac}(w^{\mu})^{-1}(w)\right]^{1/2}\cdot \chi_{w^{\mu}(\mathscr{B}\cap\mathbf{R})}(w), \qquad w \in w^{\mu}(\mathbf{R}).$$

By construction, $A(\rho^*_{\mathbf{R}}) = A(\rho_{\mathbf{R}}) \leq \ell_1(\mathbf{R})\ell_2(\mathbf{R})$.

Since the restriction of w^{μ} is K_1 quasiconformal on each blue square,

$$\ell_{\rho^*_{\mathbf{R}}}(w^{\mu}(\gamma)) \geq (1/\sqrt{K_1})\cdot \ell_{\rho_{\mathbf{R}}}(\gamma), \qquad \gamma \in \Gamma_{\updownarrow}(\mathbf{R}).$$

Combining this with (6.1), we get

$$\ell_{\rho^*_{\mathbf{R}}}(w^{\mu}(\gamma)) \geq \ell_1(\mathbf{R})/(2\sqrt{K_1}), \qquad \gamma \in \Gamma_{\updownarrow}(\mathbf{R}).$$

Since the metric $\frac{2\sqrt{K_1}}{\ell_1(\mathbf{R})}\cdot\rho^*_{\mathbf{R}}$ is admissible for $\Gamma_{\updownarrow}(w^{\mu}(\mathbf{R}))$,

$$\operatorname{Mod} w^{\mu}(\mathbf{R}) \leq \frac{4K_1}{\ell_1(\mathbf{R})^2}\cdot \ell_1(\mathbf{R})\ell_2(\mathbf{R}) = 4K_1 \operatorname{Mod}\mathbf{R}\,.$$

The proof is complete. □

6.2 Searching for the optimal direction

We say that two squares $\mathbf{S}_1, \mathbf{S}_2$ have the same *orientation* if $\mathbf{S}_2 = a\,\mathbf{S}_1 + b$, where $a > 0$ and $b \in \mathbb{C}$. To motivate our proof of Theorem 6.1, note that a complex-linear mapping of the plane preserves moduli of rectangles, while a real-linear mapping A that is not complex-linear has an extremal direction: the modulus of any geometric rectangle oriented in this direction is stretched by $K(A)$, while the modulus of any other rectangle is stretched by a strictly smaller amount. More generally, a quasiconformal mapping φ has constant dilatation $\mu_\varphi = \mu_A$ on Ω if and only if φ stretches moduli of rectangles pointing in the A direction by $K(A)$. In order to show that a random quasiconformal mapping behaves like A on Ω, we need a mechanism for identifying this direction.

For a square $\mathbf{S} \subset \Omega$, look at the probability that $\operatorname{Mod} w^\mu(\mathbf{S}) > K$ and take $\limsup$ as $\delta \to 0$. Let $K^*(\mathbf{S})$ be the infimum of $K > 0$ for which this $\limsup$ is 0. Thus $K^*(\mathbf{S})$ measures the maximal effective distortion of $\mathbf{S}$. Maximizing $K^*(\mathbf{S})$ over all squares $\mathbf{S} \subset \Omega$, we obtain the constant K^* which measures the maximal effective distortion of the model. As the product of the moduli of the "horizontal" and "vertical" families of a conformal rectangle is 1, $K^* \geq 1$. Since with high probability, a random quasiconformal mapping is roughly quasiconformal, K^* is finite.

Lemma 6.3. (i) *$K^*(\mathbf{S})$ depends only on the orientation of $\mathbf{S}$ and not its side length or location within Ω, or on the domain $\Omega \subset \mathbb{C}$.*

(ii) *For a square $\mathbf{S}$, let $\mathbf{S}_\theta$ denotes the square obtained by rotating $\mathbf{S}$ by $e^{i\theta}$ around its center. Let $\mathbf{S}$ be a square for which all $\mathbf{S}_\theta$, $\theta \in [0, 2\pi]$ are in Ω. The function $\theta \to K^*(\mathbf{S}_\theta)$ is continuous.*

(iii) *There exists $\theta^* \in [0, 2\pi]$ such that for any $\varepsilon > 0$, there exists a constant $c(\varepsilon) > 0$ and a sequence of scales $\delta_{\varepsilon,j} \to 0$ for which*

$$\mathbb{P}_{\delta_{\varepsilon,j}}\big(\operatorname{Mod} w^\mu(\mathbf{S}_{\theta^*}) > K^* - \varepsilon\big) > c(\varepsilon). \tag{6.2}$$

Proof. If $\mathbf{S}, \mathbf{S}' \subset \Omega$ are two squares with the same orientation, then the distribution of the random variable $\operatorname{Mod} w^\mu_\delta(\mathbf{S})$ is essentially the same as that of $\operatorname{Mod} w^\mu_{\delta'}(\mathbf{S}')$ with $\delta' = \delta \cdot \ell(\mathbf{S}')/\ell(\mathbf{S})$, i.e.

$$\Delta(t) \;=\; \mathbb{P}\big(\operatorname{Mod} w^\mu_\delta(\mathbf{S}) < t\big) - \mathbb{P}\big(\operatorname{Mod} w^\mu_{\delta'}(\mathbf{S}') < t\big),$$

tends weakly to 0 as $\delta \to 0$. The reason that $\Delta(t)$ could be non-zero comes from the slight discrepancy of how the δ and δ' grids intersect $\mathbf{S}$ and $\mathbf{S}'$, however, by rough quasiconformality and Lemma 2.1, this discrepancy is essentially negligible if the grids are very fine. This proves (i). The same circle of ideas also show (ii) and (iii). □

Let A_λ be the real-linear transformation with dilatation K^* which stretches all squares pointing in the $\mathbf{S}_{\theta^*}$ direction by K^* and fixes the points $0, 1$. We denote the dilatation of A_λ by μ_λ. Our goal is to show that w^μ is close to the normalized quasiconformal map Φ with dilatation μ_λ on Ω and μ_0 on $\mathbb{C} \setminus \Omega$ in $C\big(B(0,R)\big)$. Alternatively, we may show that $f := A_\lambda^{-1} \circ w^\mu$ is close to $A_\lambda^{-1} \circ \Phi$. We do this in a series of incremental improvements.

6.3 Existence of a good sequence of scales

We first promote positive probability to high probability in (6.2):

Lemma 6.4. *There is a sequence of good scales $\delta_j \to 0$ such that*

$$\mathbb{P}_{\delta_j}\big(\operatorname{Mod} w^\mu(\mathbf{S}_{\theta^*}) > K^* - 1/j\big) > 1 - 1/j. \tag{6.3}$$

The proof rests on the following lemma:

Lemma 6.5. *Suppose $\mathbf{S}$ is a square in the plane and $\varphi : \mathbf{S} \to \mathbb{C}$ is a K quasiconformal map. For an integer $n \ge 1$, divide $\mathbf{S} = S_1 \cup S_2 \cup \cdots \cup S_{n^2}$ into n^2 squares of equal size. If $\operatorname{Mod} \varphi(S_i) \le K_0$ for at least $c \cdot n^2$ of these squares, then $\operatorname{Mod} \varphi(\mathbf{S}) \le K_1$ for some constant $K_0 < K_1 < K$ which depends on K, K_0, c but not on n.*

Proof. Let x, y denote the coordinates pointing in the horizontal and vertical directions of $\mathbf{S}$ respectively. Since any path in $\Gamma_\updownarrow(\mathbf{S})$ travels $\ge \ell(\mathbf{S})$ vertically, the metric

$$\rho^*(w) = \frac{1}{\ell(\mathbf{S})} \cdot \left| \frac{\partial \varphi}{\partial y}(\varphi^{-1}(w)) \right|^{-1}, \qquad w \in \varphi(\mathbf{S}),$$

is admissible for $\Gamma_\updownarrow(\varphi(\mathbf{S}))$. Since

$$\left| \frac{\partial \varphi}{\partial y} \right|^2 \ge (1/K) \cdot \operatorname{Jac} \varphi \quad \implies \quad \left| \frac{\partial \varphi}{\partial y} \circ \varphi^{-1} \right|^{-2} \le K \cdot \operatorname{Jac} \varphi^{-1}, \tag{6.4}$$

$\operatorname{Mod}\varphi(\mathbf{S}) \le A(\rho^*) \le K$ and equality holds if and only if φ is the extremal stretch by K in the horizontal direction.

A compactness argument shows that if $\operatorname{Mod}\varphi(S_i) \le K_0$, then (6.4) has a definite defect on S_i, i.e. $A(\rho^* \cdot \chi_{\varphi(S_i)}) \le \frac{(1-\varepsilon)K}{n^2}$. If $\operatorname{Mod}\varphi(S_i) \le K_0$ for a definite proportion of the small squares S_i, then $A(\rho^*)$ is bounded away from K. The proof is complete. □

Proof of Lemma 6.4. If the lemma were false, there would exist constants $K_0 < K^*$ and $c_0 > 0$ such that

$$\mathbb{P}_\delta\big(\operatorname{Mod} w^\mu(\mathbf{S}_{\theta^*}) < K_0\big) \ge c_0, \qquad \text{for any } \delta > 0 \text{ sufficiently small.} \tag{6.5}$$

Assuming this, we will construct a sequence of quasiconformal maps φ_k with the following properties:

(i) φ_k is $(K, 1/k)$ roughly quasiconformal on $B(0, 2R)$ where K is from Lemma 6.2.

(ii) $\operatorname{Mod}\varphi_k(\sigma) \le K^* + 1/k$ for all $\sigma \in \mathcal{S}'_\Omega(1/k)$, where $\mathcal{S}'_\Omega(1/k)$ is a finite collection of squares in Ω of side length $\ge 1/k$ which was defined in Section 3.

(iii) $\operatorname{Mod}\varphi_k(\mathbf{S}_{\theta^*}) > K^* - \varepsilon > K_1 > K_0$, where K_1 is given by Lemma 6.5 with $K = K^*$ and $c = c_0/8$.

(iv) For an integer $n \ge 1$, divide $\mathbf{S}_{\theta^*} = S_1 \cup S_2 \cup \cdots \cup S_{n^2}$ into n^2 squares of equal size, where n is a positive integer that will be chosen below. For at least $(c_0/8)n^2$ of these squares, $\operatorname{Mod}\varphi_k(S_i) \le K_0$.

By (i), the sequence of mappings φ_k is precompact, (ii) implies that any subsequential limit φ is K^* quasiconformal, (iii) tells us that $\operatorname{Mod}\varphi(\mathbf{S}_{\theta^*}) \ge K^* - \varepsilon$, while (iv) ensures that $\operatorname{Mod}\varphi(S_i) < K_0$ for at least $(c_0/8)n^2$ of the small squares S_i. This contradicts Lemma 6.5.

To prove the lemma, it remains to verify that the random quasiconformal mappings $\varphi_k = w^\mu$ satisfy properties (i)-(iv) with positive probability. Clearly, (iii) holds with probability $\ge c(\varepsilon)$ for the special scales $\delta_{\varepsilon,j}$ from (6.2). According to Lemma 6.2, by requesting the mesh size $\delta_{\varepsilon,j(k)}$ to be small, we can ensure that the probability that φ_k is $(K, 1/k)$ quasiconformal on $B(0, 2R)$

exceeds $1 - c(\varepsilon)/4$. Since the number of squares in $\mathcal{S}'_\Omega(1/k)$ is finite, if $\delta_{\varepsilon,j(k)}$ is small, then

$$\mathbb{P}\Big(\operatorname{Mod} w^\mu(\sigma) \le K^* + 1/k,\ \forall \sigma \in \mathcal{S}'_\Omega(1/k)\Big) \ > \ 1 - c(\varepsilon)/4. \tag{6.6}$$

Discard $\sim 3n^2/4$ of the small squares S_i so that the remaining squares are a definite distance apart (and therefore the moduli of their images are independent). By (6.5) and the law of large numbers, we can pick n sufficiently large so that for arbitrarily small $\delta > 0$, with probability at least $1 - c(\varepsilon)/4$, $\operatorname{Mod} \varphi(S_i) \le K_0$ for at least $(c_0/2) \cdot (n^2/4) = (c_0/8)n^2$ squares S_i. □

Lemma 6.4 says that for the special sequence of scales $\delta_j \to 0$, with high probability, $\operatorname{Mod} w^\mu(\mathbf{S}_{\theta^*}) \ \ge \ K^* - 1/j$. Since the collection of background squares $\mathcal{S}'_\Omega(1/j)$ is finite, with high probability, $\operatorname{Mod} w^\mu(\sigma) \le K^* + 1/j$ for all $\sigma \in \mathcal{S}'_\Omega(1/j)$, c.f. (ii). A compactness argument shows that $w^\mu|_{\mathbf{S}_{\theta^*}}$ is close to a quasiconformal mapping which has constant dilatation $\mu_\lambda = \frac{\overline{\partial} A_\lambda}{\partial A_\lambda}$ on $\mathbf{S}_{\theta^*}$. In terms of $f := A_\lambda^{-1} \circ w^\mu$, we have:

Lemma 6.6. *There is a sequence of good scales $\delta_j \to 0$ such that the following properties hold with probability $> 1 - 1/j$:*

(i) *f is $(KK^*, \ell(\mathbf{S}_{\theta^*})/j)$ roughly quasiconformal on $\mathbf{S}_{\theta^*}$,*

(ii) $\operatorname{Mod} f(\sigma) < 1 + 1/j$ *for all $\sigma \in \mathcal{S}'_{\mathbf{S}_{\theta^*}}(\ell(\mathbf{S}_{\theta^*})/j)$,*

(iii) $\operatorname{Mod} f(\mathbf{S}_{\theta^*}) > 1 - 1/j$*.*

Remark. From the scale invariance of the model, we know that if $\mathbf{S} \subset \Omega$ has the same orientation as $\mathbf{S}_{\theta^*}$, then the scales

$$\delta'_j = \ell(\mathbf{S})/\ell(\mathbf{S}_{\theta^*}) \cdot \delta_j$$

are good for $\mathbf{S}$.

6.4 Any sequence of scales is good

We now eliminate the need to use a subsequence of scales:

Lemma 6.7. *For any square $\mathbf{S} \subset \Omega$ and $\varepsilon > 0$, when the mesh size $\delta < \delta_0(\varepsilon, \mathbf{S})$ is sufficiently small,*

$$\mathbb{P}_\delta\big(1 - \varepsilon < \operatorname{Mod} f(\mathbf{S}) < 1 + \varepsilon\big) > 1 - \varepsilon.$$

Proof. Let δ_j be a good scale from Lemma 6.6. We will use the goodness of the scale δ_j to show that any scale $0 < \delta < \delta_j$ is also good.

To that end, fix an $0 < \eta < 1$ and consider the grid $e^{i\theta^*}\beta\mathbb{Z}^2$ which consists of squares of side length $\beta = \eta \cdot \delta/\delta_j \cdot \ell(\mathbf{S}_{\theta^*})$ that have the same orientation as $\mathbf{S}_{\theta^*}$. By the remark above, if δ_j is a good scale for $\mathbf{S}_{\theta^*}$, then δ is a good scale for squares of the form $\eta^{-1}\square$ with $\square \in e^{i\theta^*}\beta\mathbb{Z}^2$.

We colour a cell $\square \in e^{i\theta^*}\beta\mathbb{Z}^2$ *blue* if $(f, \eta^{-1}\square)$ satisfies conditions (i)–(iii) of Lemma 6.6 and *yellow* otherwise. According to Lemma 6.6, the probability that any given cell is blue is at least $1 - 1/j$. Even though the colours of the cells are not independent, the colour of a cell only depends on the behaviour of the Beltrami coefficient μ in $\eta^{-1}\,\square$, cf. modification (iv) of Lemma 4.2.

Since f is close to conformal on $\{\eta^{-1}\square : \square \text{ blue}\}$, by Koebe's distortion theorem, if $\eta > 0$ is small, then f is close to linear on the blue cells $\square$. An application of Lemma 5.2 completes the proof. $\square$

6.5 Conclusion of the proof

From here, it is now a simple matter to prove Theorem 6.1:

Proof of Theorem 6.1. As noted previously, to show that w^μ is close to Φ, we can instead show that $f = A_\lambda^{-1} \circ w^\mu$ is close to $A_\lambda^{-1} \circ \Phi$. By Lemmas 6.2 and 6.7, for any $\varepsilon > 0$, if the mesh size $\delta < \delta_0(\varepsilon)$ is small, then with probability $\geq 1 - \varepsilon$,

$$f \text{ is } (KK^*, \varepsilon) \text{ roughly quasiconformal on } B(0, 2R), \tag{6.7}$$

$$1 - \varepsilon \leq \operatorname{Mod} f(\sigma) \leq 1 + \varepsilon, \qquad \forall \sigma \in \mathcal{S}'_\Omega(\varepsilon). \tag{6.8}$$

$$\frac{\overline{\partial} f}{\partial f} = \frac{\overline{\partial}(A_\lambda^{-1} \circ w^\mu)}{\partial(A_\lambda^{-1} \circ w^\mu)} = \frac{\overline{\partial}(A_\lambda^{-1} \circ \Phi)}{\partial(A_\lambda^{-1} \circ \Phi)}, \qquad \text{on } \mathbb{C} \setminus \Omega. \tag{6.9}$$

If the theorem were false, we would have a sequence of normalized quasiconformal mappings f_n which satisfy the above conditions with $\varepsilon = 1/n$, but were a definite distance away from $A_\lambda^{-1} \circ \Phi$ in $C\big(B(0, R)\big)$. This is impossible since $A_\lambda^{-1} \circ \Phi$ is the only possible limit of such a sequence. $\square$

7 Random q.c. mappings on the plane

Let μ be a random Beltrami coefficient on the plane, constructed with help of the probability measure λ. Since $K_\mu = \frac{1+|\mu|}{1-|\mu|}$ is not bounded in general, one may wonder if there is a homeomorphism w^μ with dilatation μ which fixes the points $0, 1, \infty$. Uniqueness follows from the fact that any two injective solutions of the Beltrami equation $\overline{\partial} w(z) = \mu(z) \partial w(z)$ with the same dilatation differ by a conformal automorphism of $\mathbb{C}$.

For the existence, we only need K_μ to be locally bounded, although a priori, the map w^μ may not be surjective: for any $R > 2$, we can truncate $\mu_R = \mu \cdot \chi_{B(0,R)}$ and use the measurable Riemann mapping theorem to construct a quasiconformal map w_R with dilatation μ_R that fixes $-1, 0, 1$. Compactness properties of quasiconformal maps allow us to extract a subsequential limit of the w_R as $R \to \infty$, yielding an injective map $w : \mathbb{C} \to \hat{\mathbb{C}}$ with dilatation μ. We may post-compose w with a Möbius transformation to make it fix the points $0, 1, \infty$ instead.

To show that w^μ is surjective almost surely, we choose $0 < k < 1$ so that $\lambda\big(\{z : k < |z| < 1\}\big) < r_0$ where r_0 is the constant from Lemma 4.2, and colour a cell $\square$ in $\delta\mathbb{Z}^2$ *yellow* if $|\mu(\square)| > k$ and *blue* otherwise. Since $\sum_{N=1}^{\infty} (\delta/N)^2 < \infty$, the Borel-Cantelli lemma shows thats almost surely, for all sufficiently large $N \ge N_0$, w^μ is $(K, \delta \log(N/\delta))$ roughly quasiconformal on $[-N, N] \times [-N, N]$. Since the moduli of infinitely many annuli $w^\mu\big(A(0, N, 2N)\big)$ are bounded from below, w^μ is surjective.

Fix a ball $B(0, R)$ with $R > 2$. We now show that when $\delta > 0$ is sufficiently small, then with probability at least $1 - \varepsilon$, $w^\mu(z)$ is within ε of the affine mapping $A_\lambda(z)$ on $B(0, R)$. For any $\gamma > 1$, the compactness of roughly quasiconformal mappings tells us that when the mesh size $\delta < \delta_0(R, \varepsilon, \gamma)$ is sufficiently small, $w^\mu|_{B(0,\gamma R)}$ is close to a quasiconformal map $\Phi : B(0, \gamma R) \to \mathbb{C}$ with constant dilatation μ_λ. By requesting γ to be large and applying Koebe's distortion theorem, we see that $(A_\lambda^{-1} \circ \Phi)|_{B(0,R)}$ is close to a complex-linear map. Since $A_\lambda^{-1} \circ \Phi$ fixes the points 0 and 1, it is close to the identity. Putting this together, we see that $w^\mu|_{B(0,R)}$ is close to A_λ as desired.

8 Moduli of rectangles in circle packings

For a combinatorial rectangle $\mathbf{R}$ in a circle packing, we have two different notions of moduli for curves connecting the opposite sides of $\mathbf{R}$: the discrete modulus of the triangulation and the continuous modulus of the carrier. In general, the two notions of modulus are unrelated, however, when $\mathcal{P}$ has bounded geometry, the continuous modulus and the discrete modulus agree up to a multiplicative constant.

This is facilitated by the Euclidean Ring Lemma [11, Lemma 8.2], which says that for any $N \ge 3$, there exists a constant $0 < \mathfrak{c}(N) < 1$ such that if $C = B(v,r)$ is an interior circle in $\mathcal{P}$ whose degree is at most N, then $r_i \ge \mathfrak{c}(N) \cdot r$ for any circle $C_i = B(v_i, r_i) \in \mathcal{P}$ tangent to C. Elementary geometry shows that the polygon $v_1 v_2 \dots v_n$ whose vertices are centers of circles tangent to C contains the ball $B\big(v, (1+\mathfrak{c}(N)/2)r\big)$.

Lemma 8.1. *Let* $\mathbf{R}$ *be an interior combinatorial rectangle in a circle packing* $\mathcal{P}$*. Suppose* ρ_{d} *is a discrete metric defined on the vertices of the underlying triangulation, which is supported on the set of vertices of degree at most* N*. Let* $\eta = \mathfrak{c}(N)/2$*. Define a continuous metric by the formula*

$$\rho_{\mathrm{c}}(z) := \frac{1}{\eta} \sum_{B(v_i,r_i)\in\mathcal{P}} \frac{\rho_{\mathrm{d}}(v_i)}{r_i} \cdot \chi_{B(v_i,(1+\eta)r_i)\cap\mathbf{R}}(z). \tag{8.1}$$

If ρ_d *was admissible for the vertical curve family* $\Gamma^{\mathrm{d}}_{\updownarrow}$ *in the discrete sense, then* ρ_c *will be admissible for* $\Gamma^{\mathrm{c}}_{\updownarrow}$ *in the continuous sense. Furthermore, the total area of* ρ_{c} *is controlled by the total area of* ρ_{d}*:* $A(\rho_{\mathrm{c}}) \le C(N) A(\rho_{\mathrm{d}})$*.*

Proof. The bound on the total area is clear since the sum defining $\rho_{\mathrm{c}}(z)$ has at most 3 non-zero terms: if z lies in an interior triangle $v_i v_j v_k$, then only the indices i, j, k can contribute to the sum (8.1). To check that $\ell_{\rho_{\mathrm{c}}}(\gamma) \ge 1$ for $\gamma \in \Gamma^{\mathrm{c}}_{\updownarrow}$, notice that as the sides of triangles in $\mathcal{P}$ are contained in $\bigcup \overline{C_i}$, the combinatorial progress that γ makes through the triangulation is recorded by the collection of circles it visits. Since the ρ_c cost of entering or exiting the influence of a circle is at least $\rho_{\mathrm{d}}(v_i)$, the continuous length of γ exceeds its discrete length. □

To study the boundary behaviour of maximal circle packings, we need to allow $\mathbf{R}$ to be an extended combinatorial rectangle. For a boundary circle

$C_i = B(v_i, r_i)$ in $\mathcal{P}$, we let v_i^* denote the point of tangency between C_i and the unit circle. We extend the underlying triangulation of $\mathcal{P}$ by adding edges from v_i to v_i^* and from v_i^* and v_j^* if C_{v_i} and C_{v_j} are tangent. With this definition, the extended "triangulation" also includes the quadrilaterals $v_i v_i^* v_j^* v_j$. By an *extended combinatorial rectangle*, we mean a combinatorial rectangle in the extended triangulation. If $\mathcal{P}$ is a maximal circle packing, then the Euclidean Ring Lemma also applies to boundary circles: one can see this by reflecting the packing in the unit circle and adding inscribed circles to get a triangulation of the sphere. We leave it to the reader to check that Lemma 8.1 also holds for extended combinatorial rectangles.

9 Random Delauney triangulations

In this section, we prove Theorem 1.3 which says that a circle packing of a random Delauney triangulation approximates a conformal map. For technical reasons, it is preferable to use a slightly different construction of a random Delauney triangulation where the Delauney points are chosen according to a Poisson point process of high intensity. We recall the definition. For a measurable set $E \subset \Omega$, we denote its Euclidean area by $A(E)$ and the number of Poisson points contained in E by N_E.

A *Poisson point process* of intensity λ produces a random collection of points in Ω according to the following two axioms:

(1) For any measurable set $E \subset \Omega$,

$$\mathbb{P}(N_E = n) = e^{-A(E)\lambda} \cdot \frac{(A(E)\lambda)^n}{n!}.$$

(2) If $E_1, E_2, \dots, E_k$ are disjoint measurable sets, then $N_{E_1}, N_{E_2}, \dots, N_{E_k}$ are independent random variables.

From the uniqueness of the Poisson point process, it follows that the union of two independent Poisson point processes is also a Poisson point process and that the intensities add. The law of large numbers tells us that when the intensity λ is large, then with high probability (w.h.p.) $N_E \sim A(E)\lambda$. In other words, for any $\varepsilon > 0$, when $\lambda \geq \lambda_0(\varepsilon, \Omega)$ is sufficiently large, $(1-\varepsilon)A(E)\lambda \leq N_E \leq (1+\varepsilon)A(E)\lambda$ holds with probability $\geq 1-\varepsilon$.

We will need the following estimate:

Lemma 9.1. *Suppose $\Omega \subset \mathbb{C}$ is a bounded domain. For any $0 < \varepsilon < 1$, when $\lambda \geq \lambda_0(\varepsilon, \Omega)$ is sufficiently large, with probability at least $1-\varepsilon$, the estimate*

$$(1-\varepsilon) \cdot A(\mathbf{R})\lambda \;\leq\; N_{\mathbf{R}} \;\leq\; (1+\varepsilon) \cdot A(\mathbf{R})\lambda$$

holds for every rectangle $\mathbf{R} \subset \Omega$ whose sides have length at least ε.

Proof. Take $\delta = (1/20)\varepsilon^2$ and consider all cells in the square grid $\delta\mathbb{Z}^2$ which intersect Ω. The lemma follows from the following two observations:

(i) By the law of large numbers, when λ is large, w.h.p.

$$(1-\varepsilon/3) \cdot A(\square)\lambda \;\leq\; N_{\square} \;\leq\; (1+\varepsilon/3) \cdot A(\square)\lambda$$

for any cell $\square \in \delta\mathbb{Z}^2$ that is completely contained in Ω, while the upper bound holds for any $\square \in \delta\mathbb{Z}^2$ that merely intersects Ω.

(ii) Given a rectangle $\mathbf{R} \subset \Omega$ whose sides have length at least ε, let E_1 be the union of cells in $\delta\mathbb{Z}^2$ that are completely contained in $\mathbf{R}$, and E_2 be the union of cells in $\delta\mathbb{Z}^2$ that have non-empty intersection with $\mathbf{R}$. The area of $E_2 \setminus E_1$ is bounded above by $(\varepsilon/3) \cdot A(\mathbf{R})$. □

One can deduce Theorem 1.3 for the original model where the number of Delauney points is fixed by using the following simple observation: for any $\varepsilon > 0$, when N is large, w.h.p. a collection of N random points is squeezed between Poisson point processes with intensities $N/A(\Omega) - \varepsilon$ and $N/A(\Omega) + \varepsilon$.

9.1 Basic properties of Delauney triangulations

The following lemma says that when the intensity is large, Delauney triangulations tend to have short edges and exhaust Ω:

Lemma 9.2. *Suppose $\Omega \subset \mathbb{C}$ is a Jordan domain. For any $\varepsilon > 0$ and compact set $K \subset \Omega$, when $\lambda \geq \lambda_0(\varepsilon, K, \Omega)$ is sufficiently large, with probability at least $1-\varepsilon$, we have:*

(i) *The length of any edge of $\mathcal{T}$ that intersects K is less than ε,*

(ii) $\operatorname{carr} \mathcal{T} \supset K$.

Proof. Fix a real number $\delta > 0$ with $3\sqrt{2}\delta < \operatorname{dist}(K, \partial\Omega)$, and let Ω^δ be the union of all cells in $\delta\mathbb{Z}^2$ contained in Ω. When the intensity $\lambda > 0$ is large, with probability $\geq 1-\varepsilon$, every cell in Ω^δ contains at least one point of $\mathcal{T}$.

Suppose $v \in \mathcal{T}$ is a Delauney point with $\operatorname{dist}(v, \partial\Omega) > 2\sqrt{2}\delta$. We first show that the Vorononi cell $F_v \subset \overline{B(v, \sqrt{2}\delta)}$. To see this, note that if $y \in \partial B(v, \sqrt{2}\delta)$, then y cannot lie in the same cell of the grid $\delta\mathbb{Z}^2$ as v. By assumption, there is a Delauney point $v' \neq v$ in the cell that contains y. As the cells of $\delta\mathbb{Z}^2$ have diameter $\sqrt{2}\delta$, $|v' - y| \leq |v - y|$, which shows that $y \notin \operatorname{Int} F_v$. This proves the claim.

Let $v_1, v_2, v_3, \dots, v_d \in \mathcal{T}$ be the vertices connected to v by an edge, listed in counter-clockwise order. Since the midpoint of the edge vv_i lies on $F_v \cap F_{v_i}$, $|v - v_i| \leq 2\sqrt{2}\delta$. In particular, the polygon $v_1 v_2 \dots v_d$ is contained in $\overline{B(v, 2\sqrt{2}\delta)}$. Since any point in F_v belongs to one of the Delauney triangles vv_iv_{i+1}, $F_v \subset \operatorname{carr} \mathcal{T}$.

Finally, if $\operatorname{dist}(z, \partial\Omega) > 3\sqrt{2}\delta$, then z is contained in a Voronoi cell F_v with $\operatorname{dist}(v, \partial\Omega) > 2\sqrt{2}\delta$, which means that $z \in \operatorname{carr} \mathcal{T}$. □

A similar argument shows:

Lemma 9.3. *Consider a* 7×7 *square in the plane, which is naturally formed from 49 unit squares. Suppose that each of these 49 unit squares contains a vertex of the Delauney triangulation. Then each Delauney edge that passes through the middle square is contained in the* 7×7 *square.*

Proof. Suppose e is an edge of the Delauney triangulation that connects the vertices v_1, v_2 and passes through the middle 1×1 square. If the lemma were false, there would be a segment $\overline{e} \subseteq e$ of length 3 contained in the 7×7 square. Let z be the midpoint of $\overline{e}$. On one hand, since $e \subset F_{v_1} \cup F_{v_2}$, the distance from z to any Delauney vertex is at least $\min(|z - v_1|, |z - v_2|) \geq 3/2$. On the other hand, since z is a point in the 7×7 square, it is located within $\sqrt{2}$ of some vertex, which is a contradiction. □

9.2 Rough quasiconformality

Let $\mathbf{R}$ be a rectangle compactly contained in Ω. Its *exterior discrete approximation* $\mathbf{R}_+^{\mathrm{d}}$ consists of all vertices of $\mathcal{T}$ that either lie in $\mathbf{R}$ or are adjacent to a vertex that lies in $\mathbf{R}$. For each corner of $\mathbf{R}$, mark the closest point in $\mathcal{T} \cap \mathbf{R}_+^{\mathrm{d}}$. (In case of a tie, choose the marked points arbitrarily.) The four marked points turn $\mathbf{R}_+^{\mathrm{d}}$ into a discrete combinatorial rectangle. In practice,

$\mathbf{R}^{\mathrm{d}}_{+}$ is close to $\mathbf{R}$: if $\tilde{\mathbf{R}}$ is a slightly larger rectangle which contains $\mathbf{R}$ in its interior, then for λ large, w.h.p. $\mathbf{R} \subset \mathbf{R}^{+}_{\mathrm{d}} \subset \tilde{\mathbf{R}}$.

Lemma 9.4. *For any $\varepsilon > 0$, when $\lambda \geq \lambda_0(\varepsilon, \Omega)$ is sufficiently large, the probability that $\varphi_{\mathcal{P}} : \Omega \to \mathbb{D}$ is (K, ε) roughly quasiconformal on Ω is $\geq 1 - \varepsilon$.*

Proof. Let $\mathbf{R} \in \mathcal{R}_{\Omega}(\varepsilon)$ be a rectangle compactly contained in Ω whose sides have length at least ε. Consider the square grid $\delta\mathbb{Z}^2$ with mesh size $\delta = C/\sqrt{\lambda}$. From the law of large numbers, we expect a cell $\square$ in $\delta\mathbb{Z}^2$ to contain roughly C^2 points from $\mathcal{T}$. We colour a cell $\square$ in $\delta\mathbb{Z}^2$ *blue* if it contains between 1 and C^3 points from $\mathcal{T}$ and *yellow* otherwise. We call $\square$ *7-deep blue* if all cells $\hat{\square} \subset 7\square$ are blue and *13-deep blue* if all cells $\hat{\square} \subset 13\square$ are blue.

It is easy to see that any vertex of $\mathcal{T}$ in a deep blue cell has valence at most $49C^3$ since the Delauney edges emanating it from it are contained in $7\square$. By making $C > 0$ large, we can ensure that the probability that a cell is blue is at least $1 - r_0(13)$ where $r_0(13)$ is the constant from modification (iii) of Lemma 4.2.

Consider the discrete metric $\rho_{\mathrm{d}} = \chi_{\mathscr{B} \cap \mathbf{R}^{\mathrm{d}}_{+}}$ where $\mathscr{B}$ is the union of the 7-deep blue cells. By Lemma 9.1, if the intensity λ is large, then w.h.p.

$$A(\rho_{\mathrm{d}}) = \sum_{v} \rho_{\mathrm{d}}^2(v) \leq N_{\mathbf{R}^{\mathrm{d}}_{+}} \leq 2 \cdot \ell_1(\mathbf{R})\ell_2(\mathbf{R})\lambda = \frac{2C^2}{\delta^2} \cdot \ell_1(\mathbf{R})\ell_2(\mathbf{R}).$$

To estimate the ρ_{d}-length of a discrete path $\gamma_{\mathrm{d}} \in \Gamma_{\updownarrow}(\mathbf{R}^{\mathrm{d}}_{+})$, we view it as a continuous piecewise-linear curve γ by connecting the vertices with line segments. According to modification (iii) of Lemma 4.2, when λ is sufficiently large, w.h.p. every γ passes through at least $\lfloor \ell_1(\mathbf{R})/(2\delta) \rfloor$ 13-deep blue cells. By Lemma 9.3, if γ passes through a 13-deep blue cell $\square$, γ_{d} must contain a vertex in a 7-deep blue cell. We see that w.h.p. the ρ_{d}-length of every path in $\Gamma_{\updownarrow}(\mathbf{R}^{\mathrm{d}}_{+})$ is at least $c \cdot \ell_1(\mathbf{R})/\delta$ where $c > 0$ is a definite constant.

The above computations show that when the intensity is large, w.h.p. the discrete modulus of $\Gamma_{\updownarrow}(\mathbf{R}^{\mathrm{d}}_{+})$ is bounded above by a definite multiple of the continuous modulus of $\Gamma_{\updownarrow}(\mathbf{R})$. Since ρ_{d} was supported on vertices of bounded valence, Lemma 8.1 tells us that the continuous modulus of $\Gamma_{\updownarrow}(\varphi_{\mathcal{P}}(\mathbf{R}^{\mathrm{d}}_{+}))$ is also bounded by a definite multiple of the continuous modulus of $\Gamma_{\updownarrow}(\mathbf{R})$. This completes the proof. $\square$

9.3 Interior conformality

The following lemma says that the radii of the interior circles are small when the intensity λ is large:

Lemma 9.5. *For any $\varepsilon > 0$ and subdomain Ω' compactly contained in Ω, if the intensity $\lambda > \lambda_0(\varepsilon, \Omega')$ is sufficiently large, then with probability at least $1-\varepsilon$, the radii of all circles $C_v \in \mathcal{P}$ associated to Delauney points $v \in \Omega'$ are less than ε.*

Proof. We can surround a vertex $v \in \Omega'$ by an annulus $\mathbf{A} = A(v, r, r') \subset \Omega'$ of arbitrarily large modulus. When λ is large, w.h.p. all vertices adjacent to v will lie inside $B(v, r)$. Rough quasiconformality tells us that that the annulus $\varphi_{\mathcal{P}}(\mathbf{A})$ will have large modulus. Since the image of $\varphi_{\mathcal{P}}(\mathbf{A})$ is contained in the unit disk and surrounds C_v, the radius of C_v must be small. □

By a deep theorem of He and Schramm [5, Theorem 1.1], we have:

Corollary 9.6. *Let $\mathbf{S}$ be a square compactly contained in Ω and $\tilde{\mathbf{S}} \subset \Omega$ be a slightly larger square with the same center as $\mathbf{S}$. For any $\varepsilon > 0$, when the intensity $\lambda > \lambda_0(\varepsilon, \mathbf{S}, \tilde{\mathbf{S}})$ is sufficiently large, with probability at least $1-\varepsilon$, the modulus of $\varphi_{\mathcal{P}}(\mathbf{S})$ is determined by the Delauney triangulation on $\tilde{\mathbf{S}}$ within ε of its true value.*

Since the model of random Delauney triangulations does not have a preferred direction, the arguments of Section 6 show:

Lemma 9.7. *Let Ω' be a subdomain compactly contained in Ω which contains z_1, z_2. For any $\varepsilon > 0$, when the intensity $\lambda > \lambda_0(\varepsilon, \Omega', \Omega)$ is sufficiently large, with probability at least $1-\varepsilon$, the map $\varphi_{\mathcal{P}}$ is within ε of a conformal map defined on Ω'.*

9.4 Boundary behaviour

To complete the proof of Theorem 1.3, we need to show that if $\partial\Omega$ is C^1, then the image of the approximating conformal map is the unit disk:

Lemma 9.8. *For any $\varepsilon > 0$, there exists an $r > 0$ so that when the intensity $\lambda > \lambda_0(\varepsilon, r)$ is sufficiently large, with probability at least $1-\varepsilon$, the image of $\Omega_r = \{z \in \Omega : \operatorname{dist}(z, \partial\Omega) < r\}$ under $\varphi_{\mathcal{P}}$ contains $B(0, 1-\varepsilon)$.*

In particular, the above lemma implies that all circles in $\mathcal{P}$ have small radii, not just ones confined to the interior. The proof is quite similar to Lemma 9.4, so we only give a sketch of the argument and allow the reader to work out the details.

At this stage, it is important to use the modified random Delauney triangulation where one adds $\asymp \sqrt{A(\Omega)\lambda}$ equally-spaced points on $\partial\Omega$. This ensures that the diameters of the Delauney triangles near the boundary are $O(1/\sqrt{\lambda})$. If we did not do this, the Delauney triangulation could have long thin triangles of diameter $O(1)$ which would lead to large circles in the packing.

In the proof of Lemma 9.8, we will need to decompose Ω into "cells" of diameter comparable to δ and area comparable to δ^2. As in Section 9.1, let Ω_δ be the union of all cells in $\delta\mathbb{Z}^2$ contained in Ω. To define a partition of Ω, we need to distribute $\Omega \setminus \Omega^\delta$ amongst the boundary cells of Ω^δ. Since $\partial\Omega$ is C^1, for small δ, we can distribute the excess mass so that cells are connected sets with the correct area and diameter.

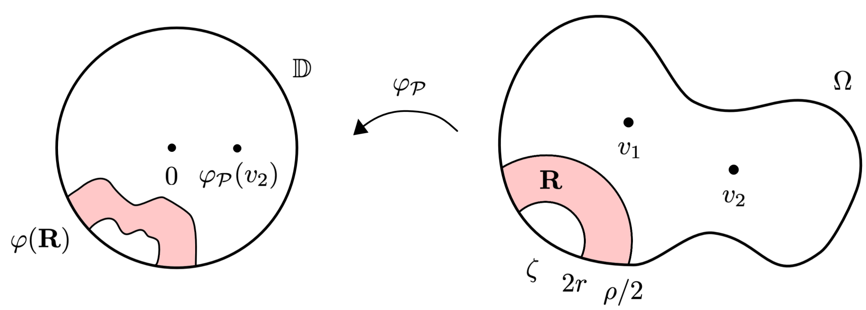

Figure 3: If $\mathbf{R}$ has large modulus, then the diam $\varphi_{\mathcal{P}}\big(B(\zeta, \frac{3r}{2})\big)$ is small.

Proof of Lemma 9.8. Since $\partial\Omega$ is C^1, there exists a number $\rho_0 > 0$ so that for any boundary point $\zeta \in \partial\Omega$ and $0 < \rho < \rho_0$, the intersection $\partial B(\zeta, \rho) \cap \Omega$ consists of a single circular arc. Shrinking ρ if necessary, we may assume that $\operatorname{dist}(z_1, \partial\Omega) \geq \rho$ and $\operatorname{dist}(z_2, \partial\Omega) \geq \rho$.

For a point $\zeta \in \partial\Omega$, consider the conformal rectangle $\mathbf{R} = A(\zeta, 2r, \rho/2) \cap \Omega$ where the round sides contained in $\partial A(\zeta, 2r, \rho/2)$ have been marked. Since the modulus of $\mathbf{R}$ can be made arbitrarily large by making r small, it is reasonable to believe that w.h.p. its image $\varphi_{\mathcal{P}}(\mathbf{R})$ also has large modulus. Assuming this temporarily, we see that the diameter of $\varphi_{\mathcal{P}}\big(B(\zeta, \frac{3r}{2})\big)$ is small since $\varphi_{\mathcal{P}}(\mathbf{R})$ separates $\varphi_{\mathcal{P}}\big(B(\zeta, \frac{3r}{2})\big)$ from C_{v_1} and C_{v_2}. The lemma follows since finitely many balls $B(\zeta_i, \frac{3r}{2})$ cover $\Omega \setminus \Omega_r$.

To estimate $\operatorname{Mod} \varphi_{\mathcal{P}}(\mathbf{R})$, we follow the strategy from the proof of rough quasiconformality (Lemma 9.4). Set $\delta = C/\sqrt{\lambda}$ as in Lemma 9.4. Since $\partial\Omega$ is C^1, we may partition Ω into cells of diameter comparable to δ and area comparable to δ^2 as described above. We colour each cell in Ω either blue or yellow as in Lemma 9.4, that is, we colour a cell *blue* if it contains between 1 and C^3 points of $\mathcal{T}$ and *yellow* otherwise. Consider the metric

$$\rho_{\mathrm{d}}(v) = \frac{1}{|v-\zeta|} \cdot \chi_{\mathscr{B} \cap \mathbf{R}^{\mathrm{d}}_{+}},$$

where $\mathscr{B}$ is the union of the deep blue cells and $\mathbf{R}^{\mathrm{d}}_{+}$ is the exterior discrete approximation of $\mathbf{R}$ (which is now an extended combinatorial rectangle). We claim that when the intensity λ is large, w.h.p.

$$A(\rho_{\mathrm{d}}) \lesssim \log \frac{\rho}{4r} \cdot (1/\delta)^2, \qquad \ell_{\rho_{\mathrm{d}}}(\gamma) \gtrsim \log \frac{\rho}{4r} \cdot (1/\delta), \qquad \gamma \in \Gamma_{\leftrightarrow}(\mathbf{R}^{\mathrm{d}}_{+}).$$

The area estimate follows from the law of large numbers, while the length estimate follows from modification (ii) of Lemma 4.1. These length-area estimates imply that

$$\operatorname{Mod} \Gamma_{\leftrightarrow}(\mathbf{R}^{\mathrm{d}}_{+}) \lesssim \left(\log \frac{\rho}{r}\right)^{-1} \qquad \Longrightarrow \qquad \operatorname{Mod} \Gamma_{\updownarrow}(\mathbf{R}^{\mathrm{d}}_{+}) \gtrsim \log \frac{\rho}{r}.$$

Since ρ_{d} is supported on vertices of bounded valence, by Lemma 8.1,

$$\operatorname{Mod} \Gamma_{\updownarrow}(\varphi_{\mathcal{P}}(\mathbf{R})) \asymp \operatorname{Mod} \Gamma_{\updownarrow}(\mathbf{R}^{\mathrm{d}}_{+}) \gtrsim \log \frac{\rho}{r}.$$

as desired. The proof is complete. □

A Weak convergence is not enough

It sounds plausible that if a sequence of Beltrami coefficients μ_n converges weakly to μ, then the quasiconformal mappings w^{μ_n} converge pointwise to w^{μ}. However, this is not true. For a counterexample, partition the plane into vertical strips of width δ and assign $\mu^{\delta} = 1/3$ on odd-numbered strips and

$-1/3$ on even-numbered strips. Clearly, the Beltrami coefficients μ_δ converge weakly to 0 as $\delta \to 0$, however, the maps μ_δ converge to the affine stretch in the horizontal direction by the factor $(2 + \frac{1}{2})/2 = 5/4$. Indeed, on each even-numbered strip, the x-coordinate is stretched by a factor of 2, while in each odd numbered strip, one contracts the x-coordinate by a factor of 2.

By the law of large numbers, if one randomly assigns the Beltrami coefficient to be $\pm 1/3$ on vertical strips, then the limit is also an affine stretch by a factor of $5/4$ in the x-coordinate.

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