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THE GOOD PANTS HOMOLOGY AND A PROOF OF THE EHRENPREIS CONJECTURE

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Abstract. We develop the notion of the good pants homology and show that it agrees with the standard homology on closed surfaces (good pants are pairs of pants whose cuffs have the length nearly equal to some large number $R > 0$). Combined with our previous work on the Surface Subgroup Theorem [5], this yields a proof of the Ehrenpreis conjecture.

1. Introduction

Let S and R denote two closed Riemann surfaces (all closed surfaces in this paper are assumed to have genus at least 2). The well-known Ehrenpreis conjecture asserts that for any $K > 1$, one can find finite degree covers S_1 and R_1, of S and R respectively, such that there exists a K-quasiconformal map $f : S_1 \to R_1$. The purpose of this paper is to prove this conjecture. Below we outline the strategy of the proof.

Let $R > 1$ and let $\Pi(R)$ be the pair of pants whose all three cuffs have the length R. We define the surface $S(R)$ to be the genus two surface that is obtained by gluing two copies of $\Pi(R)$ along the cuffs with the twist parameter equal to $+1$ (these are the Fenchel-Nielsen coordinates for $S(R)$). By $\operatorname{Orb}(R)$ we denote the quotient orbifold of the surface $S(R)$ (the quotient of $S(R)$ by the group of automorphisms of $S(R)$). For a fixed $R > 1$, we sometimes refer to $\operatorname{Orb}(R)$ as the model orbifold. The following theorem is the main result of this paper.

Theorem 1.1. *Let S be a closed hyperbolic Riemann surface. Then for every $K > 1$, there exists $R_0 = R_0(K, S) > 0$ such that for every $R > R_0$ there are finite covers S_1 and O_1 of the surface S and the model orbifold $\operatorname{Orb}(R)$ respectively, and a K-quasiconformal map $f : S_1 \to O_1$.*

The Ehrenpreis conjecture is an immediate corollary of this theorem.

Corollary 1.1. *Let S and M denote two closed Riemann surfaces. For any $K > 1$, one can find finite degree covers S_1 and M_1 of S and M respectively, such that there exists a K-quasiconformal map $f : S_1 \to M_1$.*

Proof. Fix $K > 1$. It follows from Theorem 1.1 that for R large enough, there exist

(1) Finite covers S_1, M_1, of S and M respectively, and

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(2) Finite covers O_1 and O_1' of $\mathrm{Orb}(R)$,
(3) $\sqrt{K}$-quasiconformal mappings $f : S_1 \to O_1$, and $g : M_1 \to O_1'$.

Let O_2 denote a common finite cover of O_1 and O_1' (such O_2 exists since O_1 and O_1' are covers of the same orbifold $\mathrm{Orb}(R)$). Then there are finite covers S_2 and M_2, of S_1 and M_1, respectively, and the $\sqrt{K}$-quasiconformal maps $\widetilde{f} : S_2 \to O_2$, and $\widetilde{g} : M_2 \to O_2$, that are the lifts of f and g. Then $\widetilde{g}^{-1} \circ \widetilde{f} : S_2 \to M_2$ is K-quasiconformal map, which proves the corollary. □

In the remainder of the paper we prove Theorem 1.1. This paper builds on our previous paper [5] where we used immersed skew pants in a given hyperbolic 3-manifold to prove the Surface Subgroup Theorem. We note that Lewis Bowen [1] was the first to attempt to build finite covers of Riemann surfaces by putting together immersed pairs of pants. We also note that it follows from the work of Danny Calegari [2] that the pants homology is equal to the standard homology. This means that every sum of closed curves on a closed surface S that is zero in the standard homology $\mathbf{H}_1(S)$ is the boundary of a sum of immersed pairs of pants in S.

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Outline. In our previous paper [5] we proved a theorem very similar to Theorem 1.1, namely that given a closed 3-manifold $\mathbf{M^3}$, and $K > 1$, $R > R_0(K, \mathbf{M^3})$, we can find a finite cover O_1 of $\mathrm{Orb}(R)$ and a map $f : O_1 \to \mathbf{M^3}$ that lifts to a map $\widetilde{f} : \mathbb{H}^2 \to \mathbb{H}^3$ such that $\partial \widetilde{f} : \partial \mathbb{H}^2 \to \partial \mathbb{H}^3$ has a K-quasiconformal extension. We proved that theorem by finding a large collection of "skew pairs of pants" whose cuffs have complex half-lengths close to R, and which are "evenly distributed" around every closed geodesic that appears as a boundary.

We then assemble these pants by (taking two copies of each and then) pairing off the pants that have a given geodesic as boundary, so that the resulting complex twist-bends (or reduced Fenchel-Nielsen coordinates) are close to 1. We can then construct a cover O_1 of Orb(R) and a function $f : O_1 \to M$ whose image is the closed surface that results from this pairing. The function f will then have the desired property.

We would like to proceed in the same manner in dimension 2, that is when a 3-manifold $\mathbf{M^3}$ is replaced with a closed surface $\mathbf{S}$. We can, as before, find a collection of good immersed pants (with cuff length close to R) that is "evenly distributed" around each good geodesic (of length close to R) that appears as boundary. If and when we can assemble the pants to form a closed surface with (real) twists close to 1, we will have produced a K-quasiconformal immersion of a cover of $\mathrm{Orb}(R)$ into $\mathbf{S}$.

There is only one minor detail: the unit normal bundle of a closed geodesic in $\mathbf{S}$ has two components. In other words, an immersed pair of pants that has a closed geodesic γ as a boundary can lie on one of the two sides of γ. If, in our formal sum of pants, we find we have more pants on one side of γ than the other, then we have no chance to form a closed surface out of this formal sum of pants. It is for this very reason that the Ehrenpreis Conjecture is more difficult to prove than the Surface Subgroup Theorem.

Because our initial collection is evenly distributed, there are almost the same number of good pants on both sides of any good geodesic, so it is natural to try to adjust the number of pairs of pants so that it is balanced (with the same number of pants on both sides of each geodesic). This leads us to look for a "correction function" ϕ from formal sums of (normally) oriented closed geodesics (representing the imbalance) to formal sums of good pants, such that the boundary of $\phi(X)$ is X.

The existence of this correction function then implies that "good boundary is boundary good", that is any sum of good geodesics that is a boundary in $H_1(\mathbf{S})$ (the first homology group of $\mathbf{S}$) is the boundary of a sum of good pants. Thus we define the good pants homology to be formal sums of good geodesics (with length close to R) modulo boundaries of formal sums of good pants. We would like to prove that the good pants homology is the standard homology on good curves.

The natural approach is to show that any good curve is homologous in good pants homology to a formal sum of $2g$ particular good curves that represent a basis in $H_1(\mathbf{S})$ (g is the genus of $\mathbf{S}$). That is, we want to show that there are $\{h_1, ..., h_{2g}\}$ good curves that generate $H_1(\mathbf{S})$ (here $H_1(\mathbf{S})$ is taken with rational coefficients) such that every good curve γ is homologous in the good pants homology to a formal sum $\sum a_i h_i$, for some $a_i \in \mathbb{Q}$. Then any sum of good curves is likewise homologous to a sum of good generators h_i, but if the original sum of good curves is zero in $H_1(\mathbf{S})$ then the corresponding sum of h_i's is zero as well.

To prove the Good Correction Theorem , we must first develop the theory of good pants homology. Let $*$ denote a base point on $\mathbf{S}$. For $A \in \pi_1(\mathbf{S}, *) \setminus \{\mathrm{id}\}$, we let $[A]$ denote the closed geodesic freely homotopic to A. Our theory begins with the Algebraic Square Lemma, which states that, under certain geometric conditions,

$$\sum_{i,j=0,1} (-1)^{i+j}[A_i U B_i V] = 0,$$

in the good pants homology. (The curves $[A_i U B_j V]$ must be good curves, the words $A_i U B_j V$ reasonably efficient, and the length of U and V sufficiently large). This then permits us to define, for $A, T \in \pi_1(\mathbf{S}, *)$,

$$A_T = \frac{1}{2}\left([TAT^{-1}U] - [TA^{-1}T^{-1}U]\right),$$

where U is fairly arbitrary. Then A_T in good pants homology is independent of the choice of U.

We then show through a series of lemmas that $(XY)_T = X_T + Y_T$ in good pants homology, and therefore

$$X_T = \sum \sigma(j)(g_{i_j})_T,$$

where by $X = g_{i_1}^{\sigma(1)}...g_{i_k}^{\sigma(k)}$ we have written X as a product of generators (here $g_1, ..., g_{2g}$ are the generators for $\pi_1(\mathbf{S}, *)$ and $\sigma(j) = \pm 1$). With a little more work we can show

$$[X] = \sum \sigma(i)(g_{i_j})_T$$

as well, and thus we can correct any good curve to good generators.

We are then finished except for one last step: We must show that our correction function, which gives an explicit sum of pants with a given boundary, is well-behaved in that it maps sums of curves, with bounded weight on each curve, to sums of pants, with bounded weight on each pair of pants. We call such a function semirandom, because if we pick a curve at random, the expected consumption of a given pair of pants is not much more than if we picked the pair of pants at random.

We define the correction function implicitly, through a series of lemmas, each of which asserts the existence of a formal sum of pants with given boundary, and which is in principle constructive. The notion of being semirandom is sufficiently natural to permit us to say that the basic operations, such as the group law, or forming $[A]$ from A, as well as composition and formal addition, are all semirandom. So in order to verify that our correction function is semirandom, we need only to go through each lemma observing that the function we have defined is built out of the functions that we have previously defined using the standard operations which we have proved are semirandom.

To make the the paper as easy to read as possible, we have relegated the verification of semi-randomness to the "Randomization remarks" which follow our homological lemmas, and which use the notation and results (that basic constructions are semirandom) that we have placed in the Appendix. We strongly recommend that the reader skip over these Randomization remarks in the first reading, and to interpret the word "random" in the text to simply mean "arbitrary".

2. Constructing "good" covers of a Riemann surface

2.1. The reduced Fenchel-Nielsen coordinates and the model orbifolds.

Let S^0 be an oriented closed topological surface with a given pants decomposition $\mathcal{C}$, where $\mathcal{C}$ is a maximal collection of disjoint simple closed curves that cut S^0 into the corresponding pairs of pants . Denote by $\mathcal{C}^*$ the set of oriented curves from $\mathcal{C}$ (each curve in $\mathcal{C}$ is taken with both orientations). The set of pairs (Π, C^*), where Π is a pair of pants from the pants decomposition and $C^* \in \mathcal{C}^*$ is an oriented boundary cuff of Π, is called the set of marked pants and it is denoted by $\mathbf{\Pi}(S^0)$. For $C \in \mathcal{C}^0$ there are exactly two pairs $(\Pi_i, C_i^*) \in \mathbf{\Pi}(S^0)$, $i = 1, 2$, such that C_1^* and C_2^* are the two orientations on C (note that Π_1 and Π_2 may agree as pairs of pants).

Recall the reduced Fenchel-Nielsen coordinates $(\mathbf{hl}(C), s(C))$ from [5]. Here $\mathbf{hl}(C)$ is the half-length of the geodesic homotopic to C, and $s(C) \in$

$\mathbb{R}/\mathbf{hl}\mathbb{Z}$ is the reduced twist parameter which lives in the circle $\mathbb{R}/\mathbf{hl}(C)\cdot\mathbb{Z}$ (when we write $s(C)=x\in\mathbb{R}$, we really mean $s(C)\equiv x \operatorname{mod}(\mathbf{hl}(C)\mathbb{Z})$). The following theorem was proved in [5] (see Theorem 2.1 and Corollary 2.1 in [5]).

Theorem 2.1. *There exist constants $\widehat{\epsilon},\widehat{R}>0$ such that the following holds. Let S^0 be a closed topological surface with an (unmarked) pants decomposition $\mathcal{C}$. Let S denote a marked Riemann surface whose reduced Fenchel-Nielsen coordinates satisfy the inequalities.*

$$|\mathbf{hl}(C)-R|<\epsilon, \;\; \text{and} \;\; |s(C)-1|<\frac{\epsilon}{R},$$

for some $\widehat{\epsilon}>\epsilon>0$ and $R>\widehat{R}$. Then there exists a marked surface M_R, with the reduced Fenchel-Nielsen coordinates $\mathbf{hl}(C)=R$ and $s(C)=1$, and a K-quasiconformal map $f:S\to M_R$, where

$$K=\frac{\widehat{\epsilon}+\epsilon}{\widehat{\epsilon}-\epsilon}.$$

Let $R>1$, and let $\operatorname{Orb}(R)$ denote the corresponding model orbifold (defined in the introduction). In the next subsection we will see that the significance of the above theorem comes from the observation that any Riemann surface M_R with reduced Fenchel-Nielsen coordinates $\mathbf{hl}(C)=R$ and $s(C)=1$ is a finite cover of $\operatorname{Orb}(R)$.

2.2. **A proof of Theorem 1.1.** Below we state the theorem which is then used to prove Theorem 1.1.

Theorem 2.2. *Let S denote a closed Riemann surface, and let $\epsilon>1$. There exists $R(S,\epsilon)>1$ such that for every $R>R(S,\epsilon)$ we can find a finite cover S_1 of S that has a pants decomposition whose reduced Fenchel-Nielsen coordinates satisfy the inequalities*

$$|\mathbf{hl}(C)-R|<\epsilon, \;\; \text{and} \;\; |s(C)-1|<\frac{\epsilon}{R}.$$

This theorem will be proved later. We proceed now to prove Theorem 1.1. Let $K>1$. It follows from Theorem 2.1 that for $\epsilon>0$ small enough, and every R large enough, there is a K-quasiconformal map $f:S_1\to M_R$, where M_R is a Riemann surface with reduced Fenchel-Nielsen coordinates $\mathbf{hl}(C)=R$ and $s(C)=1$, and S_1 is the finite cover of S from Theorem 2.2. Recall the corresponding model orbifold $\operatorname{Orb}(R)$ (defined in the introduction). As we observed, the surface M_R is a finite cover of $\operatorname{Orb}(R)$. This completes the proof of the theorem. We sometimes write $\gamma^*\in\Gamma$ to emphasize a choice of orientation.

2.3. **The set of immersed pants in a given Riemann surface.** From now on $\mathbf{S}=\mathbb{H}^2/\mathcal{G}$ is a fixed closed Riemann surface and $\mathcal{G}$ a suitable Fuchsian group. By Γ we denote the collection of oriented closed geodesics in $\mathbf{S}$. By $-\gamma$ we denote the opposite orientation of an oriented geodesic $\gamma\in\Gamma$.

Let Π_0 denote a hyperbolic pair of pants (that is Π is equipped with the hyperbolic metric such that the cuffs of Π_0 are geodesics). Let $f : \Pi_0 \to \mathbf{S}$ be a local isometry (such f must be an immersion). We say that $\Pi = f(\Pi_0)$ is an immersed pair of pants in $\mathbf{S}$. The set of all immersed pants in $\mathbf{S}$ is denoted by $\mathbf{\Pi}$. Let C^* denote an oriented cuff of Π_0 (the geodesic C^* is oriented as a boundary component of Π_0). Set $f(C^*) = \gamma \in \Gamma$. We say that γ is an oriented cuff of Π. The set of such pairs (Π, γ) is called the set of marked immersed pants and denoted by $\mathbf{\Pi}^*$. The half-length $\mathbf{hl}(\gamma)$ associated to the cuff γ of Π is defined as the half-length $\mathbf{hl}(C)$ associated to the cuff C of Π_0.

Let $\gamma \in \Gamma$ be a closed oriented geodesic in $\mathbf{S}$. Denote by $N^1(\gamma)$ the unit normal bundle of γ. Elements of $N^1(\gamma)$ are pairs (p, v), where $p \in \gamma$ and v is a unit vector at p that is orthogonal to γ. The bundle $N^1(\gamma)$ is a differentiable manifold that has two components which we denote by $N^1_R(\gamma)$ and $N^1_L(\gamma)$ (the right-hand side and the left-hand side components). Each component inherits the metric from the geodesic γ, and both $N^1_R(\gamma)$ and $N^1_L(\gamma)$ are isometric (as metric spaces) to the circle of length $2\mathbf{hl}(\gamma)$. By dis we denote the corresponding distance functions on $N^1_R(\gamma)$ and $N^1_L(\gamma)$.

Let $(p, v) \in N^1(\gamma)$, and denote by $(p_1, v_1) \in N^1(\gamma)$ the pair such that (p, v) and (p_1, v_1) belong to the same component of $N^1(\gamma)$, and $\operatorname{dis}(p, p_1) = \mathbf{hl}(\gamma)$. Set $A(p, v) = (p_1, v_1)$. Then A is an involution that leaves invariant each component of $N^1(\gamma)$. Define the bundle $N^1(\sqrt{\gamma}) = N^1(\gamma)/A$. The two components are denoted by $N^1_R(\sqrt{\gamma})$ and $N^1_L(\sqrt{\gamma})$, and both are isometric (as metric spaces) to the circle of length $\mathbf{hl}(\gamma)$. The disjoint union of all such bundles is denoted by $N^1(\sqrt{\Gamma})$.

Let $\Pi \in \mathbf{\Pi}$ be an immersed pants and $f : \Pi_0 \to \Pi$ the corresponding local isometry. Let C^* denote an oriented cuff of Π_0 and $\gamma = f(C^*)$. Let C_1 and C_2 denote the other two cuffs of Π_0, and let $p'_1, p'_2 \in C^*$ denote the two points that are the feet of the shortest geodesic segments in Π_0 that connect C and C_1, and C and C_2, respectively. Let v'_1 denote the unit vector at p'_1, that is orthogonal to C and points towards the interior of Π_0. We define v'_2 similarly. Set $(p_1, v_1) = f_*(p'_1, v'_1)$ and $(p_2, v_2) = f_*(p'_2, v'_2)$. Then (p_1, v_1) and (p_2, v_2) are in the same component of $N^1(\gamma)$, and the points p_1 and p_2 separate γ into two intervals of length $\mathbf{hl}(\gamma)$. Therefore, the vectors (p_1, v_1) and (p_2, v_2) represent the same point $(p, v) \in N^1(\sqrt{\Gamma})$, and we set

$$\operatorname{foot}(\Pi, \gamma) = (p, v) \in N^1(\gamma).$$

We call the vector (p, v) the foot of the immersed pair of pants Π at the cuff γ. This defines the map

$$\operatorname{foot} : \mathbf{\Pi}^* \to N^1(\sqrt{\Gamma}).$$

2.4. **Measures on pants and the $\widehat{\partial}$ operator.** By $\mathcal{M}(\mathbf{\Pi})$ we denote the space of real valued Borel measures with finite support on the set of immersed pants $\mathbf{\Pi}$, and likewise, by $\mathcal{M}(N^1(\sqrt{\Gamma}))$ we denote the space of real valued Borel measures with compact support on the manifold $N^1(\sqrt{\Gamma})$ (a measure from $\mathcal{M}^+(N^1(\sqrt{\Gamma}))$ has a compact support if and only if its support is contained in at most finitely many bundles $N^1(\sqrt{\gamma}) \subset N^1(\sqrt{\Gamma})$). By $\mathcal{M}^+(\mathbf{\Pi})$ and $\mathcal{M}^+(N^1(\sqrt{\Gamma}))$, we denote the corresponding spaces of positive measures.

We define the operator

$$\widehat{\partial} : \mathcal{M}(\mathbf{\Pi}) \to \mathcal{M}(N^1(\sqrt{\Gamma})),$$

as follows. The set $\mathbf{\Pi}$ is a countable set, so every measure from $\mu \in \mathcal{M}(\mathbf{\Pi})$ is determined by its value $\mu(\Pi)$ on every $\Pi \in \mathbf{\Pi}$. Let $\Pi \in \mathbf{\Pi}$ and let $\gamma_i \in \Gamma$, $i = 0, 1, 2$, denote the corresponding oriented geodesics so that $(\Pi, \gamma_i) \in \mathbf{\Pi}^*$. Let $\alpha_i^{\Pi} \in \mathcal{M}(N^1(\sqrt{\Gamma}))$ be the atomic measure supported at the point $\operatorname{foot}(\Pi, \gamma_i) \in N^1(\sqrt{\gamma_i})$, where the mass of the atom is equal to 1. Let

$$\alpha^{\Pi} = \sum_{i=0}^{2} \alpha_i^{\Pi},$$

and define

$$\widehat{\partial}\mu = \sum_{\Pi \in \mathbf{\Pi}} \mu(\Pi)\alpha^{\Pi}.$$

We call this the $\widehat{\partial}$ operator on measures. If $\mu \in \mathcal{M}^+(\mathbf{\Pi})$, then $\widehat{\partial}\mu \in \mathcal{M}^+(N^1(\sqrt{\Gamma}))$, and the total measure of $\widehat{\partial}\mu$ is three times the total measure of μ.

We recall the notion of equivalent measures from Section 3 in [5]. Let (X, d) be a metric space. By $\mathcal{M}^+(X)$ we denote the space of positive, Borel measures on X with compact support. For $A \subset X$ and $\delta > 0$ let

$$\mathcal{N}_\delta(A) = \{x \in X : \quad \text{there exists} \quad a \in A \quad \text{such that} \quad d(x, a) \le \delta\},$$

be the δ-neighbourhood of A.

Definition 2.1. *Let $\mu, \nu \in \mathcal{M}^+(X)$ be two measures such that $\mu(X) = \nu(X)$, and let $\delta > 0$. Suppose that for every Borel set $A \subset X$ we have $\mu(A) \le \nu(\mathcal{N}_\delta(A))$. Then we say that μ and ν are δ-equivalent measures.*

In our applications X will be either a 1-torus (a circle) or $\mathbb{R}$. In this case, $\mu(A) \le \nu(\mathcal{N}_\delta(A))$ for all Borel sets A if it holds for all intervals A.

Let $\gamma \in \Gamma$ and $\alpha \in \mathcal{M}(N^1(\sqrt{\gamma}))$. The bundle $N^1(\sqrt{\gamma})$ has the two components $N^1_+(\sqrt{\gamma})$ and $N^1_-(\sqrt{\gamma})$ (the right-hand and left-hand side components), each isometric to the circle of length $\mathbf{hl}(\gamma)$. The restrictions of α to $N^1_+(\sqrt{\gamma})$ and $N^1_-(\sqrt{\gamma})$ are denoted by α_+ and α_- respectively. In particular, by $\widehat{\partial}_+\mu$ and $\widehat{\partial}_-\mu$ we denote the decomposition of the measure $\widehat{\partial}\mu$.

Definition 2.2. *Fix $\gamma \in \Gamma$, and let $\alpha, \beta \in \mathcal{M}^+(N^1(\sqrt{\gamma}))$. We say that α and β are δ-equivalent if the pairs of measures α_+ and β_+, and α_- and β_-, are respectively δ-equivalent. Also, by $\lambda(\gamma) \in \mathcal{M}^+(N^1(\sqrt{\gamma}))$, we denote the measure whose components $\lambda_+(\gamma)$ and $\lambda_-(\gamma)$, are the standard Lebesgue* *measures on the metric spaces* $N^1(\sqrt{\gamma})^+$ *and* $N^1(\sqrt{\gamma})^-$*, respectively. In particular, the measure* $\lambda(\gamma)$ *is invariant under the full group of isometries of* $N^1(\sqrt{\gamma})$.

Let $\epsilon, R > 0$. By $\Gamma_{\epsilon,R} \subset \Gamma$ we denote the closed geodesics in the Riemann surface $\mathbf{S}$ whose half-length is in the interval $[R - \epsilon, R + \epsilon]$. We define $\mathbf{\Pi}_{\epsilon,R} \subset \mathbf{\Pi}$, as the set of immersed pants whose cuffs are in $\Gamma_{\epsilon,R}$. Our aim is to prove the following theorem, which in turn yields the proof of Theorem 2.2 stated above.

We adopt the following convention. In the rest of the paper by $P(R)$ we denote a polynomial in R whose degree and coefficients depend only on the choice of ϵ and the surface $\mathbf{S}$.

Theorem 2.3. *Let* $\epsilon > 0$*. There exists* $q > 0$ *and (the constants* q *only depends on the surface* $\mathbf{S}$ *and* ϵ*), so that for every* $R > 0$ *large enough, there exists a measure* $\mu \in \mathcal{M}^+(\mathbf{\Pi}_{\epsilon,R})$ *with the following properties. Let* $\gamma \in \Gamma_{\epsilon,R}$ *and let* $\widehat{\partial}\mu(\gamma)$ *denote the restriction of* $\widehat{\partial}\mu$ *to* $N^1(\sqrt{\gamma})$*. If* $\widehat{\partial}\mu(\gamma)$ *is not the zero measure then there exists a constant* $K_\gamma > 0$ *such that the measures* $\widehat{\partial}\mu(\gamma)$ *and* $K_\gamma \lambda(\gamma)$ *are* $P(R)e^{-qR}$*-equivalent.*

Remark. *We say that* $\alpha \in \mathcal{M}(N^1(\sqrt{\gamma}))$ *is* $\delta > 0$ *symmetric if for every isometry* $\iota : N^1(\sqrt{\gamma}) \to N^1(\sqrt{\gamma})$ *the measures* α *and* $\iota_* \alpha$ *are* δ*-equivalent. If* α *and* $K_\gamma \lambda(\gamma)$ *are* δ*-equivalent then* α *is* 2δ *symmetric because* $\lambda(\gamma)$ *is* 0 *symmetric.*

The proof of Theorem 2.2 follows from Theorem 2.3 in the same way as it was done in Section 3 in [5]. The brief outline is as follows. We may assume that the measure μ from the above theorem has integer coefficients. Then we may think of μ as a formal sum of immersed pants such that the restriction of the measure $\widehat{\partial}\mu$ on any $N^1(\sqrt{\gamma})$ is $P(R)e^{-qR}$-equivalent with some multiple of the Lebesgue measure (unless the restriction is the zero measure). Considering μ as the multiset (formally one may use the notion of a labelled collection of immersed pants) we then define a pairing between marked immersed pants, such that two marked pants (Π_1, γ_1) and (Π_2, γ_2), are paired if $\gamma_1 = -\gamma_2$, and the corresponding twist parameter between these two pairs is $P(R)e^{-pR}$ close to $+1$, for some universal constant $p > 0$. Such a pairing may be constructed using the Hall Marriage theorem (see Section 3 in [5]), and it possible because there is the same number of pants on either side of γ_1, and because the measure $\widehat{\partial}\mu$ is $P(R)e^{-qR}$-equivalent with some multiple of the Lebesgue measure. After gluing all the marked pants we have paired we obtain the finite cover from Theorem 2.2.

3. Equidistribution and self-Correction

We introduce in this section the Equidistribution Theorem (Theorem 3.1) and the Correction Theorem (Theorem 3.3), and we use them to prove Theorem 2.3. Theorem 3.1 follows from our previous work [5], and provides us with an evenly distributed collection of good pants. The Correction Theorem allows us to correct the slight imbalance (as described in the introduction) that may be found in the original collection of pants.

3.1. **Generating essential immersed pants in S.** Let Π denote a pair of pants whose three cuffs have the same length, and let $\omega : \Pi \to \Pi$ denote the standard (orientation preserving) isometry of order three that permutes the cuffs of Π. Let a and b be the fixed points of ω. Let $\gamma \subset \Pi$ denote a simple oriented geodesic arc that starts at a and terminates at b. Set $\omega_0^i(\gamma_0) = \gamma_i$. The union of two different arcs γ_i and γ_j is a closed curve in Π homotopic to a cuff. One can think of the union of these three segments as the spine of Π. Moreover, there is an obvious projection from Π to the spine $\gamma = \gamma_0 \cup \gamma_1 \cup \gamma_2$, and this projection is a homotopy equivalence.

Let p and q be two (not necessarily) distinct points in $\mathbf{S}$, and let α_0, α_1, and α_2, denote three distinct oriented geodesic arcs, each starting at p and terminating at q. We let $\alpha = \alpha_0 \cup \alpha_1 \cup \alpha_2$. Let $i(\alpha_j) \in T^1_p\mathbf{S}$ and $t(\alpha_j) \in T^1_q\mathbf{S}$ denote the initial and terminal unit tangent vectors to α_j at p and q respectively. Suppose that the triples of vectors $(i(\alpha_0), i(\alpha_1), i(\alpha_2))$ and $(t(\alpha_0), t(\alpha_1), t(\alpha_2))$, have opposite cyclic orders on the unit circle.

We define the map $f : \Pi \to \mathbf{S}$ by first projecting the pants Π onto its spine γ, and then by mapping γ_j onto α_j by a map that is a local (orientation preserving) homeomorphism. Then the induced conjugacy class of maps $f_* : \pi_1(\Pi) \to \pi_1(\mathbf{S})$ is injective.

Moreover we can homotop the map f to an immersion $g : \Pi \to \mathbf{S}$ as follows. We can write the pants Π as a (non-disjoint) union of three strips G_0, G_1, G_2, where each G_i is a fattening of the geodesic arc γ_i. Then we define a map $g_i : G_i \to \mathbf{S}$ to be a local homeomorphism on each G_i by extending the restriction of the map f on γ_i. The condition on the cyclic order of the α_i's at the two vertices enables us to define g_i and g_j on G_i and G_j respectively, so that $g_i = g_j$ on $G_i \cap G_j$. Then we set $g = g_i$.

We say that $g : \Pi \to \mathbf{S}$ is the essential immersed pair of pants generated by the three geodesic segments α_0, α_1 and α_2.

Often we will be given two geodesic segments, say α_0 and α_1, and then find a third geodesic segment α_2 so to obtain an essential immersed pair of pants. We then say that α_2 is a third connection. In this paper we will often be given a closed geodesic C on a Riemann surface $\mathbf{S}$, with two marked points $p, q \in C$. Then every geodesic arc α between p and q, that meets p and q at different sides of C will be called a third connection, since then C and α generate an immersed pair of pants as described above. In particular, this represents an efficient way of generating pants that contain a given closed geodesic C as its cuff.

3.2. **Preliminary lemmas.** Let $T^1\mathbb{H}^2$ denote the unit tangent bundle. Elements of $T^1(\mathbb{H}^2)$ are pairs (p, u), where $p \in \mathbb{H}^2$ and $u \in T^1_p\mathbb{H}^2$. Sometimes we write $u = (p, u)$ and refer to u at a unit vector in $T^1_p \mathbb{H}^2$. By $T^1\mathbf{S}$ we denote the unit tangent bundle over $\mathbf{S}$. For $u, v \in T^1_p \mathbb{H}^2$ we let $\mathbf{\Theta}(u, v)$ denote the unoriented angle between u and v. The function $\mathbf{\Theta}$ takes values in the interval $[0, \pi]$.

For $L, \epsilon > 0$, and $(p, u), (q, v) \in T^1\mathbf{S}$, we let $\mathrm{Conn}_{\epsilon,L}((p, u), (q, v))$ be the set of unit speed geodesic segments $\gamma : [0, l] \to \mathbf{S}$ such that

- $\gamma(0) = p$, and $\gamma(l) = q$,
- $|l - L| < \epsilon$,
- $\mathbf{\Theta}(u, \gamma'(0)), \mathbf{\Theta}(v, \gamma'(l)) < \epsilon$.

The next lemma will be referred to as the Connection Lemma.

Lemma 3.1. *Given $\epsilon > 0$, we can find $L_0 = L_0(\mathbf{S}, \epsilon)$ such that for any $L > L_0$, the set $\mathrm{Conn}_{\epsilon,L}$ is non-empty and*

$$\left|\mathrm{Conn}_{\epsilon,L}\right| \geq e^{L-L_0}.$$

Proof. The unit tangent bundle $T^1\mathbf{S}$ is naturally identified with $G\backslash\mathbf{PSL}(2, \mathbb{R})$, where G is a lattice. By dis we denote a distance function on $T^1\mathbf{S}$ (we defined dis explicitly later in the paper). Then we can find a neighbourhood U of the identity in $\mathbf{PSL}(2, \mathbb{R})$ so that if $(q, v) \in T^1\mathbf{S} = G\backslash\mathbf{PSL}(2, \mathbb{R})$, and $u \in U$, then $\mathrm{dis}((q, v), (q, v) \cdot u) < \frac{\epsilon}{16}$.

We let $\psi : U \to [0, \infty)$ be a C^∞ function with compact support in U, with $\int_U \psi = 1$. For any $(q, v) \in T^1\mathbf{S}$ we let $\mathcal{N}_U(q, v) = \{(q, v) \cdot s : s \in U\}$, and let $\psi_{(q,v)}((q, v) \cdot s) = \psi(s)$ on $\mathcal{N}_U(q, v)$. (If ϵ is small, then $s \mapsto (q, v) \cdot s$ is injective). The C^k norm of $\psi_{(q,v)}$ is independent of (q, v).

Let $g_t : T^1\mathbf{S} \to T^1\mathbf{S}$ be the geodesic flow. By uniform mixing for uniformly C^∞ functions on $\mathbf{S}$,

$$\int_{T^1\mathbf{S}} \psi_{(q,v)}(g_t(x, w))\psi_{(p,u)}(x, w)\, d(x, w) \to \frac{1}{2\pi^2|\chi(\mathbf{S})|}, \tag{1}$$

uniformly in (p, u) and (q, v), as $t \to \infty$. If $\psi_{(q,v)}(g_t(x, w))\psi_{(p,u)}(x, w) > 0$, then $(x, w) \in \mathcal{N}_U(p, u)$ and $g_t(x, w) \in \mathcal{N}_U(q, v)$.

The segment $g_{[0,t]}(x, w)$ is ϵ-nearly homotopic (see the the definition after this proof) to a unique geodesic segment α connecting p and q. The reader can verify that $\alpha \in \mathrm{Conn}_{\epsilon,t}((p, u), (q, v))$. We let $E_\alpha \subset \mathcal{N}_U(p, u)$ be the set of (x, w) for which $g_t(x, w) \in \mathcal{N}_U(q, v)$, and $g_{[0,t]}(x, w)$ is ϵ-homotopic to α. Then by (1),

$$\sum_\alpha \int_{E_\alpha} \psi_{(q,v)}(g_t(x, w))\psi_{(p,u)}(x, w)\, d(x, w) = \frac{1 + o(1)}{2\pi^2|\chi(\mathbf{S})|}.$$

On the other hand, we can easily verify that $V(E_\alpha) \leq K(\psi)e^{-t}$ (here $V(E_\alpha)$ is the volume of E_α). Hence

$$\int_{E_\alpha} \psi_{(q,v)}(g_t(x,w))\psi_{(p,u)}(x,w)\,d(x,w) \leq \int_{E_\alpha} (\sup_U \psi)^2\,d(x,w)$$
$$\leq K(\psi)e^{-t}.$$

So the number of good α is at least $K(\psi)e^t$, as long as t is large given $\mathbf{S}$ and ϵ.

□

Definition 3.1. *Let $E \geq 0$. We say that two geodesic segments A and B in $\mathbb{H}^2$ are E-nearly homotopic if the distance between the endpoints of the segments A and B is at most E. Two geodesic segments on the closed surface $\mathbf{S}$ are E nearly homotopic if they have lifts to $\mathbb{H}^2$ that are E nearly homotopic.*

Lemma 3.2. *Let $\epsilon > 0$. We let $\gamma \in \Gamma_{\epsilon,R}$ and let $\mathbf{\Pi}_{\epsilon,R}(\gamma)$ denote the set of pants in $\mathbf{\Pi}_{\epsilon,R}$ that contain γ as a cuff. Then*

$$|\mathbf{\Pi}_{\epsilon,R}(\gamma)| \asymp Re^R$$

where the constant for $\asymp$ depends only on $\mathbf{S}$ and ϵ.

Proof. For the upper bound, let F_γ denote a set of $\lceil 2R \rceil$ evenly distributed points on γ. If $\Pi \in \mathbf{\Pi}_{\epsilon,R}$ and if $\gamma \in \partial\Pi$, we let α be the geodesic segment in Π that is orthogonal to γ at its endpoints and simple on Π. Then we let α' be a geodesic segment connecting two points of F_γ, such that α' is $\frac{1}{2}$-nearly homotopic to α, and hence $\mathbf{l}(\alpha') \leq \mathbf{l}(\alpha) + 1$ and α'.

If two pants produce identical α', then the two pants are the same. Between any two points of F_γ there are at most Ne^R such arcs α' (the constant N depends only on $\mathbf{S}$), and the endpoints of α' are $R \pm 1$ apart, so the total number of such arcs is at most $10NRe^R$.

For the lower bound: By Lemma 3.1 we can find $N(\mathbf{S},\epsilon)e^R$ geodesic segments $\widehat{\alpha}$ connecting two given diametrically opposite points of γ, of length within $\frac{\epsilon}{100}$ of $R+\log 4$, and such that the angle between $\widehat{\alpha}$ and γ is within $\frac{\epsilon}{100}$ of $\frac{\pi}{2}$. Here we assume that the two vectors (at the two diametrically opposite points) at γ that are tangent to $\widehat{\alpha}$ are on the same side of γ. Then to any such $\widehat{\alpha}$ there is an $\frac{\epsilon}{10}$-nearly homotopic α with endpoints on γ (homotopic on $\mathbf{S}$ through arcs with endpoints on γ) and such that α is orthogonal to γ. Each such α produces a pair of pants $\Pi \in \mathbf{\Pi}_{\epsilon,R}$ that contains γ as a cuff.

Different α's give different pants. Two $\widehat{\alpha}$ with the same α must have nearby endpoints along γ and be 10ϵ nearly homotopic. So we get at least

$$\frac{2RN(\mathbf{S},\epsilon)}{10\epsilon}e^R$$

of the α's, and hence of the pants. □

Remark. *Let $M > 1$ and let $X_\gamma(M)$ denote the number of pants in $\mathbf{\Pi}_{\epsilon,R}(\gamma)$ whose two other cuffs are in $\Gamma_{\frac{\epsilon}{M},R}$. Then $X_\gamma(M) \asymp Re^R$. The upper bound* *follows from the upper bound of the lemma. If the segment $\widehat{\alpha}$ is of length within $\frac{\epsilon}{100M}$ of $2R - \mathbf{hl}(\gamma) + \log 4$, and if that the angle between $\widehat{\alpha}$ and γ is within $\frac{\epsilon}{100}$ of $\frac{\pi}{2}$, then the induced pants have the desired property that the other two cuffs are in $\Gamma_{\frac{\epsilon}{M},R}$. On the other hand it follows from the Connection Lemma that the number of such $\widehat{\alpha}$'s is $\asymp$ to $N(\mathbf{S},\epsilon,M)e^R$ for some constant $N(\mathbf{S},\epsilon,M)$.*

3.3. **The Equidistribution Theorem.** The following is the Equidistribution theorem, which was essentially proved in our previous paper [5].

Theorem 3.1. *Let $\epsilon > 0$. There exist $q > 0$ and $D_1, D_2, D_3, D_4 > 0$ (the constants q and D_i depend only on $\mathbf{S}$ and ϵ) so that for every $R > 0$ large enough, there exists a measure $\mu \in \mathcal{M}^+(\mathbf{\Pi}_{\epsilon,R})$ with the following properties:*

1. $\mu(\mathbf{\Pi}_{\epsilon,R}) = 1$,
2. *$\mu(\Pi) < D_1 e^{-3R}$ for every $\Pi \in \mathbf{\Pi}_{\epsilon,R}$, and $\mu(\Pi) > D_2 e^{-3R}$ for every $\Pi \in \mathbf{\Pi}_{\frac{\epsilon}{2},R}$,*
3. *For every $\gamma \in \Gamma_{\epsilon,R}$ the measure $\widehat{\partial}_{\pm}\mu(\gamma)$ is $P(R)e^{-qR}$ equivalent to $K^{\pm}_{\gamma}\lambda_{\pm}(\gamma)$, for some constants K^+_γ and K^-_γ that satisfy the inequality*
$$\left|\frac{K^+_\gamma}{K^-_\gamma} - 1\right| < P(R)e^{-qR},$$
4. *Moreover, $K^{\pm}_\gamma < D_3 e^{-2R}$ for $\gamma \in \Gamma_{\epsilon,R}$, and $K^{\pm}_\gamma > D_4 e^{-2R}$ for $\gamma \in \Gamma_{\frac{\epsilon}{2},R}$.*

Proof. The existence of such measure μ that satisfies the conditions (1) and (3) was proved in Theorem 3.4 in [5] (see the remark after the statement of Theorem 3.4).

□

3.4. **The Good Correction Theorem.**

Definition 3.2. *Let $s_0, s_1 \in \mathbb{R}\Gamma_{\epsilon,R}$ and let $M \geq 1$. We say that $s_0 = s_1$ in $\mathbf{\Pi}_{M\epsilon,R}$ homology if there exists $W \in \mathbb{R}\mathbf{\Pi}_{M\epsilon,R}$ such that $\partial W = s_1 - s_0$.*

Let $M > 1$. The following lemma states that any curve $\gamma \in \Gamma_{\epsilon,R}$ is homologous to some $s \in \mathbb{R}\Gamma_{\frac{\epsilon}{M},R}$ in $\mathbf{\Pi}_{\epsilon,R}$ homology.

Lemma 3.3. *Let $\epsilon, M > 0$. Then there exists $R_0 > 0$ such that for every $R > R_0$ we can find a map $q_M = q : \Gamma_{\epsilon,R} \to \mathbb{Q}^+\mathbf{\Pi}_{\epsilon,R}$ such that*

1. *For every $\gamma \in \Gamma_{\epsilon,R}$, $q(\gamma)$ is a positive sum of pants, all of which have γ as one boundary cuff (with the appropriate orientation), and two other cuffs in $\Gamma_{\frac{\epsilon}{M},R}$, that is $\gamma - \partial q(\gamma) \in \mathbb{Q}^+\Gamma_{\frac{\epsilon}{M},R}$,*
2. *For every $\Pi \in \mathbf{\Pi}_{\epsilon,R}$, we have*
$$\sum_{\gamma \in \Gamma_{\epsilon,R}} |q(\gamma)(\Pi)| \leq \frac{K}{R}e^{-R},$$
for some constant $K \equiv K(\mathbf{S},\epsilon,M) > 0$, where $q(\gamma)(\Pi) \in \mathbb{Q}^+$ is the coefficient of Π in $q(\gamma)$.

Proof. It follows from the remark after Lemma 3.2 that the number of pants which have γ as one boundary cuff, and two other cuffs in $\Gamma_{\epsilon,\frac{R}{M}}$ is of the order Re^R. Let $q(\gamma)$ be the average of these pants (the average of a finite set S is the formal sum of elements from S where each element in the sum has the weight $\frac{1}{|S|}$). The inequality in the condition (2) follows from the fact that for any $\Pi \in \mathbf{\Pi}_{1,R}$ the sum

$$\sum_{\gamma \in \Gamma_{\epsilon,R}} |q(\gamma)(\Pi)|,$$

has at most 3 non-zero terms.

□

The following theorem summarizes the main idea of this paper. It implies that every sum of good curves that is zero in the standard homology is the boundary of a sum of good pants. That is, if $s_0, s_1 \in \mathbb{R}\Gamma_{\epsilon,R}$ and $s_0 = s_1$ in the standard homology on the surface $\mathbf{S}$, then $s_0 = s_1$ in $\mathbf{\Pi}_{300\epsilon,R}$ homology. By $\mathbf{H_1}(\mathbf{S})$ we denote the first homology on $\mathbf{S}$ with rational coefficients.

Theorem 3.2. *Let $\epsilon > 0$. There exists $R_0 = R_0(\mathbf{S}, \epsilon) > 0$ such that for every $R > R_0$ the following holds. There exists a set $H = \{h_1, ..., h_{2g}\} \subset \mathbb{Q}\Gamma_{\epsilon,R}$ of linearly independent elements of $\mathbf{H_1}(\mathbf{S})$, such that for every $\gamma \in \Gamma_{\epsilon,R}$ there are $a_i \in \mathbb{Q}$ so that*

$$\gamma = \sum_{i=1}^{2g} a_i h_i$$

in $\mathbf{\Pi}_{300\epsilon,R}$ homology.

The proof of this theorem occupies the primary text of sections 4-9. But to prove the Ehrenpreis conjecture we require the following stronger result.

Theorem 3.3. *Let $\epsilon > 0$. There exists $R_0 > 0$ (that depend only on $\mathbf{S}$ and ϵ) such that for every $R > R_0$ there exists a set $H = \{h_1, ..., h_{2g}\} \subset \mathbb{Q}\Gamma_{\epsilon,R}$ and a map $\phi : \Gamma_{\epsilon,R} \to \mathbb{Q}\mathbf{\Pi}_{300\epsilon,R}$ such that*

1. *$h_1, ..., h_{2g}$ is a basis for $\mathbf{H_1}(\mathbf{S})$,*
2. *$\partial(\phi(\gamma)) - \gamma \in \mathbb{Z}H$,*
3. $$\sum_{\gamma \in \Gamma_{\epsilon,R}} |\phi(\gamma)(\Pi)| < P(R)e^{-R}.$$

Remark. *Note that the map ϕ depends on ϵ and R, so sometimes we write $\phi = \phi_{\epsilon,R}$.*

The existence of the function ϕ that satisfies the conditions (1) and (2) follows from Theorem 3.2. The condition (3) will be proved using our randomization theory (see the Appendix). As we go along, after every relevant homological statement we make Randomization remarks. Theorem 3.3 then follows from the these randomization remarks as we explain at the end of Section 9.

Remark. *Randomization remark for Theorem 3.3. The estimate (3) in the statement of the theorem can be reformulated as follows. Consider the standard measures σ_Γ on $\Gamma_{\epsilon,R}$ and $\sigma_{\mathbf{\Pi}}$ on $\mathbf{\Pi}_{300\epsilon,R}$. Then the map ϕ is $P(R)$-semirandom with respect to σ_Γ and $\sigma_{\mathbf{\Pi}}$. See the Appendix for definitions of the standard measures and the notion of semirandom maps.*

3.5. **A proof of Theorem 2.3.** First we state and prove the following lemma about equivalent measures on the circle $\mathbb{R}/2R\mathbb{Z}$, where $R > 0$ is a parameter. Recall that λ denotes the Lebesgue measure on the circle $\mathbb{R}/2R\mathbb{Z}$.

Lemma 3.4. *If α is δ equivalent to $K\lambda$ on $\mathbb{R}/2R\mathbb{Z}$, for some $K > 0$, and β is a measure on $\mathbb{R}/2R\mathbb{Z}$, then $\alpha+\beta$ is $(\frac{|\beta|}{2K}+\delta)$-equivalent to $(K+\frac{|\beta|}{2R})\lambda$ on $\mathbb{R}/2R\mathbb{Z}$.*

Proof. Recall the definition of δ-equivalent measures from the previous section. We need to prove that $(\alpha+\beta)(I) \leq (K+\frac{|\beta|}{2R})|\mathcal{N}_{\delta'}(I)|$, where $\delta' = \delta+\frac{|\beta|}{2K}$, for any interval I such that δ' neighbourhood of I is a proper subset of the circle $\mathbb{R}/2R\mathbb{Z}$.

Providing $2R > 1$, we have

$$
\begin{aligned}
(\alpha+\beta)(I) &\leq K(|I|+2\delta)+|\beta| \\
&\leq \left(K+\frac{|\beta|}{2R}\right)\left(|I|+2\delta+\frac{|\beta|}{K}\right) \\
&\leq (K+|\beta|)\left|\mathcal{N}_{\delta'}(I)\right|.
\end{aligned}
$$

□

We give the following definitions. To any measure $\alpha \in \mathcal{M}(N^1(\sqrt{\gamma}))$, we associate the number $\operatorname{sgn}(\alpha)$ defined as the total measure of α_+ minus the total measure of α_-. We also let $|\alpha| = |\alpha^+|+|\alpha^-|$.

Let ν be a measure on pants and set $\widehat{\partial}\nu(\gamma) = \alpha$. We let $\partial\nu = \operatorname{sgn}(\alpha)$, and $|\partial|\nu = |\alpha|$.

We proceed with the proof of Theorem 2.3. Let μ_0 be the measure on $\mathbf{\Pi}_{\epsilon,R}$ from Theorem 3.1. Then $|\partial\mu_0(\gamma)| < P(R)e^{-qR}|\partial|\mu_0(\gamma)$ by Theorem 3.1.

Let $\mu_1 = q_{600}(-\partial\mu_0)$, where q_{600} is the function q from Lemma 3.3. Let $\mu_2 = \phi(-\partial(\mu_0+\mu_1))$, where $\phi = \phi_{\frac{\epsilon}{100},R}$. Finally, we set $\mu = \mu_0+\mu_1+\mu_2$.

Observe that, for $\Pi \in \mathbf{\Pi}_{\epsilon,R}$

(2) $\mu_1(\Pi) \leq \frac{K}{R} e^{-R} \sup_\gamma |\partial \mu_0(\gamma)|$, by (2) of Lemma 3.3

(3) $\leq \frac{K}{R} e^{-R} P(R) e^{-qR} e^{-2R}$, by (3) and (4) of Theorem 3.1

(4) $= P(R) e^{-(3+q)R}$.

It then follows (using Lemma 3.2) that for any $\gamma \in \Gamma_{\epsilon,R}$

(5)
$$\partial \mu_1(\gamma) \leq P(R) e^{-(q+2)R}.$$

Secondly, we observe that

$\mu_2(\Pi) = \phi(-\partial(\mu_0 + \mu_1))(\Pi)$

(6) $\leq P(R) e^{-R} sup_\gamma \partial(\mu_0 + \mu_1)(\gamma)$, by (3) of Theorem 3.3

(7) $\leq P(R) e^{-R} e^{-qR} e^{-2R}$, by (5) and (3) and (4) of Theorem 3.1 .

We need to verify that μ satisfies the assumptions of Theorem 2.3. Observe that by (2) of Theorem 3.3 we have $\partial \mu = 0$, that is $\widehat{\partial}_+ \mu(\gamma)$ and $\widehat{\partial}_- \mu(\gamma)$ have the same total measure for every γ. It follows from (3) and (6) that μ is positive.

It remains to show that the restriction of $\widehat{\partial} \mu$ on $N^1(\sqrt{\gamma})$ is e^{-qR} equivalent to some multiple of the Lebesgue measure $\lambda(\gamma)$ on $N^1(\sqrt{\gamma})$. Observe

$$|(\widehat{\partial} \mu_1 + \widehat{\partial} \mu_2)(N^1(\sqrt{\gamma}))| \leq P(R) e^{-(2+q)R}.$$

If $\gamma \in \Gamma_{\epsilon,R}$ then

$$|\widehat{\partial} \mu_0(N^1(\sqrt{\gamma}))| \geq D e^{-2R}.$$

Since the restriction of $\widehat{\partial} \mu_0$ on $N^1(\sqrt{\gamma})$ is $P(R)e^{-qR}$-equivalent to some multiple of the Lebesgue measure $\lambda(\gamma)$ on $N^1(\sqrt{\gamma})$, it follows from the previous 3 inequalities that $\widehat{\partial} \mu$ is $P(R)e^{-qR}$ equivalent to some multiple of the Lebesgue measure $\lambda(\gamma)$ on $N^1(\sqrt{\gamma})$.

If $\gamma \in \Gamma_{\epsilon,R} \setminus \Gamma_{\frac{\epsilon}{10},R}$ then $\widehat{\partial} \mu_2(N^1(\sqrt{\gamma})) = 0$, and

$$\widehat{\partial} \mu_1(N^1(\sqrt{\gamma})) \leq P(R) e^{-qR} \widehat{\partial} \mu_0(N^1(\sqrt{\gamma})).$$

So $\widehat{\partial} \mu = \widehat{\partial} \mu_0 + \widehat{\partial} \mu_1$, is $P(R)e^{-qR}$ equivalent to a multiple of the Lebesgue measure, by Lemma 3.4.

4. The theory of inefficiency

In this section we develop the theory of inefficiency. This theory supports the geometric side of the correction theory that is used to prove our main technical result the Good Correction Theorem.

4.1. **The inefficiency of a piecewise geodesic arc.** By $T^1\mathbb{H}^2$ we denote the unit tangent bundle of $\mathbb{H}^2$. Elements of $T^1\mathbb{H}^2$ are pairs (p,u), where $p \in \mathbb{H}^2$ and $u \in T^1_p\mathbb{H}^2$. For $u,v \in T^1_p\mathbb{H}^2$ we let $\boldsymbol{\Theta}(u,v)$ denote the unoriented angle between u and v. Let $\alpha : [a,b] \to \mathbb{H}^2$ be a unit speed geodesic segment. We let $i(\alpha) = \alpha'(a)$, and $t(\alpha) = \alpha'(b)$.

Let $\alpha_1, ..., \alpha_n$ denote oriented piecewise geodesic arcs on $\mathbf{S}$ such that the terminal point of α_i is the initial point of α_{i+1}. By $\alpha_1\alpha_2...\alpha_n$ we denote the concatenation of the arcs α_i. If the initial point of α_1 and the terminal point of α_n are the same, by $[\alpha_1\alpha_2...\alpha_n]$ we denote the corresponding closed curve.

We define the inefficiency operator as follows. We first discuss the inefficiency of piecewise geodesic arcs and after that the inefficiency of piecewise geodesic closed curves.

Definition 4.1. *Let α be an arc on a surface. By γ we denote the appropriate geodesic arc, with the same endpoints and homotopic to α. We let $I(\alpha) = \mathbf{l}(\alpha) - \mathbf{l}(\gamma)$. We call $I(\alpha)$ the inefficiency of α (the inefficiency $I(\alpha)$ is equal to 0 if and only if α is a geodesic arc).*

We observe the monotonicity of inefficiency. Let $\alpha, \beta, \gamma \subset \mathbb{H}^2$ be three piecewise geodesic arcs in $\mathbb{H}^2$ such that $\alpha\beta\gamma$ is a well defined piecewise geodesic arc. Then $I(\alpha\beta\gamma) \geq I(\beta)$. This is seen as follows. Let η be the geodesic arc with the same endpoints as $\alpha\beta\gamma$, and let β' be the geodesic arc with the same endpoints as β. Then

$$\begin{aligned} I(\alpha\beta\gamma) &= \mathbf{l}(\alpha\beta\gamma) - \mathbf{l}(\eta) \\ &\geq \mathbf{l}(\alpha\beta\gamma) - \mathbf{l}(\alpha\beta'\gamma) \\ &= \mathbf{l}(\beta) - \mathbf{l}(\beta') \\ &= I(\beta). \end{aligned}$$

We also define the inefficiency function of an angle $\theta \in [0,\pi]$ as follows. Let α_∞ and β_∞ be two geodesic rays in $\mathbb{H}^2$ that have the same initial point, such that θ is the exterior angle between α_∞ and β_∞ (then θ is also the bending angle of the piecewise geodesic $\alpha_\infty^{-1}\beta_\infty$). We want to define $I(\theta)$ as the inefficiency of $\alpha_\infty^{-1}\beta_\infty$, but since the piecewise geodesic $\alpha_\infty^{-1}\beta_\infty$ is infinite in length we need to prove that such a definition is valid.

Consider a geodesic triangle in $\mathbb{H}^2$ with sides A, B and C, and let $\theta > 0$ be the exterior angle opposite to C (we also let $\mathbf{l}(A) = A$, $\mathbf{l}(B) = B$, $\mathbf{l}(C) = C$). Then

$$\cosh C = \cosh A \cosh B + \cos\theta \sinh A \sinh B,$$

and therefore

$$\frac{\cosh C}{e^{A+B}} = \frac{\cosh A}{e^A}\frac{\cosh B}{e^B} + \cos\theta = \frac{\sinh A}{e^A}\frac{\sinh B}{e^B}.$$

We conclude that

$$\frac{\cosh C}{e^{A+B}} \to \frac{1}{4}(\cos\theta + 1),$$

when $A, B \to \infty$. Since

$$\frac{\cosh C}{e^C} \to \frac{1}{2},\ C \to \infty,$$

we get

$$e^{C-A-B} \to \left(\cos\frac{\theta}{2}\right)^2,$$

and therefore

$$A + B - C \to 2\log\sec\frac{\theta}{2}.$$

Let $r, s > 0$ and let α_r be the geodesic subsegment of α_∞ (with the same initial point) of length r. Similarly, β_s is a geodesic subsegment of β_∞ of length s. Then we let

$$I(\theta) = \lim_{r,s\to\infty} I(\alpha_r^{-1}\beta_s).$$

It follows from the above discussion that this limit exists and

$$I(\theta) = 2\log\sec\frac{\theta}{2}. \tag{8}$$

4.2. **Preliminary lemmas.** We have the following lemma.

Lemma 4.1. *Let α denote an arc on $\mathbf{S}$, and let γ be the appropriate geodesic arc with the same endpoints as α and homotopic to α. Choose lifts of α and γ in the universal cover $\mathbb{H}^2$ that have the same endpoints and let $\pi : \alpha \to \gamma$ be the nearest point projection. Let*

$$E(\alpha) = \sup_{x\in\alpha} d(x, \pi(x)).$$

Then

$$E(\alpha) \le \frac{I(\alpha)}{2} + \log 2.$$

Proof. Let $E > 0$. The minimally inefficient arc α (that has the same endpoints as γ and is homotopic to γ), that is at the distance E from γ is given in Figure 1. Here γ is divided into two sub-segments of length L^- and L^+. Let $A^- = \mathbf{l}(\alpha^-)$ and $A^+ = \mathbf{l}(\alpha^+)$. By the monotonicity of inefficiency, and using the inefficiency for angles, we have

$$E + L^- - A^- < I(\frac{\pi}{2}),$$

and

$$E + L^+ - A^+ < I(\frac{\pi}{2}).$$

Summing up yields

$$E < \frac{I(\alpha)}{2} + I(\frac{\pi}{2}) = \frac{I(\alpha)}{2} + \log 2,$$

since by (8) we have $I(\frac{\pi}{2}) = \log 2$.

□

The following is the New Angle Lemma.

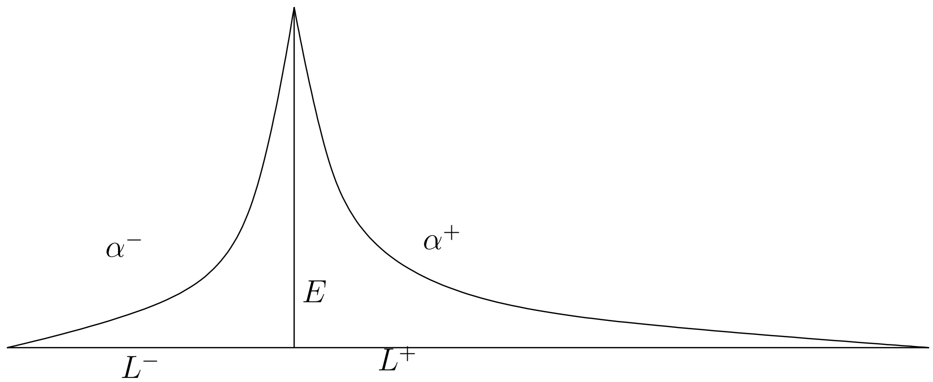

FIGURE 1. The case where γ is an arc

Lemma 4.2. *Let $\delta, \Delta > 0$ and let $\alpha\beta \subset \mathbb{H}^2$ be a piecewise geodesic arc, where α is piecewise geodesic arc and β is a geodesic arc. Suppose that γ is the geodesic arc with the same endpoints as $\alpha\beta$. There exists $L = L(\delta, \Delta) >$ such that if $\mathbf{l}(\beta) > L$ and $I(\alpha\beta) \leq \Delta$ then the unoriented angle between γ and β is at most δ.*

Proof. Denote by θ the angle between γ and β, and let h be the distance between the other endpoint of β and γ. Then

$$\frac{\sinh(h)}{\sin(\theta)} = \sinh(\mathbf{l}(\beta)).$$

The lemma follows from this equation and the fact that h is bounded in terms of $I(\alpha\beta)$ (see Lemma 4.1).

□

We also have

Lemma 4.3. *Suppose that $\alpha\beta\gamma$ is a concatenation of three geodesic arcs in $\mathbb{H}^2$, and let $\theta_{\alpha\beta}$ and $\theta_{\beta\gamma}$ be the two bending angles. Suppose that $\theta_{\alpha\beta}, \theta_{\beta\gamma} < \frac{\pi}{2}$. Then*

$$I(\alpha\beta\gamma) \leq \log(\sec(\theta_{\alpha\beta})) + \log(\sec(\theta_{\beta\gamma})).$$

Proof. Let η_1 be the geodesic that is orthogonal to β at the point where α and β meet. Similarly let η_2 be the geodesic that is orthogonal to β at the point where β and γ meet. Let A_α be the geodesic arc orthogonal to η_1 that starts at the initial point of α, and let A_γ be the geodesic arc orthogonal to η_2 that starts at the terminal point of γ.

Let η be the geodesic arc with the same endpoints as $\alpha\beta\gamma$. Then $\mathbf{l}(\eta) \leq \mathbf{l}(A_\alpha) + \mathbf{l}(\beta) + \mathbf{l}(A_\gamma)$. On the other hand, $\mathbf{l}(A_\alpha) > \mathbf{l}(\alpha) - \log(\sec(\theta_{\alpha\beta}))$, and similarly $\mathbf{l}(A_\gamma) > \mathbf{l}(\gamma) - \log(\sec(\theta_{\beta\gamma}))$. So

$$\begin{aligned} I(\alpha\beta\gamma) &< \mathbf{l}(\alpha) + \mathbf{l}(\gamma) - \mathbf{l}(A_\alpha) - \mathbf{l}(A_\gamma) \\ &< \log(\sec(\theta_{\alpha\beta})) + \log(\sec(\theta_{\beta\gamma})). \end{aligned}$$

□

4.3. **The Long Segment Lemmas for arcs.** We now state and prove several lemmas called the Long Segment Lemmas. The following is the Long Segment Lemma for angles.

Lemma 4.4. *Let $\delta >, \Delta > 0$. There exists a constant $L = L(\delta, \Delta) > 0$ such that if α and β are oriented geodesic arcs such that $I(\alpha\beta) \leq \Delta$ (assuming that the terminal point of α is the initial point of β) and $\mathbf{l}(\alpha), \mathbf{l}(\beta) > L$, then $I(\alpha\beta) < I(\boldsymbol{\Theta}(t(\alpha), i(\beta)) < I(\alpha\beta) + \delta$.*

Proof. The left hand side of the above inequality follows from the monotonicity of inefficiency. For the right hand side, let α_∞ and β_∞ denote the geodesic rays whose initial point is the point where α and β meet, and that contain α and β respectively (we also assume that α_∞ has the same orientation as α and β_∞ the same orientation as β). Let η be the geodesic arc with the same endpoints as $\alpha\beta$, and η_1 the geodesic ray with the same endpoints as $\alpha_\infty\beta$. By θ_0 we denote the angle between α and η, by θ_1 the angle between η and β, and by θ_2 the angle between η and η_1.

We observe that θ_0, θ_1 and θ_2 are small (by the New Angle Lemma), and therefore

$$\begin{aligned} I(\alpha_\infty\beta_\infty) &< I(\alpha_\infty\beta) + I(\theta_1 + \theta_2) \\ &< I(\alpha\beta) + I(\theta_0) + I(\theta_1 + \theta_2) \\ &< I(\alpha\beta) + o(1), \end{aligned}$$

where $o(1) \to 0$ for $L \to \infty$.

□

The following is the Long Segment Lemma for arcs.

Lemma 4.5. *Suppose we can write $\eta = \alpha\beta\gamma$, where α and γ are piecewise geodesic arcs, and β is a geodesic arc of length l. Then*

$$\left| I(\alpha\beta) + I(\beta\gamma) - I(\alpha\beta\gamma) \right| < \delta,$$

where $\delta \to 0$ when $l \to \infty$ and $I(\alpha\beta) + I(\beta\gamma)$ is bounded above.

Proof. If we replace α and γ by the associated geodesics arcs, then $I(\alpha\beta) + I(\beta\gamma) - I(\alpha\beta\gamma)$ will be unchanged, and $I(\alpha\beta) + I(\beta\gamma)$ will be decreased, so we can assume that α and β are geodesic arcs. We divide β at its midpoint into β^- and β^+, so $\beta = \beta^-\beta^+$, and $\alpha\beta\gamma = \alpha\beta^-\beta^+\gamma$. We will show the following estimates (for an appropriate δ):

(1) $\left| I(\alpha\beta^-) + I(\beta^+\gamma) - I(\alpha\beta\gamma) \right| < \delta$,
(2) $\left| I(\alpha\beta) + I(\alpha\beta^-)) \right| < \delta$,

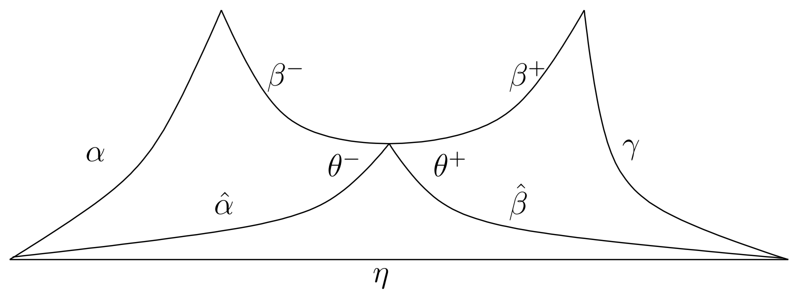

FIGURE 2. The Long Segment Lemma

(3) $\left|I(\beta\gamma) + I(\beta^+\gamma))\right| < \delta$,

The lemma then follows from (1), (2) and (3).

For (1), we refer to the Figure 2. We find that

$$0 \leq I(\widehat{\alpha}\widehat{\beta}) = I(\alpha\beta\gamma) - I(\alpha\beta^-) - I(\beta^+\gamma).$$

Moreover, when $I(\alpha\beta^-) \leq I(\alpha\beta)$ and $I(\beta^+\gamma) \leq I(\beta\gamma)$ are bounded above, and $\mathbf{l}(\beta^-) = \mathbf{l}(\beta^+) = \frac{\mathbf{l}(\beta)}{2}$ is a large, then θ^- and θ^+ are small (by the New Angle Lemma), so $I(\widehat{\alpha}\widehat{\beta}) \leq I(\theta^- + \theta^+)$ is small. Likewise, $I(\alpha\beta) - I(\alpha\beta^-) = I(\widehat{\alpha}\beta^+)$, and $0 \leq I(\widehat{\alpha}\beta^+) \leq I(\theta^-)$. This proves (1) and (2), and (3) is the same as (2).

□

4.4. **The inefficiency of a closed piecewise geodesic curve.** Let $\alpha_1, ..., \alpha_n$ denote oriented piecewise geodesic arcs on $\mathbf{S}$ such that the terminal point of α_i is the initial point of α_{i+1}. By $\alpha_1\alpha_2...\alpha_n$ we denote the concatenation of the arcs α_i. Assume that the initial point of α_1 and the terminal point of α_n are the same. By $[\alpha_1\alpha_2...\alpha_n]$ we denote the corresponding closed curve.

We define the inefficiency operator as follows.

Definition 4.2. *Let α be a closed curve on a surface. By γ we denote the appropriate closed geodesic that is freely homotopic to α. We let $I(\alpha) = \mathbf{l}(\alpha) - \mathbf{l}(\gamma)$. We call $I(\alpha)$ the inefficiency of α (the inefficiency $I(\alpha)$ is equal to 0 if and only if α is a closed geodesic).*

The following is a closed curve version of Lemma 4.1.

Lemma 4.6. *Let α denote a closed curve on $\mathbf{S}$, and let γ be the appropriate closed geodesic freely homotopic to α. Choose lifts of α and γ in the universal cover $\mathbb{H}^2$ that have the same endpoints and let $\pi : \alpha \to \gamma$ be the nearest point projection. Let*

$$E(\alpha) = \sup_{x \in \alpha} d(x, \pi(x)).$$

The providing $\mathbf{l}(\alpha) > L_0$ for some universal constant L_0 we have

$$E \leq \frac{I(\alpha)}{2} + 2.$$

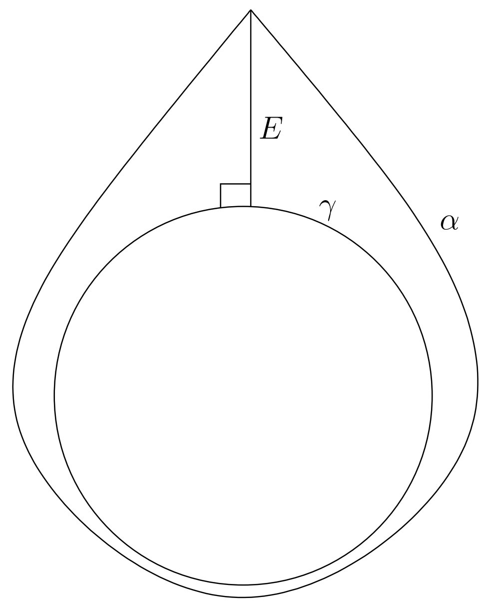

FIGURE 3. The case where γ is a closed curve

Proof. The minimally inefficient closed curve α that is freely homotopic to γ and the distance E from γ is given in Figure 3. Denote by η the corresponding geodesic segment that of length E. Then by the Long Segment Lemma and monotonicity of inefficiency

$$\begin{aligned} I(\eta\gamma\eta^{-1}) &\leq I(\eta\gamma) + I(\gamma\eta^{-1}) + \frac{1}{7} \\ &\leq I(\frac{\pi}{2}) + I(\frac{\pi}{2}) + \frac{1}{7} \\ &< 2, \end{aligned}$$

providing that $\mathbf{l}(\gamma) > L_0$, where L_0 is a universal constant. Hence

$$\mathbf{l}(\alpha) - \mathbf{l}(\gamma) \leq 2\mathbf{l}(\eta) + 2,$$

or

$$E \leq \frac{I(\alpha)}{2} + 1.$$

□

The following is the Long Segment Lemma for closed curves.

Lemma 4.7. *Let α be an piecewise geodesic arc and β an geodesic arc on $\mathbf{S}$, such that the initial point of α is the terminal point of β and the initial point of β is the terminal point of α. The*

$$|I([\alpha\beta]) - I(\beta\alpha\beta)| < \delta,$$

where $\delta \to 0$ when $\mathbf{l}(\beta) \to \infty$ and $I(\beta\alpha\beta)$ is bounded above.

Proof. The proof is similar to the proof of Lemma 4.5 and is left to reader. □

4.5. **The Sum of Inefficiencies Lemma.** The following is the Sum of Inefficiencies Lemma. Let $\mathbf{S}$ denote a closed hyperbolic Riemann surface

Lemma 4.8. *Let $\epsilon, \Delta > 0$ and $n \in \mathbb{N}$. There exists $L = L(\epsilon, \Delta, n) > 0$ such that the following holds. Let $\alpha_1, ..., \alpha_{n+1} = \alpha_1, \beta_1, ...\beta_n$, be geodesic arcs on the surface $\mathbf{S}$ such that $\alpha_1\beta_1\alpha_2\beta_2...\alpha_n\beta_n$ is a piecewise geodesic arc on $\mathbf{S}$. If $I(\alpha_i\beta_i\alpha_{i+1}) \le \Delta$, and $\mathbf{l}(\alpha_i) \ge L$, then*

$$\left| I([\alpha_1\beta_1\alpha_2\beta_2...\alpha_n\beta_n]) - \sum_{i=1}^{n} I(\alpha_i\beta_i\alpha_{i+1}) \right| \le \epsilon.$$

Proof. It directly follows from the Long Segment Lemma. □

Remark. *In particular, we can leave out the β's in the above lemma, and write*

$$\left| I([\alpha_1\alpha_2...\alpha_n]) - \sum_{i=1}^{n} I(\alpha_i\alpha_{i+1}) \right| \le \epsilon,$$

providing that $I(\alpha_i\alpha_{i+1}) \le \Delta$, and $\mathbf{l}(\alpha_i) \ge L$. Moreover, by the Long Segment Lemma for Angles (for L large enough) we have

$$\left| I([\alpha_1\alpha_2...\alpha_n]) - \sum_{i=1}^{n} I(\theta_i) \right| \le 2\epsilon,$$

where $\theta_i = \mathbf{\Theta}(t(\alpha_i), i(\alpha_{i+1}))$.

A more general version of the Sum of Inefficiencies Lemma is as follows (the proof is the same).

Lemma 4.9. *Let $\epsilon, \Delta > 0$ and $n \in \mathbb{N}$. There exists $L = L(\epsilon, \Delta, n) > 0$ such that the following holds. Let $\alpha_1, ..., \alpha_{n+1} = \alpha_1$ and $\beta_{11}, ...\beta_{1j_1}, ...\beta_{n1}, ...\beta_{nj_n}$, be geodesic segments on $\mathbf{S}$ such that $\alpha_1\beta_{11}...\beta_{1j_1}...\alpha_n\beta_{n1}...\beta_{nj_n}$ is a piecewise geodesic arc on $\mathbf{S}$. If $I(\alpha_i\beta_i\alpha_{i+1}) \le \Delta$, and $\mathbf{l}(\alpha_i) \ge L$, then*

$$\left| I([\alpha_1\beta_{11}...\beta_{1j_1}...\alpha_n\beta_{n1}...\beta_{nj_n}]) - \sum_{i=1}^{n} I(\alpha_i\beta_{i1}...\beta_{ij_i}\alpha_{i+1}) \right| \le \epsilon.$$

Proof. It directly follows from the Long Segment Lemma. □

5. The Geometric Square Lemma

From now on we can think of the surface $\mathbf{S}$ as being fixed. We also fix $\epsilon > 0$. However, for the reader's convenience we always emphasise how quantities may depend on $\mathbf{S}$ and ϵ.

5.1. **Notation and preliminary lemmas.** By an oriented closed geodesic C on $\mathbf{S}$ we will mean an isometric immersion $C : \mathbb{T}_C \to \mathbf{S}$, where $\mathbb{T}_C = \mathbb{R}/\mathbf{l}(C)$, and $\mathbf{l}(C)$ is the length of C. To simplify the notation, by $C : \mathbb{R} \to \mathbf{S}$ we also denote the corresponding lift (such a lift is uniquely determined once we fix a covering map $\pi : \mathbb{R} \to \mathbb{T}_C$). We call $\mathbb{T}_C$ the parametrising torus for C (because $\mathbb{T}_C = \mathbb{R}/\mathbf{l}(C)$ is a 1-torus). By a point on C we mean $C(p)$ where $p \in \mathbb{T}_C$, or $p \in \mathbb{R}$. Given two points $a, b \in \mathbb{R}$, we let $C[a,b]$ be the restriction of $C : \mathbb{R} \to \mathbf{S}$ to the interval $[a,b]$. If $b < a$ the the orientation of the segment $C[a,b]$ is the negative of the orientation for C. Of course $C[a + n\mathbf{l}(C), b + n\mathbf{l}(C)]$ is the same creature for $n \in \mathbb{Z}$. By $C'(p)$ we denote the unit tangent vector to C with the appropriate orientation.

Recall that $T^1\mathbb{H}^2$ denotes the unit tangent bundle, where elements of $T^1(\mathbb{H}^2)$ are pairs (p,u), where $p \in \mathbb{H}^2$ and $u \in T^1_p\mathbb{H}^2$. The tangent space T^1_p has the complex structure and given $u \in T^1\mathbf{S}$, by $\sqrt{-1}u \in T^1_p\mathbf{S}$ we denote the vector obtained from u by rotating for $\frac{\pi}{2}$.

Recall that for $u, v \in T^1_p\mathbb{H}^2$ we let $\mathbf{\Theta}(u,v)$ denote the unoriented angle between u and v. If $u \in T^1_p\mathbb{H}^2$ then $u @ q \in T^1_q\mathbb{H}^2$ denotes the vector u parallel transported to q along the geodesic segment connecting p and q. We use the similar notation for points in $T^1\mathbf{S}$, except that in this case one always has to specify the segment between p and q along which we parallel transport vectors from $T^1_p\mathbf{S}$ to $T^1_q\mathbf{S}$.

We refer to the following lemma as the Convergence Lemma. The proof is left to the reader.

Lemma 5.1. *Suppose A and B are oriented geodesics in $\mathbb{H}^2$ that are E nearly homotopic, and let*

$$a : \left[-\frac{\mathbf{l}(A)}{2}, \frac{\mathbf{l}(A)}{2}\right] \to \mathbb{H}^2, \quad b : \left[-\frac{\mathbf{l}(B)}{2}, \frac{\mathbf{l}(B)}{2}\right] \to \mathbb{H}^2,$$

denote the unit time parametrisation. Set $l = \frac{1}{2}\min(\mathbf{l}(A), \mathbf{l}(B))$. Then there exists $0 \le t_0 \le E$, such that for $t \in [-l, l]$ the following inequalities hold

1. $d(a(t), b(t+t_0)) \le e^{|t|+E+1-l}$,
2. $\mathbf{\Theta}(a'(t)@b(t+t_0), b'(t+t_0)) \le e^{|t|+E+1-l}$.

Let (p,u) and (q,v) be two vectors from $T^1(\mathbb{H}^2)$. We define the distance function

$$\operatorname{dis}((p,u),(q,v)) = \max(\mathbf{\Theta}(u @ q, v), d(p,q)).$$

(We do not insist that dis is a metric on $T^1\mathbb{H}^2$).

Let $\alpha : [a,b] \to \mathbb{H}^2$ be a unit speed geodesic segment. We let $i(\alpha) = \alpha'(a)$, and $t(\alpha) = \alpha'(b)$. We have the following lemma (we omit the proof).

Lemma 5.2. *Let $\epsilon, L > 0$. There exists a constant $\epsilon'(L)$ with the following properties. Suppose that $\alpha : [a_0, a_1] \to \mathbb{H}^2$ and $\beta : [b_0, b_1] \to \mathbb{H}^2$ are ϵ nearly homotopic, that is $d(\alpha(a_i), \beta(b_i)) \le \epsilon$. Suppose that $a_1 - a_0 > L$, and $\epsilon < 1$. Then*

$$\operatorname{dis}(\alpha'(a_i), \beta'(b_i)) \le \epsilon(1 + \epsilon'(L)),$$

with $\epsilon'(L) \to 0$ *as* $L \to \infty$.

5.2. **The preliminary Geometric Square Lemma (the PGSL).** The following is the preliminary Geometric Square Lemma. We have added the hypothesis (5) to the GSL, so as to find points in the two convergence intervals of the four curves, that are nearly diametrically opposite.

Lemma 5.3. *Let* $E, \epsilon > 0$. *There exist constants* $K = K(\epsilon, E) > 0$ *and* $R_0(\mathbf{S}, \epsilon, E) > 0$ *with the following properties. Suppose that we are given four oriented geodesics* $C_{ij} \in \Gamma_{\epsilon,R}$, $i, j = 0, 1$, *and for each* ij *we are given 4 points* $x^-_{ij} < x^+_{ij} < y^-_{ij} < y^+_{ij} < x^-_{ij} + \mathbf{l}(C_{ij})$. *Assume that*

(1) *The inequalities* $x^+_{ij} - x^-_{ij} > K$, *and* $y^+_{ij} - y^-_{ij} > K$, *hold,*
(2) *The segments* $C_{ij}[x^-_{ij}, x^+_{ij}]$ *and* $C_{i'j'}[x^-_{i'j'}, x^+_{i'j'}]$ *are* E *nearly homotopic, and likewise the segments* $C_{ij}[y^-_{ij}, y^+_{ij}]$ *and* $C_{i'j'}[y^-_{i'j'}, y^+_{i'j'}]$ *are* E *nearly homotopic, for any* $i, j, i', j' \in \{0, 1\}$,
(3) *The segments* $C_{0j}[x^-_{0j}, y^+_{0j}]$ *and* $C_{1j}[x^-_{1j}, y^+_{1j}]$ *are* E *nearly homotopic,*
(4) *The geodesic segments* $C_{i0}[y^-_{i0}, x^+_{i0} + \mathbf{l}(C_{ij})]$ *and* $C_{i1}[y^-_{i1}, x^+_{i1} + \mathbf{l}(C_{ij})]$ *are* E *nearly homotopic,*
(5) $y^+_{00} - x^-_{00} \geq R + K$, *and* $x^+_{00} + \mathbf{l}(C_{00}) - y^-_{00} \geq R + K$.

Then for $R > R_0$, *we have*

$$\sum_{i,j=0,1} (-1)^{i+j} C_{ij} = 0, \tag{9}$$

in $\mathbf{\Pi}_{10\epsilon,R}$ *homology.*

Remark. *The condition (5) is satisfied provided that* $y^+_{00} - y^-_{00} \geq R$.

Proof. Set $K = 20 + 2\log\frac{E}{\epsilon}$. For simplicity we write $\mathbf{l}(C_{ij}) = \mathbf{l}_{ij}$. We claim that we can find $x_{00} \in [x^-_{00} + \frac{K}{2}, x^+_{00} - \frac{K}{2} + 1]$ and $y_{00} \in [y^-_{00} + \frac{K}{2}, y^+_{00} - \frac{K}{2}]$ such that $y_{00} - x_{00} = R$. If $y^-_{00} \leq x^-_{00} + R$, we let $x_{00} = x^-_{00} + \frac{K}{2}$ and $y_{00} = x_{00} + R$. If $y^-_{00} \geq x^-_{00} + R$, we let $y_{00} = y^-_{00} + \frac{K}{2}$, and $x_{00} = y_{00} - R$.

By the Convergence Lemma (Lemma 5.1), and by the choice of the constant K, we can find $x^-_{ij} < x_{ij} < x^+_{ij}$ and $y^-_{ij} < y_{ij} < y^+_{ij}$ so that

$$\operatorname{dis}(C'_{ij}(x_{ij}), C'_{00}(x_{00})), \operatorname{dis}(C'_{ij}(y_{ij}), C'_{00}(y_{00})) \leq \epsilon,$$

and the pairs of geodesic segments $C_{i0}[x_{i0}, y_{i0}]$ and $C_{i0}[x_{i1}, y_{i1}]$, and $C_{0j}[y_{0j}, x_{0j} + \mathbf{l}_{ij}]$ and $C_{0j}[y_{1j}, x_{1j} + \mathbf{l}_{ij}]$ are ϵ nearly homotopic.

Let $I_{ij} = y_{ij} - x_{ij}$ and $J_{ij} = x_{ij} + \mathbf{l}_{ij} - y_{ij}$, so $I_{ij} + J_{ij} = \mathbf{l}_{ij}$. Then $I_{00} = R$ and $J_{00} = \mathbf{l}_{00} - R$, so $|J_{00} - R| < 2\epsilon$.

Also, by the triangle inequality we have $|I_{01} - R| = |I_{01} - I_{00}| < 2\epsilon$. So

$$|J_{01} - R| \leq |I_{01} - R| + |\mathbf{l}_{01} - 2R| \leq 4\epsilon.$$

Then

$$|J_{1j} - R| \leq |J_{0j} - R| + |J_{1j} - J_{0j}| < 6\epsilon,$$

so

$$|I_{1j} - R| \leq |J_{1j} - R| + |\mathbf{l}_{1j} - 2R| < 8\epsilon.$$

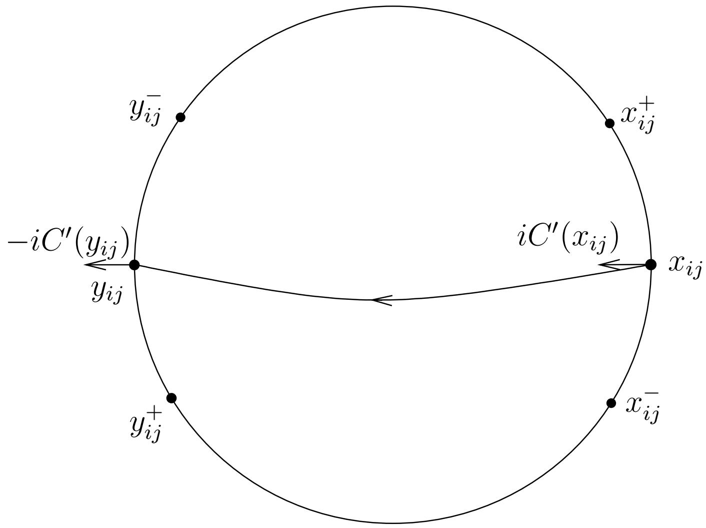

FIGURE 4. The Preliminary Geometric Square Lemma

Therefore we get $|I_{ij} - R|, |J_{ij} - R| < 8\epsilon$ for $i, j \in \{0, 1\}$.

We take

$$\alpha_{00} \in \mathrm{Conn}_{\epsilon, R+\log 4}(\sqrt{-1}C'(x_{00}), -\sqrt{-1}C'(y_{00})),$$

and let α_{ij} be the geodesic arc connecting x_{ij} and y_{ij} that is ϵ nearly homotopic to α_{00} (see Figure 4). Then $\mathrm{dis}(i(\alpha_{ij}), i(\alpha_{00})), \mathrm{dis}(t(\alpha_{ij}), t(\alpha_{00})) \leq 2\epsilon$. Therefore, because $\mathrm{dis}(C'_{ij}(x_{ij}), C'_{00}(x_{00})) \leq \epsilon$ and $\mathrm{dis}(C'_{ij}(y_{ij}), C'_{00}(y_{00})) \leq \epsilon$, we have

$$\alpha_{ij} \in \mathrm{Conn}_{3\epsilon, R+\log 4}(\sqrt{-1}C'(x_{ij}), -\sqrt{-1}C'(y_{ij})),$$

Define Π_{ij} as the pants generated from C_{ij} by adding the third connection α_{ij}. Denote by A_{ij} and B_{ij} the other two oriented cuffs of Π_{ij}, that is

$$\partial \Pi_{ij} = C_{ij} + A_{ij} + B_{ij},$$

where A_{ij} is freely homotopic to the closed broken geodesic $C_{ij}[x_{ij}, y_{ij}]\alpha_{ij}^{-1}$, and B_{ij} to $C_{ij}[y_{ij}, x_{ij} + \mathbf{l}_{ij}]\alpha_{ij}$.

Applying Lemma 4.8, we obtain

$$|\mathbf{l}(A_{ij}) - 2R| < |I_{ij} - R| + 10\epsilon < 20\epsilon$$

and similarly $|\mathbf{l}(B_{ij} - 2R| < 20\epsilon$, so $\Pi_{ij} \in \Gamma_{10\epsilon, R}$. Finally, $A_{i0} = A_{i1}$, and $B_{0j} = B_{1j}$, so

$$0 = \sum_{i,j=0,1} (-1)^{i+j} \partial \Pi_{ij} = \sum_{i,j=0,1} (-1)^{i+j} C_{ij},$$

in $\mathbf{\Pi}_{10\epsilon, R}$ homology, which proves the lemma. □

5.3. **The Geometric Square Lemma.**

Lemma 5.4. *Let $E, \epsilon > 0$. There exist constants $K_1 = K_1(\mathbf{S}, \epsilon, E) > 0$ and $R_0(\mathbf{S}, \epsilon, E) > 0$ with the following properties. Suppose that we are given four oriented geodesics $C_{ij} \in \Gamma_{\epsilon,R}$, $i, j = 0, 1$, and for each ij we are given 4 points $x^-_{ij} < x^+_{ij} < y^-_{ij} < y^+_{ij} < x^-_{ij} + \mathbf{l}(C_{ij})$. Assume that*

1. *The inequalities $x^+_{ij} - x^-_{ij} > K_1$, and $y^+_{ij} - y^-_{ij} > K_1$, hold,*
2. *The segments $C_{ij}[x^-_{ij}, x^+_{ij}]$ and $C_{i'j'}[x^-_{i'j'}, x^+_{i'j'}]$, are E nearly homotopic, and likewise the segments $C_{ij}[y^-_{ij}, y^+_{ij}]$ and $C_{i'j'}[y^-_{i'j'}, y^+_{i'j'}]$, are E nearly homotopic, for any $i, j, i', j' \in \{0, 1\}$,*
3. *The segments $C_{0j}[x^-_{0j}, y^+_{0j}]$ and $C_{1j}[x^-_{1j}, y^+_{1j}]$ are E nearly homotopic,*
4. *The geodesic segments $C_{i0}[y^-_{i0}, x^+_{i0} + \mathbf{l}(C_{ij})]$ and $C_{i1}[y^-_{i1}, x^+_{i1} + \mathbf{l}(C_{ij})]$ are E nearly homotopic,*

Then for $R > R_0$, we have

$$\sum_{i,j=0,1} (-1)^{i+j} C_{ij} = 0, \tag{10}$$

in $\mathbf{\Pi}_{100\epsilon,R}$ homology.

Proof. Below we use $L_0 = L_0(\mathbf{S}, \epsilon, E)$ and $K_0 = (\mathbf{S}, \epsilon, E)$ to denote two sufficiently large constants whose values will be determined in the course of the argument. The constant Q_0 can depend on K_0 and L_0. The constants K_1 and R_0 (from the statement of the GSL) can depend K_0 and L_0 and Q_0. Each of these constants will be implicitly defined as a maximum of expressions in terms of constants which precede the given constant in the partial order of dependence which we just have described.

If we cannot apply the PGSL, then possibly interchanging the roles of the x' and the y's, we find that

$$\begin{aligned} x^+_{00} &\leq y^-_{00} - \mathbf{l}_{00} + R + K(\epsilon, E) \\ &< y^-_{00} - R + K(\epsilon, E) + 1, \end{aligned}$$

where $K = K(\epsilon, E)$ is the constant from the previous lemma. We then let $y_{00} = y^-_{00} + Q_0$, and let $w_{00} = y_{00} - R$ (we assume that $Q_0 > K$). Then

$$w_{00} > x^+_{00} + 10, \tag{11}$$

provided $Q_0 > K + 11$, and

$$y^-_{00} + Q_0 \leq y_{00} \leq y^+_{00} + K_1 - Q_0,$$

which implies

$$y^-_{00} + \log\frac{E}{\epsilon} + 10 \leq y_{00} \leq y^+_{00} - (\log\frac{E}{\epsilon}), \tag{12}$$

provided $Q_0 \geq \log\frac{E}{\epsilon} + 10$ and $K_1 \geq Q_0 + \log\frac{E}{\epsilon} + 10$.

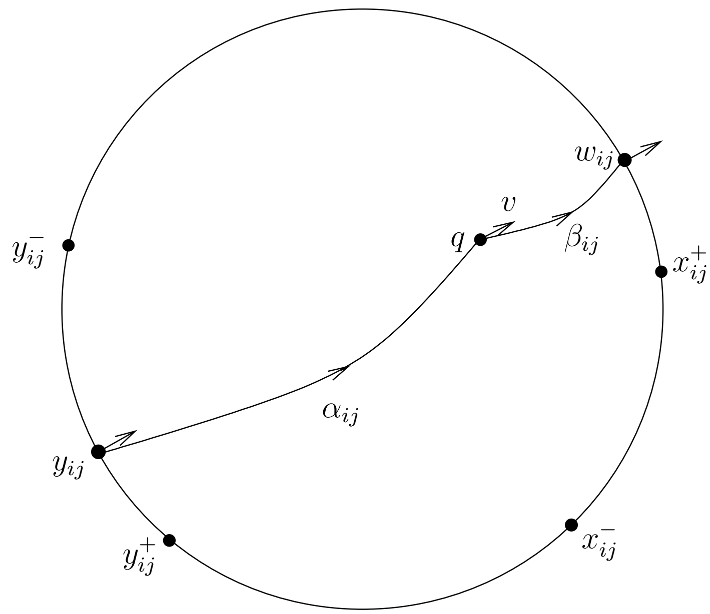

FIGURE 5. The Geometric Square Lemma

Therefore by the Convergence Lemma we can find y_{ij} in the interval $[y_{ij}^-, y_{ij}^+]$ such that $\operatorname{dis}(C'_{ij}(y_{ij}), C'_{00}(y_{00})) \le \epsilon$. We then let

$$\begin{aligned} w_{ij} &= y_{ij} - R \\ &\ge x_{ij}^+ + 10 \\ (\text{provided}\, Q_0 &> E + K + 12) \\ &\ge x_{ij}^- + K_1 \\ &\ge x_{ij}^- + \log\frac{E}{\epsilon} + 10 \\ (\text{provided}\, K_1 &> \log\frac{E}{\epsilon} + 10). \end{aligned}$$

Then $C_{i0}[w_{i0}, y_{i0}]$ and $C_{i1}[w_{i1}, y_{i1}]$ are ϵ C^1 nearly homotopic.

Let $(q, v) \in T^1\mathbf{S}$, and take $\beta_{i0} \in \operatorname{Conn}_{\epsilon, L_0}(v, -\sqrt{-1}C'_{i0}(w_{i0}))$ (where we assume that $L_0 > L_0(\epsilon, \mathbf{S})$ and $L_0(\epsilon, \mathbf{S})$ is the constant from the Connection Lemma). We take $\alpha_{00} \in \operatorname{Conn}_{\epsilon, R+\log 4 - L_0}(\sqrt{-1}C'_{00}(y_{00}), v)$ (see Figure 5). Then we find $\alpha_{ij} \in \operatorname{Conn}_{3\epsilon, R+\log 4 - L_0}(\sqrt{-1}C'_{ij}(y_{ij}), v)$, and $\beta_{ij} \in \operatorname{Conn}_{3\epsilon, L_0}(v, -\sqrt{-1}C'_{ij}(w_{ij})$ such that α_{00} and α_{ij} are ϵ nearly homotopic and β_{i0} and β_{ij} are 2ϵ nearly homotopic for every $i, j = 0, 1$.

We let Π_{ij} be the pair of pants generated by the geodesic segment $C_{ij}[w_{ij}, y_{ij}]$, the broken geodesic segment $\beta^{-1}\alpha_{ij}^{-1}$, and the geodesic segment $(C_{ij}[y_{ij}, w_{ij} + \mathbf{l}_{ij}])^{-1}$. The reader can verify that it is a topological pair of pants.

We let A_{ij} be the closed geodesic freely homotopic to $\alpha_{ij}\beta_{ij}C_{ij}[w_{ij}, y_{ij}]$ and let B_{ij} be the one for $C_{ij}[y_{ij}, w_{ij}+\mathbf{l}_{ij}]\beta_{ij}^{-1}\alpha_{ij}^{-1}$. Then $\partial\Pi_{ij} = C_{ij} - A_{ij} - B_{ij}$.

Using the Sum of Inefficiencies Lemma (see Lemma 4.8), we find that $|\mathbf{l}(A_{ij}) - 2R| \leq 13\epsilon$, and $|\mathbf{l}(B_{ij}) - 2R| \leq 15\epsilon$. Hence $\Pi_{ij} \in \mathbf{\Pi}_{10\epsilon,R}$.

Observe that $A_{i0} = A_{i1}$, so

$$\sum_{i,j=0,1} (-1)^{i+j} C_{ij} - \sum_{i,j=0,1} (-1)^{i+j} \partial\Pi_{ij} = \sum_{i,j=0,1} (-1)^{i+j} B_{ij}.$$

Let the a's and b's be real numbers and $B_{ij} : \mathbb{R} \to B_{ij}$ be a parametrisation of the geodesic B_{ij} so that $B_{ij}(a_{ij}^-)$, $B_{ij}(a_{ij}^+)$, $B_{ij}(b_{ij}^-)$, $B_{ij}(b_{ij}^+)$ are the projections of points q, $C_{ij}(y_{ij}^+)$, $C_{ij}(x_{ij}^-)$ and $C_{ij}(x_{ij}^+)$ respectively onto the geodesic B_{ij}. The points q, $C_{ij}(y_{ij}^+)$, $C_{ij}(x_{ij}^-)$ and $C_{ij}(x_{ij}^+)$ belong to the broken geodesic $C_{ij}[y_{ij}, w_{ij} + \mathbf{l}_{ij}]\beta_{ij}^{-1}\alpha_{ij}^{-1}$, and we project them to B_{ij} by choosing lifts of B_{ij} and $C_{ij}[y_{ij}, w_{ij} + \mathbf{l}_{ij}]\beta_{ij}^{-1}\alpha_{ij}^{-1}$ in $\mathbb{H}^2$ that have the same endpoints and then use the standard projection onto the lift of B_{ij}.

It follows from the Convergence Estimate that each of q, $C_{ij}(y_{ij}^+)$, $C_{ij}(x_{ij}^-)$ and $C_{ij}(x_{ij}^+)$ are within distance 1 of the corresponding projections on B_{ij}. Then

$$\begin{aligned} b_{ij}^+ - b_{ij}^- &\geq x_{ij}^+ - x_{ij}^- - 2 \\ &\geq K_0 \\ (\text{provided}\, K_1 &> K_0 + 2), \end{aligned}$$

and

$$\begin{aligned} a_{ij}^+ - a_{ij}^- &\geq R - L_0 - 3 + K_1 - Q_0 - E - 1 \\ &\geq R + K_0 \\ (\text{provided}\, K_1 &> K_0 + Q_0 + L_0 + E + 4). \end{aligned}$$

Assuming that $K_0 \geq K(10\epsilon, E+2)$ (where K is the constant from the PGSL) we find that the differences $b_{ij}^+ - b_{ij}^-$ and $a_{ij}^+ - a_{ij}^-$ satisfy the lower bound from the PGSL (observe that b_{ij}^+ and b_{ij}^- are $E+2$ close and similarly for the a's).

Also, the B_{ij}'s are in $\Gamma_{10\epsilon,R}$. So we apply the PGSL to show that

$$\sum_{i,j=0,1} (-1)^{i+j} B_{ij} = 0$$

in $\mathbf{\Pi}_{100\epsilon,R}$ homology. □

Randomization. *Randomization remarks for the GSL. Let $\epsilon, E > 0$. Every constant K below may depend only on ϵ, $\mathbf{S}$ and E.*

Below we will define a partial map $g : \left(\ddddot{\Gamma}_{1,R}\right)^4 \to \mathbb{R}\mathbf{\Pi}_{100\epsilon,R}$ *(then* g *is also a partial map* $g : \left(\dot{\Gamma}_{1,R}\right)^{16} \to \mathbb{R}\mathbf{\Pi}_{100\epsilon,R}$*) such that*

(1) g *is defined on any input* $(C_{ij}, x_{ij}^{\pm}, y_{ij}^{\pm})$ *that satisfies the hypothesis* $1-4$ *of GSL,*
(2) $\sum(-1)^{i+j}C_{ij} = \partial g(C_{ij}, x_{ij}^{\pm}, y_{ij}^{\pm})$,
(3) g *is* K*-semirandom with respect to measures classes* $\Sigma_{\ddddot{\Gamma}}^{\boxtimes 4}$ *on* $\left(\ddddot{\Gamma}_{1,R}\right)^4$ *and* $\sigma_{\mathbf{\Pi}}$ *on* $\mathbf{\Pi}_{1,R}$.

We first define a partial function $g_0 : \left(\ddddot{\Gamma}_{1,R}\right)^4 \to \mathbb{R}\mathbf{\Pi}_{10\epsilon,R}$ *that is defined on inputs* $(C_{ij}, x_{ij}^{\pm}, y_{ij}^{\pm})$ *that satisfy the extra hypothesis* (5) *from the PGSL. Given such an input, we follow the construction of the PGSL to construct* x_{ij} *and* y_{ij}*, and we observe that because these new points are bounded distance from the old ones, the map* $(C_{ij}, x_{ij}^{\pm}, y_{ij}^{\pm}) \to (C_{ij}, x_{ij}, y_{ij})$ *is* K*-semirandom as a partial map from* $\left(\ddddot{\Gamma}_{1,R}\right)^4$ *to* $\left(\ddot{\Gamma}_{1,R}\right)^4$*, with respect to the measure classes* $\Sigma_{\ddddot{\Gamma}}^{\boxtimes 4}$ *on* $\left(\ddddot{\Gamma}_{1,R}\right)^4$ *and* $\Sigma_{\ddot{\Gamma}}^{\boxtimes 4}$ *on* $\left(\ddot{\Gamma}_{1,R}\right)^4$.

Then we take a random third connection

$$\alpha_{00} \in \mathrm{Conn}_{\epsilon,R+\log 4}(\sqrt{-1}C'_{00}(x_{00}), -\sqrt{-1}C'_{00}(y_{00})).$$

Likewise for α_{ij}*. Adding the third connection* α_{ij} *to* C_{ij} *we obtain the pants* $\Pi_{ij}(\alpha_{ij})$*. We claim that distinct* α_{ij} *lead to distinct pants* $\Pi_{ij}(\alpha_{ij})$*. The third connection* α_{ij} *is* ϵ*-close to the unique simple geodesic arc on* $\Pi_{ij}(\alpha_{ij})$ *that is orthogonal to* γ_{ij} *at both ends. On the other hand, no two distinct* α_{ij} *are* ϵ*-close, so assuming that the injectivity radius of the surface* $\mathbf{S}$ *is at least* 2ϵ *we find that distinct* α_{ij} *give distinct* $\Pi_{ij}(\alpha_{ij})$.

So, for each input (C_{ij}, x_{ij}, y_{ij})*, by adding a random third connection* α_{ij} *we construct the pants* $\Pi_{ij}(\alpha_{ij})$*. So far, we have been using the term "random" to mean arbitrary. In these randomization remarks we will also interpret the phrase "a random element of a finite set* S*" as "the random element of* $\mathbb{R}S$*, namely* $\frac{1}{|S|}\sum_{x\in S} x$*.*

We can then think of every map $f : S \to T$ *that we have implicitly constructed in the text as the associated linear map* $f : \mathbb{R}S \to \mathbb{R}T$ *defined by* $f(\sum a_i x_i) = \sum a_i f(x_i)$*. So, for example, we let*

$$\underline{\alpha}_{ij} \in \mathrm{Conn}_{\epsilon,R+\log 4}(\sqrt{-1}C'_{ij}(x_{ij}), -\sqrt{-1}C'_{ij}(y_{ij}))$$

be the random element of

$$\mathrm{Conn}_{\epsilon,R+\log 4}(\sqrt{-1}C'_{ij}(x_{ij}), -\sqrt{-1}C'_{ij}(y_{ij})),$$

and then $\Pi_{ij}(\underline{\alpha}_{ij})$ *is the image of* $\underline{\alpha}_{ij}$ *by the linear form of the map* $\alpha_{ij} \to \Pi_{ij}(\alpha_{ij})$.

In this manner we have constructed a partial map from $\ddot{\Gamma}_{1,R} \to \Pi_{1,R}$ *(defined by* $(C_{ij}, x_{ij}, y_{ij}) \to \Pi_{ij}(\underline{\alpha}_{ij})$*, compare with Lemma 3.2), and we* *claim that it is K-semirandom with respect to $\Sigma_{\ddot{\Gamma}}$ and $\sigma_{\mathbf{\Pi}}$. To verify this claim we need to show that for any given pants $\Pi \in \mathbf{\Pi}_{1,R}$, the weight of Π is at most Ke^{-3R}. Let C be a cuff of Π, and choose points $x, y \in C$ which lie in certain unit length intervals on C. Let* Conn *be the set of all good third connections between x and y (by this we mean all connections α so that C and α produce a pair of pants in $\mathbf{\Pi}_{1,R}$). The set* Conn *has e^{R-K} elements. Moreover, there is a unique third connection $\alpha \in$ Conn so that α and C yield the given pair of pants Π. So, the total weight of Π is at most e^{K-R} times the total weight for the three choices of $C \in \partial\Pi$ (with associated unit intervals), and we conclude that that the total weight for Π is at most $3e^{K-R}e^{-2R} = Ke^{-3R}$.*

Also, the map $\left(\Pi_{ij}\right)_{i,j\in\{0,1\}} \mapsto \sum(-1)^{i+j}\Pi_{ij}$, is of course 4-semirandom from $(\mathbf{\Pi}^4_{1,R}, \Sigma^{\boxtimes 4}_{\mathbf{\Pi}})$ to $(\mathbb{R}\mathbf{\Pi}_{1,R}, \Sigma_{\mathbf{\Pi}})$. Composing the above maps we construct the map g_0 and see that g_0 is K-semirandom.

For the general case, similarly as above we first define the map

$$h : (C_{ij}, x^{\pm}_{ij}, y^{\pm}_{ij}) \to \mathbb{R}\mathbf{\Pi}_{1,R}$$

according to our second construction, on every input $(C_{ij}, x^{\pm}_{ij}, y^{\pm}_{ij})$ that satisfies conditions $(1)-(4)$ of the GSL, but not condition (5) of the PGSL.

We construct y_{ij} and w_{ij} as before. The map $(C_{ij}, x^{\pm}_{ij}, y^{\pm}_{ij}) \to (C_{ij}, y_{ij}, w_{ij})$ is K-semirandom. Then we find the connections α_{ij} and β_{ij}. There are at least e^{R-K} of the α_{ij} (we only fix a single β_{ij}), and each third connection $\alpha_{ij}\beta_{ij}$ leads to a new pair of pants $\Pi_{ij}(\alpha_{ij}\beta_{ij})$. Let N denote the number of connections α_{ij} (by construction, the number N does not depend on i and j). This defines the map

$$h(C_{ij}, x^{\pm}_{ij}, y^{\pm}_{ij}) = \sum \frac{1}{N}\Pi_{ij}(\alpha_{ij}\beta_{ij}),$$

and we can verify that h is K-semirandom.

Then we observe that $\partial_B : \Pi_{ij} \to B_{ij}$ formed by taking the appropriate boundary curve of the Π_{ij} we constructed is K-semirandom, so the induced map $\widetilde{h} : (C_{ij}, x^{\pm}_{ij}, y^{\pm}_{ij}) \to (B_{ij}, a_{ij}, b_{ij})$ is as well. So the map g_1 defined by $g_1(C_{ij}, x^{\pm}_{ij}, y^{\pm}_{ij}) = \sum(-1)^{i+j}\Pi_{ij} + g_0(B_{ij}, a^{\pm}_{ij}, b^{\pm}_{ij})$ is K-semirandom, and hence $g = g_0 + g_1$ is as well.

6. The Algebraic Square Lemma

We prove the Algebraic square lemma which will be building block in constructing good pants and proving that the standard homology agrees with the good pants homology.

6.1. **Notation.** Let $* \in \mathbf{S}$ denote a point that we fix once and for all. By $\pi_1(\mathbf{S}, *)$ we denote the fundamental group of a pointed surface. If $A \in \pi_1(\mathbf{S}, *)$, we let $[A]$ denote the closed geodesic on $\mathbf{S}$ that is freely homotopic to we let $\cdot A \cdot$ be the geodesic segment from $*$ to $*$ homotopic to A. If $A_1, ..., A_n \in \pi_1(\mathbf{S}, *)$ we let $\cdot A_1 \cdot A_2 ... \cdot A_n \cdot$ be the piecewise geodesic arc that is the concatenation of the arcs $\cdot A_i \cdot$. We let $[\cdot A_1 \cdot A_2 ... \cdot A_n \cdot]$ be the closed piecewise geodesic that arises from the arc $\cdot A_1 \cdot A_2 ... \cdot A_n \cdot$ by noticing that the starting and the the ending point of $\cdot A_1 \cdot A_2 ... \cdot A_n \cdot$ are the same. If $A \in \pi_1(\mathbf{S}, *)$, we let $[A]$ denote the closed geodesic on $\mathbf{S}$ that represents the free homotopy class for A. By $\mathbf{l}([A])$ is the length of the closed geodesic $[A]$. By $\mathbf{l}(\cdot A \cdot)$ we mean of course the length of the geodesic arc $\cdot A \cdot$, and in general by $\mathbf{l}(\cdot A_1 \cdot ... \cdot A_n \cdot)$ the length of the corresponding piecewise geodesic arc.

Remark. *Observe that for any $X_i \in \pi_1(\mathbf{S}, *)$, $i = 0, ..., n$, the closed geodesics $[X_j, X_{j+1}, ..., X_{n+j-1}]$ are one and the same. We will call this rotation and often use it without warning.*

We remind the reader that $\cdot AB \cdot$ is a geodesic arc from $*$ to $*$ representing AB, while $\cdot A \cdot B \cdot$ is a concatenation of two geodesic arcs. Similarly $\cdot AB \cdot C \cdot$ is a concatenation of two geodesic arcs, while $\cdot A \cdot B \cdot C \cdot$ is a concatenation of three, and so on.

In particular, we have the following statements about the inefficiency function,

$$I(\cdot A_1 \cdot ... \cdot A_n \cdot) = \sum \mathbf{l}(\cdot A_i \cdot) - \mathbf{l}(\cdot A_1 A_2 .... A_n \cdot),$$

and

$$I([\cdot A_1 \cdot ... \cdot A_n \cdot]) = \sum \mathbf{l}(\cdot A_i \cdot) - \mathbf{l}([A_1 .... A_n]).$$

Notice that we may have (and will usually have)

$$I([\cdot A_1 \cdot .... \cdot A_n \cdot]) > I(\cdot A_1 \cdot .... \cdot A_n \cdot),$$

6.2. **The Algebraic Square Lemma (the ASL).** The following is the Algebraic Square Lemma.

Lemma 6.1. *Let $\epsilon, \Delta > 0$. There exists a constant $K(\mathbf{S}, \epsilon, \Delta) = K$ and $R_0 = R_0(\mathbf{S}, \epsilon, \Delta)$ so that for $R > R_0$ the following holds. Let $A_i, B_i, U, V \in \pi_1(\mathbf{S}, *)$, $i = 0, 1$, be such that*

(1) $\left|\mathbf{l}([A_i U B_j V]) - 2R\right| < 2\epsilon$, $i, j = 0, 1$,
(2) $I([\cdot A_i \cdot U \cdot B_j \cdot V \cdot]) < \Delta$,
(3) $\mathbf{l}(\cdot U \cdot), \mathbf{l}(\cdot V \cdot) > K$.

Then

$$\sum_{ij} (-1)^{i+j} [A_i U B_j V] = 0$$

in $\mathbf{\Pi}_{100\epsilon, R}$ homology.

Proof. For each $i, j \in \{0, 1\}$ we project the closed piecewise geodesic $[\cdot A_i \cdot U \cdot B_j \cdot V \cdot]$ onto the closed geodesic $\gamma_{ij} = [A_i U B_j V]$. By Lemma 4.6 we find that each appearance of $*$ is moved at most distance $E = \frac{\Delta}{2} + 1$ by the projections. Let $\gamma_{ij}(x^{\pm}_{ij})$ and $\gamma_{ij}(y^{\pm}_{ij})$ be the projections of $*$ on γ_{ij} before and after U, and before and after V, respectively. Then providing that our K is at least $2E$ plus the corresponding constant from the GSL, we have

$x_{ij}^- < x_{ij}^+ < y_{ij}^- < y_{ij}^+ < x_{ij}^- + \mathbf{l}(\gamma_{ij})$ and the hypotheses of the Geometric Square Lemma. We conclude that

$$\sum_{ij}(-1)^{i+j}[A_iUB_jV] = 0$$

in $\mathbf{\Pi}_{100\epsilon,R}$ homology.

□

Randomization. *Randomization remarks for the ASL. Let $\epsilon, \delta > 0$. By K we denote any constant that may depend only on ϵ, $\mathbf{S}$, and Δ.*

Below we will define a partial map

$$f : G^2 \times G \times G^2 \times G \to \mathbb{R}\mathbf{\Pi}_{1,R},$$

such that

1. *f is defined on any input (A_i, U, B_j, V) that satisfies the assumptions of the ASL,*
2. *$\sum(-1)^{i+j}[A_iUB_jV] = \partial f(A_i, U, B_j, V)$,*
3. *f is K-semirandom with respect to the classes of measures $\Sigma_G^{\boxtimes 2} \times \Sigma_G \times \Sigma_G^{\boxtimes 2} \times \Sigma_G$ on $G^2 \times G \times G^2 \times G$ and $\sigma_{\mathbf{\Pi}}$ on $\mathbf{\Pi}_{1,R}$.*

Let h be a partial map

$$h : G^2 \times G \times G^2 \times G \to \left(\ddot{\ddot{\Gamma}}_{1,R}\right)^4$$

defined by letting $h(A_i, U, B_j, V) = (C_{ij}, x_{ij}^{\pm}, y_{ij}^{\pm})$, where $C_{ij} = [A_iUB_jV]$, and $x_{ij}^{\pm}$ and $y_{ij}^{\pm}$ are the points on the parametrizing torus for C_{ij} such that the points $C_{ij}(x_{ij}^{\pm})$ and $C_{ij}(y_{ij}^{\pm})$ are the corresponding projections of the 4 copies of the base point $*$ (that belong to the closed piecewise geodesic $[\cdot A_i \cdot U \cdot B_j \cdot V \cdot]$) to the closed geodesic C_{ij} (these projections were defined above). It follows from Lemma 9.5 and Lemma 9.6 that h is K-semirandom. Let $g : \left(\ddot{\ddot{\Gamma}}_{1,R}\right)^4 \to \mathbb{R}\mathbf{\Pi}_{1,R}$ be the K-semirandom map from the previous section (see the Randomization remarks for the GSL). Then $f = g \circ h$ is K-semirandom.

6.3. **The Sum of Inefficiencies Lemma in the algebraic notation.** The following lemma follows from Lemma 4.8.

Lemma 6.2. *Let $\epsilon, \Delta > 0$ and $n \in \mathbb{N}$. There exists $L = L(\epsilon, \Delta, n) > 0$ such that is $U_1, ..., U_{n+1} = U_1, X_1, ...X_n \in \pi_1(\mathbf{S}, *)$, and $I(\cdot U_i \cdot X_i \cdot U_{i+1}) \leq \Delta$, and $\mathbf{l}(\cdot U_i \cdot) \geq L$, then*

$$\left| I([\cdot U_1 \cdot X_1 \cdot U_2 \cdot X_2 \cdot ... \cdot U_n \cdot X_n \cdot]) - \sum_{i=1}^{n} I(U_iX_iU_{i+1}) \right| \leq \epsilon.$$

Remark. *In particular, we can leave out the X's in the above lemma, and write*

$$\left| I([\cdot U_1 \cdot U_2 \cdot ... \cdot U_n \cdot]) - \sum_{i=1}^{n} I(U_iU_{i+1}) \right| \leq \epsilon,$$

providing that $I(U_iU_{i+1}) \leq \Delta$, *and* $\mathbf{l}(\cdot U_i \cdot) \geq L$. *Moreover, by the Long Segment Lemma for Angles (for L large enough) we have*

$$\left| I([\cdot U_1 \cdot U_2 \cdot ... \cdot U_n \cdot]) - \sum_{i=1}^{n} I(\theta_i) \right| \leq 2\epsilon,$$

where $\theta_i = \mathbf{\Theta}(t(\cdot U_i \cdot), i(\cdot U_{i+1} \cdot))$.

Similarly, the following lemma follows from Lemma 4.9

Lemma 6.3. *Let* $\epsilon, \Delta > 0$ *and* $n \in \mathbb{N}$. *There exists* $L = L(\epsilon, \Delta, n) > 0$ *such that is* $U_1, ..., U_{n+1} = U_1, \in \pi_1(\mathbf{S}, *)$ *and* $X_{11}, ...X_{1j_1}, ...X_{n1}, ...X_{nj_n} \in \pi_1(\mathbf{S}, *)$, *and* $I(\cdot U_i \cdot X_i \cdot U_{i+1}) \leq \Delta$, *and* $\mathbf{l}(\cdot U_i \cdot) \geq L$, *then*

$$\left| I([\cdot U_1 \cdot X_{11} \cdot ... \cdot X_{1j_1} \cdot ... \cdot U_n \cdot X_{n1} \cdot ... \cdot X_{nj_n} \cdot]) - \sum_{i=1}^{n} I(U_i X_{i1} ... X_{ij_i} U_{i+1}) \right| \leq \epsilon.$$

Finally, we have the Flipping Lemma.

6.4. **The Flipping lemma.** For $X \in \pi_1(\mathbf{S}, *)$ we let $\bar{X} = X^{-1}$ denote the inverse of X.

Lemma 6.4. *Let* $\epsilon, \Delta > 0$. *There exists a constant* $L = L(\epsilon, \Delta) > 0$ *with the following properties. Suppose* $A, B, T \in \pi_1(\mathbf{S}, *)$, *and*

$$I(\cdot T \cdot A \cdot \bar{T} \cdot), I(\cdot \bar{T} \cdot B \cdot T \cdot) \leq \Delta,$$

and $\mathbf{l}(\cdot T \cdot) \geq L$. *Then*

$$\left| I([\cdot T \cdot A \cdot \bar{T} \cdot B \cdot]) - I([\cdot T \cdot \bar{A} \cdot \bar{T} \cdot B \cdot]) \right| < \epsilon,$$

and therefore

$$\left| \mathbf{l}([TA\bar{T}B]) - \mathbf{l}([T\bar{A}\bar{T}B]) \right| < \epsilon.$$

Proof. By the Long Segment Lemmas,

$$\left| I(\cdot T \cdot A \cdot \bar{T} \cdot B \cdot T \cdot) - I(\cdot T \cdot A \cdot \bar{T} \cdot) - I(\cdot \bar{T} \cdot B \cdot T \cdot) \right| < \frac{\epsilon}{4}$$

and

$$\left| I(\cdot T \cdot A \cdot \bar{T} \cdot B \cdot) - I(\cdot T \cdot A \cdot \bar{T} \cdot B \cdot T \cdot) \right| < \frac{\epsilon}{4}$$

Likewise

$$\left| I(\cdot T \cdot \bar{A} \cdot \bar{T} \cdot B \cdot T \cdot) - I(\cdot T \cdot \bar{A} \cdot \bar{T} \cdot) - I(\cdot \bar{T} \cdot B \cdot T \cdot) \right| < \frac{\epsilon}{4}$$

and

$$\left| I(\cdot T \cdot \bar{A} \cdot \bar{T} \cdot B \cdot) - I(\cdot T \cdot \bar{A} \cdot \bar{T} \cdot B \cdot T \cdot) \right| < \frac{\epsilon}{4}$$

But $I(\cdot T \cdot A \cdot \bar{T} \cdot) = I(\cdot T \cdot \bar{A} \cdot \bar{T} \cdot)$, because $\cdot T \cdot A \cdot \bar{T} \cdot$ is the same as $\cdot T \cdot \bar{A} \cdot \bar{T} \cdot$ with reversed orientation. So

$$\left| I([\cdot T \cdot A \cdot \bar{T} \cdot B \cdot]) - I([\cdot T \cdot \bar{A} \cdot \bar{T} \cdot B \cdot]) \right| < \epsilon.$$

Similarly we conclude $\left| \mathbf{l}([TA\bar{T}B]) - \mathbf{l}([T\bar{A}\bar{T}B]) \right| < \epsilon$.

□

7. Applications of the Algebraic Square Lemma

In the next two sections we state several results definitions (notably the definition of A_T in the next lemma), which depend on an element $T \in \pi_1(\mathbf{S}, *)$ and $\Delta > 0$. We treat both T and Δ as parameters, and the exact value of both T and Δ (that are then used in the proof of the main theorem) will be determined in Section 9.

The main purpose of this section is to define A_T and prove the Itemization Lemma (and the Simple Itemization Lemma).

7.1. **The definition of A_T.** For $A, T \in \pi_1(\mathbf{S}, *)$, and $\epsilon, R > 0$, we let $\mathcal{F}\mathrm{Conn}_{\epsilon,R}(A,T)$ be the set all $B \in \pi_1(\mathbf{S}, *)$ such that $[TA\bar{T}B], [T\bar{A}\bar{T}B] \in \Gamma_{\epsilon,R}$, and $I(\cdot\bar{T} \cdot B \cdot T\cdot) < 1$.

Lemma 7.1. *Let $\epsilon, \Delta > 0$. There exists a constant $L = L(\mathbf{S}, \epsilon, \Delta)$ such that if $A, T \in \pi_1(\mathbf{S}, *)$ and $I(\cdot T \cdot A \cdot \bar{T}\cdot) \leq \Delta$, and $\mathbf{l}(\cdot T \cdot) \geq L$, and $2R - \mathbf{l}(\cdot A\cdot) - 2\mathbf{l}(\cdot T\cdot) \geq L$, then*

1. *$\mathcal{F}\mathrm{Conn}_{\epsilon,R}(A,T)$ is non-empty, and $\log \left|\mathcal{F}\mathrm{Conn}_{\epsilon,R}(A,T)\right| \geq 2R - \mathbf{l}(\cdot A\cdot) - 2\mathbf{l}(\cdot T\cdot) - \Delta - L$,*
2. *$[TA\bar{T}B] - [T\bar{A}\bar{T}B] = [TA\bar{T}B'] - [T\bar{A}\bar{T}B']$ in $\mathbf{\Pi}_{100\epsilon,R}$ homology for any $B, B' \in \mathcal{F}\mathrm{Conn}_{\epsilon,R}(A,T)$.*

We then let

$$A_T = \frac{1}{2}\big([TA\bar{T}B] - [T\bar{A}\bar{T}B]\big)$$

for a random $B \in \mathcal{F}\mathrm{Conn}_{\epsilon,R}(A,T)$.

Remark. *It follows from the Algebraic Square Lemma that for any given $B \in \mathcal{F}\mathrm{Conn}_{\epsilon,R}(A,T)$ we have*

$$[TA\bar{T}B] - [T\bar{A}\bar{T}B] = A_T,$$

in $\mathbf{\Pi}_{100\epsilon,R}$ homology. Also, it is important to note that $[A]$ is equal to A_T in the standard homology $\mathbf{H}_1$.

Proof. Suppose that $\cdot B\cdot \in \mathrm{Conn}_{\epsilon,R'}(-i(\cdot T\cdot), i(\cdot T\cdot))$, where $R' = 2R - \mathbf{l}(\cdot A\cdot) - 2\mathbf{l}(\cdot T\cdot) - I(\cdot T \cdot A \cdot \bar{T}\cdot)$. The set $\mathrm{Conn}_{\epsilon,R'}(-i(\cdot T\cdot), i(\cdot T\cdot))$ will be non-empty (by the Connection lemma) provided L is large. Then, by the Sum of Inefficiencies Lemma,

$$\left|\mathbf{l}([TA\bar{T}B]) - 2R\right| < \epsilon + O(\epsilon^2),$$

and

$$\left|\mathbf{l}([T\bar{A}\bar{T}B]) - 2R\right| < \epsilon + O(\epsilon^2),$$

provided $\mathbf{l}(\cdot T\cdot)$ is large. Thus, with slight abuse of notation we have

$$\mathrm{Conn}_{\epsilon,R}(-i(\cdot T\cdot), i(\cdot T\cdot)) \subset \mathcal{F}\mathrm{Conn}_{\epsilon,R}(A,T),$$

and

$$\log|\mathrm{Conn}_{\epsilon,R}(-i(\cdot T\cdot), i(\cdot T\cdot)| \geq 2R - \mathbf{l}(\cdot A\cdot) - 2\mathbf{l}(\cdot T\cdot) - L$$

if L is large, so we have proved the statement (1) of the lemma.

The statement (2) then follows, provided L (and hence $(\mathbf{l}(\cdot T\cdot))$ is large, from the Algebraic Square Lemma. □

Randomization. *Randomization remarks for A_T. All constants K may depend only on ϵ, Δ, and $\mathbf{S}$ and $T \in \pi_1(\mathbf{S}, *)$.*

Letting $\underline{B}_A \in \mathbb{R}G$ denote the random element of $\mathcal{F}\mathrm{Conn}_{\epsilon,R}(A,T)$, we consider the map $A \to \underline{B}_A$ from G to $\mathbb{R}G$. If $\mathbf{l}(\cdot A \cdot) \in [a, a+1]$, we find that $\mathbf{l}(\cdot B \cdot) \in [L_a, R_a]$, where $L_a = 2R - a - 2\mathbf{l}(\cdot T \cdot) - \Delta - 4$ and $R_a = 2R - a - 2\mathbf{l}(\cdot T \cdot)$, for all $B \in \mathcal{F}\mathrm{Conn}_{\epsilon,R}(A,T)$. Because

$$\left|\mathcal{F}\mathrm{Conn}_{\epsilon,R}(A,T)\right| > e^{L_a - L}$$

(where $L = L(\epsilon, \mathbf{S})$ from the Connection Lemma), and $\sigma_a(G) \le K$ (see Appendix for the definition of σ_a), we find that for any $X \in G$

$$(A \to \underline{B}_A)_* \sigma_a(X) \le e^{L - L_a},$$

if $\mathbf{l}(\cdot X \cdot) \in [L_a, R_a]$, and $(A \to \underline{B}_A)_* \sigma_a(X) = 0$ otherwise. This implies

$$(A \to \underline{B}_A)_* \sigma_a \le K \sum_{k = \lfloor L_a \rfloor}^{\lfloor R_b \rfloor} e^{k + L - L_a} \sigma_k,$$

which in turn implies that the map $A \to \underline{B}_A$ is K-semirandom with respect to Σ_G and Σ_G.

We define $[A\bar{T}\underline{B}_A T]$ by

$$[A\bar{T}\underline{B}_A T] = \frac{1}{|\mathcal{F}\mathrm{Conn}_{\epsilon,R}(A,T)|} \sum_{B \in \mathcal{F}\mathrm{Conn}_{\epsilon,R}(A,T)} [A\bar{T}BT]$$

Then the partial maps from G to $\mathbb{R}G^4$ defined by $A \to (A, \bar{T}, \underline{B}_A, T)$ and $A \to (\bar{A}, \bar{T}, \underline{B}_A, T)$, are K-semirandom with respect to Σ_G and Σ_G^4 and hence the map

$$A \to A_T = \frac{1}{2}\left([A\bar{T}\underline{B}_A T] - [\bar{A}\bar{T}\underline{B}_A T]\right)$$

is K-semirandom with respect to Σ_G and Σ_G^4.

The map $(A, B') \to (A, \underline{B}_A, B')$ is K-semirandom with respect to $\Sigma_G^{\times 2}$ and $\Sigma_G^{\times 3}$, and $(A, \underline{B}_A, B') \to (A, \bar{A}, \bar{T}, \underline{B}_A, B', T)$ is K-semirandom with respect to $\Sigma_G^{\times 3}$ and $\Sigma_G^{\times 2} \times \Sigma_G \times \Sigma_G^{\times 2} \times \Sigma_G$.

Also, by the Algebraic Square Lemma, the map $(A, \bar{A}, \bar{T}, B, B', T) \to \Pi \in \mathbb{R}\mathbf{\Pi}_{1,R}$, such that

$$\partial \Pi = [TA\bar{T}B] - [T\bar{A}\bar{T}B] - [TA\bar{T}B'] + [T\bar{A}\bar{T}B'],$$

is K-semirandom from $G^2 \times G \times G^2 \times G$ to $\mathbf{\Pi}_{1,R}$, with respect to the measure classes $\Sigma_G^{\boxtimes 2} \times \Sigma_G \times \Sigma_G^{\boxtimes 2} \times \Sigma_G$ and $\sigma_{\mathbf{\Pi}}$. Composing the above mappings we find a K-semirandom map $g : G^2 \to \mathbb{R}\mathbf{\Pi}_{1,R}$ such that

$$\partial g(A, B') = A_T - \frac{1}{2}\left([TA\bar{T}B'] - [T\bar{A}\bar{T}B']\right).$$

7.2. **The Simple Itemization Lemma.** The following lemma is a corollary of the previous one and we refer to it as the Simple Itemization Lemma.

Lemma 7.2. *Let $\epsilon, \Delta > 0$. There exists a constant $L = L(\mathbf{S}, \epsilon, \Delta) > 0$ such that for any $A, B, T \in \Gamma_{\epsilon,R}$ we have $[TA\bar{T}B] = A_T + B_{\bar{T}}$ in $\mathbf{\Pi}_{100\epsilon,R}$ homology, provided that $\mathbf{l}(\cdot T \cdot), \mathbf{l}(\cdot A \cdot), \mathbf{l}(\cdot B \cdot) > L$ and $I([\cdot T \cdot A \cdot \bar{T} \cdot B \cdot]) \leq \Delta$.*

Proof. We observe

$$
\begin{aligned}
[TA\bar{T}B] &= \frac{1}{2}\big([TA\bar{T}B] - [\bar{B}T\bar{A}\bar{T}]\big) \\
&= \frac{1}{2}\big([TA\bar{T}B] - [T\bar{A}\bar{T}\bar{B}]\big) \\
&= \frac{1}{2}\big([TA\bar{T}B] - [T\bar{A}\bar{T}B]\big) \\
&+ \frac{1}{2}\big([T\bar{B}T\bar{A}] - [\bar{T}\bar{B}T\bar{A}]\big) \\
&= A_T + B_{\bar{T}},
\end{aligned}
$$

in $\mathbf{\Pi}_{100\epsilon,R}$ homology.

□

Randomization. *The randomization remarks for the Simple Itemization Lemma. We have implicitly defined a map $g : G^2 \to \mathbb{R}\mathbf{\Pi}_{100\epsilon,R}$ such that $\partial g(A, B) = A_T + B_T - [A\bar{T}BT]$. The map g is $Ke^{\Delta + 2\mathbf{l}(\cdot T \cdot T)}$-semirandom with respect to $\Sigma_G \times \Sigma_G$ and $\sigma_{\mathbf{\Pi}}$.*

Remark. *In fact it should be true that*

$$
[TA_1\bar{T}B_1...TA_n\bar{T}B_n] = \sum_{i=1}^{n} (A_i)_T + (B_i)_{\bar{T}}, \tag{13}
$$

provided $\mathbf{l}(\cdot T \cdot)$ is large given $I(TA_i\bar{T})$ and $I(\bar{T}B_iT)$. Above we proved this when $n = 1$ (provided $\mathbf{l}(\cdot A \cdot)$ and $\mathbf{l}(\cdot B \cdot)$ are large) and we will prove it in the rest of this section for $n = 2$, using the $ADCB$ lemma which we prove next. The general case can be proved by induction using the cases $n = 1$ and $n = 2$ (but we will only need this statement for $n = 1, 2$).

We also observe that under the usual conditions we have $A_{TU} = A_U$ in $\mathbf{\Pi}_{100\epsilon,R}$ homology. This follows from the fact that $2A_{TU} = [TUA\bar{U}\bar{T}B] + [TUA\bar{U}\bar{T}\bar{B}] = [UA\bar{U}\bar{T}BT] + [UA\bar{U}\bar{T}\bar{B}T] = 2A_U$.

7.3. **The $ADCB$ Lemma.**

Claim. *Let $\delta, \Delta > 0$. There exists $L = L(\Delta, \delta) > 0$ with the following properties. Let $A_i, B_j, T \in \pi_1(\mathbf{S}, *)$, $i, j = 0, 1$. If $\mathbf{l}(\cdot T \cdot) > L$ then*

$$
\left| \mathbf{l}([A_0TB_0\bar{T}A_1TB_1\bar{T}]) - \sum_{i=0}^{1} \mathbf{l}(\cdot \bar{T}A_iT \cdot) - \sum_{j=0}^{1} \mathbf{l}(\cdot \bar{T}B_jT \cdot) + 4\mathbf{l}(\cdot T \cdot) \right| < \delta.
$$

Proof. It follows from the Sum of Inefficiencies Lemma. □

Lemma 7.3. *Let $\epsilon, \Delta > 0$. There exists $L = L(\mathbf{S}, \epsilon, \Delta) > 0$ and $R_0 = R_0(\mathbf{S}, \epsilon, \Delta) > 0$ with the following properties. Let $A, B, C, D, T \in \pi_1(\mathbf{S}, *)$ such that $\mathbf{l}(\cdot B \cdot), \mathbf{l}(\cdot D \cdot), \mathbf{l}(\cdot T \cdot) > L$. If $R > R_0$ and*

$$I(\cdot T \cdot A \cdot \bar{T} \cdot), I(\cdot T \cdot C \cdot \bar{T} \cdot), I(\cdot T \cdot B \cdot \bar{T} \cdot), I(\cdot T \cdot D \cdot \bar{T} \cdot) \leq \Delta$$

then $[ATB\bar{T}CTD\bar{T}] = [ATD\bar{T}CTB\bar{T}]$ in $\mathbf{\Pi}_{200\epsilon,R}$ homology provided that the curves in questions are in $\Gamma_{\epsilon,R}$.

Proof. Let $\langle X, Y \rangle = [ATX\bar{T}CTY\bar{T}]$, for $X, Y \in \pi_1(\mathbf{S}, *)$. We claim that

$$\langle X_0, Y_0 \rangle - \langle X_0, Y_1 \rangle = \langle Y_0, X_1 \rangle - \langle Y_1, X_1 \rangle \tag{14}$$

in $\mathbf{\Pi}_{100\epsilon,R}$ whenever $I(TX_i\bar{T}), I(TY_i\bar{T}) \leq \Delta$, and the curves in question are in $\Gamma_{\epsilon,R}$. To verify (14) we let $A_i = Y_i$, $B_0 = ATX_0\bar{T}C$, $B_1 = CTX_1\bar{T}A$, and $U = T$ and $V = \bar{T}$, where A_i, B_i, U, V are from the statement of the Algebraic Square Lemma. Since by rotation

$$\begin{aligned}
\langle X_0, Y_0 \rangle &= [Y_0\bar{T}ATX_0\bar{T}CT] \\
\langle Y_0, X_1 \rangle &= [Y_0\bar{T}CTX_1\bar{T}AT] \\
\langle X_0, Y_1 \rangle &= [Y_1\bar{T}ATX_0\bar{T}CT] \\
\langle Y_1, X_1 \rangle &= [Y_1\bar{T}CTX_1\bar{T}AT],
\end{aligned}$$

the equation (14) follows from the Algebraic Square Lemma.

In order to prove the lemma we we first suppose that $|\mathbf{l}(\cdot TB\bar{T}\cdot) - \mathbf{l}(\cdot TD\bar{T}\cdot)| < \frac{\epsilon}{4}$. If L is large enough (and hence $\mathbf{l}(\cdot B\cdot)$,$\mathbf{l}(\cdot D\cdot)$ and $\mathbf{l}(\cdot T\cdot)$ are large enough), it follows from the Connection Lemma that we can find a random geodesic arc

$$\cdot E \cdot \in \text{Conn}_{\epsilon, \mathbf{l}(\cdot TB\bar{T}\cdot) - 2\mathbf{l}(\cdot T\cdot)}(-i(\cdot T\cdot), i(\cdot T\cdot)).$$

For any such E we have $|\mathbf{l}(\cdot TB\bar{T}\cdot) - \mathbf{l}(\cdot TE\bar{T}\cdot)| < \epsilon + O(\epsilon^2)$. Therefore, by the previous Claim we have that the curves $\langle B, E \rangle$, $\langle E, B \rangle$, $\langle D, E \rangle$, and $\langle E, D \rangle$ are in $\Gamma_{2\epsilon,R}$

On the other hand, the following equations follow from (14)

$$\begin{aligned}
\langle B, D \rangle - \langle B, E \rangle - \langle D, B \rangle + \langle E, B \rangle &= 0, \\
\langle B, D \rangle - \langle E, D \rangle - \langle D, B \rangle + \langle D, E \rangle &= 0, \\
\langle D, E \rangle - \langle B, E \rangle - \langle E, D \rangle + \langle E, B \rangle &= 0,
\end{aligned}$$

in $\mathbf{\Pi}_{200\epsilon,R}$ homology, and hence (adding the first two equations and subtracting the third) we get $2\langle B, D \rangle - 2\langle D, B \rangle = 0$ in $\mathbf{\Pi}_{200\epsilon,R}$ homology and we are finished.

More generally, if $\mathbf{l}(\cdot B\cdot) > \mathbf{l}(\cdot D\cdot) > L$ let k be smallest integer such that

$$k > 4\frac{|\mathbf{l}(\cdot TD\bar{T}\cdot) - \mathbf{l}(\cdot TB\bar{T}\cdot)|}{\epsilon}.$$

Set

$$r_i = \frac{i}{2k}\mathbf{l}(\cdot TD\bar{T}\cdot) + \frac{2k - i}{2k}\mathbf{l}(\cdot TB\bar{T}\cdot) - 2\mathbf{l}(\cdot T\cdot).$$

We can find random $\cdot E_i \cdot \in \mathrm{Conn}_{\epsilon,r_i}(t(\cdot T \cdot), i(\bar{T}))$ (observe that $r_i > L - \Delta$). In a similar fashion as above, for $0 < i \leq k$ we derive the equations

$$\langle E_i, E_{2k-i} \rangle - \langle E_{i+1}, E_{2k-i} \rangle - \langle E_{2k-i}, E_i \rangle + \langle E_{2k-i}, E_{i+1} \rangle = 0,$$

$$\langle E_i, E_{2k+1-i} \rangle - \langle E_i, E_{2k-i} \rangle - \langle E_{2k+1-i}, E_i \rangle + \langle E_{2k-1}, E_i \rangle = 0,$$

in $\mathbf{\Pi}_{200\epsilon,R}$ homology. Adding these we get

$$\langle E_0, E_{2k} \rangle - \langle E_k, E_{k+1} \rangle - \langle E_{2k}, E_0 \rangle + \langle E_{k+1}, E_k \rangle = 0,$$

in $\mathbf{\Pi}_{200\epsilon,R}$ homology. But $\langle E_k, E_{k+1} \rangle - \langle E_{k+1}, E_k \rangle = 0$ in $\mathbf{\Pi}_{200\epsilon,R}$ homology by the first case so we are finished.

□

Randomization. *The randomization remarks for the ADCB Lemma. All constants K may depend only on ϵ, $\mathbf{S}$ and Δ. We have defined a map $g : G^4 \to \mathbb{R}\mathbf{\Pi}_{1,R}$ such that*

$$\partial g(A, B, C, D) = [ATB\bar{T}CTD\bar{T}] - [ATD\bar{T}CTB\bar{T}].$$

In particular, we defined $h : G^6 \to \mathbb{R}\mathbf{\Pi}_{1,R}$ so

$$\partial h(A, C, X_0, X_1, Y_0, Y_1) = \langle X_0, Y_0 \rangle - \langle X_0, Y_1 \rangle - \langle Y_0, X_1 \rangle + \langle Y_1, X_1 \rangle .$$

This map h is $e^{4\mathbf{l}(\cdot T \cdot)}K$-semirandom with respect to the measure classes $\Sigma_G^{\times 2} \times \Sigma_G^{\boxtimes 2} \times \Sigma_G^{\boxtimes 2}$ and $\Sigma_{\mathbf{\Pi}}$.

Then $g(A, B, C, D)$ is a sum of $2k$ terms of the form $\partial h(A, C, X_0, X_1, Y_0, Y_1)$, where each of X_0, X_1, Y_0, Y_1 is either B or D, or E_i, which is a K-semirandom element of G with respect to Σ_G. Moreover, while we may take $X_0 = X_1$, or $Y_0 = Y_1$, the X_i are always independent from Y_i. Therefore, for each choice we make of X_0, X_1, Y_0, Y_1 (such as $X_0 = E_i$, $X_1 = E_{i+1}$, $Y_0 = Y_1 = E_{2k-i}$, or $X_0 = X_1 = B$, and $Y_0 = D$, $Y_1 = E$) the map from (A, B, C, D) to $(A, C, X_0, X_1, Y_0, Y_1)$ is K-semirandom with respect to $\Sigma_G^{\times 4}$ and $\Sigma^{\times 2} \times \Sigma_G^{\boxtimes 2} \times \Sigma_G^{\boxtimes 2}$. Therefore, noting that $k < \frac{\lfloor 8R \rfloor}{\epsilon}$, we find that g is $KRe^{4\mathbf{l}(\cdot T \cdot)}$-semirandom, with respect to $\Sigma_G^{\times 4}$.

Remark. *This is a remark to the previous randomization remark. Where B and D are close in length, we can write $\langle B, D \rangle - \langle B, B \rangle = \langle D, B \rangle - \langle B, B \rangle$ by (14), and hence $\langle B, D \rangle = \langle D, B \rangle$. But we are letting (X_0, X_1, Y_0, Y_1) be (B, B, D, B), and the map $(B, D) \to (B, B, D, B)$ is not 1-semirandom for $\Sigma_G^{\times 2}$ and $\Sigma_G^{\boxtimes 2} \times \Sigma_G^{\boxtimes 2}$ (because X_0 and Y_1 are not independent). This map is only $e^{\mathbf{l}(\cdot B \cdot)}$-semirandom, which is no good. It is for this reason that we introduce E.*

The following lemma is a corollary of the *ADCB* Lemma. We call it the Itemisation Lemma.

Lemma 7.4. *Let $\epsilon, \Delta > 0$. There exists $L = L(\epsilon, \Delta) > 0$ such that for any $A, B, C, D, T \in \pi_1(\mathbf{S}, *)$ we have*

$$[ATB\bar{T}CTD\bar{T}] - [T\bar{D}\bar{T}\bar{C}T\bar{B}\bar{T}A] = 2(A_{\bar{T}} + B_T + C_{\bar{T}} + D_T)$$

in $\mathbf{\Pi}_{200\epsilon,R}$ *homology provided that* $\mathbf{l}(\cdot T\cdot) > L$ *and* $I(\cdot A\cdot T\cdot B\cdot\bar{T}\cdot C\cdot T\cdot D\cdot\bar{T}\cdot) < \Delta$*, and the curve* $[ATB\bar{T}CTD\bar{T}]$ *is in* $\Gamma_{\epsilon,R}$*.*

Proof. Recall the remark after the statement of Lemma 7.1. We have

$$[ATB\bar{T}CTD\bar{T}] - [\bar{A}TB\bar{T}CTD\bar{T}] = 2A_{\bar{T}}$$
$$[\bar{A}TB\bar{T}CTD\bar{T}] - [\bar{A}T\bar{B}\bar{T}CTD\bar{T}] = 2B_T$$
$$[\bar{A}T\bar{B}\bar{T}CTD\bar{T}] - [\bar{A}T\bar{B}\bar{T}\bar{C}TD\bar{T}] = 2C_{\bar{T}}$$
$$[\bar{A}T\bar{B}\bar{T}\bar{C}TD\bar{T}] - [\bar{A}T\bar{B}\bar{T}\bar{C}T\bar{D}\bar{T}] = 2D_T,$$

in $\mathbf{\Pi}_{100\epsilon,R}$ homology. So

$$[ATB\bar{T}CTD\bar{T}] - [\bar{A}T\bar{B}\bar{T}\bar{C}T\bar{D}\bar{T}] = 2(A_{\bar{T}} + B_T + C_{\bar{T}} + D_T)$$

in $\mathbf{\Pi}_{100\epsilon,R}$ homology. But

$$[\bar{A}T\bar{B}\bar{T}\bar{C}T\bar{D}\bar{T}] - [\bar{A}T\bar{D}\bar{T}\bar{C}T\bar{B}\bar{T}] = 0$$

in $\mathbf{\Pi}_{200\epsilon,R}$ homology by the $ADCB$ Lemma so we are finished.

□

Randomization. *The randomization remark for the Itemization Lemma. We have defined* $g : G^4 \to \mathbb{R}\mathbf{\Pi}_{1,R}$ *such that*

$$\partial g(A,B,C,D) = [ATB\bar{T}CTD\bar{T}] - (A_{\bar{T}} + B_T + C_{\bar{C}} + D_T).$$

This map is $KRe^{4\mathbf{l}(\cdot T\cdot)}$*-semirandom with respect to* $\sigma^{\times 4}$ *and* $\sigma_{\mathbf{\Pi}}$*, for some* $K = K(\epsilon, \mathbf{S})$*.*

8. The XY Theorem

The purpose of this section is to prove the XY Theorem. The proof of this theorem will follow from the two Rotation Lemmas we prove first.

8.1. **The Rotation Lemmas.** Let $X, Y, Z \in \pi_1(\mathbf{S}, *)$. Then we have the three geodesic arcs $\cdot X\cdot$, $\cdot Y\cdot$, and $\cdot Z\cdot$. Consider the union of these three geodesic arcs as a θ-graph on the surface $\mathbf{S}$. This θ-graph generates an immersed pair of pants in $\mathbf{S}$ if and only if the triples of unit vectors $i(\cdot X\cdot), i(\cdot Y\cdot), i(\cdot Z\cdot)$ and $t(\cdot X\cdot), t(\cdot Y\cdot), t(\cdot Z\cdot)$, have the opposite cyclic orderings.

The following is the First Rotation Lemma.

Lemma 8.1. *Let* $\epsilon, \Delta > 0$*. There exists* $K = K(\epsilon, \Delta) > 0$ *with the following properties. Let* $R_i, S_i, T \in \pi_1(\mathbf{S}, *)$*,* $i = 0, 1, 2$*, such that*

(1) $I(\cdot T\cdot R_i \cdot \bar{R}_{i+1} \cdot \bar{T}\cdot), I(\cdot T\cdot S_i\cdot \bar{S}_{i+1}\cdot\bar{T}\cdot) < \Delta$*,*
(2) $\mathbf{l}(\cdot T\cdot) \geq K$*,*
(3) $\mathbf{l}(\cdot R_i\cdot) + \mathbf{l}(\cdot S_i\cdot) + 2\mathbf{l}(\cdot T\cdot) < R - K$*,*
(4) *The triples of vectors* $\big(t(\cdot TR_i\cdot)\big)$ *and* $\big(t(\cdot TS_i\cdot)\big)$*,* $i = 0, 1, 2$*, have opposite cyclic ordering in* $T^1_*\mathbf{S}$ *(one of them is clockwise and the other one anti-clockwise).*

Then

$$\sum_{i=0}^{2}(R_{i+1}\bar{R}_i)_T + \sum_{i=0}^{2}(S_i\bar{S}_{i+1})_T = 0, \tag{15}$$

in $\mathbf{\Pi}_{300\epsilon,R}$ *homology.*

Remark. *It follows by relabelling that if* $\max(\mathbf{l}(\cdot R_i\cdot))+\max(\mathbf{l}(\cdot S_i\cdot))+2\mathbf{l}(\cdot T\cdot) < R-K$ *and the triples of vectors* $\big(t(\cdot TR_i\cdot)\big)$ *and* $\big(t(\cdot TS_i\cdot)\big)$, $i=0,1,2$, *have the same cyclic ordering in* $T^1_*\mathbf{S}$, *then*

$$\sum_{i=0}^{2}(R_i\bar{R}_{i+1})_T + \sum_{i=0}^{2}(S_i\bar{S}_{i+1})_T = 0, \tag{16}$$

in $\mathbf{\Pi}_{300\epsilon,R}$ *homology.*

Proof. Let $r_i \geq 0$, $i=0,1,2$, be the solutions of the equations

$$r_i + r_{i+1} = 2R - \mathbf{l}(\cdot TR_{i+1}\bar{R}_i\bar{T}\cdot) - \mathbf{l}(\cdot TS_i\bar{S}_{i+1}\bar{T}\cdot). \tag{17}$$

Then we let A_i be a random element of $\text{Conn}_{\epsilon,r_i}(-i(\cdot T\cdot), i(\cdot T\cdot))$.

Consider the three elements $\bar{R}_i\bar{T}A_iTS_i$ of $\pi_1(\mathbf{S},*)$ and the corresponding geodesic arcs $\cdot\bar{R}_i\bar{T}A_iTS_i\cdot$. We will show that the corresponding θ-graph generates an immersed pair of pants Π_A in $\mathbf{S}$. The three cuffs of Π_A are the closed curves $[\bar{R}_{i+1}\bar{T}A_{i+1}TS_{i+1}\bar{S}_i\bar{T}\bar{A}_iTR_i]$. We will also show that these closed geodesics have the length 3ϵ close to $2R$, which implies that $\Pi_A \in \mathbf{\Pi}_{3\epsilon,R}$.

We finish the argument as follows. Taking the boundary of Π_A, we obtain

$$\sum_{i=0}^{2}[\bar{R}_i\bar{T}A_iTS_i\bar{S}_{i+1}\bar{T}\bar{A}_{i+1}TR_{i+1}] = 0$$

in $\mathbf{\Pi}_{3\epsilon,R}$ homology. Applying the Itemization Lemma we find

$$\begin{aligned} 0 &= \sum_{i=0}^{2}[R_{i+1}\bar{R}_i\bar{T}A_iTS_i\bar{S}_{i+1}\bar{T}\bar{A}_{i+1}T] \\ &= \sum_{i=0}^{2}\big((R_{i+1}\bar{R}_i)_T + (A_i)_{\bar{T}} + (S_i\bar{S}_{i+1})_T + (\bar{A}_{i+1})_{\bar{T}}\big) \\ &= \sum_{i=0}^{2}(R_i\bar{R}_{i+1})_T + \sum_{i=0}^{2}(S_i\bar{S}_{i+1})_T, \end{aligned}$$

in $\mathbf{\Pi}_{300\epsilon,R}$, because $(A_i)_{\bar{T}} = -(\bar{A}_i)_{\bar{T}}$.

We now verify that $[\bar{R}_i\bar{T}A_iTS_i\bar{S}_{i+1}\bar{T}\bar{A}_{i+1}TR_{i+1}] \in \Gamma_{\epsilon,R}$. By the New Angle Lemma (Lemma 4.2), for K large enough (and therefore $\mathbf{l}(\cdot T\cdot)$ large) the angle $\mathbf{\Theta}\big(i(\cdot T\cdot), i(\cdot TR_{i+1}\bar{R}_i\bar{T}\cdot)\big) \leq \frac{\epsilon}{2}$, and likewise $\mathbf{\Theta}\big(t(\cdot\bar{T}\cdot), i(\cdot TR_{i+1}\bar{R}_i\bar{T}\cdot)\big) \leq \frac{\epsilon}{10}$, and for the same with R_i replaced with S_i. It follows that $\mathbf{\Theta}\big(t(\cdot\bar{A}_{i+1}\cdot), i(\cdot TR_{i+1}\bar{R}_iT\cdot)\big) < 2\epsilon$, and so on, so by the Sum of Inefficiencies Lemma

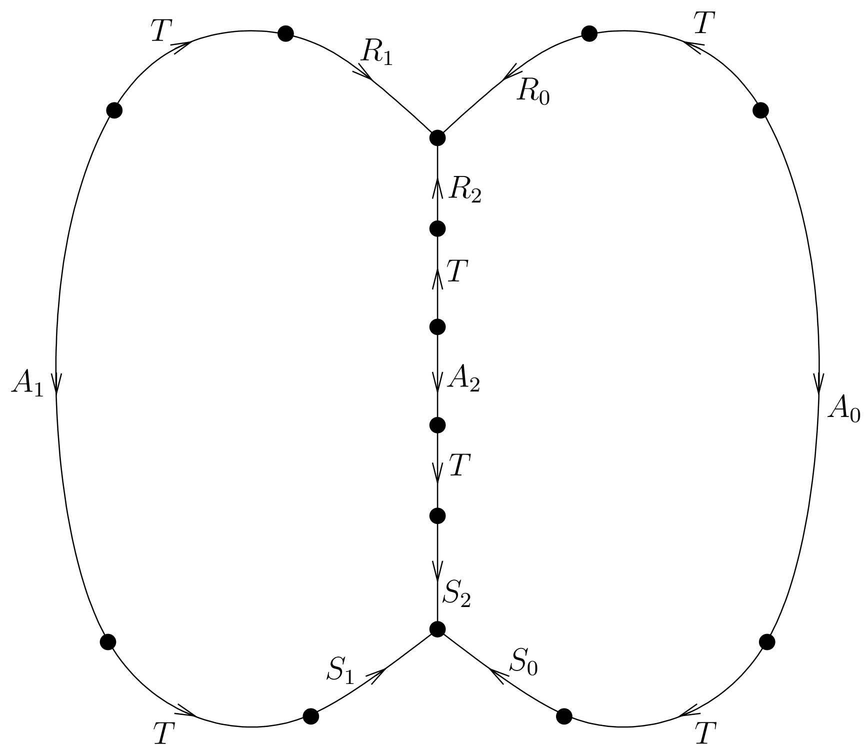

FIGURE 6. The Rotation Lemma

$$\Big|\mathbf{l}([\bar{R}_i\bar{T}A_iTS_i\bar{S}_{i+1}\bar{T}\bar{A}_{i+1}TR_{i+1}]) - \mathbf{l}(\cdot A_i\cdot) - \mathbf{l}(\cdot TR_{i+1}\bar{R}_i\bar{T}\cdot) - \mathbf{l}(\cdot A_{i+1}\cdot) - \\ - \mathbf{l}(\cdot TS_i\bar{S}_{i+1}\bar{T}\cdot)\Big| < O(\epsilon^2),$$

and moreover by (17) we have

$$\Big|\mathbf{l}(\cdot A_i\cdot) + \mathbf{l}(\cdot TR_{i+1}\bar{R}_i\bar{T}\cdot) + \mathbf{l}(\cdot A_{i+1}\cdot) + \mathbf{l}(\cdot TS_i\bar{S}_{i+1}\bar{T}\cdot) - 2R\Big| < 2\epsilon,$$

which proves the claim.

We now verify that the θ-graph associated to the geodesic arcs $\cdot\bar{R}_i\bar{T}A_iTS_i\cdot$ generates an immersed pair of pants Π_A in $\mathbf{S}$. We find a unique $\theta_0 \in [0,\pi]$ such that $I(\theta_0) = \Delta + 1$ (that is $\theta_0 = 2\sec^{-1}(e^{\frac{\Delta+1}{2}})$). Observe that $I(\cdot\bar{R}_i\cdot\bar{T}\cdot) < I(\cdot T\cdot R_{i+1}\cdot\bar{R}_i\cdot\bar{T}\cdot) \le \Delta$. Then $\mathbf{l}(\cdot\bar{R}_i\bar{T}\cdot) > \mathbf{l}(\cdot T\cdot) - \Delta > K - \Delta$. By the Sum of Inefficiencies Lemma for Angles

$$I(\mathbf{\Theta}\big(i(\cdot\bar{R}_i\bar{T}\cdot), i(\cdot\bar{R}_{i+1}\bar{T}\cdot)\big) - 1 < I(\cdot TR_{i+1}\cdot\bar{R}_i\bar{T}\cdot) \le I(\cdot T\cdot R_{i+1}\cdot\bar{R}_i\cdot\bar{T}\cdot) < \Delta.$$

Therefore $\mathbf{\Theta}\big(i(\cdot\bar{R}_i\bar{T}\cdot), i(\cdot\bar{R}_{i+1}\bar{T}\cdot)\big) > \theta_0$.

On the other hand, by the New Angle Lemma, because the geodesic arc $\cdot\bar{R}_i\bar{T}\cdot$ is long for large enough K (we showed above that $\mathbf{l}(\cdot\bar{R}_i\bar{T}\cdot) > K - \Delta$), we have $\mathbf{\Theta}\big(i(\cdot\bar{R}_i\bar{T}\cdot), i(\cdot\bar{R}_i\bar{T}A_iTS_i\cdot)\big) < \frac{\theta_0}{2}$, so the cyclic order of the triple of vectors $i(\cdot\bar{R}_i\bar{T}A_iTS_i\cdot)$, $i = 0,1,2$, is the same as of the triple of vectors $i(\cdot \bar{R}_i \bar{T} \cdot)$, and likewise the cyclic order of the triple of vectors $t(\cdot \bar{R}_i \bar{T} A_i T S_i \cdot)$, $i = 0, 1, 2$, is the same as of the triple of vectors $t(\cdot T S_i \cdot)$. So the corresponding cyclic orderings are opposed and we are finished.

□

Randomization. *The randomization remark for the First Rotation Lemma. We let $K = K(\epsilon, \mathbf{S})$. We have defined $g : G^6 \to \mathbb{R}\mathbf{\Pi}_{1,R}$ such that*

$$\partial g(R_0, R_1, R_2, S_0, S_1, S_2) = \sum_{i=0}^{2} (R_{i+1} \bar{R}_i)_T + (S_i \bar{S}_{i+1})_T.$$

Let Π denote the pants whose θ-graph is made out of the three connections $\cdot \bar{R}_i \bar{T} A_i T S_i \bar{S}_{i+1} \bar{T} A_{i+1} T R_{i+1} \cdot$, $i = 0, 1, 2$. We can write

$$g((R_i), (S_i)) = \Pi + \sum_{i-0}^{2} g_1(R_{i+1} \bar{R}_i, A_i, S_i S_{i+1}, \bar{A}_{i+1}),$$

where g_1 is the map from the ADCB Lemma (see the randomization remark). So g is $K(e^{12\mathbf{l}(\cdot T \cdot)} + Re^{4\mathbf{l}(\cdot T \cdot)})$-semirandom with respect to $\Sigma_G^{\times 6}$ and $\Sigma_{\mathbf{\Pi}}$.

The Second Rotation Lemma is:

Lemma 8.2. *Let $\epsilon, \Delta > 0$. There exists $K = K(\epsilon, \Delta) > 0$ with the following properties. Let $R_i, T \in \pi_1(\mathbf{S}, *)$, $i = 0, 1, 2$, such that*

1. *$I(\cdot T \cdot R_i \cdot \bar{R}_{i+1} \cdot \bar{T} \cdot) < \Delta$,*
2. *$\mathbf{l}(\cdot T \cdot) \geq K$.*

Then

$$\sum_{i=0}^{2} (R_i \bar{R}_{i+1})_T = 0, \qquad (18)$$

in $\mathbf{\Pi}_{300\epsilon,R}$ homology.

Proof. Given T; We choose $v \in T^1_* \mathbf{S}$ and let $\rho = e^{\frac{2\pi i}{3}}$. We take L sufficiently large so that $\mathrm{Conn}_{\epsilon,L}(t(\cdot T \cdot), \rho^i v)$ is non-empty, for $i = 0, 1, 2$. Then we choose $\cdot S_i \cdot \in \mathrm{Conn}_{\epsilon,L}(t(\cdot T \cdot), \rho^i v)$. Then $I(\cdot T \cdot S_i \cdot \bar{S}_{i+1} \cdot \bar{T} \cdot) \leq \log \frac{4}{3} + O(\epsilon) \leq 1$, by the the Sum of Inefficiencies for Angles Lemma, so when $\mathbf{l}(\cdot T \cdot)$ is large we can apply the previous Lemma (see the Remark after Lemma 8.1) to obtain

$$\sum_{i=0}^{2} (S_i \bar{S}_{i+1})_T = 0$$

in $\mathbf{\Pi}_{300\epsilon,R}$ homology.

Then given R_i as in the hypothesis to this lemma, we obtain

$$\sum_{i=0}^{2} (S_i \bar{S}_{i+1})_T + (R_i R_{i+1})_T = 0,$$

so

$$\sum_{i=0}^{2}(R_iR_{i+1})_T = 0.$$

□

Randomization. *The randomization remark for the Second Rotation Lemma. All constants K may only depend on ϵ, Δ and $\mathbf{S}$. We have defined $g : G^3 \to \mathbb{R}\mathbf{\Pi}_{1,R}$ such that*

$$\partial g(R_0, R_1, R_2) = \sum_{i=0}^{2}(R_iR_{i+1})_T.$$

We are fixing S_0, S_1, S_2 of length L, so the triple (S_0, S_1, S_2) is e^{3L}-semirandom. Let $\Pi = g_1(S_0, S_1, S_2, S_2, S_1, S_0) \in \mathbb{R}\mathbf{\Pi}_{1,R}$, where g_1 is the map from the First Rotation Lemma. Then Π is $RKe^{6L+12\mathbf{l}(\cdot T\cdot)}$-semirandom for $\sigma_{\mathbf{\Pi}}$ (in the sense that the unit atom on Π is bounded by that multiple of $\sigma_{\mathbf{\Pi}}$).

Then

$$g(R_0, R_1, R_2) = g_1((R_i), (S_i)) - \frac{1}{2}\Pi,$$

so g is $RKe^{6L+12\mathbf{l}(\cdot T\cdot)}$-semirandom with respect to $\Sigma_G^{\times 3}$ and $\sigma_{\mathbf{\Pi}}$.

8.2. **The XY Theorem.** The following theorem follows from the Second Rotation Lemma. We call it the XY Theorem.

Theorem 8.1. *Let $\epsilon, \Delta > 0$. There exists $K = K(\epsilon, \Delta) > 0$ with the following properties. Let $X, Y, T \in \pi_1(\mathbf{S}, *)$, $i = 0, 1, 2$, such that*

1. $I(\cdot T \cdot X \cdot Y \cdot \bar{T}\cdot), I(\cdot T \cdot X \cdot \bar{T}\cdot), I(\cdot T \cdot Y \cdot \bar{T}\cdot) < \Delta$,
2. $\mathbf{l}(\cdot T\cdot) \geq K$.

Then $(XY)_T = X_T + Y_T$ in $\mathbf{\Pi}_{300\epsilon,R}$ homology.

Proof. Set $R_0 = \mathrm{id}$, $R_1 = X$, and $R_2 = \bar{Y}$, and apply the previous lemma. □

Randomization. *We have defined the map $g_{XY} : (X, Y) \to \mathbb{R}\mathbf{\Pi}_{1,R}$ such that $\partial g_{XY}(X, Y) = (XY)_T - X_T - Y_T$ (the map g_{XY} is defined on the appropriate subset of Σ_G^2 described in the statement of Theorem 8.1). This map is $RKe^{6\mathbf{l}(\cdot T\cdot)}$-semirandom with respect to $\Sigma_G^{\times 2}$ and $\sigma_{\mathbf{\Pi}}$, where $K = K(\epsilon, \Delta)$.*

9. The Endgame

9.1. **The good pants homology of short words.** The following is the Good Direction Lemma.

Lemma 9.1. *For any finite set $W \subset \pi_1(\mathbf{S}, *)$, we can find $\Delta = \Delta(\mathbf{S}, W)$, such that for any L we can find $T \in \pi_1(\mathbf{S}, *)$ such that $\mathbf{l}(\cdot T\cdot) > L$ and $I(\cdot T \cdot X \cdot \bar{T}\cdot) < \Delta$, when $X \in W$.*

Proof. For any $v \in T^1_* \mathbf{S}$, and $t > 0$, we let $\alpha_t(v)$ be the geodesic segment of length t such that $i(\alpha_t(v)) = v$, and we let $\alpha_\infty(v)$ be the corresponding infinite geodesic ray. We claim that for any $X \in \pi_1(\mathbf{S}, *)$, and $X \neq id$, there are at most two $v \in T^1_* \mathbf{S}$ such that

$$\lim_{t \to \infty} I(\alpha_t^{-1}(v) \cdot X \cdot \alpha_t(v)) = \infty. \tag{19}$$

To prove the claim we lift $\cdot X \cdot$ to the universal cover $\mathbb{H}^2$, and thus get two lifts of $*$, and hence two lifts of v. We observe that (19) holds if and only if the two lifts of $\alpha_\infty(v)$ end at the same point of $\partial \mathbb{H}^2$. The map from the unit circle of $T^1_* \mathbf{S}$ to the endpoint α_∞ (this map maps v to the endpoint of a given lift of $\alpha_\infty(v)$) is a Möbius transformation M. Therefore the relation (19) holds if and only if v is the fixed point of the Möbius transformation M, so if M is not the identity then we are finished.

We prove that that M is not the identity as follows. Suppose that M is the identity. Then if we take $v = i(\cdot X \cdot)$, then v has to be the terminal vector $t(\cdot X \cdot)$ as well. But then if we take $v = \sqrt{-1} i(\cdot X \cdot)$ then we would find that the geodesic rays that start at the lifts of $\sqrt{-1} i(\cdot X \cdot)$ and $\sqrt{-1} t(\cdot X \cdot)$ have the same endpoint on $\partial \mathbb{H}^2$ which is impossible since then we would produce a triangle in $\mathbb{H}^2$ whose sum of angles is equal to π.

We write $I(\alpha_\infty^{-1}(v) \cdot X \cdot \alpha_\infty(v))$ for $\lim_{t \to \infty} I(\alpha_t^{-1}(v) \cdot X \cdot \alpha_t(v))$. Then we observe that for each $X \in \pi_1(\mathbf{S}, *)$ the function $v \to I(\alpha_\infty^{-1}(v) \cdot X \cdot \alpha_\infty(v))$ is a continuous map from $T^1_* \mathbf{S}$ to $[0, \infty]$. Therefore, for any closed subinterval J of the unit circle in $T^1_* \mathbf{S}$ that is disjoint from the set of bad directions for $X \in W$ (we have proved that there are at most $2|W|$ bad directions), there exists Δ such that $I(\alpha_t^{-1}(v) \cdot X \cdot \alpha_t(v)) \leq \Delta$ for all $X \in W$, $t > 0$, and $v \in J$.

We then let v_0 be the midpoint of J and let $\delta = \frac{|J|}{2}$. If we take $T \in \pi_1(\mathbf{S}, *)$ such that $\cdot T \cdot \in \text{Conn}_{\delta, L+1}(v_0, v_0)$, then $L < \mathbf{l}(\cdot T \cdot) < L + 2$, and $I(\cdot T \cdot X \cdot \bar{T} \cdot) < \Delta$, for all $X \in W$.

□

In the remainder of this section we fix a set of standard generators $g_1, ..g_{2n}$ of $\pi_1(\mathbf{S}, *)$ (here n is the genus of $\mathbf{S}$). Recall that $\mathbf{H}_1$ denotes the standard homology on $\mathbf{S}$. Let $[g_i]$ denote the corresponding closed curves. For any closed curve $\gamma \subset \mathbf{S}$ there are unique $a_1, ...a_{2n}$ such that $\gamma = \sum a_i [g_i]$ in $\mathbf{H}_1$. We define $q : \Gamma \to \mathbb{R}\pi_1(\mathbf{S}, *)$ by $q(\gamma) = \sum a_i g_i$, where Γ is the set of all closed curves on $\mathbf{S}$. We extend the definition of q to a map $q : \pi_1(\mathbf{S}, *) \to \mathbb{R}\{g_1, .., g_{2n}\}$ by $q(X) = q([X])$.

For $l \in \mathbb{N}$, we define the set W_l as the set of elements $X \in \pi_1(\mathbf{S}, *)$ that can be written as a product of at most l generators (or their inverses).

Theorem 9.1. *Let $\epsilon > 0$. For all $l \in \mathbb{N}$, and $L > 0$, we can find $T \in \pi_1(\mathbf{S}, *)$ and R_0 such that $\mathbf{l}(\cdot T \cdot) > L$, and for $R > R_0$, and $X \in W_l$, we have*

$$X_T = (q(X))_T$$

in $\mathbf{\Pi}_{300\epsilon, R}$ homology.

Remark. *Here we extended the partial map* $(\cdot)_T : \pi_1(\mathbf{S}, *) \to \mathbb{R}\Gamma_{\epsilon,R}$ *(given by* $X \mapsto X_T$*) to a partial map* $(\cdot)_T : \mathbb{R}\pi_1(\mathbf{S}, *) \to \mathbb{R}\Gamma_{\epsilon,R}$*. We remind the reader that* X_T *depends implicitly on* R *and* ϵ*.*

Proof. We take $\Delta = \Delta(W_l)$ and $T = T(W_l, L)$ from the previous lemma, so $\mathbf{l}(\cdot T \cdot) > L$ and $I(\cdot T \cdot X \cdot \bar{T}) < \Delta$, for all $X \in W_l$. If $X \in W_1$, then $q(X) = X$ or $q(X) = -\bar{X}$, so $X_T = (q(X))_T$.

Take $1 \le k < l$, and assume $X_T = ((q(X))_T$ in $\mathbf{\Pi}_{300\epsilon,R}$ homology for all $X \in W_k$. Then for any $X \in W_{k+1}$ we can write $X = g^\sigma Y$, for some $i \in \{1, .., 2n\}$, and $\sigma = \pm 1$, and $Y \in W_k$. Then $X_T = (g_i^\sigma)_T + Y_T$ by the XY Theorem (see Theorem 8.1) which requires

$$I(\cdot T \cdot X \cdot \bar{T} \cdot), I(\cdot T \cdot g_i^\sigma \cdot \bar{T} \cdot), I(\cdot T \cdot Y \cdot \bar{T} \cdot) < \Delta,$$

and $Y_T = (q(Y))_T$ by assumption, so $X_T = ((q(X))_T$. We conclude the theorem by induction.

□

Randomization. *The Randomization remarks for Theorem 9.1. Given* l, L, T *and* R *(and* ϵ*) we have implicitly defined the map* $g_W : W_l \to \mathbf{\Pi}_{300\epsilon,R}$ *such that* $\partial g_W(X) = X_T - (q(X))_T$*. The map* g_W *arises from a sum of at most* l *applications of the* XY *Theorem so* g_W *is* $K(\mathbf{S})RKe^{6\mathbf{l}(\cdot T \cdot)}$*-semirandom, because every measure in* Σ_G *has total mass at most* $K(\mathbf{S})$*.*

9.2. **Preliminary lemmas.**

Lemma 9.2. *There exists a universal constant* $\widehat{\epsilon} > 0$ *such that for every* $0 < \epsilon < \widehat{\epsilon}$*, there exists a constant* $L = L(\epsilon, \mathbf{S}) > 0$*, with the following properties. For any* $\gamma \in \Gamma_{\epsilon,R}$ *and* $T \in \pi_1(\mathbf{S}, *)$*,* $\mathbf{l}(\cdot T \cdot) > L$*, we can find* $X_0, X_1 \in \pi_1(\mathbf{S}, *)$ *such that*

(1) $|\mathbf{l}(\cdot X_i \cdot) - (R + 2L - \log 4)| < \frac{1}{2}$,
(2) $\mathbf{\Theta}(t(\cdot T \cdot), i(\cdot X_i \cdot))$, $\mathbf{\Theta}(t(\cdot X_i \cdot), i(\cdot \bar{T} \cdot)) \le \frac{\pi}{6}$,
(3) $\gamma = (X_0)_T + (X_1)_T$ *in* $\mathbf{\Pi}_{300\epsilon,R}$ *homology.*

Proof. We take at random two points x_0 and x_1 on the parametrising torus $\mathbb{T}_\gamma$ that are $\mathbf{hl}(\gamma)$ apart and we let $w_i \in T^1_{x_i}\mathbf{S}$ be $-\sqrt{-1}\gamma'(x_i)$. We let γ_i be the subsegment of γ from x_i to x_{i+1} (where $x_2 = x_0$).

For $i = 0, 1$ we take $\alpha_i \in \mathrm{Conn}_{\frac{\epsilon}{10},L}(t(\cdot T \cdot), w_i)$, where $L = L(\epsilon, \mathbf{S})$ is the constant from the Connection Lemma (that is, we choose L so that the set $\mathrm{Conn}_{\frac{\epsilon}{10},L}(t(\cdot T \cdot), w_i)$ is non-empty). Observe that the piecewise geodesic arc $\alpha_0\gamma_0\alpha_1^{-1}$ begins and end at the point $*$, so we let $X_0 \in \pi_1(\mathbf{S}, *)$ denote the corresponding element of $\pi_1(\mathbf{S}, *)$. Similarly we let $X_1 \in \pi_1(\mathbf{S}, *)$ be the element that corresponds to the curve $\alpha_1\gamma_1\alpha_0^{-1}$.

It follows from the Sum of Inefficiencies for Angles Lemma that the inequality (1) of the statement of the lemma holds. On the other hand, by the New Angle Lemma the angle $\mathbf{\Theta}(i(\cdot X_0 \cdot), i(\alpha_0))$ is as small as we want providing that $\mathbf{l}(\alpha_0) > L$ is large enough (here we use that the inefficiency $I(\alpha_0\gamma_0\alpha_1^{-1})$ is bounded above). Since by construction the angle

$\mathbf{\Theta}(i(\alpha_0), t(\cdot T\cdot))$ is less than $\frac{\epsilon}{10}$ we conclude that for L large enough we have $\mathbf{\Theta}(t(\cdot T\cdot), i(\cdot X_0\cdot)) < \frac{\pi}{6}$. Other cases are treated similarly.

Let $\cdot A\cdot$ be a random element of $\mathrm{Conn}_{\frac{\epsilon}{10},R'}(-i(\cdot T\cdot), i(\cdot T\cdot))$, where $R' = R + \log 4 - 2L - 2\mathbf{l}(\cdot T\cdot)$. Then

$$\left|\mathbf{l}([X_0\bar{T}\bar{A}T]) - 2R\right| < \epsilon,$$

$$\left|\mathbf{l}([X_1\bar{T}AT]) - 2R\right| < \epsilon,$$

so $\gamma = [X_0\bar{T}\bar{A}T] + [X_1\bar{T}AT]$ in $\mathbf{\Pi}_{\epsilon,R}$ homology.

Moreover, $[X_0\bar{T}\bar{A}T] = (X_0)_T + (\bar{A})_{\bar{T}}$, and $[X_1\bar{T}AT] = (X_1)_T + A_{ovT}$ in $\mathbf{\Pi}_{100\epsilon,R}$ homology by the Simple Itemization Lemma. Since $(\bar{A})_{\bar{T}} = -A_{\bar{T}}$ we conclude $\gamma = (X_0)_T + (X_1)_T$ in $\mathbf{\Pi}_{300\epsilon,R}$ homology.

□

Randomization. *The randomization remarks for Lemma 9.2. We have defined the maps* $q_C : \Gamma_{1,R} \to \mathbb{R}G$ *(by* $q_C(\gamma) = X_0 + X_1$*) and* $g_C : \Gamma_{1,R} \to \mathbb{R}\mathbf{\Pi}_{1,R}$*, such that* $\partial g_C(\gamma) = \gamma - (q_C(\gamma))_T$ *(where* $A \to A_T$ *maps* $\mathbb{R}G \to \mathbb{R}\Gamma_{1,R}$*).*

The map q_C *is* $e^L K$*-semirandom with respect to* σ_Γ *and* Σ_G*. The map* g_C *is* $e^{2\mathbf{l}(\cdot T\cdot)}K(\mathbf{S}, \epsilon)$ *semirandom with respect to* σ_Γ *and* $\sigma_{\mathbf{\Pi}}$*, where* $K = K(\mathbf{S}, \epsilon)$*.*

We have the following definition. For any $X, T \in \pi_1(\mathbf{S}, *)$, $X \neq \mathrm{id}$, we let

$$\theta_X^T = \max\{\mathbf{\Theta}(t(\cdot T\cdot), i(\cdot X\cdot)), \mathbf{\Theta}(t(\cdot X\cdot), i(\cdot \bar{T}\cdot))\}.$$

Lemma 9.3. *For* $L > L_0(\mathbf{S})$*, and* $X, T \in \pi_1(\mathbf{S}, *)$*,* $X \neq \mathrm{id}$*, then we can write* $X = X_0X_1$*, for some* $X_0, X_1 \in \pi_1(\mathbf{S}, *)$*, such that*

1. $\left|\mathbf{l}(\cdot X_i\cdot) - (\frac{\mathbf{l}(\cdot X\cdot)}{2} + L - \log 2)\right| < \frac{1}{2}$,
2. $I(\cdot X_0 \cdot X_1\cdot) \leq 2L + 3$,
3. $\theta_{X_i}^T \leq \max\{\theta_X^T + e^{L+4}e^{-\mathbf{l}(\cdot X_i\cdot)}, \frac{\pi}{6}\}$.

Proof. We let $\alpha = \cdot X\cdot$, then $\alpha : [0, \mathbf{l}(\cdot X\cdot)] \to \mathbf{S}$ is the unit speed parametrization with $\alpha(0) = \alpha(\mathbf{l}(\cdot X\cdot)) = *$. We let $y = \frac{\mathbf{l}(\cdot X\cdot)}{2}$. Then for L large enough, we can find $\beta \in \mathrm{Conn}_{\frac{1}{20},L}(t(\cdot T\cdot), \sqrt{-1}\alpha'(y))$ (as always, L is determined by the Connection Lemma).

Then $\alpha[0, y]\beta^{-1}$ begins and ends at $*$, so it represents some $X_0 \in \pi_1(\mathbf{S}, *)$. Likewise $\beta\alpha[y, \mathbf{l}(\cdot X\cdot)]$ represents some $X_1 \in \pi_1(\mathbf{S}, *)$, and $X = X_0X_1$. Moreover,

$$\left|\mathbf{l}(\cdot X_i\cdot) - \left(\frac{\mathbf{l}(\cdot X\cdot)}{2} + L - \log 2\right)\right| < \frac{1}{2},$$

The condition (2) follows immediately from (1).

Let $\theta = \mathbf{\Theta}(i(\cdot X\cdot), i(\cdot X_0\cdot))$. Then by the hyperbolic law of sines, assuming that $\mathbf{l}(\cdot X_i\cdot) \geq 1$ (which follows if we assume that $\mathbf{l}(\alpha) \geq L - 1$ is at least 1) we obtain

$$\sin(\theta) \leq \frac{\sinh(L+1)}{\sinh(\mathbf{l}(\cdot X_0\cdot))} \leq e^{L+2-\mathbf{l}(\cdot X_0\cdot)},$$

Therefore

$$\mathbf{\Theta}(t(\cdot T\cdot), i(\cdot X_0\cdot)) \leq \mathbf{\Theta}(t(\cdot T\cdot), i(\cdot X\cdot)) + e^{L+4-\mathbf{l}(\cdot X_0\cdot)}.$$

By similar reasoning we find that $\mathbf{\Theta}(t(\cdot X_0\cdot), -i(\beta)) \leq e^{2-L} < \frac{\pi}{12}$, assuming that L is large enough. Also by construction $\mathbf{\Theta}(-i(\beta), t(\cdot T\cdot)) < \frac{1}{20} < \frac{\pi}{12}$, so $\mathbf{\Theta}(t(\cdot X_0\cdot), i(\cdot \bar{T}\cdot)) \leq \frac{\pi}{6}$. We proceed similarly for X_1.

□

Randomization. *The randomization remarks for Lemma 9.3. We have defined* $\widehat{q}_D : G \to G^2$ *such that* $\widehat{q}_D(X) = (X_0, X_1)$*. If* $\mathbf{l}(\cdot X\cdot) \in [a, a+1]$*, then* $\mathbf{l}(\cdot X_0\cdot), \mathbf{l}(\cdot X_1\cdot) \in [\frac{a}{2} + L', \frac{a}{2} + L' + 1]$*, where* $L' = L - \log 2 - \frac{1}{2}$*.*

Moreover, given $(X_0, X_1) \in G^2$ *there is at most one* X *such that* $\widehat{q}_D(X) = (X_0, X_1)$ *(because* $X = X_0X_1$*). We conclude that*

$$(\widehat{q}_D)_*\sigma_a \leq e^{2L'+2}\sigma_{\frac{a}{2}+L'} \times \sigma_{\frac{a}{2}+L'},$$

and hence $\widehat{q}_D$ *is* $e^{2L'+2}$*-semirandom. It follows that the map* $q_D : G \to \mathbb{R}G$ *defined by* $X \to X_0 + X_1$*, is* $2e^{2L'+2}$*-semirandom.*

9.3. **Proof of Theorem 3.2.** The following theorem implies Theorem 3.2. Recall $\{g_1, ..., g_{2n}\}$ denotes a standard basis for $\pi_1(\mathbf{S}, *)$, where n is the genus of $\mathbf{S}$.

Theorem 9.2. *Let* $\epsilon > 0$*. There exists* $R_0 = R_0(\mathbf{S}, \epsilon) > 0$ *with the following properties. There exists* $T \in \pi_1(\mathbf{S}, *)$*, where* T *depends only on* ϵ *and* $\mathbf{S}$*, such that for every* $R > R_0$ *and every* $\gamma \in \Gamma_{\epsilon,R}$ *we have*

$$\gamma = \sum_{i=1}^{2g} a_i (g_i)_T,$$

in $\mathbf{\Pi}_{300\epsilon,R}$ *homology, for some* $a_i \in \mathbb{Q}$*.*

Remark. *To prove Theorem 3.2 we take* $h_i = (g_i)_T$*. Since* $(g_i)_T$ *is equal to the closed curve on* $\mathbf{S}$ *that corresponds to* g_i *in the standard homology* $\mathbf{H}_1$*, it follows that* h_i *is a basis for* $\mathbf{H}_1$ *(with rational coefficients).*

Proof. We take L that is sufficiently large for Lemma 9.2 and Lemma 9.3. We let $l \in \mathbb{N}$ be such that $X \in W_l$ whenever $\mathbf{l}(\cdot X\cdot) < 2L + 5$. Then by Theorem 9.1 we can find T such that $\mathbf{l}(\cdot T\cdot) > L$ and $\mathbf{l}(\cdot T\cdot) > K(\epsilon, 2L+3)$, where $K(\epsilon, \Delta)$ is the constant from Theorem 8.1, and such that $X_T = (q(X))_T$ for all $X \in W_l$. We take $R > R_0(\mathbf{S}, \epsilon, L)$ from Lemma 9.2, and $R > R_0(L, T)$ from Theorem 9.1.

Fix any $\gamma \in \Gamma_{\epsilon,R}$.

By Lemma 9.2 we can find $X_0, X_1 \in \pi_1(\mathbf{S}, *)$ such that $|\mathbf{l}(\cdot X_i\cdot) - (R + 2L - \log 4)| < \frac{1}{2}$, and

$$\gamma = (X_0)_T + (X_1)_T \tag{20}$$

in $\mathbf{\Pi}_{\epsilon,R}$ homology. Observe that $q(\gamma) = q(X_0) + q(X_1)$.

By Lemma 9.3 we can write $X_0 = X_{00}X_{01}$, where

$$\mathbf{l}(\cdot X_{0i}\cdot) \in \Big[\frac{R}{2}+2L, \frac{R}{2}+2L+1\Big] \tag{21}$$

and the conclusions of Lemma 9.3 hold. And likewise for X_1.

Let $N = \lfloor \log_2 R \rfloor - 1$. For every $0 \leq k \leq N$, we define sets $\mathcal{X}_k$ by letting $\mathcal{X}_0 = \{X_0, X_1\}$ and the set $\mathcal{X}_{k+1}$ is the set of children of elements of $\mathcal{X}_k$. Each set $\mathcal{X}_k$ has 2^{k+1} elements and the elements of $\mathcal{X}_k$ are not necessarily distinct. Moreover, for any $X \in \mathcal{X}_k$ we have

$$\mathbf{l}(\cdot X\cdot) \in \big[R2^{-k}+2L, R2^{-k}+2L+1\big].$$

We claim that

$$\theta_X^T < \frac{\pi}{3}$$

for every X in any $\mathcal{X}_k$. For any such X we can find a sequence $Y_0, Y_1, ..., Y_k$, so that $Y_0 = X_0$ or $Y_0 = X_1$ and $Y_k = X$, and where Y_{i+1} is a child of Y_i. It follows from the equation (21) that $\mathbf{l}(\cdot Y_{i+1}\cdot) \leq \mathbf{l}(\cdot Y_i\cdot) - 1$, and $\mathbf{l}(\cdot Y_k\cdot) \geq 2L-2$.

$$\begin{aligned}\theta_{Y_k}^T &\leq \frac{\pi}{6} + \sum_{i=0}^{k} e^{L+4-\mathbf{l}(\cdot Y_i\cdot)} \\ &\leq \frac{\pi}{6} + \frac{e}{e-1}e^{L+4-(2L-3)} < \frac{\pi}{3},\end{aligned}$$

assuming $L > 8$.

By Lemma 4.3 we have $I(\cdot T \cdot X \cdot \bar{T}\cdot) \leq \log 4$ for every X in every $\mathcal{X}_k$. Therefore, we can apply The XY Theorem (see Theorem 8.1) and conclude that

$$Y_T = (Y_0Y_1)_T \tag{22}$$

whenever Y is a non-trivial node of our tree and Y_0 and Y_1 are its two children.

It follows by (20) and (22) applied recursively that

$$\gamma = \sum_{X\in\mathcal{X}_N} X_T,$$

in $\mathbf{\Pi}_{300\epsilon,R}$ homology. We know that if $X \in \mathcal{X}_N$ then $X \in W_l$, so $X_T = (q(X))_T$. Therefore

$$\gamma = \sum_{X\in\mathcal{X}_N} (q(X))_T = \sum((q(\gamma))_T,$$

so we are finished.

□

Randomization. *Randomization remarks for the proof of Theorem 3.3. We have determined $T \equiv T(\mathbf{S}, \epsilon)$, so $e^{\mathbf{l}(\cdot T\cdot)} = K(\mathbf{S}, \epsilon)$. We have implicitly defined the map $g : \Gamma_{\epsilon,R} \to \mathbb{R}\mathbf{\Pi}_{300\epsilon,R}$ such that $\partial g(\gamma) = \gamma - (q(\gamma))_T$. We* *note that* $q = q \circ q_D^N \circ q_C$*, where* $q_C(\gamma) = X + X'$ *from Lemma 9.2,* $q_D(X) = X_0 + X_1$*, from Lemma 9.3.*

Moreover,

$$g(\gamma) = g_C(\gamma) + \sum_{i=0}^{N-1} g_{XY}(\widehat{q}_D(q_D^i(q_C(\gamma)))) + g_W(q_D^N(q_C(\gamma)))$$

where g_C *is the map from Lemma 9.2,* q_D *and* $\widehat{q}_D$ *are the maps from Lemma 9.3,* g_{XY} *is the map from Theorem 8.1,* N *is the number of times we iterate the division (the application of Lemma 9.3), and* g_W *is from Theorem 9.1.*

By far the most important point is that q_D *is* $K = K(\mathbf{S}, \epsilon)$*-semirandom, so* q_D^i *is* K^i*-semirandom, for any* $i \leq N$ *(recall that* $N \leq \lfloor \log_2 R \rfloor$*) and therefore* $K^i \leq R^{\log_2 K}$ *so the map* q_D^i *is* $P(R)$*-semirandom, where* $P(R)$ *denotes a polynomial in* R*).*

9.4. **The proof of Theorem 3.3.** The map ϕ from Theorem 3.3 is defined to be equal to the map g from the Randomization remarks for Theorem 9.2. We take $h_i = (g_i)_T$. Then $\partial\phi(\gamma) = \gamma - (q(\gamma))_T$, and $(q(\gamma))_T \in \mathbb{R}\{h_1, .., h_{2n}\}$, for any $\gamma \in \mathbb{R}\Gamma_{\epsilon,R}$. Moreover, the map ϕ is $P(R)$-semirandom as shown in those Randomization remarks. This implies the estimate (3) of the statement of Theorem 3.3 and we are finished.

Appendix

Introduction to randomization. Let (X, μ) and (Y, ν) denote two measure spaces (where μ and ν are positive measures).

Definition 9.1. *We say that a map* $g : (X, \mu) \to (Y, \nu)$ *is* K*-semirandom with respect to* μ *and* ν *if* $g_*\mu \leq K\nu$*.*

By $\mathbb{R}X$ we denote the vector space of finite formal sums (with real coefficients) of points in X. There is a natural inclusion map $\iota : X \to \mathbb{R}X$, where $\iota(x) \in \mathbb{R}X$ represents the corresponding sum. Then every map $\widetilde{f} : \mathbb{R}X \to S$, where S is any set, induces the map $f : X \to S$ by letting $f = \widetilde{f} \circ \iota$.

Let $f : X \to \mathbb{R}Y$ be a map. Then we can write $f(x) = \sum_y f_x(y)y$, where the function $f_x : Y \to \mathbb{R}$ is non-zero for at most finitely many points of Y. We define $|f| : X \to \mathbb{R}Y$ by

$$|f|(x) = \sum_y |f_x(y)|y.$$

We define the measure $|f|_*\mu$ on Y by

$$|f|_*\mu(V) = \int_X \left(\sum_y |f_x(y)|\chi_V(y) \right) d\mu(x),$$

for any measurable set $V \subset Y$, and $\chi_V(y) = 1$, if $y \in V$ and $\chi_V(y) = 0$, if $y \notin V$.

Definition 9.2. *Let (X,μ) and (Y,ν) be two measure spaces (with positive measures μ and ν). A map $f : X \to \mathbb{R}Y$ is K-semirandom if $|f|_*\mu \le K\nu$. A linear map $\widetilde{f} : \mathbb{R}X \to \mathbb{R}Y$ is K-semirandom with respect to measures μ and ν on X and Y respectively, if the induced map $f : X \to \mathbb{R}Y$ is K-semirandom.*

The following propositions are elementary.

Proposition 9.1. *Let X, Y and Z denote three measure spaces. If $f : \mathbb{R}X \to \mathbb{R}Y$ is K-semirandom, and $f : \mathbb{R}Y \to \mathbb{R}Z$ is L-semirandom, then $g \circ f : \mathbb{R}X \to \mathbb{R}Z$ is KL-semirandom.*

Proposition 9.2. *If $f_i : \mathbb{R}X \to \mathbb{R}Y$ is K_i-semirandom, $i = 1, 2$, and $\lambda_i \in \mathbb{R}$, then the map $(\lambda_1 f_1 + \lambda_2 f_2) : \mathbb{R}X \to \mathbb{R}Y$ is $(|\lambda_1|K_1 + |\lambda_2|K_2)$-semirandom.*

Remark. *We say that $f : X \to Y$ is a partial map if it is defined on some measurable subset $X_1 \subset X$. The notion of a semirandom maps generalizes to the case of partial maps by letting a partial map $f : X \to Y$ be K-semirandom if the restriction $f : X_1 \to Y$ is K-semirandom, where the corresponding measure on X_1 is the restriction of the measure from X. Every statement we make about semirandom maps has its version for a partial semirandom map. In particular, if $f : X \to Y$ is K-semirandom then the restriction of f onto any $X_1 \subset X$ is K-semirandom. Moreover, trivial partial maps (those that are defined on an empty set) are K-semirandom for any $K \ge 0$.*

A measure class on a space X is a subset of $\mathcal{M}(X)$.

Definition 9.3. *Let X and Y be measure spaces and let $\mathcal{M} \subset \mathcal{M}(X)$ and $\mathcal{N} \subset \mathcal{M}(Y)$ be measures classes on X and Y respectively (all measures from $\mathcal{M}$ and $\mathcal{N}$ are positive measures). We say $f : X \to Y$ is K semirandom with respect to $\mathcal{M}$ and $\mathcal{N}$ if for every $\mu \in \mathcal{M}$ there is $\nu \in \mathcal{N}$ such that f is K-semirandom with respect to μ and ν, that is $f_*\mu \le K\nu$.*

In a similar fashion as above we define the notion of a semirandom map $f : \mathbb{R}X \to \mathbb{R}Y$ with respect to classes of measures $\mathcal{M}$ and $\mathcal{N}$ on X and Y respectively. The following proposition follows from Proposition 9.1.

Proposition 9.3. *Let X, Y and Z denote three measure spaces, with classes of measures $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{Z}$ respectively. If $f : \mathbb{R}X \to \mathbb{R}Y$ is K-semirandom with respect to $\mathcal{M}$ and $\mathcal{N}$, and $f : \mathbb{R}Y \to \mathbb{R}Z$ is L-semirandom with respect to $\mathcal{N}$ and $\mathcal{Z}$, then $g \circ f : \mathbb{R} \to \mathbb{R}Z$ is KL-semirandom with respect to $\mathcal{M}$ and $\mathcal{Z}$.*

We say that a class of measures $\mathcal{M}$ is convex if it contains all convex combinations of its elements. The following proposition then follows from Proposition 9.2

Proposition 9.4. *If $f_i : \mathbb{R}X \to \mathbb{R}Y$ is K_i-semirandom with respect to classes of measures $\mathcal{M}$ and $\mathcal{N}$, $i = 1, 2$, and if $\mathcal{N}$ is convex, then for $\lambda_i \in \mathbb{R}$, the map $(\lambda_1 f_1 + \lambda_2 f_2) : \mathbb{R}X \to \mathbb{R}Y$ is $(|\lambda_1|K_1 + |\lambda_2|K_2)$-semirandom with respect to $\mathcal{M}$ and $\mathcal{N}$.*

Remark. *The space $\mathbb{R}X$ is naturally contained in the space $\mathcal{M}(X)$, and in a similar way we can define the notion of a semirandom map $f : \mathcal{M}(X) \to \mathcal{M}(Y)$.*

Natural measure classes. Let X_i, $i = 1, .., k$, denote measure spaces with measure classes $\mathcal{M}_i$. Let $X_1 \times ... \times X_k$ denote the product space and by $\pi_i : (X_1 \times ... \times X_k) \to X_i$ we denote the coordinate projections. By $\mathcal{M}_1 \times \mathcal{M}_2 ... \times \mathcal{M}_k$ we denote the set of measures on $X_1 \times ... \times X_k$ that arise as the convex combinations of all standard products $\mu_1 \times ... \times \mu_k$ with $\mu_i \in \mathcal{M}_i$. We also define a natural class of measures $\mathcal{N}_1 \boxtimes \mathcal{N}_2 ... \boxtimes \mathcal{N}_k$ on $X_1 \times ... \times X_k$ as

$$\mathcal{N}_1 \boxtimes \mathcal{N}_2 ... \boxtimes \mathcal{N}_k = \{\mu \in \mathcal{M}(X_1 \times ... \times X_k) : (\forall i)(\exists \mu_i \in \mathcal{M}_i)((\pi_i)_* \mu \leq \mu_i)\}.$$

This produces a large class of measures even if each $\mathcal{M}_i$ consists of a single measure. If each $\mathcal{M}_i$ is convex then $\mathcal{N}_1 \boxtimes \mathcal{N}_2 ... \boxtimes \mathcal{N}_k$ is as well. If $X_i = X$ and $\mathcal{M}_i = \mathcal{M}$, then the standard product measure on X^k is $\mathcal{M}^{\times k}$ and the other class of measures is denoted by $\mathcal{M}^{\boxtimes k}$.

We define the class $\mathcal{L}_1$ of Borel measures on $\mathbb{R}$ by saying that $\mu \in \mathcal{L}_1$ if $\mu[x, x+1) \leq 1$, for all $x \in \mathbb{R}$. This is a closed convex class of measures. Likewise we define the class of measures $\mathcal{L}_1$ on $\mathbb{R}/\lambda\mathbb{Z}$, for $\lambda > 1$, by saying that $\mu \in \mathcal{L}_1$ if $\mu[x, x+1) \leq 1$, for all $x \in \mathbb{R}/\lambda\mathbb{Z}$. The class of measures $\mathcal{L}_1$ is the class of measures that are controlled by the Lebesgue measure at the unit scale.

We consider the following spaces and their measure classes. In this paper, we define several maps (or partial maps) between these spaces (or their powers) and prove they are semirandom. We have

(1) The space of curves $\Gamma_{1,R}$ with the measure class containing the single measure σ_Γ which is defined by setting $\sigma_\Gamma(\gamma) = Re^{-2R}$, for every $\gamma \in \Gamma_{1,R}$. We may assume that ϵ is small enough so that $\Gamma_{\epsilon,R} \subset \Gamma_{1,R}$.

(2) The space of pants $\mathbf{\Pi}_{1,R}$ with the measure class containing the single measure $\sigma_{\mathbf{\Pi}}$ given by $\sigma_{\mathbf{\Pi}}(\Pi) = e^{-3R}$. We may assume that ϵ is small enough so that $\mathbf{\Pi}_{300\epsilon,R} \subset \mathbf{\Pi}_{1,R}$.

(3) Let $\dot{\Gamma}_{1,R} = \{(x, \gamma) : \gamma \in \Gamma_{1,R}, x \in \mathbb{T}_\gamma\}$ denote the space of pointed curves (recall that $\mathbb{T}_\gamma = \mathbb{R}/\mathbf{l}(\gamma)\mathbb{Z}$ is the parametrising torus for γ). The space $\dot{\Gamma}_{1,R}$ is really just the union of parametrizing tori $\mathbb{T}_\gamma$ for curves $\gamma \in \Gamma_{1,R}$. By $\Sigma_{\dot{\Gamma}}$ we denote the measure class on $\dot{\Gamma}_{1,R}$, such that $\mu \in \Sigma_{\dot{\Gamma}}$ if the restriction $\mu_\gamma = \mu|_{\mathbb{T}_\gamma}$ is in $e^{-2R}\mathcal{L}_1$, where $\mathcal{L}_1$ is the measure class on the circle $\mathbb{T}_\gamma$ that was defined above.

(4) Let $\overset{.k}{\Gamma}_{1,R} = \{(x_1, ..., x_k, \gamma) : \gamma \in \Gamma_{1,R}, x_i \in \mathbb{T}_\gamma\}$ denote the space of curves with k marked points. The space $\overset{.k}{\Gamma}_{1,R}$ is canonically contained in $\left(\dot{\Gamma}_{1,R}\right)^k$. The measure class $\Sigma_{\overset{.k}{\Gamma}}$ on $\overset{.k}{\Gamma}_{1,R}$ is the restriction of $\Sigma_{\dot{\Gamma}}^{\boxtimes k}$ on the image of $\overset{.k}{\Gamma}_{1,R}$ in $\left(\dot{\Gamma}_{1,R}\right)^k$.

(5) The space $G = \pi_1(\mathbf{S}, *)$ with the measure class Σ_G that is the convex closure of the collection of measures σ_a on G, where σ_a is defined so that for $X \in G$ we have $\sigma_a(X) = \nu_a(\mathbf{l}(\cdot X \cdot))e^{-\mathbf{l}(\cdot X \cdot)}$, where $\nu_a(x) = 1$, if $x \in [a, a+1]$, and $\nu_a(x) = 0$ otherwise.

We observe that there exists a constant $K = K(\mathbf{S})$ such that for any measure μ in any of the above defined measure classes, the total measure of μ is bounded by K.

Finally we consider the map $\partial : \mathbf{\Pi}_{1,R} \to \Gamma_{1,R}$ defined by $\partial \Pi = \gamma_0 + \gamma_1 + \gamma_2$, where γ_i are the three oriented boundary curves of Π. We observe that ∂ is $K(\mathbf{S})$-semirandom from $\sigma_{\mathbf{\Pi}}$ to σ_Γ.

Standard maps are semirandom. We consider several standard mappings and prove they are semirandom.

Lemma 9.4. *Let $l > 0$ and $a, b \leq l - 1$. Then for any $Z \in G = \pi_1(\mathbf{S}, *)$ such that $\mathbf{l}(\cdot Z \cdot) = l$, there are at most $Ke^{\frac{a+b-l}{2}}$ ways of writing $Z = XY$, with $\mathbf{l}(\cdot X \cdot) \in [a, a+1]$ and $\mathbf{l}(\cdot Y \cdot) \in [b, b+1]$, for some $K = K(\mathbf{S})$.*

Proof. Suppose that X and Y satisfy the given conditions. Consider a triangle in $\mathbb{H}^2$ whose sides are lifts of $\cdot X \cdot$, $\cdot Y \cdot$ and $\cdot Z \cdot$ (these lifts are denoted the same as the arcs we are lifting). Then we drop the perpendicular t from the vertex opposite to $\cdot Z \cdot$ to the side $\cdot Z \cdot$, and let a' and b' be the lengths of the subintervals of $\cdot Z \cdot$ that meet at the endpoint of t on $\cdot Z \cdot$ (then $a' + b' = \mathbf{l}(\cdot Z \cdot)$). For simplicity, set $t = \mathbf{l}(t)$ We find that

$$a \leq \mathbf{l}(\cdot X \cdot) \leq t + a' \leq \mathbf{l}(\cdot X \cdot) + \log 2 \leq a + 2,$$

and likewise $b \leq t + b' \leq b + 2$. So

$$t \in \left[\frac{a+b-l}{2}, \frac{a+b-l}{2} + 2\right],$$

and

$$a' \in \left[\frac{a+b-l}{2} - 2, \frac{a+b-l}{2} + 2\right].$$

Therefore, the lift of the lifts of $\cdot X \cdot$ and $\cdot Y \cdot$ must lie in a disc of radius $\frac{a+b-l}{2} + 4$ around the foot of t at $\cdot Z \cdot$. It follows that there are at most $K(\mathbf{S})e^{\frac{a+b-l}{2}}$ lift of the base point in this disc, and we are finished.

□

Let $p : G \times G \to G$ be the product map, that is $g(X, Y) = XY$.

Lemma 9.5. *The map $p : G \times G \to G$ is K-semirandom with respect to $\Sigma_G \times \Sigma_G$ on G^2 and Σ_G on G, for some $K = K(\mathbf{S})$*

Proof. Let $a, b \in [0, \infty)$, and assume $b \geq a$. Recall the measures σ_a on G, and let $\sigma = p_*(\sigma_a \times \sigma_b)$. We must show that $\sigma \leq K\Sigma_G$.

Let $Z \in G$, and let $l = \mathbf{l}(\cdot Z \cdot)$. If $a \leq b \leq l - 1$, then

$$\sigma(Z) \leq Ke^{\frac{a+b-l}{2}} e^{-a} e^{-b} = Ke^{-l} e^{-\frac{a+b-l}{2}}.$$

(If $l > a+b+2$ then $\sigma(Z) = 0$).

If $l-1 \leq b$, then because there are at most Ke^a X's in G for which $\sigma_a(X) > 0$, we find

$$\sigma(Z) \leq Ke^a e^{-a} e^{-b} = Ke^{-l} e^{-(b-l)}.$$

Then we see that

$$\frac{1}{K}\sigma \leq \sum_{k=\lfloor b-a-1\rfloor}^{\lfloor b+1\rfloor} e^{-(b-k)}\sigma_k + \sum_{k=\lfloor b\rfloor}^{\lfloor a+b+3\rfloor} e^{-\frac{(a+b-k)}{2}}\sigma_k,$$

so $\sigma \leq K\Sigma_G$.

□

We define a partial map $\text{proj} : G \to \dot{\Gamma}_{1,R}$ as follows. Given $A \in G$, we let $\gamma = [A]$, and $z \in \gamma$ be the projection of the base point $*$ to γ. As always, the projection is defined by choosing lifts of $\cdot A \cdot$ and γ in $\mathbb{H}^2$ that have the same endpoints and then we project a lift of $*$ to the lift of γ, where the lift of $*$ belongs to the lift of $\cdot A \cdot$. We let $\text{proj}(A) = (\gamma, z)$.

Lemma 9.6. *The map* $\text{proj} : G \to \dot{\Gamma}_{1,R}$ *is* $K(\mathbf{S})$*-semirandom with respect to* Σ_G *and* $\Sigma_{\dot{\Gamma}}$.

Proof. Let J be a unit interval on a curve $\gamma \in \Gamma_{1,R}$. We have seen in the two previous proofs that there are at most $Ke^{\frac{l-2R}{2}}$ many $Z \in G$ for which $\mathbf{l}(\cdot Z \cdot) \leq l$, and $\text{proj}(Z) \in J$. Therefore, if $\sigma \in \Sigma_G$, then

$$\text{proj}_*\sigma(\gamma, J) \leq K \sum_{k=\lfloor 2R\rfloor}^{\infty} e^{\frac{k-2R}{2}} e^{-k} \leq Ke^{-2R},$$

and we are finished.

□

Another standard map we consider is the projection map $\dot{\Gamma}_{1,L} \to \Gamma_{1,L}$ given by $(\gamma, x) \to \gamma$. This map is clearly 1-semirandom. Going in the opposite direction, we have the map $\gamma \to (\gamma, x)$ which assigns to $\gamma \in \Gamma_{1,R}$ a random point $x \in \gamma$. This map is really defined as a map $\mathcal{M}(\Gamma_{1,R}) \to \mathcal{M}(\dot{\Gamma}_{1,R})$, and we observe that it is 1-semirandom as well.

The principles of randomization. After almost every lemma or theorem we prove in Sections 4-9, we have added a "Randomization remark" which considers the functions that we have implicitly defined, states their domain and range, and argues that the functions are semirandom with respect to a certain measure class. In the remarks we have followed the following principles:

1. When we write "a random element" (of a finite set S) which the reader was previously told to read as "an arbitrary element", we now mean "the

random element" of $\mathbb{R}S$, namely

$$\frac{1}{|S|}\sum_{x\in S} x.$$

If $a \in \mathbb{R}S \subset \mathcal{M}(S)$ and $\mathcal{M}$ is a measure class on S we say that a is a K-semirandom element of S, with respect to $\mathcal{M}$, if there exists $\mu \in \mathcal{M}$ such that $a \leq K\mu$.

2. We can replace at will any map $f : X \to Y$ (or $f : X \to \mathbb{R}Y$) by the linear extension $f : \mathbb{R}X \to \mathbb{R}Y$. This can cause confusion if you think about it the wrong way so we offer the following example to clarify what is going on.

In the hard case of the GSL, we take a random third connection (meaning the random third connection), and then cancel out one square (A_{ij}), $i, j = 0, 1$, of boundaries to get a formal sum of squares (B_{ij}) of curves. We then find, for each new square (in the formal sum) a second third connection at random from a set depending on (B_{ij}), to complete the argument. The right way to think of the randomization (and linearization) is that the first operation defines a partial map

$$q_1 : \big(\ddddot{\Gamma}_{1,R}\big)^4 \to \mathbb{R}\big(\ddddot{\Gamma}_{1,R}\big)^4$$

and the second operation defines

$$g_0 : \Gamma^4_{1,R} \to \mathbb{R}\mathbf{\Pi}_{1,R},$$

so we can write $g_0 \circ q_1$ by extending g_0 to a map from $\mathbb{R}\Gamma^4_{1,R}$ to $\mathbb{R}\mathbf{\Pi}_{1,R}$ linearly. The danger is that one may try to imagine g_0 acting on a formal sum of curves by taking the random element from $\mathbb{R}\mathrm{Conn}_{\epsilon,R}(\cdot,\cdot)$.

So we will imagine that we are defining functions from X to Y, or from X to $\mathbb{R}Y$, and only think of them as functions from $\mathbb{R}X$ to $\mathbb{R}Y$ when we want to compose them.

3. We want to use the measure class $\Sigma_G \times \Sigma_G = \Sigma_G^{\times 2}$ on $G^2 = \{(X, Y)\}$ when we want to form the product XY. We want to use the measure class $\Sigma_G \boxtimes \Sigma_G = \Sigma_G^{\boxtimes 2}$ on G^2 if we want to be able to let $X = Z$ and $Y = Z$ for some $Z \in G$.

For example, for the ASL, we use the measure class $\Sigma_G^{\boxtimes 2} \times \Sigma_G \times \Sigma_G^{\boxtimes 2} \times \Sigma_G$ on six-tuples $(A_0, A_1, U, B_0, B_1, V)$ in $G^6 = G^2 \times G \times G^2 \times G$. This is basically the largest measure class for which the maps $\pi_{ij} : G^6 \to G^4$ defined by $\pi_{ij}(A_0, A_1, U, B_0, B_1, V) = (A_i, U, B_j, V)$ are 1-semirandom with respect to the measure class $\Sigma_G^{\times 4}$ on G^4.

This is exactly what we want, because we have to form the words $A_i U B_j V$, but we need the freedom to assign to A_0 and A_1 (or B_0 and B_1) the same value.

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