

GEOMETRICALLY AND TOPOLOGICALLY RANDOM SURFACES IN A CLOSED HYPERBOLIC THREE MANIFOLD

JEREMY KAHN, VLADIMIR MARKOVIĆ, AND ILIA SMILGA

ABSTRACT. We study the distribution of geometrically and topologically nearly geodesic random surfaces in a closed hyperbolic 3-manifold \mathbf{M} . In particular, we describe $\mathbf{PSL}(2, \mathbb{R})$ invariant measures on the Grassmann bundle $\mathcal{G}_2(\mathbf{M})$ which arise as limits of random minimal surfaces. It is showed that if \mathbf{M} contains at least one totally geodesic subsurface then every topological limiting measure is totally scarring (i.e. supported on the totally geodesic locus), while we prove that geometrical limiting measures are never totally scarring.

1. INTRODUCTION

Fix a closed hyperbolic 3-manifold \mathbf{M} . We begin by defining the notions of geometrically and topologically random surfaces in \mathbf{M} .

Definition 1.1. *We let \mathcal{S} denote the set of conjugacy classes of surface subgroups of $\pi_1(\mathbf{M})$. For $\epsilon > 0$, we let \mathcal{S}_ϵ denote the subset of \mathcal{S} consisting of conjugacy classes of quasifuchsian surface subgroups whose limit set is a $(1 + \epsilon)$ -quasicircle.*

The following well known proposition follows from the works of Uhlenbeck [19] and Seppi [18].

Proposition 1.2. *There exists a universal constant $\hat{\epsilon} > 0$ such that when $\epsilon < \hat{\epsilon}$ there exists a unique minimal surface in the homotopy class defined by $\Sigma \in \mathcal{S}_\epsilon$, and this minimal surface is immersed in \mathbf{M} .*

Thus, when $\epsilon < \hat{\epsilon}$ we can define the function $\text{Area}_{\mathbf{M}}: \mathcal{S}_\epsilon \rightarrow (0, \infty)$ by letting $\text{Area}_{\mathbf{M}}(\Sigma)$ be the area of the (unique) minimal surface in the corresponding homotopy class. In the remainder of the paper we assume that $\epsilon < \hat{\epsilon}$.

Definition 1.3. *Let $g \geq 2$, and $T > 0$. We set*

$$\mathcal{S}_\epsilon^{\text{top}}(g) = \{\Sigma \in \mathcal{S}_\epsilon : \mathbf{g}(\Sigma) \leq g\},$$

and

$$\mathcal{S}_\epsilon^{\text{geo}}(T) = \{\Sigma \in \mathcal{S}_\epsilon : \text{Area}_{\mathbf{M}}(\Sigma) \leq T\},$$

2000 *Mathematics Subject Classification.* Primary 20H10.

This work was supported by the *Simons Investigator Award* 409735 from the Simons Foundation

where $\mathbf{g}(\Sigma)$ denotes the genus. (To alleviate the notation we often write $\mathcal{S}_\epsilon(g)$ instead of $\mathcal{S}_\epsilon^{\text{top}}(g)$, and $\mathcal{S}_\epsilon(T)$ instead of $\mathcal{S}_\epsilon^{\text{geo}}(T)$).

A typical element of $\mathcal{S}_\epsilon(g)$ is called a topologically random surface, and a typical element of $\mathcal{S}_\epsilon(T)$ a geometrically random surface. The study of topologically random surfaces was initiated by Kahn-Marković [8], while the study of geometrically random ones was initiated by Calegari-Marques-Neves [4], and Labourie [10].

1.1. The motivation. In the next subsection we define probability measures on the Grassmann bundle $\mathcal{G}_2(\mathbf{M})$ which capture the distribution of random surfaces. Topological and geometrical limiting measures inform our knowledge regarding the distribution of random surfaces. Establishing the uniqueness of such measures (or at least describing the range of such limits) would tell us how a random surface distributes in \mathbf{M} .

As observed in [10], the motivation for studying geometrical limiting measures comes from the classical case dealing with the limiting measures arising as limits (when $T \rightarrow \infty$) of measures supported on closed geodesics of length at most $T > 0$. It was proved by Bowen and Margulis that in this case the limiting measure is unique and equal to the corresponding Liouville measure on the tangent bundle of \mathbf{M} . The motivation for studying topological limiting measures is clear.

1.2. Invariant limiting measures. Let $\mathcal{G}_2(\mathbf{M})$ denote the Grassmann bundle (the 2-plane bundle) over \mathbf{M} . We begin by explaining how each $\Sigma \in \mathcal{S}_\epsilon$ yields a probability measure on $\mathcal{G}_2(\mathbf{M})$. Let $f: S_\Sigma \rightarrow \mathbf{M}$ be the minimal immersion representing the homotopy class of Σ (here S_Σ is the Riemann surface determined by the requirement that f is conformal and harmonic). Since f is an immersion, we have the induced map $f: S_\Sigma \rightarrow \mathcal{G}_2(\mathbf{M})$ (to simplify the notation, we use the same letter to denote both maps). Let m_S denote the hyperbolic area measure on S_Σ normalised to be a probability measure. We define the pushforward measure

$$(1) \quad \mu(\Sigma) = f_* m_S$$

on the Grassmann bundle $\mathcal{G}_2(\mathbf{M})$.

Remark 1. One can define another probability measure $\mu_1(\Sigma)$ on $\mathcal{G}_2(\mathbf{M})$ using the induced area form on the subsurface $f(S)$. Unless the minimal surface $f(S)$ is totally geodesic, the measure $\mu_1(\Sigma)$ may not be the same as $\mu(\Sigma)$. However, when ϵ is small the minimal map f is uniformly locally close to a Möbius map. Thus, the distance between the probability measures $\mu(\Sigma)$ and $\mu_1(\Sigma)$ uniformly tends to zero when $\epsilon \rightarrow 0$, which means that the limiting measures we define below do not depend whether we use $\mu(\Sigma)$ or $\mu_1(\Sigma)$ to define them.

Definition 1.4. We say that a probability measure $\mu_\epsilon^{\text{top}}$ on $\mathcal{G}_2(\mathbf{M})$ is an ϵ -topological limiting measure if there exists a sequence $g_n \rightarrow \infty$ such that

$$\mu_\epsilon^{\text{top}} = \lim_{g_n \rightarrow \infty} \frac{1}{|\mathcal{S}_\epsilon(g_n)|} \sum_{\Sigma \in \mathcal{S}_\epsilon(g_n)} \mu(\Sigma).$$

We say that a probability measure $\mu_\epsilon^{\text{geo}}$ on $\mathcal{G}_2(\mathbf{M})$ is an ϵ -geometrical limiting measure if there exists a sequence $T_n \rightarrow \infty$ such that

$$\mu_\epsilon^{\text{geo}} = \lim_{T_n \rightarrow \infty} \frac{1}{|\mathcal{S}_\epsilon(T_n)|} \sum_{\Sigma \in \mathcal{S}_\epsilon(T_n)} \mu(\Sigma).$$

Definition 1.5. We say that a probability measure μ^{top} on $\mathcal{G}_2(\mathbf{M})$ is a topological limiting measure if there exists a sequence $\epsilon_n \rightarrow 0$ such that

$$\mu^{\text{top}} = \lim_{n \rightarrow \infty} \mu_{\epsilon_n}^{\text{top}},$$

where $\mu_{\epsilon_n}^{\text{top}}$ is some ϵ_n -topological limiting measure. Likewise, we say that a probability measure μ^{geo} on $\mathcal{G}_2(\mathbf{M})$ is a geometrical limiting measure if there exists a sequence $\epsilon_n \rightarrow 0$ such that

$$\mu^{\text{geo}} = \lim_{n \rightarrow \infty} \mu_{\epsilon_n}^{\text{geo}},$$

where $\mu_{\epsilon_n}^{\text{geo}}$ is some ϵ_n -geometrical limiting measure.

1.3. The main results. Each topological or geometrical limiting measure is invariant under the natural left $\mathbf{PSL}(2, \mathbb{R})$ action on $\mathcal{G}_2(\mathbf{M})$. The classical Ratner theorem says that each such measure is a linear combination of the probability Liouville measure on $\mathcal{G}_2(\mathbf{M})$, and the probability measures supported on the images of totally geodesic surfaces in $\mathcal{G}_2(\mathbf{M})$. In particular, if \mathbf{M} does not contain a totally geodesic subsurface then there exists a unique limiting measure (either topological or geometrical) and it is equal to the Liouville measure on $\mathcal{G}_2(\mathbf{M})$. Therefore, it remains to understand the case when \mathbf{M} contains totally geodesic subsurfaces.

Definition 1.6. Let μ denote a $\mathbf{PSL}(2, \mathbb{R})$ -invariant probability measure on $\mathcal{G}_2(\mathbf{M})$. Then by $\mu_{\mathcal{L}}$ we denote its Liouville part. Moreover, we say that μ is totally scarring if $\mu_{\mathcal{L}}$ is the zero measure.

Theorem 1.7. Suppose \mathbf{M} is a closed hyperbolic 3-manifold. There exists a constant $0 < q$ such that if μ^{geo} is a geometrical limiting measure then $q < |\mu_{\mathcal{L}}^{\text{geo}}|$.

The previous theorem says that a geometrical limiting measure is never totally scarring. The opposite is true for topological limiting measures as the following theorem attests.

Theorem 1.8. If \mathbf{M} is a closed hyperbolic 3-manifold with a totally geodesic surface then every topological limiting measure is totally scarring.

We finish this subsection with two conjectures.

Conjecture 1.9. *Every geometrical limiting measure is Liouville.*

Conjecture 1.10. *Suppose that \mathbf{M} is a closed hyperbolic 3-manifold with a totally geodesic surface. Then every topological limiting measure is supported on \mathcal{H}_{h_0} where $h_0 = h_0(\mathbf{M})$ is the genus of the totally geodesic surface of the smallest genus in \mathbf{M} .*

Answering these two conjectures would complete the description of how random surfaces distribute in \mathbf{M} .

Remark 2. The proof of Theorem 1.8 should go through in any symmetric space without many changes. The proof of Theorem 1.7 should likewise go through given the analog of Rao's theorem. Conjectures 1.9 and 1.10 should presumably apply as well in this more general setting (see Kassel's survey article [9]).

Remark 3. Labourie [10] proved that the Liouville measure on $\mathcal{G}_2(\mathbf{M})$ is the limit of μ_{f_n} for some sequence of ϵ_n -nearly geodesic minimal (possibly disconnected) surfaces $f_n: S_n \rightarrow \mathbf{M}$. Lowe and Neves [11] showed that one can find such a sequence consisting of connected surfaces. Al Assal [1] then proved that the same is true for any convex combination of invariant probability measures on $\mathcal{G}_2(\mathbf{M})$.

Remark 4. We expect both Theorem 1.7 and Theorem 1.8 to hold if instead of conjugacy classes \mathcal{S}_ϵ one considers commensurability classes of $(1 + \epsilon)$ -quasifuchsian surfaces in \mathbf{M} .

1.4. The totally geodesic locus. We let $\mathcal{H} \subset \mathcal{G}_2(\mathbf{M})$ be the union of the totally geodesic subsurfaces. For any $h \in \mathbb{N}$, we let $\mathcal{H}_h \subset \mathcal{H}$ be the union of the totally geodesic subsurfaces of genus at most h , and let $\mathcal{H}'_h = \mathcal{H} \setminus \mathcal{H}_h$. We let $h_0(\mathbf{M})$ be the genus of the least genus totally geodesic surface in \mathbf{M} .

Remark 5. In Theorem 1.8 we can prove the explicit estimate $\mu^{\text{top}}(\mathcal{H}'_h) \leq 1/(C \log h)$ for large h , and constant $C = C(\mathbf{M})$.

By the results of Margulis-Mohammadi [14], and Bader-Fisher-Miller-Stover [2], we know that a non-arithmetic hyperbolic 3-manifold has at most finitely many totally geodesic subsurfaces. We observe that in this case $\mathcal{H} = \mathcal{H}_h$ for some h .

1.5. Outline of the proof of Theorem 1.7. The proof of this theorem follows readily from two key lemmas. In the first lemma (which is Lemma 3.2 below) we show that there exists a constant $C_1 > 0$ such that a geometrically random surface $\Sigma \in \mathcal{S}_\epsilon(T)$ satisfies the bound

$$(2) \quad 4\pi(\mathbf{g}(\Sigma) - 1) \geq (1 + C_1\epsilon^2)T$$

when ϵ is small enough. Here C_1 is a constant which only depends on \mathbf{M} (and in particular it does not depend on ϵ or T). Note that if Σ is a totally geodesic surface of area T then $4\pi(\mathbf{g}(\Sigma) - 1) = T$. Therefore, the inequality (2) shows that the genus of a geometrically random surface is significantly

larger than the genus of a totally geodesic surface of the same area. This is the key factor in the proof of Theorem 1.7, and the key difference between geometrically and topologically random surfaces.

The second key lemma bounds from above the genus of a geometrically random surface $\Sigma \in \mathcal{S}_\epsilon(T)$ in terms of the percentage of its area which concentrates near the totally geodesic locus \mathcal{H}_h . In Lemma 3.3, for every $h \in \mathbb{N}$, and every $\epsilon < \epsilon_0(h)$, we establish the estimate

$$(3) \quad 4\pi(\mathbf{g}(S) - 1) \leq \left(1 + C_2 \left(1 - \frac{|S(\delta_1(h), h)|}{|S|}\right)\epsilon^2\right) T.$$

Here $f: S \rightarrow \mathbf{M}$ is the ϵ -nearly geodesic minimal surface representing the class Σ , and $S(\delta_1(h), h)$ is the subset of S such that $f(S(\delta_1(h), h))$ is close to the geodesic locus \mathcal{H}_h . Note that the constant C_2 does not depend on h .

Combing the two lemmas yields the estimate

$$0 < \frac{C_1}{C_2} \leq |\mu_{\mathcal{L}}^{\text{geo}}|$$

which then proves Theorem 1.7.

1.6. Outline of the proof of Theorem 1.8. Before outlining the basic idea of the proof we briefly recall the proof of the upper bound on the number of homotopy classes of genus \mathbf{g} surfaces in \mathbf{M} . Let \mathcal{D} be a finite cover of $\mathcal{G}_2(\mathbf{M})$ by open sets of sufficiently small diameter.

Given a π_1 -injective map $f: S \rightarrow \mathbf{M}$ of a closed surface of genus \mathbf{g} , we can realize it as a pleated surface and find a bounded geometry triangulation of S . We can then take a spanning tree $\hat{\tau}$ of the triangulation, along with a set of another $2\mathbf{g}$ *distinguished* edges, denoted by $e(\tau)$, so that the inclusion of $\hat{\tau} \cup e(\tau)$ into S is surjective on H_1 and hence on π_1 . For each vertex v of τ , we choose $D \in \mathcal{D}$ such that $f(v) \in D$ and denote D by $\psi(v)$. We thereby obtain a \mathcal{D} -valued \mathbf{g} -*polygonalization* consisting of the pair (τ, ψ) where $\tau = (\hat{\tau}, e(\tau))$ is a *graph pair* that we call a \mathbf{g} -polygonalization because it efficiently divides a genus \mathbf{g} surface into polygons.

The point is that we can reconstruct the homotopy class of f from the \mathcal{D} -valued \mathbf{g} -polygonalization, where we think of $\tau = (\hat{\tau}, e(\tau))$ as an abstract graph (with a cyclic ordering of the edges at each vertex), and hence as purely combinatorial data. We then need only bound the number of possible such data, and in that way we obtain the $(Cg)^{2g}$ upper bound in [8].

In this paper¹ we refine this bound in the case where f is ϵ -nearly geodesic and at least q of the induced measure f_*m_S is away from \mathcal{H} , to obtain a bound of $(cg)^{2g}$, where $c \equiv c(q, \epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$ and $q > 0$ remains fixed. To do so we let \mathcal{D} be a system of “wafer-thin disks” that efficiently cover $\mathcal{G}_2(\mathbf{M})$. Namely, a (two-dimensional) disk in \mathbf{M} lifts to a disk in $\mathcal{G}_2(\mathbf{M})$, and each wafer-thin disk will be the thickening of such a disk in $\mathcal{G}_2(\mathbf{M})$ by a small orthogonal radius η to form an open set in $\mathcal{G}_2(\mathbf{M})$. These wafer-thin

¹In this introduction we assume that \mathbf{M} is non-arithmetic so $\mathcal{H} = \mathcal{H}_h$ for some h ; the proof in the general case is very similar.

disks all have small volume, but travel between them is carefully restricted for an ϵ -nearly geodesic surface (with ϵ small given η): if two points on such a surface are at bounded distance, lying respectively in $D_0, D_1 \in \mathcal{D}$, then D_0 and D_1 are *mutually compatible*, and there are only a bounded number of disks in \mathcal{D} that are compatible with a given disk, where this bound is independent of the choice of $\eta > 0$.

Then given a nearly geodesic $f: S \rightarrow \mathbf{M}$ of genus \mathbf{g} , we

- (1) lift f to $f: S \rightarrow \mathcal{G}_2(\mathbf{M})$,
- (2) find a bounded triangulation and spanning tree for S just as in [8],
- (3) find our $2\mathbf{g}$ distinguished edges so that at least $q\mathbf{g}$ of them are some distance from \mathcal{H} , and
- (4) assign, as before, an element $\psi(v) \in \mathcal{D}$ to each vertex v so that $f(v) \in \psi(v)$.

Our additional constraint in Step 3 on the choice of distinguished edges implies that the endpoints of the $q\mathbf{g}$ edges (that are away from \mathcal{H}) are roughly evenly distributed in $\mathcal{G}_2(\mathbf{M})$, and hence with respect to \mathcal{D} . (This follows from Ratner's theorem.) The small volume of each disk in \mathcal{D} , combined with this equidistribution, implies that only a small proportion of these endpoints can be in any $D \in \mathcal{D}$. So we obtain a \mathcal{D} -valued \mathbf{g} -polygonalization, along with some constraint on how many v can satisfy $\psi(v) = D$ for a given $D \in \mathcal{D}$, and for which the vertices of a given edge lie in mutually compatible elements of \mathcal{D} . We now need only bound the number of such combinatorial structures, and these constraints are sufficient to obtain the desired bound of $(c\mathbf{g})^{2\mathbf{g}}$, where c is small when ϵ is.

On the other hand, since $\mathcal{H} \neq \emptyset$, the Müller-Puchta [15] estimate implies that the number of *totally geodesic* surfaces in \mathbf{M} of genus $\mathbf{g} < g$ grows as $(Cg)^{2g}$. (These are all just covers of a component of \mathcal{H} .) We can now compare this to the number of nearly geodesic surfaces where a definite part of the area lies *away* from \mathcal{H} , and see that the latter number is much smaller than the former. Thus the only significant contributors to the measure for a topologically random nearly geodesic surface of genus at most g are surfaces where most of their area lies close to \mathcal{H} . This implies Theorem 1.8.

1.7. How to read this paper. After Section 2 the paper is divided into two halves; the first half proves Theorem 1.7, and the second half proves Theorem 1.8. Section 2 provides some preliminaries and background, including a smoothing lemma and isoperimetric theorem for subsurfaces of a hyperbolic surface. After Section 2, the two halves can be read in either order, starting with Sections 3 or 12. Both halves have a top-down structure, so the main theorems are proven in one section assuming one or more lemmas, which are then proven in the subsequent sections.

2. PRELIMINARIES

Once and for all we fix a closed hyperbolic 3-manifold \mathbf{M} . In this section we discuss the Grassmann bundle of \mathbf{M} , and define and discuss elementary properties of ϵ -nearly geodesic minimal surfaces and how they relate to the totally geodesic locus \mathcal{H} in \mathbf{M} . In particular, we fix two constants $\delta_0(h)$ and $\delta_1(h)$ which capture the geometry of the locus \mathcal{H}_h in \mathbf{M} and its interaction with nearly isometric minimal maps.

2.1. Notation and conventions. If we define a constant without specifying what it depends on, it means that it only depends on \mathbf{M} . If X is a hyperbolic manifold (which in this paper is either a surface or a 3-manifold) then d_X denotes the hyperbolic distance on X . We use $|Y|$ to denote the hyperbolic area of a subset $Y \subset X$, and $|\alpha|$ the hyperbolic length of a smooth curve $\alpha \subset X$. Furthermore, $B_r(x)$ denotes the ball of radius r centred at x , and $\mathcal{N}_r(Y)$ the r -neighbourhood of a set Y .

2.2. The Grassmann bundle. The Grassmann bundle $\mathcal{G}_2(\mathbf{M})$ is the 2-plane bundle over \mathbf{M} . A point in $\mathcal{G}_2(\mathbf{M})$ is a pair (x, Π) where $x \in \mathbf{M}$, and $\Pi < T_x\mathbf{M}$ a 2-plane. The universal cover of $\mathcal{G}_2(\mathbf{M})$ is $\mathcal{G}_2(\mathbb{H}^3)$. By $\xi: \mathcal{G}_2(\mathbf{M}) \rightarrow \mathbf{M}$ we denote the obvious fibration. The Levi-Civita connection on $T\mathbf{M}$ induces a connection on $\mathcal{G}_2(\mathbf{M})$ and hence a canonical complement in $T\mathcal{G}_2(\mathbf{M})$ to the fibers of $T\xi: T\mathcal{G}_2(\mathbf{M}) \rightarrow T\mathbf{M}$. The standard Riemannian metric $\langle \cdot, \cdot \rangle_{\mathcal{G}}$ on $\mathcal{G}_2(\mathbf{M})$ is characterised by the following properties:

- (1) within each fibre $\langle \cdot, \cdot \rangle_{\mathcal{G}}$ coincides with the usual Riemannian metric on $\mathcal{G}_2^3 = \mathbb{R}P^2$,
- (2) the map ξ is a Riemannian submersion, and
- (3) the fibres described in (1) are orthogonal to their canonical complements.

These properties imply that the canonical complement to the fiber at each point of $T\mathcal{G}_2(\mathbf{M})$ is mapped isometrically by $T\xi$; this in turn implies that the lift of a path in \mathbf{M} to $\mathcal{G}_2(\mathbf{M})$ by parallel transport (with some starting 2-plane at the initial point) is an isometric lift.

Obviously the lift of this metric to $\mathcal{G}_2(\mathbb{H}^3)$ is invariant under the action by $\text{Isom}(\mathbb{H}^3)$, and agrees with the hyperbolic metric on totally geodesic planes in $\mathcal{G}_2(\mathbb{H}^3)$. Moreover, for each $(x_1, \Pi_1), (x_2, \Pi_2) \in \mathcal{G}_2(\mathbb{H}^3)$, we have

$$\mathbf{d}_{\mathbb{H}^3}(x_1, x_2) \leq \mathbf{d}_{\mathcal{G}}((x_1, \Pi_1), (x_2, \Pi_2)).$$

2.3. The geodesic locus. Given $h \in \mathbb{N}$, we let $\mathcal{H}_h \subset \mathcal{G}_2(\mathbf{M})$ denote the union of totally geodesic surfaces of genus at most h . We refer to \mathcal{H}_h as the totally geodesic locus of \mathbf{M} of level h . By $\tilde{\mathcal{H}}_h \subset \mathcal{G}_2(\mathbb{H}^3)$ we denote the preimage of \mathcal{H}_h by the covering from $\mathcal{G}_2(\mathbb{H}^3)$ to $\mathcal{G}_2(\mathbf{M})$. We observe that each connected component of $\tilde{\mathcal{H}}_h$ is a geodesic plane.

Proposition 2.1. *There exists $\delta_0(h) > 0$ such that*

- $|\overline{\mathcal{N}_{\delta_0(h)}(\mathcal{H}_h)}| \leq \frac{1}{2}|\mathcal{G}_2(\mathbf{M})|$ and

- $\mathbf{d}_{\mathcal{G}}(P_1, P_2) \geq 10\delta_0(h)$ when P_1 and P_2 are two different components of $\widetilde{\mathcal{H}}_h$.

Remark 6. In particular, the geometry of the components of $\mathcal{N}_{\delta_0(h)}(\mathcal{H}_h)$ depends only on $\delta_0(h)$ and the geometry of the components of \mathcal{H}_h .

Proof. The first property of $\delta_0(h)$ holds because \mathcal{H}_h is a closed subset of $\mathcal{G}_2(\mathbf{M})$ of measure 0. The second property follows from the fact that \mathcal{H}_h has finitely many components, each of which is compact. \square

2.4. Nearly geodesic minimal maps. We begin with a definition.

Definition 2.2. *We say that $f: S \rightarrow \mathbf{M}$ is an ϵ -nearly geodesic minimal map if:*

- (1) S is a closed hyperbolic surface,
- (2) f is a minimal map,
- (3) $f_*: \pi_1(S) \rightarrow \pi_1(\mathbf{M})$ is a quasifuchsian surface subgroup whose limit set is a $(1 + \epsilon)$ -quasicircle.

Since every ϵ -nearly geodesic minimal map $f: S \rightarrow \mathbf{M}$ is an immersion assuming ϵ is small, the induced map $f: S \rightarrow \mathcal{G}_2(\mathbf{M})$ is well defined. We now define the subset of the minimal surface which is close to the geodesic locus \mathcal{H}_h .

Definition 2.3. *Let $f: S \rightarrow \mathbf{M}$ be an ϵ -nearly geodesic minimal map. For $\delta > 0$, we let*

$$S(\delta, h) = \{p \in S : \mathbf{d}_{\mathcal{G}}(f(p), \mathcal{H}_h) < \delta\}.$$

Next, we observe that ϵ -nearly geodesic minimal maps are nearly locally isometric, meaning that the restriction of f to any ball of a bounded radius is close to an actual isometry. The following proposition readily follows from the compactness principle for minimal maps which is stated as Theorem 2.5 in [6].

Proposition 2.4. *For every $\eta > 0$ there exists a constant $\epsilon(\eta) > 0$ with the following properties. Let $f: S \rightarrow \mathbf{M}$ be a ϵ -nearly geodesic minimal map where $\epsilon < \epsilon(\eta)$, and let $\iota: B_{10}(x_0) \rightarrow \mathbf{M}$ be a local isometry such that $\iota(x_0) = f(x_0)$ as points in $\mathcal{G}_2(\mathbf{M})$. Then $\mathbf{d}_{\mathcal{G}}(\iota(x), f(x)) < \eta$ for every $x \in B_{10}(x_0)$.*

We finish the section by proving the following proposition relating sets $S(\delta, h)$ for different values of δ .

Proposition 2.5. *For all $\delta > 0$, and $h \in \mathbb{N}$, there exists $\delta' = \delta'(\delta, h) > 0$ and $\epsilon_0 = \epsilon_0(\delta, h) > 0$ such that for every ϵ -minimal surface $f: S \rightarrow \mathbf{M}$,*

$$(4) \quad \mathcal{N}_{10}(S(\delta', h)) \subset S(\delta, h)$$

assuming $\epsilon < \epsilon_0$.

Remark 7. For future reference, we define $\delta_1(h) = \delta'(\delta_0(h), h)$, and $\epsilon_0(h) = \epsilon_0(\delta_0(h), h)$.

Remark 8. We could actually make δ' and ϵ_0 independent of h , but in our context δ will depend on h , so we state and prove the proposition in its current form.

Proof. The proof is by contradiction. Fix $\delta > 0$, and $h \in \mathbb{N}$, and assume that for each $n \in \mathbb{N}$ there exists ϵ_n -nearly geodesic minimal map $f_n : S_n \rightarrow \mathbf{M}$, where $\epsilon_n \rightarrow 0$, and sequences of points $x_n, y_n \in S_n$, such that

$$(5) \quad \mathbf{d}_{\mathcal{G}}(f_n(x_n), \mathcal{H}_h) \geq \delta, \quad \mathbf{d}_{\mathcal{G}}(f_n(y_n), \mathcal{H}_h) \leq \frac{1}{n},$$

and

$$(6) \quad d_{S_n}(x_n, y_n) \leq 10.$$

From (5) we find that $f_n(x_n) \rightarrow (x, \Pi) \notin \mathcal{H}_h$, and $f_n(y_n) \rightarrow (y, \Pi') \in \mathcal{H}_h$. Combining (6) with the assumption $\epsilon_n \rightarrow 0$, and by applying Proposition 2.4, we conclude that there exists a totally geodesic plane $P \subset \mathcal{G}_2(\mathbf{M})$ which contains both (x, Π) and (y, Π') . This implies that $(x, \Pi) \in \mathcal{H}_h$. This contradiction proves the proposition. \square

2.5. A smoothing lemma for subsets of S . Here's a simple version of the Vitali Covering Lemma that will suffice in this paper:

Lemma 2.6. *Suppose X is a totally bounded metric space (e.g. a subset of a compact metric space), and $r > 0$. Then we can find points x_1, \dots, x_m such that the balls $B_r(x_i)$ are pairwise disjoint, and $X \subset \bigcup_i B_{2r}(x_i)$.*

Proof. Since X is totally bounded, there is a finite set x_1, \dots, x_m such that the $B_r(x_i)$ are pairwise disjoint, and this set cannot be extended to a larger one with the same property. Take any $x \in X$; there must be an x_i such that $B_r(x_i) \cap B_r(x)$ is nonempty, and then $x \in B_{2r}(x_i)$. \square

We can use 2.6 to prove a smoothing lemma for subsurfaces of a closed hyperbolic surface S :

Lemma 2.7. *Suppose $R \subset S$. Then we can find R' with piecewise smooth boundary such that $R \subset R' \subset \mathcal{N}_1(R)$, and $|\partial R'| \leq 25|\mathcal{N}_1(\partial R)|$.*

Proof. In what follows, for any $r > 0$, we let $|\partial B_r| = 2\pi \sinh r$ denote the perimeter of a hyperbolic ball in \mathbb{H}^2 of radius r , and we let $|B_r| = 2\pi(\cosh r - 1)$ denote its area. We will estimate some of these values with whole numbers.

By Lemma 2.6 we can find $x_1, \dots, x_m \in \partial R$ such that the $B_{1/2}(x_i)$ are pairwise disjoint, but $\partial R \subset Q$, where $Q := \bigcup_i B_1(x_i)$. We let $R' = R \cup Q$, and observe that $R \subset R' \subset \mathcal{N}_1(R)$, $\partial R' \subset \partial Q$ and $Q \subset \mathcal{N}_1(\partial R)$. Moreover

$$(7) \quad |\partial Q| \leq m|\partial B_1|.$$

If $y \in B_1(x)$, then $B_{1/2}(x) \subset B_{3/2}(y)$. From the disjointness of the $B_{1/2}(x_i)$ we then infer that any $y \in Q$ can lie in $B_1(x_i)$ for at most

$$\left\lfloor \frac{|B_{3/2}|}{|B_{1/2}|} \right\rfloor = 10$$

values of i . We then have

$$(8) \quad m|B_1| \leq 10|Q| \leq 10|\mathcal{N}_1(\partial R)|;$$

combining (7) and (8), we obtain

$$(9) \quad |\partial Q| \leq 10 \frac{|\partial B_1|}{|B_1|} |\mathcal{N}_1(\partial R)| \leq 25|\mathcal{N}_1(\partial R)|,$$

as required. \square

2.6. An isoperimetric inequality. Recall the classical hyperbolic isoperimetric inequality proved by Schmidt [17].

Theorem 2.8. *For any $r > 0$, any region enclosed by a curve at length $2\pi \sinh r$ has area at most $2\pi(\cosh r - 1)$.*

These functions of r are of course the perimeter and area of a hyperbolic disk of radius r . Since $\cosh r - \sinh r < 1$ for any $r > 0$ we immediately have

Theorem 2.9. *If $Q \subset \mathbb{H}^2$ is a domain then $|Q| \leq |\partial Q|$.*

We can use this to prove the following

Lemma 2.10. *Suppose β_0 and β_1 are two disjoint homotopic curves on a hyperbolic surface S that together bound a compact annular region A . Then*

$$(10) \quad |A| \leq |\beta_0| + |\beta_1|.$$

Proof. Draw a simple smooth arc γ in A connecting β_0 to β_1 . Then $A \setminus \gamma$ is connected and simply connected, and the universal cover of A in \mathbb{H}^2 is an infinite concatenation of closures of $A \setminus \gamma$; the area of n of consecutive regions is $n|A|$, and the perimeter of this concatenation of n consecutive regions is $2|\gamma| + n(|\beta_0| + |\beta_1|)$. Letting $n \rightarrow \infty$ and applying Theorem 2.9, we obtain (10). \square

We can now easily prove the fundamental estimate of this subsection.

Proposition 2.11. *Let S be a hyperbolic surface, and supposed we are given a closed geodesic $\alpha \subset S$, and a closed smooth simple curve $\beta \subset S$ homotopic to α . Let $Q \subset S$ be the (possibly disconnected) domain bounded by the two curves. Then $|Q| \leq |\beta|$.*

Proof. First, we can approximate β with simple smooth curves such that the intersection $\alpha \cap \beta$ is a finite set. In the rest of the proof we assume this to be the case. We have two cases.

Case 1. Suppose α and β intersect each other. Then Q is a disjoint union of simply connected domains Q_1, \dots, Q_k . Each Q_i is bounded by segments α_i and β_i of α and β respectively, and no two distinct β_i 's intersect. We can double Q_i over α_i to obtain $2|Q_i| \leq 2|\beta_i|$ and hence

$$(11) \quad |Q_i| \leq |\beta_i|$$

by Theorem 2.9. Summing (11) over all i , we obtain

$$(12) \quad |Q| = \sum_{i=1}^k |Q_i| \leq \sum_{i=1}^k |\beta_i| \leq |\beta|.$$

Case 2. Suppose $\alpha \cap \beta = \emptyset$. Then $Q \subset S$ is an embedded annulus. Doubling Q over α and applying Lemma 2.10, we obtain the desired result. \square

Suppose $R \subset S$ is a subsurface (with smooth boundary) of a closed hyperbolic surface S . Let $\Omega \subset S$ be convex core of U . That is, Ω is the surface with geodesic boundary obtained by filling in the holes (contractible components of the complement) of R , and replacing each boundary curve of R with its geodesic representative.

Proposition 2.12. *Suppose $R \subset S$ is subsurface (with smooth boundary) of a closed hyperbolic surface S , and let Ω be its convex core. Then*

$$|\Omega| - |\partial R| \leq |R| \leq |\Omega| + |\partial R|.$$

Proof. As above we may assume that $\partial R \cap \partial\Omega$ is a finite set. Denote by β_1, \dots, β_m the connected components of ∂R . Suppose β_i bounds a simply connected domain Q_i . Then $|Q_i| \leq |\beta_i|$ by Theorem 2.9.

Suppose now that β_i is homotopic to a closed geodesic α_i , and denote by Q_i the union of all domains whose boundary is contained in $\alpha_i \cup \beta_i$. Then, $|Q_i| \leq |\beta_i|$ by Proposition 2.11. Summing up these inequalities over $i = 1, \dots, m$ proves the proposition. \square

3. THE MAIN LEMMAS AND THE PROOF OF THEOREM 1.7

The proof of Theorem 1.7 is built on two lemmas which we state next. We close the section with the proof of theorem assuming the lemmas.

3.1. Genera of geometrically random surfaces. The first lemma estimates from below the genus of a geometrically random surface. Before stating the lemma we adopt the following definition.

Definition 3.1.

$$\mathcal{S}_\epsilon(T, C) = \{\Sigma \in \mathcal{S}_\epsilon(T) : 4\pi(\mathbf{g}(\Sigma) - 1) \geq (1 + C\epsilon^2) \text{Area}_M(\Sigma)\}.$$

Lemma 3.2. *There exists a constant $C > 0$ so that*

$$\lim_{T \rightarrow \infty} \frac{|\mathcal{S}_\epsilon(T, C)|}{|\mathcal{S}_\epsilon(T)|} = 1$$

when ϵ is sufficiently small.

Remark 9. If Σ is a totally geodesic surface of area T then $4\pi(\mathbf{g}(\Sigma) - 1) = T$. Therefore, the previous lemma shows that the genus of a geometrically random surface is significantly larger than the genus of a totally geodesic surface of the same area. This is the key factor in the proof of Theorem 1.7.

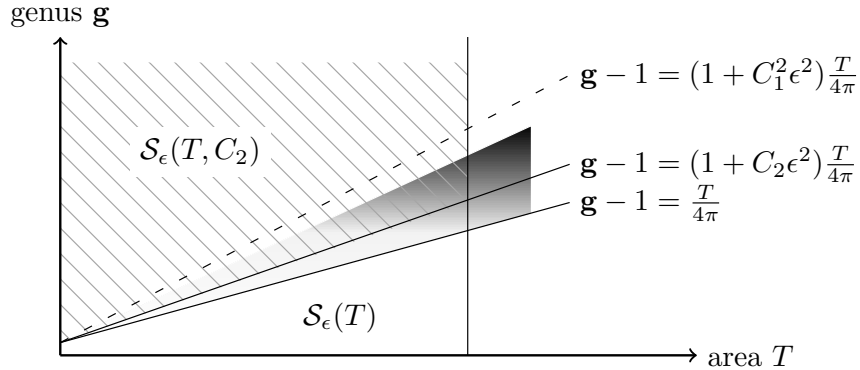


FIGURE 1.

To put this into context, we refer to Figure 1 which (schematically) shows the distribution of minimal surfaces by their area and genus. The key features of this plot are as follows:

- All surfaces lie on or above the $g - 1 = \frac{T}{4\pi}$ line (which is the locus of the totally geodesic surfaces of genus g). This immediately follows from the Gauss-Bonnet formula (and from the formula for the Gaussian curvature of a minimal surface).
- All surfaces lie below the $g - 1 = (1 + C_1^2 \epsilon^2) \frac{T}{4\pi}$ line. This follows from Seppi's curvature estimate (Theorem 8.1), where C_1 is the constant from that theorem.
- Kahn–Marković's upper bound [8] tells us that the count density of surfaces increases superexponentially with their genus (but depends much more weakly on their area); moreover Müller–Puchta's lower bound [15] tells us that this superexponential growth can be observed even among the covers of a single surface (which have constant ratio of $g - 1$ to area).
- From this it is not hard to deduce that the subset $S_\epsilon(T, C_2)$ (where C_2 is the constant of Lemma 3.2) dominates the count of S_ϵ , as long as it is nonempty. Verifying this latter condition (Proposition 4.1) is the most substantial part of the proof.

3.2. Nearly geodesic minimal surfaces and the totally geodesic locus. The second key lemma bounds from above the genus of an ϵ -nearly geodesic minimal surface which concentrates too much on the totally geodesic locus \mathcal{H} . Recall the constant $\delta_1(h)$ (see the remark after Proposition 2.5).

Lemma 3.3. *There exist constants $C > 0$, and $\epsilon_0(h) > 0$, such that for every $\Sigma \in \mathcal{S}_\epsilon(T)$, the inequality*

$$(13) \quad 4\pi(\mathbf{g}(S) - 1) \leq \left(1 + C \left(1 - \frac{|S(\delta_1(h), h)|}{|S|}\right)\right) \epsilon^2 T$$

holds for every $h \in \mathbb{N}$ assuming $\epsilon < \epsilon_0(h)$. Here $f: S \rightarrow \mathbf{M}$ is the ϵ -nearly geodesic minimal surface in the class Σ .

3.3. Proof of Theorem 1.7. Let $\mu_\epsilon^{\text{geo}}$ be an ϵ -geometrical limiting measure on $\mathcal{G}_2(\mathbf{M})$. That is, there exists a sequence $T_n \rightarrow \infty$ such that

$$\mu_\epsilon^{\text{geo}} = \lim_{T_n \rightarrow \infty} \frac{1}{|\mathcal{S}_\epsilon(T_n)|} \sum_{\Sigma \in \mathcal{S}_\epsilon(T_n)} \mu(\Sigma)$$

(recall that $\mu(\Sigma)$ is the push forward of the hyperbolic area measure from S to $\mathcal{G}_2(\mathbf{M})$). Let C_1 be the constant from Lemma 3.2. Then

$$(14) \quad \mu_\epsilon^{\text{geo}} = \lim_{T_n \rightarrow \infty} \frac{1}{|\mathcal{S}_\epsilon(T_n, C_1)|} \sum_{\Sigma \in \mathcal{S}_\epsilon(T_n, C_1)} \mu(\Sigma).$$

Let $\Sigma \in \mathcal{S}_\epsilon(T_n, C_1)$ and denote by $f: S \rightarrow \mathbf{M}$ the minimal surface representing Σ . Let $h \in \mathbb{N}$, and denote by C_2 and $\epsilon_0(h)$ the constants from Lemma 3.3. Applying Lemma 3.3 to Σ yields the inequality

$$(15) \quad \frac{|S(\delta_1(h), h)|}{|S|} \leq 1 - \frac{C_1}{C_2} < 1$$

assuming $\epsilon < \epsilon_0(h)$.

In turn, the inequality (15) yields the estimate

$$\mu(\Sigma)(\mathcal{N}_{\delta_1(h)}(\mathcal{H}_h)) \leq \frac{|S(\delta_1(h), h)|}{|S|} < 1 - \frac{C_1}{C_2}$$

for every $\Sigma \in \mathcal{S}_\epsilon(T_n, C)$ assuming $\epsilon < \epsilon_0(h)$. Combining this with (14) gives the estimate

$$\mu_\epsilon^{\text{geo}}(\mathcal{N}_{\delta_1(h)}(\mathcal{H}_h)) \leq 1 - \frac{C_1}{C_2}$$

for every ϵ -geometrical limiting measure on $\mathcal{G}_2(\mathbf{M})$ when $\epsilon < \epsilon_0(h)$. In turn, this shows that

$$(16) \quad \mu^{\text{geo}}(\mathcal{N}_{\delta_1(h)}(\mathcal{H}_h)) \leq 1 - \frac{C_1}{C_2}$$

for every geometrical limiting measure μ^{geo} on $\mathcal{G}_2(\mathbf{M})$. We observe that (16) holds for every $h \in \mathbb{N}$, and that the constant on the right hand side does not depend on h . Define $q = \frac{C_1}{C_2}$, and note that q only depends on \mathbf{M} . Then (16) implies

$$\mu^{\text{geo}}(\mathcal{H}_h) \leq 1 - q.$$

Since \mathcal{H}_h is an increasing sequence of sets, we have

$$\mu^{\text{geo}}(\mathcal{H}) \leq \lim_{h \rightarrow \infty} \mu^{\text{geo}}(\mathcal{H}_h) \leq 1 - q.$$

Thus, $q \leq |\mu_{\mathcal{L}}^{\text{geo}}|$, and the proof is complete.

4. GENERA OF GEOMETRICALLY RANDOM SURFACES I

The proof of Lemma 3.2 occupies Sections 4 to 7. The proof is carried out in Section 5, by combining two contrasting statements:

- Proposition 5.1, which gives an upper bound on the rate of growth of the complement of $\mathcal{S}_\epsilon(T, C)$, and easily follows from Kahn–Marković’s upper bound [8];
- and Proposition 5.2, which gives a lower bound on the rate of growth of $\mathcal{S}_\epsilon(T, C)$ itself. We obtain this bound in Section 6, just by counting the covers of a single element of $\mathcal{S}_\epsilon(T_0, C)$ (using Müller–Puchta’s lower bound [15]).

However, this last argument relies on $\mathcal{S}_\epsilon(T, C)$ being nonempty:

Proposition 4.1. *There exists a universal constant $C > 0$ such that for each small enough ϵ there exists $T = T(\epsilon)$ so that $\mathcal{S}_\epsilon(T, C) \neq \emptyset$.*

The goal of this section is to prove Proposition 4.1. To do so, we need to construct a surface satisfying the defining inequality (2) of $\mathcal{S}_\epsilon(T, C)$. This is done in two steps:

- First, we use a recent result of Rao [16] to reduce it to the simpler problem of finding a “model” surface that satisfies (a stronger version of) that inequality, but which exists independently of \mathbf{M} .
- The existence of such a “model surface” is the content of Proposition 4.2. We construct it in Section 7, essentially by hand.

4.1. The model quasifuchsian manifold. Recall that a quasifuchsian 3-manifold is K -quasifuchsian if its limit set is a K -quasicircle. The following proposition is proved in Section 7.

Proposition 4.2. *There exists a universal constant $C > 0$ with the following property. For every small enough $\epsilon > 0$ there exists a $(1 + \epsilon)$ -quasifuchsian manifold N of genus two such that*

$$(17) \quad T \leq \frac{4\pi}{1 + C\epsilon^2},$$

where T is the area of the minimal surface homotopic to N .

4.2. Rao’s theorem. In [16] Rao proved a version of the Ehrenpreis conjecture for quasifuchsian manifolds of genus two. We state it in a form that is convenient for our purposes.

Theorem 4.3. *Let \mathbf{M} be a closed 3-manifold, and let $\eta > 0$. Then given any quasifuchsian 3-manifold N_0 of genus two, we can find:*

- a finite cover $N \rightarrow N_0$,
- a map $F: N \rightarrow \mathbf{M}$ which is locally $(1 + \eta)$ -bilipschitz.

4.3. Proof of Proposition 4.1. Let N_0 be the quasifuchsian manifold, and C_0 the constant, from Proposition 4.2. We choose N_0 so it is a $(1 + \frac{\epsilon}{2})$ -quasifuchsian manifold. Let $g_0: R_0 \rightarrow N_0$ be the minimal surface homotopy equivalent to N_0 (here R_0 is a Riemann surface of genus two). Then from (17) we get

$$(18) \quad \text{Area}_{\mathbf{M}}(g_0(R_0)) \leq \frac{4\pi}{1 + C_0\epsilon^2}.$$

Set $\eta = \epsilon^3$. We apply Theorem 4.3 and get the corresponding finite cover $N \rightarrow N_0$, and the $(1 + \eta)$ -bilipschitz map $F: N \rightarrow \mathbf{M}$. Let $g: R \rightarrow N$ be the lift of the map $g_0: R_0 \rightarrow N_0$, where R is the corresponding cover $R \rightarrow R_0$. Then $g: R \rightarrow N$ is the minimal surface homotopy equivalent to N , and from (18) we get

$$(19) \quad \text{Area}_{\mathbf{M}}(g(R)) \leq \frac{4\pi(\mathbf{g}(R) - 1)}{1 + C_0\epsilon^2}.$$

On the other hand, consider the composition $F \circ g: R \rightarrow \mathbf{M}$. Note that the quasifuchsian group $(F \circ g)_*: \pi_1(R) \rightarrow \pi_1(\mathbf{M})$ is $(1 + \epsilon)$ -quasifuchsian when η is small enough. Moreover, since F is $(1 + \eta)$ -bilipschitz we have

$$\text{Area}_{\mathbf{M}}((F \circ g)(R)) \leq (1 + \eta)^2 \text{Area}(g(R)).$$

Combining this with (19) we get

$$(20) \quad \text{Area}_{\mathbf{M}}((F \circ g)(R)) \leq (1 + \eta)^2 \frac{4\pi(\mathbf{g}(R) - 1)}{1 + C_0\epsilon^2}.$$

Since $\eta = \epsilon^3$ we get

$$\frac{(1 + \eta)^2}{1 + C_0\epsilon^2} = \frac{(1 + \epsilon^3)^2}{1 + C_0\epsilon^2} \leq \frac{1}{1 + C_1\epsilon^2}$$

when ϵ is small enough and $C_1 = \frac{C_0}{2}$. Replacing this into (20) gives

$$\text{Area}_{\mathbf{M}}((F \circ g)(R)) \leq \frac{4\pi(\mathbf{g}(R) - 1)}{1 + C_1\epsilon^2}.$$

Let Σ be the homotopy class of $F \circ g: R \rightarrow \mathbf{M}$. Since $\text{Area}_{\mathbf{M}}(\Sigma) \leq \text{Area}_{\mathbf{M}}((F \circ g)(R))$, we get

$$\text{Area}_{\mathbf{M}}(\Sigma) \leq \frac{4\pi(\mathbf{g}(\Sigma) - 1)}{1 + C_1\epsilon^2}.$$

Thus, $\Sigma \in \mathcal{S}_{\epsilon}(T, C_1)$, and the proposition is proved.

5. GENERA OF GEOMETRICALLY RANDOM SURFACES II

In this section we prove Lemma 3.2 assuming the following two propositions which we prove in the next section.

Proposition 5.1. *There exists a constant $\alpha > 0$ such that for every $C > 0$ the inequality*

$$|\mathcal{S}_\epsilon(T)| - |\mathcal{S}_\epsilon(T, C)| \leq (\alpha r)^{2r}$$

holds, where

$$r = r(T, C) = \frac{(1 + C\epsilon^2)T}{4\pi} + 2.$$

Proposition 5.2. *There exist constants $\beta, C_0 > 0$ such that for every ϵ small enough there exists $T_0 = T_0(\epsilon)$ so that the inequality*

$$(\beta s)^{2s} \leq |\mathcal{S}_\epsilon(T, C_0)|$$

holds for every large enough T . Here

$$s = s(T, C_0) = \frac{(1 + C_0\epsilon^2)(T - T_0)}{4\pi}.$$

5.1. Proof of Lemma 3.2. Let $0 < C_1 < C_0$, where C_0 is the constant from Proposition 5.2. In the remainder of the proof we show

$$(21) \quad \lim_{T \rightarrow \infty} \frac{|\mathcal{S}_\epsilon(T, C_1)|}{|\mathcal{S}_\epsilon(T)|} = 1$$

which proves the lemma.

We have

$$(22) \quad \frac{|\mathcal{S}_\epsilon(T)| - |\mathcal{S}_\epsilon(T, C_1)|}{|\mathcal{S}_\epsilon(T)|} \leq \frac{(\alpha r)^{2r}}{(\beta s)^{2s}}$$

where we used the above propositions to estimate the numerator and denominator respectively. Here $r = r(T, C_1)$, and $s = s(T, C_0)$. Replacing the values of $r(T, C_1)$ and $s(T, C_0)$ from the statements of the propositions, we get

$$\lim_{T \rightarrow \infty} \frac{r(T, C_1)}{s(T, C_0)} = \frac{(1 + C_1\epsilon^2)}{(1 + C_0\epsilon^2)} < 1$$

because $C_1 < C_0$. But this implies that the right hand side of the inequality (22) tends to 0 when $T \rightarrow \infty$ which proves (21).

6. GENERA OF GEOMETRICALLY RANDOM SURFACES III

We prove the two propositions from the previous section.

6.1. Proof of Proposition 5.1. The proof is based on the counting result by Kahn-Marković [8] which states that there exists $\alpha > 0$ such that the number of homotopy classes of all surfaces of genus at most g in \mathbf{M} is smaller than $(\alpha g)^{2g}$. In particular this enables us to bound the number of ϵ -quasifuchsian surfaces of genus at most g . That is, we have

$$(23) \quad |\mathcal{S}_\epsilon(g)| \leq (\alpha g)^{2g},$$

where we recall that $\mathcal{S}_\epsilon(g)$ is the set of classes in \mathcal{S}_ϵ of genus at most g .

Now, from the definition of the set $\mathcal{S}_\epsilon(T, C)$ we derive the inclusion

$$(\mathcal{S}_\epsilon(T) \setminus \mathcal{S}_\epsilon(T, C)) \subset \mathcal{S}_\epsilon(g)$$

where

$$g = \left\lceil \frac{(1 + C\epsilon^2)T}{4\pi} \right\rceil + 1 \leq \frac{(1 + C\epsilon^2)T}{4\pi} + 2 = r(T, C).$$

Combining this with (23) yields

$$|\mathcal{S}_\epsilon(T)| - |\mathcal{S}_\epsilon(T, C)| \leq |\mathcal{S}_\epsilon(g)| \leq (\alpha g)^{2g} \leq (\alpha r)^{2r},$$

which proves the proposition.

6.2. Proof of Proposition 5.2. The proof of this proposition is based on Proposition 4.1 and the counting result by Müller-Puchta [15] (see also [8]) that the number of different degree n covers of a closed hyperbolic surface is at least $(\beta g_n)^{2g_n}$, for every n large enough, and for some $\beta > 0$ depending on the surface. Here g_n denotes the genus of the covering surface of degree n .

Fix ϵ which is small enough so that Proposition 4.1 holds. Let $\Sigma_0 \in \mathcal{S}_\epsilon(T_0)$, where $T_0 = T(\epsilon)$, denote the class from Proposition 4.1. From the definition of $\mathcal{S}_\epsilon(T_0, C_0)$ we get

$$(24) \quad \frac{(1 + C_0\epsilon^2)T_0}{4\pi} + 1 \leq \mathbf{g}(\Sigma_0)$$

where C_0 is the constant from Proposition 4.1.

Let Σ be a degree n cover of Σ_0 . Then $\Sigma \in \mathcal{S}_\epsilon(nT_0, C_0)$, and the genus of Σ is given by $\mathbf{g}(\Sigma) = n(\mathbf{g}(\Sigma_0) - 1) + 1$. Therefore by Müller-Puchta, for large n we have

$$(25) \quad (\beta g_n)^{2g_n} \leq |\mathcal{S}_\epsilon(nT_0, C_0)|$$

where $g_n = n(\mathbf{g}(\Sigma_0) - 1) + 1$.

Let T be large, and n an integer such that $nT_0 \leq T \leq (n+1)T_0$. Then from (25) we get

$$(26) \quad (\beta g_n)^{2g_n} \leq |\mathcal{S}_\epsilon(nT_0, C_0)| \leq |\mathcal{S}_\epsilon(T, C_0)|.$$

From (24) we find

$$\frac{(1 + C_0\epsilon^2)nT_0}{4\pi} \leq n(\mathbf{g}(\Sigma_0) - 1) = g_n - 1,$$

implying that

$$s(T, C_0) = \frac{(1 + C_0\epsilon^2)(T - T_0)}{4\pi} \leq \frac{(1 + C_0\epsilon^2)nT_0}{4\pi} \leq g_n$$

where we used the fact that $T - T_0 \leq nT_0$. Replacing this into (26) proves the proposition.

7. THE MODEL QUASIFUCHSIAN MANIFOLD OF GENUS TWO

In this section we construct the model quasifuchsian manifold from Proposition 4.2.

7.1. Complex earthquakes. Let R denote a hyperbolic Riemann surface and suppose (Λ, μ) is a measured lamination. This means that Λ is a lamination on R , and μ a complex valued transverse measure. Let $G < \text{Isom}(\mathbb{H}^2)$ be the Fuchsian group such that $R \approx \mathbb{H}^2/G$. Recall the complex earthquake developing map $E_{(\Lambda, \mu)}: \mathbb{H}^2 \rightarrow \mathbb{H}^3$. This map is equivariant with respect to G , and it induces a homomorphism $e: G \rightarrow \text{Isom}(\mathbb{H}^3)$.

The norm $\|\mu\|$ is the supremum of the total measure of $|\mu|$ over all transverse intervals of length one. Assuming $\|\mu\|$ is small enough, it follows that the homomorphism $e: G \rightarrow \text{Isom}(\mathbb{H}^3)$ is quasifuchsian, that is the image group $e(G)$ is quasifuchsian. The following quantitative version of this claim is well known (for example, see Lemma 4.14 in [5]).

Theorem 7.1. *There exist universal constants $C, a > 0$ such that the group $e(G)$ is $(1 + C\|\mu\|)$ -quasifuchsian when $\|\mu\| \leq a$. Here R is any hyperbolic surface, and (Λ, μ) any measured lamination on R .*

7.2. Bending along a closed geodesic. Now fix some closed Riemann surface of genus two. We name it R . Fix a simple closed geodesic $\gamma \subset R$, and let Λ be the lamination whose only leaf is γ . Given $t > 0$, we let μ_t be the transverse measure on Λ whose weight on γ is equal to $t\mathbf{i}$ (here \mathbf{i} is the imaginary unit). By $e_t: G \rightarrow \text{Isom}(\mathbb{H}^3)$ we denote the corresponding homomorphism induced by the measured lamination (Λ, μ_t) .

Let $D_1 = \|\mu_1\|$. Then $\|\mu_t\| = tD_1$. From Theorem 7.1 we get that when t is small enough, the group $e_t(G)$ is a $(1 + C_1D_1t)$ -quasifuchsian group of genus two (here C_1 denotes the constant from Theorem 7.1). Let N_t denote the quasifuchsian manifold corresponding to $e_t(G)$. We have proved the following claim.

Claim 7.2. *There exists a constant $C > 0$ such that the quasifuchsian manifold N_t is a $(1 + Ct)$ -quasifuchsian manifold.*

It remains to estimate the area of the minimal surface homotopic to N_t .

7.3. The area of the minimal surface in N_t . To prove Proposition 4.2 we need to find a suitable upper bound on the area of the minimal surface in N_t . We obtain such a bound by producing an explicit surface (not necessarily minimal) homotopic to N_t , and estimating its area from above.

One obvious surface homotopic to N_t is the pleated surface $R_t \subset N_t$. Let $E_t: R \rightarrow N_t$ be the induced pleating map. The map E_t is locally an isometry, except at γ where it bends the surface R by the angle t . Set $E_t(R) = R_t$. Then the area of R_t is given by

$$(27) \quad \text{Area}_{N_t}(R_t) = 4\pi.$$

Therefore, to prove the lemma we need to find a surface of smaller area than R_t . We do that by bevelling the surface R_t near the geodesic $E_t(\gamma) \subset N_t$.

Fix $\alpha, \beta \subset R$ which are equidistant, and embedded, lines either side of the geodesic γ . By $U \subset R$ we denote the annulus (embedded in R) bounded by α and β . The annulus U is foliated by geodesic segments $\{s_x\}_{x \in \gamma}$, where

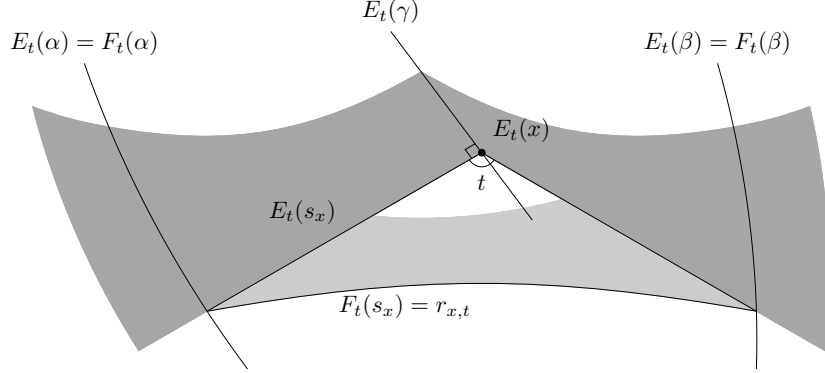


FIGURE 2. The hyperbolic bevel $F_t(U)$ associated to the hyperbolic dihedral angle $E_t(U)$.

each s_x intersects γ orthogonally at x , and the endpoints of s_x are on α and β respectively.

Define a new map $F_t: R \rightarrow N_t$ as follows. We let $F_t = E_t$ away from the annulus U . On U , we define F_t so it maps the segment s_x onto the geodesic segment $r_{x,t}$ which has the same endpoints as the piecewise geodesic arc $E_t(s_x)$. Let $S_t = F_t(R)$. We show that the area of S_t admits the required upper bound.

Let $V_t = F_t(U)$. Then V_t is a hyperbolic bevel which is placed to make a triangular shape together with the sloping surface $E_t(U)$ (see Figure 2). Let $d_0 = \frac{|s_x|}{2}$, and $d_t = \frac{|r_{x,t}|}{2}$ (obviously d_0 and d_t do not depend on the choice of $x \in \gamma$). From the hyperbolic law of sines we compute

$$(28) \quad \sinh(d_t) = \cos\left(\frac{t}{2}\right) \sinh(d_0).$$

Moreover, using elementary hyperbolic geometry we compute the hyperbolic areas of U and V_t , and get

$$(29) \quad \text{Area}_{\mathbf{M}}(E_t(U)) = 2|\gamma| \sinh(d_0), \quad \text{Area}_{N_t}(V_t) = 2|\gamma| \tanh(d_t) \cosh(d_0).$$

We have

$$\begin{aligned} \text{Area}_{N_t}(S_t) &= \text{Area}_{N_t}(R_t) - (\text{Area}_{\mathbf{M}}(E_t(U)) - \text{Area}_{N_t}(V_t)) \\ &= 4\pi - 2|\gamma|(\sinh(d_0) - \tanh(d_t) \cosh(d_0)) \\ &= 4\pi(1 - C_1(1 - \tanh(d_t) \coth(d_0))) \end{aligned}$$

where $C_1 = \frac{2|\gamma| \sinh(d_0)}{4\pi}$. Then

$$\text{Area}_{N_t}(S_t) \leq \frac{4\pi}{1 + C_1(1 - \tanh(d_t) \coth(d_0))}.$$

But from (28) we derive $d_t \leq d_0 - C_2 t^2$ for some $C_2 > 0$, which gives $C_3 t^2 \leq (1 - \tanh(d_t) \coth(d_0))$ for some $C_3 > 0$. Thus,

$$\text{Area}_{N_t}(S_t) \leq \frac{4\pi}{1 + C_1 C_3 t^2}.$$

Therefore we have proved the following claim (note that the minimal surface homotopic to N_t minimises the area in its homotopy class).

Claim 7.3. *There exists a constant $C > 0$ such that if T_t denotes the area of the minimal surface homotopic to N_t , then for every small enough t we have*

$$T_t \leq \frac{4\pi}{1 + C t^2}.$$

Combining this with Claim 7.2 proves Proposition 4.2.

8. LOCAL AND GLOBAL PROPERTIES OF MINIMAL MAPS

The proof of Lemma 3.3 occupies Sections 8 to 11. We roughly outline the idea of the proof. As usual, $f: S \rightarrow \mathbf{M}$ denotes an ϵ -nearly geodesic minimal surface. The application of the Gauss-Bonnet formula yields

$$4\pi(\mathbf{g}(S) - 1) = \int_S (1 + \lambda^2(x)) dA_\sigma \leq \left(1 + \frac{2}{|S|} \int_S \lambda^2(x) dA\right) T,$$

where T is the area of the minimal surface $f(S)$, and $\lambda(x)$ the principal curvature at the point $f(x)$. Thus, to prove the lemma it suffices to prove the bound

$$(30) \quad \frac{2}{|S|} \int_S \lambda^2(x) dA \leq C \left(1 - \frac{|S(\delta_1(h), h)|}{|S|}\right) \epsilon^2.$$

The two main ingredients in proving this bound are as follows:

- Seppi's universality estimate $|\lambda(x)| \leq C_1 \epsilon$ (see Theorem 8.1 below).
- Each connected component of $S(\delta_1(h), h)$ has a rigid convex core Ω (see Definition 10.1 below). This implies that for $x \in \text{int}(\Omega)$ the relation $|\lambda(x)| = o(1)\epsilon$ holds, where $o(1) \rightarrow 0$ when $\mathbf{d}_S(x, \partial\Omega) \rightarrow \infty$.

The purpose of the next three sections is to prove Lemma 10.2 which estimates the integral of λ^2 over rigid subsurfaces. The proof of Lemma 3.3 is completed in Section 11.

8.1. Global bound on principal curvatures. Let $f: S \rightarrow \mathbf{M}$ be an ϵ -nearly geodesic minimal surface. By $\tilde{f}: \mathbb{H}^2 \rightarrow \mathbb{H}^3$ we denote its lift (here we identify \mathbb{H}^2 and \mathbb{H}^3 with the universal covers of S and \mathbf{M} respectively). Clearly \tilde{f} is a quasiisometry (because it is a developing map of a quasifuchsian representation). But we need a more quantitative version of this which we derive first. Then we recall the basic properties of the distance function between \tilde{S} and any totally geodesic plane $P \subset \mathbb{H}^3$. These results are used

in the next two sections where we obtain an upper bound on the principal curvature of S on its subsurfaces which are rigid.

Let $\lambda: S \rightarrow [0, \infty)$ be the non-negative eigenvalue of the shape operator corresponding to the minimal map $f: S \rightarrow \mathbf{M}$. The significance of the function λ is that the principal curvatures of a minimal surface $f(S)$ at the point $f(x)$ are equal to $\lambda(x)$ and $-\lambda(x)$ respectively. The following global estimate was shown by Seppi [18].

Theorem 8.1. *There exists a universal constant C such that*

$$(31) \quad \|\lambda\|_\infty \leq C\epsilon$$

for every ϵ -nearly geodesic minimal surface $f: S \rightarrow \mathbf{M}$ when ϵ is small enough.

The first corollary we derive from this theorem concerns the quasigeodesic properties of the minimal map \tilde{f} .

Proposition 8.2. *There exists a constant $C > 0$ such that if $\alpha \subset \mathbb{H}^2$ is any geodesic, and $\beta \subset \mathbb{H}^3$ the geodesic with the same endpoints as the quasigeodesic $\tilde{f}(\alpha)$, then $\tilde{f}(\alpha) \subset \mathcal{N}_{C\epsilon}(\beta)$.*

Proof. Let $x \in \alpha$. Then the infinitesimal bending of the curve $\tilde{f}(\alpha)$ at the point $\tilde{f}(x)$ is bounded above by the maximum of principal curvatures at $\tilde{f}(x)$ which is equal to $\lambda(x)$. Let C_1 denote the constant from Theorem 8.1. Then the infinitesimal bending of the curve $\tilde{f}(\alpha)$ at all of its points is bounded above by $C_1\epsilon$.

Let $g: \alpha \rightarrow \tilde{f}(\alpha)$ be the path-length reparametrization of the quasigeodesic $\tilde{f}(\alpha)$. From Lemma 4.4 in [5] we conclude that $g(\alpha) \subset \mathcal{N}_{C_2\epsilon}(\beta)$ which proves the proposition. \square

8.2. The induced metric on S . Denote by $\sigma^2(z)|dz|^2$ the conformal metric on S obtained by pulling back the hyperbolic metric from \mathbf{M} by the minimal map f (here z is a local complex parameter on S). The Gauss curvature of this metric σ is equal to $-(1 + \lambda^2)$. Let $\rho^2(z)|dz|^2$ denote the density of the hyperbolic metric on S . Recall the Ahlfors estimate:

$$(32) \quad \frac{\rho^2(z)}{1 + \lambda^2(x)} \leq \sigma^2(x) \leq \rho^2(z).$$

The following is an immediate corollary of (32).

Proposition 8.3. *Let $\phi: \mathbb{H}^2 \rightarrow \mathbb{R}$ be any function, and let $\nabla_\sigma\phi$ and $\Delta_\sigma\phi$ denote respectively the gradient and the laplacian of ϕ with respect to the lift of the metric σ to \mathbb{H}^2 . Then*

$$|\nabla\phi| \leq |\nabla_\sigma\phi| \leq 2|\nabla\phi|,$$

and

$$|\Delta\phi| \leq |\Delta_\sigma\phi| \leq 2|\Delta\phi|,$$

assuming ϵ is small enough.

Remark 10. Here, and in the rest of the paper, $\nabla\phi$, $\Delta\phi$ denote the gradient and the laplacian of ϕ with respect to the hyperbolic metric.

8.3. The distance function. Seppi also obtained local estimates on λ which we utilise below. Let $P \subset \mathbb{H}^3$ be a geodesic plane and define the function $u: \mathbb{H}^2 \rightarrow \mathbb{R}$ by

$$(33) \quad u(x) = \sinh(\mathbf{d}_{\mathbf{M}}(\tilde{f}(x), P)).$$

Here $\mathbf{d}_{\mathbf{M}}(\tilde{f}(x), P)$ is the signed distance from the point $\tilde{f}(x)$ to the plane P . A significant property of minimal surfaces is that the distance function u satisfies the second order elliptic PDE

$$(34) \quad \Delta_{\sigma} u = 2u$$

on \mathbb{H}^2 (for example see [18]).

For a function $\phi: \mathbb{H}^2 \rightarrow \mathbb{R}$, we let

$$\|\phi\|_{C^0(B_r(x))} = \max\{|\phi(y)| : y \in B_r(x)\}.$$

The following local estimates were established in [18] by applying the standard Schauder theory to the formula (34).

Proposition 8.4. *There exists a constant $C > 0$ such that for every $x \in \mathbb{H}^2$ the inequalities*

$$|\nabla u(x)| \leq C \|u\|_{C^0(B_1(x))},$$

and

$$\lambda(x) \leq \frac{C \|u\|_{C^0(B_1(x))}}{\sqrt{1 - C \|u^2\|_{C^0(B_1(x))}}},$$

hold when ϵ is small enough.

Remark 11. Clearly, the second estimate makes sense only when $\|u\|_{C^0(B_1(x))}$ is small enough so that $C \|u^2\|_{C^0(B_1(x))} < 1$.

Proof. The first estimate is a special case of the formula (27) in [18], while the second one is the estimate (32) from [18]. \square

9. AN AUXILIARY FUNCTION

The purpose of this section is to establish some basic properties of the auxiliary function $v: \mathbb{H}^2 \rightarrow [0, \infty)$ defined as the averaging of the square of the distance function u from the previous section. The auxiliary function v is used in the proof of Lemma 10.2 in the next section.

9.1. The square of the distance function. With a view towards applications that follow, we consider the square u^2 . The advantage of considering the function u^2 is that it is non-negative and subharmonic.

Proposition 9.1. *The inequality*

$$(35) \quad 0 \leq u^2 \leq \Delta u^2$$

holds on \mathbb{H}^2 .

Proof. Computing the laplacian of u^2 in terms of the function u yields the inequality

$$2u\Delta_\sigma u \leq \Delta_\sigma u^2,$$

which together with (34) yields the estimate

$$4u^2 \leq \Delta_\sigma u^2.$$

Applying Proposition 8.3 we get $\Delta_\sigma u^2 \leq 2\Delta u^2$ when ϵ is small enough. Replacing this in the previous displayed inequality proves that $2u^2 \leq \Delta u^2$ which in turn proves the proposition. \square

The fact that u^2 is subharmonic enables us to control the (local) supremum norm of u^2 in terms of its (local) L^1 norm.

Proposition 9.2. *For every $x \in \mathbb{H}^2$, the inequality*

$$(36) \quad \|u^2\|_{C^0(B_1(x))} \leq \int_{B_2(x)} u^2(y) dA$$

holds, where dA is the hyperbolic area form.

Proof. Let $y \in B_1(x)$. Since $\Delta u^2 \geq 0$, the Mean Value Theorem yields the estimate

$$u^2(y) \leq \frac{1}{|B_1(y)|} \int_{B_1(y)} u^2(z) dA(z).$$

From $|B_1(y)| > 1$, we get

$$\frac{1}{|B_1(y)|} \int_{B_1(y)} u^2(z) dA(z) \leq \int_{B_1(y)} u^2(z) dA(z) \leq \int_{B_2(x)} u^2(z) dA(z),$$

where in the last inequality we used the fact that $B_1(y) \subset B_2(x)$ for every $y \in B_1(x)$. This together with the mean value estimate proves the proposition. \square

On the other hand, computing ∇u^2 in terms of u , and applying Proposition 8.4, yields the following proposition.

Proposition 9.3. *There exist constants $C, \eta > 0$ such that assuming ϵ is small enough, for every $x \in \mathbb{H}^2$ the inequality*

$$(37) \quad |\nabla u^2(x)| \leq C \|u^2\|_{C^0(B_1(x))}$$

holds. Moreover, if $\|u^2\|_{C^0(B_1(x))} < \eta$, then

$$(38) \quad \lambda^2(x) \leq C \|u^2\|_{C^0(B_1(x))}.$$

Proof. Let C_1 be the constant from Proposition 8.4. Since $\nabla u^2 = 2u\nabla u$, it follows that (37) holds for $C = 2C_1^2$. On the other hand, if $\|u^2\|_{C^0(B_1(x))} < \frac{1}{4C_1^2}$, then (38) follows from the second inequality in Proposition 8.4 by letting $C = 2C_1^2$. Thus, the proposition holds for $C = 2C_1^2$, and $\eta = \frac{1}{4C_1^2}$. \square

9.2. The averaging function and its properties. For $x \in \mathbb{H}^2$, we set

$$v(x) = \int_{B_2(x)} u^2(y) dA(y).$$

The following propositions list important properties of the function v .

Proposition 9.4. *The inequality*

$$(39) \quad 0 \leq v \leq \Delta v$$

holds on \mathbb{H}^2 .

Proof. Since the hyperbolic laplacian Δ is invariant under the isometries of \mathbb{H}^2 it follows that

$$\int_{B_2(x)} \Delta u^2(y) dA = \Delta v(x).$$

Combining this with Proposition 9.1 yields (39). \square

Proposition 9.5. *There exists $C > 0$ such that*

$$(40) \quad |\nabla v(x)| \leq C \|u^2\|_{C^0(B_3(x))}$$

for every $x \in \mathbb{H}^2$ assuming ϵ is small enough.

Proof. Let $x_1 \in \mathbb{H}^2$ be such that $\mathbf{d}_{\mathbb{H}^2}(x, x_1) < 1$. We can choose an isometry Q of \mathbb{H}^2 such that $Q(x) = x_1$, and so that

$$(41) \quad \max_{y \in B_2(x)} \mathbf{d}_{\mathbb{H}^2}(y, Q(y)) \leq L \mathbf{d}_{\mathbb{H}^2}(x, x_1)$$

for some universal constant $L > 0$. Then

$$\begin{aligned} |v(x_1) - v(x)| &\leq \int_{B_2(x)} |u^2(Q(y)) - u^2(y)| dA(y) \\ &\leq (1 + o(1)) \int_{B_2(x)} |\nabla u^2(y)| \mathbf{d}_{\mathbb{H}^2}(Q(y), y) dA(y), \end{aligned}$$

where $o(1) \rightarrow 0$ when $x_1 \rightarrow x$. Combining this with (41) gives

$$\frac{|v(x_1) - v(x)|}{\mathbf{d}(x, x_1)} \leq L \int_{B_2(x)} |\nabla u^2(y)| dA(y),$$

which yields

$$|\nabla v(x)| \leq L \max_{y \in B_2(x)} |\nabla u^2(y)|.$$

Combining this with (37) proves (39). \square

Proposition 9.6. *There exist $C, \eta > 0$ such that when ϵ is small enough the estimate*

$$(42) \quad \lambda^2(x) \leq Cv(x)$$

holds for every $x \in \mathbb{H}^2$ assuming $\|u^2\|_{C^0(B_1(x))} < \eta$.

Proof. Applying first (38), and then (36), we get

$$\lambda^2(x) \leq C \|u^2\|_{C^0(B_1(x))} \leq C \int_{B_2(x)} u^2(y) dA = Cv(x),$$

which proves (42). □

10. RIGID SUBSURFACES OF A MINIMAL SURFACE

Let $\Omega \subset S$ be a subsurface with geodesic boundary. In this section we show that if the lift of $f(\Omega)$ to \mathbb{H}^3 is closely aligned with a geodesic plane $P \subset \mathbb{H}^3$, then the size of the integral of the principal curvature is controlled by the length of the boundary of Ω . This is the content of Lemma 10.2 which plays the key part in the proof of Lemma 3.3.

Definition 10.1. *Let $f: S \rightarrow \mathbf{M}$ be a minimal map, and $\Omega \subset S$ an essential subsurface with geodesic boundary. We say that Ω is f -rigid if*

- *there exists $D > 0$,*
- *there exists a geodesic plane $P \subset \mathbb{H}^3$,*

such that $\mathbf{d}_{\mathbf{M}}(\tilde{f}(x), P) < D$ for every point $x \in \tilde{\Omega}$. Here $\tilde{\Omega}$ is a connected component of the lift of Ω to the universal cover \mathbb{H}^2 .

Lemma 10.2. *There exists $C > 0$ such that for every ϵ -nearly geodesic subsurface the estimate*

$$(43) \quad \int_{\Omega} \lambda^2 dA \leq C |\partial\Omega| \epsilon^2$$

holds providing that Ω is f -rigid, and ϵ is small enough.

10.1. Rigid subsurfaces are ϵ rigid. In Definition 10.1 we assume that the minimal subsurface $\tilde{f}(\tilde{\Omega})$ is D away from a geodesic plane P . The purpose of this subsection is to promote D to ϵ , meaning that if we assume that $\tilde{f}(\tilde{\Omega})$ is D away from P , then it is actually $C\epsilon$ away from P .

Let $G < \pi_1(S)$ be a subgroup corresponding to the subsurface $\Omega \subset S$. Then $f_*(G) < \pi_1(\mathbf{M}) < \mathbf{PSL}(2, \mathbb{C})$. Let $\tilde{\Omega}$ be the component of the lift of Ω which is invariant under $f_*(G)$. Since Ω is f -rigid, it follows that the plane P is invariant under the group $f_*(G)$. This implies that the function $u: \tilde{\Omega} \rightarrow \mathbb{R}$ defined by (33) is equivariant and therefore well defined on Ω .

Proposition 10.3. *There exists $C > 0$ such that*

$$(44) \quad \|u^2\|_{C^0(\Omega)} \leq C\epsilon^2$$

when ϵ is small enough.

Proof. Since Ω is a subsurface with geodesic boundary, it follows that Ω is equal to its convex core. Suppose first that α is a geodesic which is entirely contained in $\tilde{\Omega}$. Let $\beta \subset \mathbb{H}^3$ be the geodesic with the same endpoints as

$\tilde{f}(\alpha)$. Since Ω is f -rigid it follows that β is a finite distance away from the plane P . But then $\beta \subset P$ since P is a geodesic plane. On the other hand, from Proposition 8.2 we have that $\mathbf{d}_{\mathbf{M}}(f(\alpha), \beta) \leq C_1\epsilon$, where C_1 is the constant from Proposition 8.2. Therefore, we derived the inequality

$$\mathbf{d}_{\mathbf{M}}(f(x), P) \leq C_1\epsilon$$

for every $x \in \alpha$. But $\sinh(C_1\epsilon) \leq 2C_1\epsilon$ for ϵ small enough. Thus, the inequality

$$|u^2(x)| \leq C\epsilon^2, \quad x \in \alpha,$$

holds for $C = 4C_1^2$. Note that $\partial\tilde{\Omega}$ consists of geodesics which are entirely contained in $\tilde{\Omega}$. Thus, we get

$$|u^2(x)| \leq C\epsilon^2, \quad x \in \partial\tilde{\Omega}.$$

On the other hand, we know that u^2 is subharmonic. Since Ω is f -rigid we know that u^2 is a bounded function on $\tilde{\Omega}$. Therefore, (44) follows from the last inequality and the generalised maximum principle. \square

10.2. Proof of Lemma 10.2. We start by defining

$$\Omega' = \{x \in \Omega : \mathbf{d}_{\Omega}(x, \partial\Omega) \geq 3\}.$$

Observe that there exists a universal constant L such that

$$(45) \quad |\Omega \setminus \Omega'| \leq L|\partial\Omega|,$$

and

$$(46) \quad |\partial\Omega'| \leq L|\partial\Omega|.$$

Furthermore, without loss of generality we may assume that $\partial\Omega'$ is a union of finitely many smooth curves (the point is that Ω' can be approximated by such domains).

We first estimate the integral on the left hand side of (43) over $\Omega \setminus \Omega'$. Let C_0 be the constant from Theorem 8.1. Then from (45) we get

$$(47) \quad \int_{\Omega \setminus \Omega'} \lambda^2 dA \leq C_0|\Omega \setminus \Omega'|\epsilon^2 \leq C_0L|\partial\Omega|\epsilon^2.$$

It remains to estimate this integral over Ω' which we do in the remainder of the proof.

Let \mathbf{n} be the inward-pointing unit normal vector field on the boundary $\partial\Omega'$. Green's formula gives

$$(48) \quad \int_{\Omega'} \Delta v dA = \int_{\partial\Omega'} \frac{\partial v}{\partial \mathbf{n}} dl,$$

where dl is the hyperbolic length form.

Firstly, let C_1 , and η_1 , be the constants from Proposition 9.6. When $x \in \Omega'$, then $B_1(x) \subset \Omega$. Thus, by (44) the inequality

$$\|u^2\|_{C^0(\Omega)} \leq \eta_1$$

holds when ϵ is small enough. Then from Proposition 9.6 we derive the inequality

$$(49) \quad \lambda^2(x) \leq C_1 v(x), \quad x \in \Omega'.$$

Combining this with the fact that $v \leq \Delta v$, we obtain the inequality

$$(50) \quad \int_{\Omega'} \lambda^2 dA \leq C_1 \int_{\Omega'} \Delta v dA$$

assuming ϵ is small enough.

On the other hand, let C_2 be the constant from Proposition 9.5. Then

$$(51) \quad \begin{aligned} \int_{\partial\Omega'} \frac{\partial v}{\partial \mathbf{n}} dl &\leq \int_{\partial\Omega'} |\nabla v| dl \leq \max_{x \in \partial\Omega'} |\nabla v(x)| |\partial\Omega'| \\ &\leq C_2 \|u^2\|_{C^0(\Omega)} |\partial\Omega'| \leq C_2 C_3 |\partial\Omega'| \epsilon^2 \end{aligned}$$

where C_3 is the constant from Proposition 10.3. Replacing (50) and (51) in the Green's formula (48) proves

$$\int_{\Omega'} \lambda^2 dA \leq C_1 C_2 C_3 |\partial\Omega'| \epsilon^2 \leq L C_1 C_2 C_3 |\partial\Omega'| \epsilon^2$$

where we applied (46) in the last inequality. Together with (47) this gives

$$\int_{\Omega} \lambda^2 dA \leq L(C_0 + C_1 C_2 C_3) |\partial\Omega'| \epsilon^2.$$

This proves the lemma by letting $C = L(C_0 + C_1 C_2 C_3)$.

11. PROOF OF LEMMA 3.3

11.1. Smoothing out the boundary of $S(\delta_1(h), h)$. In the remainder of this section we fix $h \in \mathbb{N}$. To alleviate the notation we let $\delta_0 = \delta_0(h)$, $\delta_1 = \delta_1(h)$, and $\epsilon_0 = \epsilon_0(h)$ (see Remark 7).

Proposition 11.1. *Let $f: S \rightarrow \mathbf{M}$ be any ϵ -nearly geodesic minimal surface. There exists a (possibly disconnected) subsurface $R \subset S$ (with piecewise smooth boundary) with the following properties:*

- (1) $S(\delta_1, h) \subset R \subset S(\delta_0, h)$,
- (2) $|\partial R| \leq 25(|S| - |S(\delta_1, h)|)$,

assuming $\epsilon < \epsilon_0$.

Proof. We apply Lemma 2.7 to $S(\delta_0, h)$ to obtain R such that

$$S(\delta_1, h) \subset R \subset \mathcal{N}_1(S(\delta_1, h)) \subset S(\delta_0, h),$$

and

$$|\partial R| \leq 25|\mathcal{N}_1(\partial S(\delta_0, h))| \leq 25(|S| - |S(\delta_1, h)|). \quad \square$$

11.2. The endgame. Let R_1, \dots, R_k be the connected components of R . By $\Omega_i \subset S$ we denote the convex core of R_i . In other words, Ω_i is the subsurface whose boundary curves are geodesics homotopic to the homotopically non-trivial components of ∂R_i .

Proposition 11.2. *Each Ω_i is f -rigid.*

Proof. Let \tilde{R}_i be a component of the lift of R_i to \mathbb{H}^2 . Then there exists a component $\tilde{\Omega}_i$ of the lift of Ω_i such that every point on $\tilde{\Omega}_i$ is at most distance $D_1 > 0$ away from \tilde{R}_i . On the other hand, since $R_i \subset S(\delta_0, h)$ it follows from Proposition 2.1 that for each point $p \in \tilde{R}_i$ there exists a unique connected component of $\tilde{\mathcal{H}}$ which is δ_0 away from $\tilde{f}(x)$. Since \tilde{R}_i is connected, this component is the same for all $x \in \tilde{R}_i$, and we conclude that there exists a geodesic plane $P_i \subset \mathbb{H}^3$ such that every point on \tilde{R}_i is δ_0 away from P_i . Thus, every point on $\tilde{\Omega}_i$ is $(D_1 + \delta_0)$ away from P_i and we are done. \square

We apply the Gauss-Bonnet formula and get

$$\begin{aligned} 4\pi(\mathbf{g}(S) - 1) &= \int_S (1 + \lambda^2(x)) dA_\sigma = \int_S dA_\sigma + \int_S \lambda^2(x) dA_\sigma \leq \\ (52) \quad &\leq \text{Area}_\sigma(S) \left(1 + \frac{2}{|S|} \int_S \lambda^2(x) dA \right), \end{aligned}$$

where in the last inequality we used the inequality (32) which implies that $\frac{1}{2}dA \leq dA_\sigma \leq dA$ when ϵ is small enough. It remains to find a suitable upper bound for the last term in (52).

Let C_1 and C_2 be the constants from Lemma 10.2 and Theorem 8.1 respectively. Then

$$\begin{aligned} \int_S \lambda^2(x) dA &= \sum_{i=1}^k \int_{\tilde{\Omega}_i} \lambda^2(x) dA + \int_{S \setminus \Omega} \lambda^2(x) dA \\ &\leq C_1 \sum_{i=1}^k |\partial \Omega_i| \epsilon^2 + C_2^2 |S \setminus \Omega| \epsilon^2 \end{aligned}$$

where $\Omega = \Omega_1 \cup \dots \cup \Omega_k$. Since $\sum_{i=1}^k |\partial\Omega_i| \leq |\partial R|$, from the previous inequality we get

$$\begin{aligned} \int_S \lambda^2(x) dA &\leq C_1 |\partial R| \epsilon^2 + C_2^2 (|S| - |\Omega|) \epsilon^2 \\ &\leq C_1 |\partial R| \epsilon^2 + C_2^2 ((|S| - |R|) + ||R| - |\Omega||) \epsilon^2 \\ &\leq C_1 |\partial R| \epsilon^2 + C_2^2 ((|S| - |S(\delta_1, h)|) + |\partial R|) \epsilon^2, \end{aligned}$$

where in the last inequality we used the fact that $S(\delta_1, h) \subset R$, and also Proposition 2.12 to estimate $||R| - |\Omega||$.

Let $C_3 = 25$ be the constant from Proposition 11.1. We now use the estimate $|\partial R| \leq C_3 (|S| - |S(\delta_1, h)|)$, and get

$$\begin{aligned} \int_S \lambda^2(x) dA &\leq C_1 C_3 (|S| - |S(\delta_1, h)|) \epsilon^2 + C_2^2 ((|S| - |S(\delta_1, h)|) + C_3 (|S| - |S(\delta_1, h)|)) \epsilon^2 \\ &= (C_1 C_3 + C_2^2 + C_2^2 C_3) (|S| - |S(\delta_1, h)|) \epsilon^2. \end{aligned}$$

Thus,

$$\frac{1}{|S|} \int_S \lambda^2(x) dA \leq C_4 \left(1 - \frac{|S(\delta_1, h)|}{|S|} \right) \epsilon^2.$$

Replacing this back into (52) we get

$$4\pi(\mathbf{g}(S) - 1) \leq \text{Area}_\sigma(S) \left(1 + 2C_4 \left(1 - \frac{|S(\delta_1, h)|}{|S|} \right) \epsilon^2 \right),$$

and the lemma is proved.

12. THE PROOF OF THEOREM 1.8

We start by defining the set of minimal surfaces which have a definite part of their area located outside a $\delta_0 \equiv \delta_0(h)$ neighbourhood of the locus \mathcal{H}_h of totally geodesic surfaces of genus at most h . (We remind the reader that δ_0 was defined in Section 2.3.)

We remind the reader of our standing assumption that $\epsilon \leq \hat{\epsilon}$, where $\hat{\epsilon} > 0$ is the universal constant from Proposition 1.2. This ensures that each conjugacy class $\Sigma \in \mathcal{S}_\epsilon$ corresponds to a unique minimal surface.

To simplify our constructions we will work mostly with one genus at a time; accordingly we let $\mathcal{S}_\epsilon(\mathbf{g}) \subset \mathcal{S}_\epsilon(g)$ be the surfaces with exactly genus \mathbf{g} , so that

$$\mathcal{S}_\epsilon(g) = \bigcup_{\mathbf{g} \leq g} \mathcal{S}_\epsilon(\mathbf{g}).$$

The reader should not be overly concerned with this abuse of notation, because after this section we will talk only about $\mathcal{S}_\epsilon(\mathbf{g})$.

Definition 12.1. *Let $0 \leq q \leq 1$. We say $\Sigma \in \mathcal{S}_\epsilon(g, h, q) \subset \mathcal{S}_\epsilon(g)$ if*

$$\frac{|S(\delta_0, h)|}{|S|} \leq 1 - q$$

where $f: S \rightarrow \mathbf{M}$ is the ϵ -nearly geodesic minimal map representing the class Σ .

We also define $\mathcal{S}_\epsilon(\mathbf{g}, h, q)$ in analogy to $\mathcal{S}_\epsilon(\mathbf{g})$, so that

$$\mathcal{S}_\epsilon(g, h, q) = \bigcup_{\mathbf{g} \leq g} \mathcal{S}_\epsilon(\mathbf{g}, h, q).$$

In the remainder of this section we prove Theorem 1.8 assuming the following lemma, which will be proven in Section 14 after the introduction and discussion in Section 13.

Lemma 12.2. *For all $q > 0$ and $c > 0$, there exists h and $\epsilon_0 = \epsilon_0(q, c)$ such that*

$$(53) \quad \lim_{\mathbf{g} \rightarrow \infty} \frac{|\mathcal{S}_\epsilon(\mathbf{g}, h, q)|}{(c\mathbf{g})^{2\mathbf{g}}} = 0$$

for every $\epsilon < \epsilon_0$.

The reader can easily verify that this implies the unboldfaced version:

Corollary 12.3. *For all $q > 0$ and $c > 0$, there exists h and $\epsilon_0 = \epsilon_0(q, c)$ such that*

$$(54) \quad \lim_{g \rightarrow \infty} \frac{|\mathcal{S}_\epsilon(g, h, q)|}{(cg)^{2g}} = 0$$

for every $\epsilon < \epsilon_0$.

Combining Corollary 12.3 with the Müller-Puchta result we derive the following proposition.

Proposition 12.4. *For every $0 < q \leq 1$, there exists h such that we have*

$$\lim_{g \rightarrow \infty} \frac{|\mathcal{S}_\epsilon(g, h, q)|}{|\mathcal{S}_\epsilon(g)|} = 0,$$

when ϵ is small enough.

Proof. We are assuming (throughout the paper) that \mathbf{M} contains a totally geodesic surface; we denote by $\Sigma_0 \in \mathcal{S}_\epsilon(g_0)$ the corresponding conjugacy class (here $g_0 = \mathbf{g}(\Sigma_0)$). By the counting result by Müller-Puchta [15], there exists $\beta > 0$ such that the number of different degree n covers of Σ_0 is at least $(\beta g_n)^{2g_n}$, where $g_n = (n(g_0 - 1) + 1)$ is the genus of degree n covering surface.

Suppose g is large, and let n be an integer such that $g_n \leq g \leq g_{n+1}$. Then

$$(\beta g_n)^{2g_n} \leq |\mathcal{S}_\epsilon(g_n)| \leq |\mathcal{S}_\epsilon(g)|.$$

Since $g_n \geq (g - g_0)$, we derive the estimate

$$(55) \quad (\beta(g - g_0))^{2(g - g_0)} \leq |\mathcal{S}_\epsilon(g)|.$$

Let $c < \beta$ in Corollary 12.3. Then using (55) and (54), we get

$$\limsup_{g \rightarrow \infty} \frac{|\mathcal{S}_\epsilon(g, q)|}{|\mathcal{S}_\epsilon(g)|} \leq \limsup_{g \rightarrow \infty} \frac{(cg)^{2g}}{(\beta(g - g_0))^{2(g - g_0)}} = 0$$

when $\epsilon < \epsilon_0(q, c)$. This proves the proposition. \square

Let $\mu_\epsilon^{\text{top}}$ be an ϵ -topological limiting measure on $\mathcal{G}_2(\mathbf{M})$. That is, there exists a sequence $g_n \rightarrow \infty$ such that

$$\mu_\epsilon^{\text{top}} = \lim_{g_n \rightarrow \infty} \frac{1}{|\mathcal{S}_\epsilon(g_n)|} \sum_{\Sigma \in \mathcal{S}_\epsilon(g_n)} \mu(\Sigma)$$

(recall that $\mu(\Sigma)$ is the pushforward of the hyperbolic area measure from S to $\mathcal{G}_2(\mathbf{M})$). Let μ^{top} be a limit of the $\mu_\epsilon^{\text{top}}$ as $\epsilon \rightarrow 0$. We then have

$$\mu^{\text{top}} = \mu_{\mathcal{H}} + \mu_{\mathcal{L}}$$

where $\mu_{\mathcal{H}}$ is supported in \mathcal{H} , and $\mu_{\mathcal{L}}$ is a multiple of the Liouville measure. We will assume that $\mu_{\mathcal{L}}$ is nonzero and obtain a contradiction.

We choose $q = |\mu_{\mathcal{L}}|/4$ and choose $h > 0$ to satisfy Proposition 12.4 with this value of q .

We set

$$\widehat{\mathcal{S}}_\epsilon(g_n, h, q) = \mathcal{S}_\epsilon(g_n) \setminus \mathcal{S}_\epsilon(g_n, h, q).$$

Then from Proposition 12.4 we conclude that

$$(56) \quad \mu_\epsilon^{\text{top}} = \lim_{g_n \rightarrow \infty} \frac{1}{|\widehat{\mathcal{S}}_\epsilon(g_n, h, q)|} \sum_{\Sigma \in \widehat{\mathcal{S}}_\epsilon(g_n, h, q)} \mu(\Sigma).$$

Let $\Sigma \in \widehat{\mathcal{S}}_\epsilon(g_n, h, q)$, and denote by $f: S \rightarrow \mathbf{M}$ the minimal surface representing Σ . Then

$$(57) \quad 1 - q \leq \frac{|S(\delta_0, h)|}{|S|}.$$

The inequality (57) yields the estimate

$$1 - q \leq \mu(\Sigma)(\mathcal{N}_{\delta_0}(\mathcal{H}))$$

for every $\Sigma \in \widehat{\mathcal{S}}_\epsilon(g_n, h, q)$. Combining this with (56) gives the estimate

$$1 - q \leq \mu_\epsilon^{\text{top}}(\overline{\mathcal{N}_{\delta_0}(\mathcal{H})})$$

for every ϵ -topological limiting measure on $\mathcal{G}_2(\mathbf{M})$. In turn, this shows that

$$(58) \quad 1 - q \leq \mu^{\text{top}}(\overline{\mathcal{N}_{\delta_0}(\mathcal{H})})$$

for every topological limiting measure μ^{top} on $\mathcal{G}_2(\mathbf{M})$.

On the other hand, by the first property of δ in Proposition 2.1, taking our particular limiting measure $\mu^{\text{top}} = \mu_{\mathcal{H}} + \mu_{\mathcal{L}}$, we have

$$\begin{aligned} \mu^{\text{top}}(\mathcal{H} \setminus \overline{\mathcal{N}_{\delta_0}(\mathcal{H})}) &\geq \mu_{\mathcal{L}}(\mathcal{G}_2(\mathbf{M}) \setminus \overline{\mathcal{N}_{\delta_0}(\mathcal{H})}) \\ &\geq \frac{1}{2} \mu_{\mathcal{L}}(\mathcal{G}_2(\mathbf{M})) = 2q > q. \end{aligned}$$

This and (58) are a contradiction.

13. FROM GEOMETRY TO COMBINATORIAL DATA

In this section we describe a set of allowable combinatorial data records, such that for any ϵ -nearly geodesic $f: S \rightarrow \mathbf{M}$, we can produce such a data record, and also reconstruct f from the data. This reduces the upper bound on $\mathcal{S}(\epsilon, \mathbf{g}, h)$ to an upper bound on the number of allowable data records, and this bound is provided in Section 14.

13.1. The original construction and count. We begin with a summary of the proof of the upper bound in [8]. While there is nothing mathematically new in this subsection, it contains lemmas and definitions that will be used in the remainder of the paper.

In the setting of [8], we also have a closed hyperbolic 3-manifold \mathbf{M} , and we let $\mathcal{S}(\mathbf{g})$ denote the set of conjugacy classes of genus \mathbf{g} surface subgroups of $\pi_1(\mathbf{M})$. We say that a π_1 -injective map $f: S \rightarrow \mathbf{g}$ represents $\Sigma \in \mathcal{S}(\mathbf{g})$ if $\Sigma = [f_*\pi_1(S)]$ (where $[f]$ is the homotopy class of f), and observe that for each $\Sigma \in \mathcal{S}(\mathbf{g})$, there is a closed hyperbolic surface S of genus \mathbf{g} and a pleated surface (and hence 1-Lipshitz) map $f: S \rightarrow \mathbf{M}$ that represents Σ .

The upper bound in [8] then goes as follows.

Theorem 13.1. *There exists $C = C(\mathbf{M})$ such that $|\mathcal{S}(\mathbf{g})| \leq (C\mathbf{g})^{2\mathbf{g}}$.*

To prove Theorem 13.1, we begin by proving the following theorem, which appears as (or follows immediately from) Lemma 2.1 of [8]. Let $r_{\mathbf{M}}$ denote the injectivity radius of \mathbf{M} .

Theorem 13.2. *There exists an $L \equiv L(r_{\mathbf{M}})$ such that any closed hyperbolic surface S of injectivity radius at least $r_{\mathbf{M}}/2$ has a geodesic triangulation such that*

- *the edges have length at most $\min(1, r_{\mathbf{M}}/20)$,*
- *there are at most $L\mathbf{g}(S)$ vertices*
- *each vertex has degree at most L , and*
- *there are no more than L vertices in any ball of radius 1 in S .*

We will fix such an $L \equiv L(r_{\mathbf{M}}) \equiv L(\mathbf{M})$ and call such a triangulation a bounded geometry triangulation.

In Section 2.2 of [8] we also prove the following:

Lemma 13.3. *Let S be a closed hyperbolic surface of genus \mathbf{g} and let σ be a graph embedded in S . Then we can find $(\hat{\tau}, e(\tau))$ where*

- (1) *$\hat{\tau}$ is a spanning tree for σ ,*
- (2) *$e(\tau)$ is an additional set of $2\mathbf{g}$ “distinguished” edges of σ , and*
- (3) *the inclusion of $\hat{\tau} \cup e(\tau)$ into S is π_1 -surjective.*

We call $\underline{\tau} \equiv (\hat{\tau}, e(\tau))$ an *effective graph pair* for S , and let $\tau = \hat{\tau} \cup e(\tau)$. We observe that Property 3 is implied by the inclusion being surjective on H_1 , and implies that $S \setminus \tau$ is a union of disjoint polygons.

We can now abstract out the properties of an effective graph pair of a surface of genus \mathbf{g} . We say that $\underline{\tau} = (\hat{\tau}, e(\tau))$ is an (abstract) graph pair if $\hat{\tau}$ is an (abstract) tree, and $e(\tau)$ an additional set of edges whose endpoints are vertices of $\hat{\tau}$. We call $\hat{\tau}$ the *spanning tree* of $\underline{\tau}$, and $e(\tau)$ the *distinguished edges* of $\underline{\tau}$, and let $\tau = \hat{\tau} \cup e(\tau)$.

Definition 13.4. *We say that a graph pair $\underline{\tau} = (\hat{\tau}, e(\tau))$ is a \mathbf{g} -polygonalization if:*

- (1) *the set $e(\tau)$ has exactly $2\mathbf{g}$ elements,*
- (2) *the graph τ is equipped with a cyclic ordering of edges round each vertex,*
- (3) *the ribbon graph $R(\tau)$ is a surface of genus \mathbf{g} .*

We furthermore say that a \mathbf{g} -polygonalization $\underline{\tau}$ is L -bounded if τ has at most $L\mathbf{g}$ vertices, and each vertex has degree at most L .

Remark 12. Recall that the ribbon graph $R(\tau)$ is obtained by thickening the edges of the graph τ . In particular, $R(\tau)$ is a surface with (polygonal) boundary; we can then add polygons to this boundary to form a closed surface. So we can reconstruct our surface (with an effective graph pair) from the abstract \mathbf{g} -polygonalization. Moreover, if a \mathbf{g} -polygonalization is embedded into a closed surface of genus \mathbf{g} (so as to preserve the cyclic ordering around the vertices), then the complementary components are polygons, canonically associated to the boundary components of $R(\tau)$.

In [8] we then choose a cover \mathcal{D} of \mathbf{M} (with small open balls), and use it to turn the effective graph pair into combinatorial data. Accordingly, given \mathcal{D} , we have the following definition.

Definition 13.5. *We say that the pair $(\underline{\tau}, \psi)$ is a \mathcal{D} -valued \mathbf{g} -polygonalization if $\underline{\tau}$ is a \mathbf{g} -polygonalization, and $\psi: V(\tau) \rightarrow \mathcal{D}$ a map (colouring). Here $V(\tau)$ is the set of vertices of τ .*

We denote the set of all \mathcal{D} -valued $L(\mathbf{M})$ -bounded \mathbf{g} -polygonalizations by $\mathcal{T}_{\mathcal{D}}^{\mathbf{g}}$ (where $L(\mathbf{M})$ is given by Theorem 13.2).

Definition 13.6. *Suppose that $\Sigma \in \mathcal{S}(\mathbf{g})$. We say that Σ is compatible with $(\tau, \psi) \in \mathcal{T}_{\mathcal{D}}^{\mathbf{g}}$ if there exists a 2-Lipschitz map $f: S \rightarrow \mathbf{M}$ representing Σ and an associated embedding $b: \tau \rightarrow S$ such that $b(\tau)$ has bounded geometry and $f(b(v)) \in \psi(v)$ for all $v \in V(\tau) \subset S$.*

Given any $\Sigma \in \mathcal{S}(\mathbf{g})$, we can find a pleated surface map $f: S \rightarrow \mathbf{M}$ representing Σ , find a bounded geometry triangulation of S by Theorem 13.2, and then find an effective graph pair $\underline{\tau}$ for S by applying Lemma 13.3 to this triangulation. We can then define a labelling $\psi: V(\tau) \rightarrow \mathcal{D}$ by letting $\psi(v)$ be any $D \in \mathcal{D}$ that contains $f(v)$; the result is an element of $\mathcal{T}_{\mathcal{D}}^{\mathbf{g}}$ that is compatible with Σ .

We then observe the following, which was essentially shown in Section 2 of [8].

Proposition 13.7. *Suppose*

$$\max_{D \in \mathcal{D}} \text{diam } D \leq \frac{\mathbf{r}_M}{10}.$$

Then each $(\tau, \psi) \in \mathcal{T}_{\mathcal{D}}^{\mathbf{g}}$ is compatible with at most one $\Sigma \in \mathcal{S}_{\epsilon}(\mathbf{g})$.

Remark 13. Although each $(\tau, \psi) \in \mathcal{T}_{\mathcal{D}}^{\mathbf{g}}$ is compatible with at most one Σ , a given Σ can be compatible with many pairs $(\tau, \psi) \in \mathcal{T}_{\mathcal{D}}^{\mathbf{g}}$.

Proof of Proposition 13.7. Suppose, for $i = 1, 2$, that $f_i: S_i \rightarrow \mathbf{M}$ is an ϵ -nearly geodesic minimal map, and (τ, ψ) a \mathcal{D} -polygonalization which is compatible with each f_i , and let $b_i: \tau \rightarrow S_i$, $i = 1, 2$ be an associated embedding (for the compatibility with f_i). As observed in Remark 12, the complementary components to the images of the b_i are polygons, so we can find a homeomorphism $h: S_1 \rightarrow S_2$ such that $h \circ b_1 = b_2$. Moreover, for every $v \in V(\tau)$, the points $f_1(b_1(v))$ and $f_2(h(b_1(v)))$ belong to a common $D \in \mathcal{D}$. Then by Lemma 2.4 in [12], the maps f_1 and $f_2 \circ h$ are homotopic. Thus, they determine the same conjugacy class $\Sigma \in \mathcal{S}_{\epsilon}(\mathbf{g})$. \square

Since every $\Sigma \in \mathcal{S}(\mathbf{g})$ is compatible with at least one (τ, ψ) , and each (τ, ψ) is compatible with at most one Σ , it follows that

$$(59) \quad |\mathcal{S}(\mathbf{g})| \leq |\mathcal{T}_{\mathcal{D}}^{\mathbf{g}}|.$$

A simple combinatorial argument then shows that $|\mathcal{T}_{\mathcal{D}}^{\mathbf{g}}| \leq (C_{\mathbf{M}\mathbf{g}})^{2\mathbf{g}}$, which completes the proof of Theorem 13.1.

13.2. The new ingredients. We now introduce the refinements needed to prove Lemma 12.2. First, we must choose a cover \mathcal{D} , not of \mathbf{M} , but of $\mathcal{G}_2(\mathbf{M})$. Moreover, it must be a cover with wafer-thin disks, which we now make precise.

Let $(y, \Pi) \in \mathcal{G}_2(\mathbf{M})$. We define the set $\text{Hull}_{r,\epsilon}(y, \Pi) \subset \mathcal{G}_2(\mathbf{M})$, called the (r, ϵ) -hull of (y, Π) , by letting

$$\text{Hull}_{r,\epsilon}(y, \Pi) = \bigcup_f f(B_r(x)),$$

where (f, x) varies over all ϵ -nearly geodesic maps $f: S \rightarrow \mathbf{M}$ with $x \in S$ and $f(x) = (y, \Pi)$. We define the (r, ϵ) -hull of a subset $D \subset \mathcal{G}_2(\mathbf{M})$ by

$$\text{Hull}_{r,\epsilon}(D) = \bigcup_{(y,\Pi) \in D} \text{Hull}_{r,\epsilon}(y, \Pi).$$

Definition 13.8. *We say that sets $D_1, D_2 \subset \mathcal{G}_2(\mathbf{M})$ are mutually ϵ -accessible if $\text{Hull}_{1,\epsilon}(D_1) \cap D_2 \neq \emptyset$.*

The existence of our desired cover \mathcal{D} is summarized in the following, which we will prove in Section 16.

Lemma 13.9. *There exists $C \equiv C_{13.9}(r_{\mathbf{M}}) \in \mathbb{N}$ such that for all $c > 0$ there exists $h \in \mathbb{N}$ such that for all $\delta > 0$, there exist $\epsilon > 0$ and a finite open covering \mathcal{D} of $\mathcal{G}_2(\mathbf{M})$, with the following properties for all $D \in \mathcal{D}$:*

- (1) $\text{diam}_{\mathbf{M}}(\xi(D)) \leq \frac{r_{\mathbf{M}}}{10}$,
- (2) there are at most C elements of \mathcal{D} that are mutually ϵ -accessible with D ,
- (3) if $D \not\subset \mathcal{N}_{\delta}(\mathcal{H}_h)$ and $f: S \rightarrow \mathbf{M}$ is an ϵ -nearly geodesic minimal map, then $\mu_f(\overline{\text{Hull}_{1,\epsilon}(D)}) \leq c$.

A finite covering \mathcal{D} satisfying the conditions of the lemma is called a (c, δ) -covering. The crucial new property is Property 3; when $\mathcal{H}_h = \mathcal{H}$, this follows from D being a wafer-thin neighborhood of some $P_r(p)$ that is disjoint from \mathcal{H} . Here $P_r(p) \subset \mathcal{G}_2(\mathbf{M})$ denotes the geodesic disc of radius r that is tangent to the pointed plane p at 0.

We can naturally relate the properties of a cover \mathcal{D} (such as the one given in Lemma 13.9) to properties of a compatible \mathcal{D} -labeled triangulation.

Proposition 13.10. *Suppose $f: S \rightarrow \mathbf{M}$ is an ϵ -nearly geodesic minimal map, τ is a bounded geometry graph on S (e.g. a subgraph of a bounded geometry triangulation), \mathcal{D} is a cover of $\mathcal{G}_2(\mathbf{M})$, and $\psi: V(\tau) \rightarrow \mathcal{D}$ satisfies $f(v) \in \psi(v)$ for all $v \in V(\tau)$. Then*

- (1) if v_1 and v_2 are two vertices connected by an edge from τ , then $\psi(v_1), \psi(v_2) \in \mathcal{D}$ are mutually ϵ -accessible.
- (2) for any $D \in \mathcal{D}$, we have

$$|V_D| \leq 4\pi L \mathbf{g} \mu_f(\text{Hull}_{1,\epsilon}(D)),$$

where V_D is the set of vertices of τ that are mapped by ψ to D .

Proof. For Statement 1, let $D_i = \psi(v_i)$, so $f(v_i) \in D_i$, for $i = 1, 2$. Since $\mathbf{d}_S(v_1, v_2) \leq 1$ it follows that $\text{Hull}_{1,\epsilon}(D_1) \cap D_2 \neq \emptyset$. Therefore D_1 and D_2 are mutually ϵ -accessible.

Let us now prove Statement 2 for a given $D \in \mathcal{D}$. We have

$$(60) \quad \bigcup_{v \in V_D} f(B_1(v)) \subset \text{Hull}_{1,\epsilon}(D).$$

Set

$$X_D = \bigcup_{v \in V_D(\tau)} B_1(v).$$

Then (60) implies

$$(61) \quad |X_D| \leq \mu_f(\text{Hull}_{1,\epsilon}(D))|S|.$$

On the other hand, we know the number of vertices of τ in any ball of radius 1 is at most L . Therefore every point $x \in X_D$ belongs to at most L different balls $B_1(v)$, $v \in V_D$. We conclude

$$|V_D| \leq L|X_D| \leq Lc|S| \leq 4\pi cL \mathbf{g},$$

where $c := \mu_f(\text{Hull}_{1,\epsilon}(D))$. □

The second new ingredient needed to prove Lemma 12.2 is the following refinement of Lemma 13.3, which we will prove in Section 15.

Lemma 13.11. *For every $h \in \mathbb{N}$ and $0 < q \leq 1$ there exists $\delta \equiv \delta_{13.11}(h, q) > 0$ and $\epsilon_0 > 0$ with the following properties. Suppose $f: S \rightarrow \mathbf{M}$ is the minimal map representing $\Sigma \in \mathcal{S}_\epsilon(g, h, q)$. Then we can find an effective graph pair $(\hat{\tau}, e(\tau))$ for S such that $q\mathbf{g}$ of the edges in $e(\tau)$ lie outside of $S(\delta, h)$.*

13.3. Proportionately compatible polygonalizations. We now refine our notion of a compatible \mathcal{D} -valued \mathbf{g} -polygonalization as follows. Given any cover \mathcal{D} of $\mathcal{G}_2(\mathbf{M})$, we let $\mathcal{D}'_{h,q}$ be the $D \in \mathcal{D}$ which are not subsets of $\mathcal{N}_\delta(\mathcal{H}_h)$, where $\delta = \delta_{13.11}(h, q)$.

Definition 13.12. *Suppose that $\Sigma \in \mathcal{S}_\epsilon(\mathbf{g}, h, q)$. We say that $(\underline{\tau}, \psi) \in \mathcal{T}_{\mathcal{D}}^{\mathbf{g}}$ is proportionately compatible with Σ if it is compatible with Σ and there are at least $q\mathbf{g}$ edges in $e(\tau)$ whose endpoints are both mapped by ψ to elements of $\mathcal{D}'_{h,q}$.*

This is a mouthful, but we have an immediate corollary to Lemma 13.11:

Proposition 13.13. *For every $h \in \mathbb{N}$ and $q > 0$, there exists ϵ_0 such that for all $\epsilon < \epsilon_0$, any \mathcal{D} , and $\Sigma \in \mathcal{S}_\epsilon(\mathbf{g}, h, q)$, there exists $(\underline{\tau}, \psi) \in \mathcal{T}_{\mathcal{D}}^{\mathbf{g}}$ that is proportionately compatible with Σ .*

We say that $(\underline{\tau}, \psi) \in \mathcal{T}_{\mathcal{D}}^{\mathbf{g}}$ is a $(q, h, \mathbf{g}, \epsilon)$ -allowable \mathcal{D} -valued polygonalization if it is proportionately compatible with some $\Sigma \in \mathcal{S}_\epsilon(\mathbf{g}, h, q)$, and we denote by $\mathcal{T}_{\mathcal{D}}^{\mathbf{g}}(\epsilon, h, q) \subset \mathcal{T}_{\mathcal{D}}^{\mathbf{g}}$ the set of all $(q, h, \mathbf{g}, \epsilon)$ -allowable \mathcal{D} -valued polygonalizations. Since every ϵ -nearly geodesic minimal map is 2-Lipshitz for ϵ small enough, we can combine Proposition 13.7 and Proposition 13.13 to immediately conclude:

Proposition 13.14. *For any \mathcal{D} , the estimate*

$$(62) \quad |\mathcal{S}_\epsilon(\mathbf{g}, h, q)| \leq |\mathcal{T}_{\mathcal{D}}^{\mathbf{g}}(\epsilon, h, q)|$$

holds when $\epsilon < \epsilon_0(h, q)$.

We will show Lemma 12.2 in Section 14 by estimating the size of $\mathcal{T}_{\mathcal{D}}^{\mathbf{g}}(\epsilon, h, q)$ for a suitable choice of \mathcal{D} .

14. COUNTING ALLOWABLE \mathcal{D} -VALUED POLYGONALIZATIONS

Now suppose we are given $c_0 > 0$ and $q \in (0, 1]$. We let $h = h_{13.9}(c_0)$ (so h is the h provided in 13.9 that depends on the c in 13.9, that we take to be c_0), and let $\delta = \delta_{13.11}(h, q)$, and then let ϵ_0 and \mathcal{D} be given (in terms of c_0 and δ) by Lemma 13.9, and take any $\epsilon < \epsilon_0$. We will show that for any $c > 0$, and an appropriate choice of c_0 , that $|\mathcal{T}_{\mathcal{D}}^{\mathbf{g}}(\epsilon, q, h)| = o((c\mathbf{g})^{2\mathbf{g}})$ as $\mathbf{g} \rightarrow \infty$, which implies Lemma 12.2.

We let $\mathcal{D}' \subset \mathcal{D}$ be those $D \in \mathcal{D}$ for which $D \not\subset \mathcal{N}_\delta(\mathcal{H}_h)$. Before computing our upper bound for $|\mathcal{T}_{\mathcal{D}}^{\mathbf{g}}(\epsilon, q, h)|$, we will summarize the properties of every $(\tau, \psi) \in \mathcal{T}_{\mathcal{D}}^{\mathbf{g}}(\epsilon, q, h)$.

- (1) The graph τ has at most $L\mathbf{g}$ vertices.
- (2) The degree of every vertex of τ is at most L .

- (3) If v_1 and v_2 are two vertices connected by an edge from τ then $\psi(v_1), \psi(v_2) \in \mathcal{D}$ are mutually accessible.
- (4) For every $D \in \mathcal{D}'$, the number of vertices of τ which are mapped by ψ to D is at most $c_1 \mathbf{g}$, where $c_1 = 4\pi Lc_0$.
- (5) There are at least $q\mathbf{g}$ distinguished edges from the set $e(\tau)$ whose endpoints are mapped by ψ into \mathcal{D}' .

Properties 1 and 2 follow from the definition of an allowable \mathbf{g} -polygonalization. Property 3 follows from Statement 1 in Proposition 13.10, and Property 4 follows from Property 3 of \mathcal{D} , the definition of \mathcal{D}' , and Statement 2 of Proposition 13.10. Property 5 follows immediately from the definitions of $\mathcal{T}_{\mathcal{D}}^{\mathbf{g}}(\epsilon, q, h)$ and \mathcal{D}' .

We can obtain every \mathcal{D} -valued \mathbf{g} -polygonalization of bounded geometry $(\tau, \psi) \in \mathcal{T}_{\mathcal{D}}^{\mathbf{g}}(\epsilon, q, h)$ as follows:

- First choose a spanning tree $\hat{\tau}$.
- Then chose a colouring $\psi: V(\hat{\tau}) \rightarrow \mathcal{D}$.
- Then add the $2\mathbf{g}$ distinguished edges to the $\hat{\tau}$.
- Then equip the graph τ with the cyclic ordering around each vertex.

Of course our choices will be constrained by Properties 1–5 listed above. Let us write down the estimate that comes out from the description above:

$$(63) \quad |\mathcal{T}_{\mathcal{D}}^{\mathbf{g}}(\epsilon, q, h)| \leq A_1 A_2 A_3 A_4,$$

where

- A_1 is the number of different trees T which agree with the tree $\hat{\tau}$ of $\mathcal{T} = (\hat{\tau}, e)$ of some pair $(\mathcal{T}, \psi) \in \mathcal{T}_{\mathcal{D}}^{\mathbf{g}}(\epsilon, q, h)$,
- A_2 is the (maximal) number of ways we can define the colouring $\psi: V(\hat{\tau}) \rightarrow \mathcal{D}$ for a fixed $\hat{\tau}$ (here $\hat{\tau}$ is a spanning tree of some pair $(\mathcal{T}, \psi) \in \mathcal{T}_{\mathcal{D}}^{\mathbf{g}}(\epsilon, q, h)$),
- A_3 is the (maximal) number of different pairs $(\mathcal{T}, \psi) \in \mathcal{T}_{\mathcal{D}}^{\mathbf{g}}(\epsilon, q, h)$ which share the same spanning tree $\hat{\tau}$ and colouring ψ ,
- A_4 is the number of cyclic orderings of edges of τ .

14.1. Estimating A_1, A_2, A_3 , and A_4 . We will compute upper bounds for A_1, A_2, A_3 , and A_4 in turn.

The number of different unlabelled trees on n vertices is $\leq C_1 12^n$, for some universal constant $C_1 > 0$ (for example see [8]). It follows from Property 1 that

$$A_1 \leq C_1 12^{L\mathbf{g}}.$$

Next, we bound A_2 . Thus, we consider a fixed allowable tree $\hat{\tau}$, and estimate the number of allowable maps $\psi: V(\hat{\tau}) \rightarrow \mathcal{D}$. Choose a base vertex $v_0 \in V(\hat{\tau})$, and record $\hat{\tau}$ as a rooted tree. Then every vertex $v \in V(\hat{\tau})$ (besides the base vertex) has a predecessor denoted by v' .

There are at most $|\mathcal{D}|$ choices for the value of $\psi(v_0)$. On the other hand, if the value $\psi(v')$ has already been specified, then there are at most C_2 possible values for $\psi(v)$, where $C_2 = C_{13.9}$. This is because the open sets

$\psi(v), \psi(v') \in \mathcal{D}$ are mutually accessible by Property 3. We conclude that there are at most $C_2^{|V(\hat{\tau})|} \leq C_2^{L\mathbf{g}}$ ways of defining ψ on the set $V(\hat{\tau}) \setminus \{v_0\}$. Putting this all together yields the bound

$$A_2 \leq |\mathcal{D}|C_2^{L\mathbf{g}}.$$

We will now bound A_3 . We fix an allowable tree $\hat{\tau}$, and an allowable map $\psi: V(\hat{\tau}) \rightarrow \mathcal{D}$. By Properties 5 and 3, we can generate all $(\mathbf{g}, \epsilon, h, q)$ -allowable sets of distinguished edges $e(\tau)$ by generating a set e_1 of $\lceil q\mathbf{g} \rceil$ edges that connect two mutually accessible D 's in \mathcal{D}' , and then generating a set e_2 of $2\mathbf{g} - \lceil q\mathbf{g} \rceil$ edges that connect mutually accessible D 's in \mathcal{D} .

When we choose an edge in e_1 , there are $|V(\tau)|$ ways to choose the first vertex v . By Property 3 above and Property 2 of \mathcal{D} (from Lemma 13.9), there are at most C_2 ways to choose the label $\psi(w) \in \mathcal{D}'$ of the second vertex w , and by Property 4 above there are at most $c_1\mathbf{g}$ choices for w given its label. Therefore there are at most $(|V(\tau)|c_1C_2\mathbf{g})^{\lceil q\mathbf{g} \rceil}$ choices of sets e_1 along with an ordering of the edges, and hence at most $(|V(\tau)|c_1C_2\mathbf{g})^{\lceil q\mathbf{g} \rceil} / \lceil q\mathbf{g} \rceil!$ ways of choosing the set e_1 . Similarly, but without any constraints on the choice of the second vertex, there are $(|V(\tau)|^2)^{2\mathbf{g} - \lceil q\mathbf{g} \rceil} / (2\mathbf{g} - \lceil q\mathbf{g} \rceil)!$ ways to choose e_2 . We therefore have

$$A_3 \leq \frac{(|V(\tau)|c_1C_2\mathbf{g})^{\lceil q\mathbf{g} \rceil} (|V(\tau)|^2)^{2\mathbf{g} - \lceil q\mathbf{g} \rceil}}{\lceil q\mathbf{g} \rceil!(2\mathbf{g} - \lceil q\mathbf{g} \rceil)!} \leq \frac{(c_1C_2L\mathbf{g}^2)^{\lceil q\mathbf{g} \rceil} (L^2\mathbf{g}^2)^{2\mathbf{g} - \lceil q\mathbf{g} \rceil} 2^{2\mathbf{g}} 4^{2\mathbf{g}}}{\mathbf{g}^{2\mathbf{g}}}$$

where we used the following estimates

$$|V(\tau)| \leq L\mathbf{g} \quad \left(\frac{\mathbf{g}}{4}\right)^{2\mathbf{g}} < (2\mathbf{g})! \quad \frac{(a+b)!}{a!b!} \leq 2^{a+b}.$$

Compiling like terms we get

$$A_3 \leq C_4^{\mathbf{g}} c_1^{q\mathbf{g}} \mathbf{g}^{2\mathbf{g}}$$

for some constant $C_4 > 0$.

It remains to bound A_4 . We easily find (as in [8])

$$A_4 \leq (\deg(\tau))!^{|V(\tau)|} \leq (L!)^{L\mathbf{g}}.$$

Now, replacing these estimates in (63) we get

$$|\mathcal{T}_{\mathcal{D}}^{\mathbf{g}}(\epsilon, q, h)| \leq C_5^{\mathbf{g}} |\mathcal{D}| c_1^{q\mathbf{g}} \mathbf{g}^{2\mathbf{g}}$$

for some constant C_5 , and every small enough ϵ . Then for every $c > 0$, and every fixed $c_1 > 0$, we have

$$\frac{|\mathcal{T}_{\mathcal{D}}^{\mathbf{g}}(\epsilon, q, h)|}{(c\mathbf{g})^{2\mathbf{g}}} \leq \frac{C_5^{\mathbf{g}} |\mathcal{D}| c_1^{q\mathbf{g}} \mathbf{g}^{2\mathbf{g}}}{(c\mathbf{g})^{2\mathbf{g}}} = \left(C_5 (1 + o(1)) \frac{c_1^q}{c^2} \right)^{\mathbf{g}}$$

where $o(1)$ is a function of \mathbf{g} which tends to zero when $\mathbf{g} \rightarrow \infty$. Taking $c_1 < (c^2/C_5)^{1/q}$, we obtain the desired result.

15. PROOF OF LEMMA 13.11

In [8] the authors constructed a bounded geometry triangulation, and then found an embedded graph pair and resulting \mathcal{D} -valued polygonalization. Here we assume that $|S(\delta_0(h), h)| \leq (1 - q)|S|$, and use it to construct an embedded graph pair with at least qg edges lying outside of $S(\delta)$, for some δ that will depend only on \mathcal{H}_h and q . We will first provide an upper bound for the portion of $H_1(S)$ that is generated by $S(\delta)$, and then use that bound when we build the graph pair. In what follows, we will assume we are given a value of h and write δ_0 for $\delta_0(h)$ and $S(\delta)$ for $S(\delta, h)$.

15.1. The size of $\partial S(\delta)$. As usual, $f: S \rightarrow \mathbf{M}$ denotes an ϵ -nearly geodesic minimal map. We first prove a version of Proposition 11.1 where the resulting subsurface has small boundary compared to $|S|$.

Proposition 15.1. *For any $\alpha > 0$ there exist $\delta > 0$, $\epsilon_0 > 0$, such that for every ϵ -nearly geodesic minimal surface $f: S \rightarrow \mathbf{M}$ (with $\epsilon < \epsilon_0$), there exists a subsurface $R \subset S$ which satisfies*

- (1) $\mathcal{N}_{10}(S(\delta)) \subset R \subset S(\delta_0)$,
- (2) $|\partial R| \leq \alpha|S|$.

Proof. We let $n = \lceil 25/\alpha \rceil$.

Inductively applying Proposition 2.5, we see that there exists $\epsilon_0 = \epsilon_0(n) = \epsilon(\alpha)$ and $\delta_{2n} < \delta_{2n-1} < \dots < \delta_1 < \delta_0$, such that for every ϵ -nearly geodesic minimal surface $f: S \rightarrow \mathbf{M}$ with $\epsilon < \epsilon_0$, we have $\mathcal{N}_{10}(S(\delta_{k+1})) \subset S(\delta_k)$ (for $0 \leq k < 2n$), and hence, for $0 < k < 2n$,

$$(64) \quad \mathcal{N}_{10}(\partial(S(\delta_k))) \subset S(\delta_{k-1}) \setminus S(\delta_{k+1}).$$

We show that the proposition holds for $\delta := \delta_{2n}$ and the ϵ_0 we have chosen.

Notice that the sets $S(\delta_{k-1}) \setminus S(\delta_{k+1})$ are mutually disjoint for different odd values of k . Since there are n such sets, at least one of them has the area at most $|S|/n$. Thus we can choose l odd such that $|S(\delta_{l-1}) \setminus S(\delta_{l+1})| \leq |S|/n$. By Lemma 2.7 (applied to $S(\delta_l)$) and our choices of the δ_k , l , and α , we can find R such that

$$\mathcal{N}_{10}(S(\delta_{2n})) \subset S(\delta_l) \subset R \subset S(\delta_{l-1}) \subset S(\delta_0)$$

and

$$|\partial R| \leq 25|S|/n \leq \alpha|S|.$$

□

15.2. The homology of $S(\delta)$. Let S denote a closed hyperbolic surface. By $\text{sys}(S)$ we denote the length of shortest closed geodesic on S (note that $\text{sys}(S) = 2\mathbf{r}(S)$). If $T \subset S$ is closed, we let $b_1(T \rightarrow S)$ denote the dimension of the image of $H_1(T; \mathbb{R})$ in $H_1(S; \mathbb{R})$. If $R \subset S$ is a subsurface we let $\#\text{Comp}(\partial R)$ denote the number of components of the boundary ∂R of R . We begin with the following corollary of the Gauss-Bonnet theorem and the isoperimetric inequality.

Proposition 15.2. *There exists a constant $C > 0$ with the following properties. Let S be a closed hyperbolic Riemann surface, and suppose $R \subsetneq S$ is a subsurface with smooth boundary. Then*

$$(65) \quad b_1(R \rightarrow S) \leq \frac{|R| + |\partial R|}{2\pi} + \# \text{Comp}(\partial R)$$

$$(66) \quad \leq \frac{|R|}{2\pi} + \left(1 + \frac{1}{\text{sys}(S)}\right) |\partial R|.$$

Proof. It suffices to prove (65) for each component of R , so we will assume that R is connected. First suppose that R has geodesic boundary. By Gauss-Bonnet we have

$$(67) \quad b_1(R \rightarrow S) = b_1(R) = -\chi(R) + 1 = \frac{|R|}{2\pi} + 1 \leq \frac{|R|}{2\pi} + \# \text{Comp}(R)$$

since ∂R is nonempty.

For the case of a general R , let $\Omega \subset S$ be convex core of R , i.e. the surface with geodesic boundary obtained by filling in the holes (contractible components of the complement) of R , and replacing each boundary curve of R with its geodesic representative. (Note that we may have $\Omega \not\subset R$.) On one hand, by Proposition 2.12 we know that

$$(68) \quad |\Omega| \leq |R| + |\partial R|.$$

On the other hand, $b_1(R \rightarrow S) = b_1(\Omega \rightarrow S)$ because filling in the holes (and homotoping the boundary) does not change the image of H_1 in $H_1(S)$. Combining this with (67) and (68), we get

$$b_1(R \rightarrow S) \leq \frac{|\Omega|}{2\pi} + \# \text{Comp}(\partial\Omega) \leq \frac{|R| + |\partial R|}{2\pi} + \# \text{Comp}(\partial R). \quad \square$$

Combining Proposition 15.1 and Proposition 15.2 enables us to estimate the homology of $S(\delta)$.

Proposition 15.3. *For every $\eta > 0$ there exist $\delta(\eta) > 0$, $\epsilon_0(\eta) > 0$, such that for every ϵ -nearly geodesic minimal surface $f: S \rightarrow \mathbf{M}$, assuming $\epsilon < \epsilon_0(\eta)$, there is a subsurface (with smooth boundary) $R_\eta \subset S$ for which*

$$(1) \quad \mathcal{N}_{10}(S(\delta(\eta))) \subset R_\eta \subset S(\delta_0),$$

$$(2) \quad b_1(R_\eta \rightarrow S) \leq \frac{|R_\eta|}{2\pi} + \eta|S|.$$

Proof. Let $\alpha = \eta/(1 + 2/\text{sys}(\mathbf{M}))$, let $\delta(\eta)$ be the $\delta(\alpha)$ from Proposition 15.1, and let $\epsilon_0(\eta)$ be the minimum of $\epsilon_0(\alpha)$ given by Proposition 15.1 and a universal ϵ_1 given below.

Suppose we are given $f: S \rightarrow \mathbf{M}$ that is $\epsilon_0(\eta)$ -nearly geodesic, with $\epsilon_0(\eta) \leq \epsilon_0(\alpha)$. Let $R = R(\alpha)$ be the surface from the statement of Proposition 15.1. First, from (66) we get

$$(69) \quad b_1(R \rightarrow S) \leq \frac{|R|}{2\pi} + \left(1 + \frac{1}{\text{sys}(S)}\right) |\partial R| \leq \frac{|R|}{2\pi} + \left(1 + \frac{2}{\text{sys}(\mathbf{M})}\right) |\partial R|$$

because $\mathbf{sys}(S) \geq \mathbf{sys}(\mathbf{M})/2$ when $\epsilon_0 \leq \epsilon_1$, where ϵ_1 is a universal constant. We now apply the estimate (2) from Proposition 15.1, and get

$$(70) \quad |\partial R| \leq \alpha|S|.$$

Putting together (69), (70), and the definition of α , we obtain Property (2) of this proposition. Property 1 follows immediately from Property 1 of Proposition 15.1. \square

15.3. Proof of Lemma 13.11. Fix an ϵ -nearly geodesic minimal map $f: S \rightarrow \mathbf{M}$. We may assume $\mathbf{sys}(\mathbf{M}) \leq 2\mathbf{sys}(S)$. By Theorem 13.2 we can find a bounded geometry triangulation T of S .

Let $\eta = \frac{q}{4\pi}$. Let $R = R_\eta$ be the subsurface from Proposition 15.3, and let $\delta \equiv \delta(q, h) = \delta(\eta)$. Then from (1) in Proposition 15.3 we get $R \subset S(\delta_0)$ which implies

$$(71) \quad |R| \leq (1 - q)|S|.$$

We let T_0 be the vertices and edges of T that lie entirely in R . We can then extend T_0 to a connected subgraph T_1 of T that includes all the vertices of T ; any minimal such extension will add no new cycles to those of T_0 , and hence we'll have $b_1(T_1 \rightarrow S) = b_1(T_0 \rightarrow S)$. We observe that

$$\begin{aligned} b_1(T_1 \rightarrow S) &= b_1(T_0 \rightarrow S) \leq b_1(R \rightarrow S) \\ &\leq \frac{|R|}{2\pi} + \eta|S| \leq \frac{(1 - q)|S|}{2\pi} + \eta|S| \\ &\leq \left((1 - q) + \frac{q}{2}\right)2(\mathbf{g} - 1) \\ &\leq (2 - q)\mathbf{g}. \end{aligned}$$

Here we used (2) from Proposition 15.3 in the second inequality, (71) in the third inequality, and the choice of η in the fourth. Thus, we have shown

$$(72) \quad b_1(T_1 \rightarrow S) \leq (2 - q)\mathbf{g}.$$

We then let \widehat{T} be a spanning tree for T_1 . Then, as in [8], we find a set $e(T)$ of $2\mathbf{g}$ edges of $T \setminus \widehat{T}$ such that $b_1(e(T) \cup \widehat{T} \rightarrow S) = 2\mathbf{g}$. Let $e_0(T) \subset e(T)$ be the edges that are contained in R . Then $(\widehat{T} \cup e_0(T)) \subset T_1$, and by (72) we have

$$|e_0(T)| = b_1(\widehat{T} \cup e_0(T) \rightarrow S) \leq b_1(T_1 \rightarrow S) \leq (2 - q)\mathbf{g}.$$

Letting $e_1(T)$ be $e(T) \setminus e_0(T)$, we have

$$(73) \quad |e_1(T)| \geq q\mathbf{g}.$$

We now let $\widehat{\tau} = \widehat{T}$, and $e(\tau) = e(T)$. In light of (73) we have thus produced a bounded geometry embedding of τ where at least $q\mathbf{g}$ distinguished edges have the property required by the Lemma.

16. RESCALED RIEMANNIAN METRICS AND A COVER WITH WAFER-THIN DISKS

16.1. **A cover with wafer-thin disks.** In this section we use a rescaling of the Riemannian metric on $\mathcal{G}_2(\mathbf{M})$ to construct the cover \mathcal{D} needed for Lemma 13.9.

We will write \mathcal{G}_2 when the given operation applies to both $\mathcal{G}_2(\mathbf{M})$ and $\mathcal{G}_2(\mathbb{H}^3)$. In \mathcal{G}_2 we have already defined an invariant Riemannian metric which we will just denote by $\langle \cdot, \cdot \rangle$. We also have the tautological foliation \mathcal{F} of \mathcal{G}_2 by hyperbolic planes. This gives us an invariant splitting $T\mathcal{G}_2 = T\mathcal{F} \oplus T\mathcal{F}^\perp$. If $v \in T\mathcal{G}_2$, we say that v respects the splitting if $v \in T\mathcal{F} \cup T\mathcal{F}^\perp$. If $v_0, v_1 \in T\mathcal{G}_2$ respect the splitting, we let $m(v_0, v_1)$ be the number of $i \in \{0, 1\}$ such that $v_i \in T\mathcal{F}^\perp$. We now define $\langle \cdot, \cdot \rangle_\eta$ by

$$\langle v, w \rangle_\eta = \eta^{-m(v,w)} \langle v, w \rangle$$

whenever v and w respect the splitting. Thus the norm of a vector in $T\mathcal{F}$ is unchanged, while the norm of a vector in $T\mathcal{F}^\perp$ is rescaled by η^{-1} . We call this Riemannian metric (and associated point-pair metric d_η) the η -metric. We note that $\langle v, w \rangle_\eta$ and $d_\eta(p, q)$ are both nondecreasing functions of η . We let $B_r^\eta(p)$ be the r -ball in the η -metric around a pointed plane p . We let $|B_r^\eta(p)|_\eta$ denote the volume of this ball in the η -metric (when it is embedded), and we observe that

$$(74) \quad |B_r^\eta(p)| = \eta^3 |B_r^\eta(p)|_\eta,$$

where the left-hand side is the volume of the same ball in the original metric.

We can prove the following lemma with a simple compactness argument.

Lemma 16.1. *For any $r < r'$ and $\eta > 0$, there exists ϵ such that for $p \in \mathcal{G}_2$,*

$$(75) \quad \overline{\text{Hull}_{r,\epsilon}(p)} \subset B_{r'}^\eta(p),$$

and hence, for sufficiently small $\epsilon \equiv \epsilon(r_0, r_1, r')$,

$$(76) \quad \text{Hull}_{r_0,\epsilon}(B_{r_1}^\eta(p)) \subset B_{r'}^\eta(p)$$

whenever $r' > r_0 + r_1$.

Proof. The second statement follows from the first, so we will just prove the first. Because of the homogeneity, we can assume that p is at the origin of \mathbb{H}^3 . Suppose that for a given $r < r'$ and η , there is no ϵ . Then we have $\bigcap_{\epsilon > 0} \overline{\text{Hull}_{r,\epsilon}(p)} \not\subset B_{r'}^\eta(p)$, because $\text{Hull}_{r,\epsilon}(p)$ is nested as ϵ decreases to 0, and has compact closure. On the other hand,

$$\bigcap_{\epsilon > 0} \overline{\text{Hull}_{r,\epsilon}(p)} = \overline{P_r(p)} \subset B_{r'}^\eta(p). \quad \square$$

The following lemma may seem obvious, but requires a little thought to prove, so the proof is included.

Lemma 16.2. *For any $R, \delta > 0$ there exists an $\eta_0 > 0$ such that $B_r^\eta(p) \subset \mathcal{N}_\delta(P_r(p))$ for all $r \leq R$ and $\eta \leq \eta_0$.*

Proof. We work in $\mathcal{G}_2(\mathbb{H}^3)$; it is then a simple matter to conclude the same for $\mathcal{G}_2(\mathbf{M})$.

For a sufficiently small u , we can form the radius u orthogonal hyper-disk V to $P_r(p)$ at p with the exponential map (for the standard metric) applied to the orthogonal complement to the tangent subspace in $\mathcal{G}_2(\mathbf{M})$ of $P_r(p)$. We can define a projection π to V on every hyperbolic plane in $\mathcal{G}_2(\mathbb{H}^3)$ that intersects V (where we project each plane to its intersection with V). We then let $B' = B_R(p) \cap \pi^{-1}(V)$. We observe that, by compactness of $\overline{B'}$, there exists $\alpha > 0$ such that $\|w\| \geq \alpha \|D\pi(w)\|$ for every vector w based at a point in B' and orthogonal to \mathcal{F} . Then for any path γ in B' , we have

$$(77) \quad l_\eta(\gamma) \geq \frac{\alpha}{\eta} l(\pi(\gamma)).$$

Therefore² any path γ starting at p of length at most R lies in B' when $\eta < \alpha u/R$, and $\pi(\gamma)$ then has length at most $\eta R/\alpha$. So for η sufficiently small, every point in $B_R^\eta(p)$ is connected to p by a path of length R that lies in the preimage by π of a small ball around p in V . This must then lie in $\mathcal{N}_\delta(P_R(p))$ ³.

For $r \in [\delta, R]$ we can apply the same reasoning, and obtain a uniform η_0 by compactness. For $r < \delta$, we can take $\eta_0 = 1$. \square

We now proceed to constructing the desired cover \mathcal{D} with balls in the η -metric. We first observe

Lemma 16.3. *The sectional curvatures of \langle, \rangle_η are uniformly bounded for $\eta \in (0, 1]$.*

Corollary 16.4. *There are functions $c(r)$ and $C(r)$ such that for all $\eta \in (0, 1]$ and $r > 0$,*

$$(78) \quad c(r) \leq |B_r^\eta(p)|_\eta \leq C(r).$$

The proofs of Lemma 16.3 and Corollary 16.4 involve a modicum of Riemannian geometry and are deferred until Section 16.2.

Corollary 16.5. *For any $r, R \in \mathbb{R}^+$, there is an $N \equiv N(r, R)$ such that for all $\eta \in (0, 1]$, any R -ball in the η -metric contains at most N disjoint r -balls (in the η -metric).*

Proof. We let $N(r, R) = C(R)/c(r)$ where C and c are taken from Corollary 16.4. \square

We can now easily prove the main estimate for this section:

Theorem 16.6. *For all $r, R \in \mathbb{R}^+$, there is an $N \in \mathbb{Z}^+$ such that for all $\eta \in (0, 1]$, there is a finite set $F \subset \mathcal{G}_2(\mathbf{M})$ such that $\mathcal{G}_2(\mathbf{M}) \subset \bigcup_{p \in F} B_r^\eta(p)$, and every point of $\mathcal{G}_2(\mathbf{M})$ lies in $B_R^\eta(p)$ for at most N distinct $p \in F$.*

²By “continuous induction”

³Because the hyperbolic planes in $\mathcal{G}_2(\mathbb{H}^3)$ are locally length-minimizing.

Proof. Given r, R and η , by Lemma 2.6, we can find a finite set $F \subset \mathcal{G}_2(\mathbf{M})$ such that $B_{r/2}(p)$ for $p \in F$ are mutually disjoint, and $\mathcal{G}_2(\mathbf{M}) \subset \bigcup_{p \in F} B_r^\eta(p)$.

For any $q \in \mathcal{G}_2(\mathbf{M})$, if $q \in B_R^\eta(p)$, then $B_{r/2}(p) \subset B_{R+r}(q)$. Therefore q lies in $B_R^\eta(p)$ for at most $N(r/2, r+R)$ distinct $p \in F$, where $N(\cdot, \cdot)$ is given by Corollary 16.5. \square

Equipped with this theorem we can now prove Lemma 13.9.

Proof of Lemma 13.9. We take $r \in (0, 1]$ such that $\text{diam } \xi(B_r(p)) \leq r_{\mathbf{M}}/10$, and let $R = 4$. We then let $C_{13.9} = N_{16.6}(r, R)$. We then observe that for any $\eta > 0$ we can apply Theorem 16.6 to obtain a finite cover \mathcal{D} by disks of the form $B_r^\eta(p)$ (for p in a finite set F) that satisfies Properties 1 and 2 of Lemma 13.9 for ϵ sufficiently small. Property 1 is immediate. To see Property 2, suppose that

$$(79) \quad \text{Hull}_{1,\epsilon}(B_r^\eta(p)) \cap B_r^\eta(q) \neq \emptyset.$$

Since $\text{Hull}_{1,\epsilon}(B_r^\eta(p)) \subset B_3^\eta(p)$ for ϵ sufficiently small, (79) implies that $B_r^\eta(q) \subset B_4^\eta(p)$. But the conclusion of Theorem 16.6 is that this can happen, given $p \in F$, for at most C values of q in F .

So now we need only find $h \in \mathbb{N}$, $\eta > 0$, and $\epsilon > 0$ that satisfy Property 3. Given c , we let $h = \lceil 2/c \rceil$. We then claim that for $D = B_r^\eta(p) \not\subset \mathcal{N}_\delta(\mathcal{H}_h)$, and $f: S \rightarrow \mathbf{M}$ an ϵ -nearly geodesic map, we will have $\mu_f(\overline{\text{Hull}_{1,\epsilon}(D)}) \leq c$ for η and ϵ sufficiently small given $\delta > 0$ (and independent of p).

Note that $\text{Hull}_{1,\epsilon}(B_r^\eta(p)) \subset B_2^\eta(p)$ when ϵ is sufficiently small given η (we can always choose ϵ after η), and, assume, for the sake of contradiction, that there is a $\delta > 0$ and sequences $\eta_n \rightarrow 0$, $p_n \in \mathcal{G}_2(\mathbf{M}) \setminus \mathcal{N}_\delta(\mathcal{H}_h)$, and ϵ_n -nearly geodesic maps $f_n: S_n \rightarrow \mathbf{M}$, where $\epsilon_n \rightarrow 0$, such that

$$\mu_{f_n}(B_2^{\eta_n}(p_n)) \geq c.$$

Passing to a subsequence, we may assume that $\mu_{f_n} \rightarrow \mu$ and $p_n \rightarrow p$ with $p \notin \mathcal{H}_h$; by Lemma 16.2, $\overline{B_2^{\eta_n}(p_n)} \rightarrow \overline{P_2(p)}$ in the Hausdorff metric. Letting $P = P_2(p)$ we then have

$$(80) \quad \mu(P) \geq c.$$

Since μ is $\mathbf{PSL}(2, \mathbb{R})$ invariant, by Ratner's theorem we can write $\mu = \mu_{\mathcal{H}_h} + \mu_{\mathcal{H}'_h} + \mu_{\mathcal{L}}$, where $\mu_{\mathcal{H}_h}$ and $\mu_{\mathcal{H}'_h}$ are supported on \mathcal{H}_h and \mathcal{H}'_h respectively, and $\mu_{\mathcal{L}}$ is a multiple of the Liouville measure. Since $p \notin \mathcal{H}_h$, we must have $P \cap \mathcal{H}_h = \emptyset$, and hence

$$(81) \quad \mu_{\mathcal{H}_h}(P) = 0.$$

We also have

$$(82) \quad \mu_{\mathcal{L}}(P) = 0$$

because P is infinitely thin. Finally, letting $|P| = 2\pi(\cosh 2 - 1) < 8\pi$ be the area of the hyperbolic disk of radius 2, we have

$$(83) \quad \mu_{\mathcal{H}'_h}(P) \leq \frac{|P|}{4\pi h} < 2/h \leq c$$

because P can only intersect one totally geodesic surface in \mathcal{H}'_h , and it represents at most the given fraction of the total area of that surface. Combining (81), (82), and (83), we obtain $\mu(D) < c$, a contradiction. \square

16.2. Estimates of curvature and volume for the η -metric. We observe that around every point p of \mathcal{G}_2 we can find an orthonormal basis (e_i) of vector fields in $T\mathcal{G}_2$ of a neighborhood U of p that respect the splitting. (This basis will not actually be invariant under the isometry group of \mathbb{H}^3 , but this will not be a problem.) We then define the *structure functions* α_{ijk} by $\alpha_{ijk} = \langle [e_i, e_j], e_k \rangle$. Because $\langle e_i, e_j \rangle = \delta_{ij}$ is constant, the Koszul formula for $\langle \nabla_{e_i} e_j, e_k \rangle$ will just be a sum and difference of the α_{ijk} . It then follows that the curvature tensor R^{ijkl} is bounded when the α_{ijk} are, and then the sectional curvatures are as well.

So, we take $e_i^\eta = e_i$ for $e_i \in T\mathcal{F}$, and $e_i^\eta = \eta e_i$ for $e_i \in T\mathcal{F}^\perp$, so that (e_i^η) is an invariant orthonormal basis for the η -metric (and respects the splitting). We let α_{ijk}^η be the corresponding structure constants. We then observe (letting $m(i, j) = m(e_i, e_j)$)

- (1) $\alpha_{ijk}^\eta = 0$ when $e_i, e_j \in T\mathcal{F}$, and $e_k \in T\mathcal{F}^\perp$,
- (2) $\alpha_{ijk}^\eta = \eta^{m(i,j)-1} \alpha_{ijk}$ when $e_k \in T\mathcal{F}^\perp$, and
- (3) $\alpha_{ijk}^\eta = \eta^{m(i,j)} \alpha_{ijk}$ when $e_k \in T\mathcal{F}$.

Hence the exponent for η is non-negative whenever α_{ijk} is non-zero. This completes the proof of Lemma 16.3.

Corollary 16.4 follows immediately from Lemma 16.3 and the following comparison theorem (see Corollary 3 in Section 11.9 of the book [3]).

Theorem 16.7. *Suppose all sectional curvatures have absolute value less than or equal to K . Then the volume of any r -ball $B_r(p)$ satisfies*

$$|B_r(p)| \leq V(r, -K),$$

and also, for $r < \pi/K$,

$$|B_r(p)| \geq V(r, K),$$

where $V(r, K)$ denotes the volume of the r -ball in constant curvature K .

Theorem 16.7 is an immediate corollary to the Rauch comparison theorem. More precisely, Theorem 16.7 follows from the analogous comparison theorem for r -spheres in the two metrics; this in turn follows from the analogous comparison between the volume forms of the Riemannian metrics on the r -spheres in the natural coordinates (parametrized by the unit spheres in the tangent spaces at the centers of the balls). The comparison of the volume forms follows from the comparison of the norms, and this is literally a special case of the Rauch comparison theorem.

REFERENCES

- [1] F. Al Assal, *Limits of asymptotically Fuchsian surfaces in a closed hyperbolic 3-manifold*. arXiv:2309.02164

- [2] U. Bader, D. Fisher, N. Miller, M. Stover, *Arithmeticity, superrigidity and totally geodesic submanifolds*. Annals of Mathematics volume 193, pages 837-861 (2021)
- [3] R. Bishop, R. Crittenden, *Geometry of manifolds*. Pure and Applied Mathematics, Vol. XV. Academic Press, New York-London, (1964)
- [4] D. Calegari, F. Marques and A. Neves, *Counting minimal surfaces in negatively curved 3-manifolds*. Duke Math. J. 171 (2022), 1615-1648.
- [5] D. Epstein, A. Marden, V. Markovic, *Quasiconformal homeomorphisms and the convex hull boundary*. Annals of Math. (2) 159 (2004), no. 1, 305-336.
- [6] Z. Huang, B. Lowe, A. Seppi, *Uniqueness and non-uniqueness for the asymptotic Plateau problem in hyperbolic space*. arXiv:2309.00599
- [7] J. Kahn, V. Marković, *Immersing almost geodesic surfaces in a closed hyperbolic three manifold*. Annals of Mathematics 175 (3), pp. 1127-1190
- [8] J. Kahn, V. Marković, *Counting essential surfaces in a closed hyperbolic three-manifold*. Geom. Topol.16 (2012), no.1, 601-624.
- [9] F. Kassel, *Groupes de surface dans les réseaux des groupes de Lie semi-simples*. Seminaire Bourbaki 74e annee, 2021-2022, no 1181 (2021)
- [10] F. Labourie, *Asymptotic counting of minimal surfaces in hyperbolic 3-manifolds*. Seminaire Bourbaki 73e annee, 2020-2021, no 1179 (2021)
- [11] B. Lowe, A. Neves, *Minimal Surface Entropy and Average Area Ratio*. arXiv:2110.09451
- [12] J. Masters, *Counting immersed surfaces in hyperbolic 3-manifolds*. Algebraic Geometry Topology 5 (2005), 835-864
- [13] J. Milnor, *Curvatures of left invariant metrics on Lie groups*. Advances in Math. 21 (1976), no. 3, 293-329.
- [14] A. Mohammadi, G. Margulis, *Arithmeticity of hyperbolic 3-manifolds containing infinitely many totally geodesic surfaces*. Ergodic Theory Dynam. Systems 42 , no.3, 1188-1219 (2022)
- [15] T. Müller, J-C. Puchta, *Character theory of symmetric groups and subgroup growth of surface groups*. Journal London Math. Soc. (2) 66 (2002) 623-640
- [16] Z. Rao, *Subgroups of Genus-2 Quasi-Fuchsian groups and Cocompact Kleinian Groups*. arXiv:2302.01995
- [17] E. Schmidt, *Beweis der isoperimetrischen Eigenschaft der Kugel im hyperbolischen und spharischen Raum jeder Dimensionenzahl*. Mathematische Zeitschrift volume 49, pages 1-109 (1943)
- [18] A. Seppi, *Minimal discs in hyperbolic space bounded by a quasicircle at infinity*. Comment. Math. Helv. 91 (4), pp. 807-839 (2016)
- [19] K. Uhlenbeck, *Closed minimal surfaces in hyperbolic 3-manifolds*. Vol. 103. Ann. of Math. Stud. Princeton Univ. Press, Princeton, NJ, pp. 147-168 (1983)

MATHEMATICS DEPARTMENT
BROWN UNIVERSITY
PROVIDENCE, RI 02912, USA
Email address: `jeremy_kahn@brown.edu`

ALL SOULS COLLEGE
UNIVERSITY OF OXFORD
UNITED KINGDOM
Email address: `markovic@maths.ox.ac.uk`

MATHEMATICAL INSTITUTE
UNIVERSITY OF OXFORD
UNITED KINGDOM
Email address: `ilia.smilga@normalesup.org`