February 29, 2024

TANGLE FREE PERMUTATIONS AND THE PUTMAN-WIELAND PROPERTY OF RANDOM COVERS

ADAM KLUKOWSKI AND VLADIMIR MARKOVIĆ

ABSTRACT. Let Σ_g^p denote a surface of genus g and with p punctures. Our main result is that the fraction of degree n covers of Σ_g^p which have the Putman-Wieland property tends to 1 as $n \to \infty$. In addition, we show that the monodromy of a random cover of Σ_g^p is asymptotically almost surely tangle free.

1. Introduction

1.1. **The Putman-Wieland conjecture.** Let Σ_g^p denote a smooth surface of genus $g \geq 2$ with $p \geq 0$ points removed (which we call cusps or punctures). Once and for all, we fix a basepoint $\star \in \Sigma_g^p$ and consider the fundamental group $\pi_1(\Sigma_g^p, \star)$. Denote by Mod_g^p the corresponding pure mapping class group.

By considering the basepoint $\star \in \Sigma_g^p$ as another puncture, we obtain a standard action of Mod_g^{p+1} on $\pi_1(\Sigma_g^p, \star)$. We associate the following two objects to each finite index subgroup $K < \pi_1(\Sigma_g^p, \star)$:

(1) Let $\Gamma_K < \mathrm{Mod}_g^{p+1}$ denote the finite index subgroup which leaves K invariant (as a set) with respect to the aforementioned action of Mod_g^{p+1} on $\pi_1(\Sigma_g^p, \star)$.

(2) We say that a pointed cover $\pi : (\Sigma', \star') \to (\Sigma_g^p, \star)$ corresponds to K if $\pi_*\big(\pi_1(\Sigma', \star')\big) = K$, where $\pi_* : \pi_1(\Sigma', \star') \to \pi_1(\Sigma_g^p, \star)$ is the induced homomorphism. By $\widehat{\Sigma}$ we denote the closed surface obtained by filling in the punctures on Σ'.

Definition 1.1. *We say that a finite index subgroup $K < \pi_1(\Sigma_g^p, \star)$ has the Putman-Wieland property if for each nonzero vector $v \in H^1(\widehat{\Sigma}, \mathbb{Q})$, the Γ_K-orbit of v is infinite, where $\widehat{\Sigma}$ is the compactification of the corresponding Σ'.*

Putman and Wieland made the following conjecture (see Conjecture 1.2. in [16]).

Conjecture 1.2. *Let $g \geq 2$ and $p \geq 0$. Then every finite index subgroup $K < \pi_1(\Sigma_g^p, \star)$ has the Putman-Wieland property.*

2010 *Mathematics Subject Classification.* Primary 20H10, 20P05.

This work was supported by the *Simons Investigator Award* 409745 from the Simons Foundation

The importance of Conjecture 1.2 stems from its close connections with the Ivanov conjectures about the virtual cohomology of mapping class groups. It was shown in [12] that Conjecture 1.2 does not hold when $g = 2$.

1.2. **A random subgroup has the Putman-Wieland property.** Looijenga [8], Grunewald-Larsen-Lubotzky-Malestein [4], Landesman-Litt [7], and Marković-Tošić [14], verified that various types of finite index subgroups of $\pi_1(\Sigma_g^p, \star)$ have the Putman-Wieland property. The purpose of this paper is to prove that among all subgroups of index n, the fraction of these which have the Putman-Wieland property tends to 1 as $n \to \infty$.

Definition 1.3. *We let $\mathcal{K}_{g,p,n}$ denote the set of index n subgroups of $\pi_1(\Sigma_g^p, \star)$. By $\mathcal{K}_{g,p,n}^{PW}$ we denote the subset of $\mathcal{K}_{g,p,n}$ consisting of subgroups satisfying the Putman-Wieland property.*

Our main result says that the fraction of index n subgroups of $\pi_1(\Sigma_g^p, \star)$ which have the Putman-Wieland property tends to 1 when $n \to \infty$ provided the genus g is large enough.

Theorem 1.4. *For each $p \geq 0$ there exists $g_0 \in \mathbb{N}$ such that*

$$\lim_{n\to\infty} \frac{|\mathcal{K}_{g,p,n}^{PW}|}{|\mathcal{K}_{g,p,n}|} = 1, \tag{1}$$

when $g \geq g_0$.

In fact, we prove a stronger statement:

Theorem 1.5. *For each $\kappa < \frac{1}{2}$ there exists $g_0 \in \mathbb{N}$ such that (1) holds when $g \geq g_0$, and $p \leq g^\kappa$.*

1.3. **Random permutations are tangle free.** The Symmetric group $\mathbf{S}_n$ is the group of permutations of the set $[n] = \{1, \dots, n\}$. Let $\mathbf{F}_m$ denote the free group on m generators, and $\mathrm{Hom}_{m,n}$ the set of homomorphisms from $\mathbf{F}_m$ to the symmetric group $\mathbf{S}_n$.

Definition 1.6. *Let $R > 0$. We say that $w_1, w_2 \in \pi_1(\Sigma_g^p, \star)$ are R-tangled by $\phi \in \mathrm{Hom}_{m,n}$ if there exists $k \in [n]$ such that*

$$|\mathrm{Orb}(\phi(w_1), k)| + |\mathrm{Orb}(\phi(w_2), k)| \leq R,$$

where $\mathrm{Orb}(\phi(w_i), k) \subset [n]$ is the orbit of k under the iterates of $\phi(w_i)$.

One of the key ingredients in the proof of Theorem 1.5 is the following theorem.

Theorem 1.7. *Let $w_1, w_2 \in \mathbf{F}_m$ be two elements whose nontrivial powers are all distinct. Then for every $R > 0$ the equality*

$$\frac{|\{\phi \in \mathrm{Hom}_{m,n} : w_1, w_2 \text{ are } R\text{-tangled by } \phi\}|}{|\mathrm{Hom}_{m,n}|} \leq \frac{C}{n}$$

holds, where C depends only on w_1, w_2, R, and m.

Remark 1. It was shown by Monk-Thomas [14] that for any $R > 0$ there exists g_0 such that a random Riemann surface of genus $g > g_0$, picked with respect to the Weil-Petersson probability measure, is R-tangle-free. Applying Theorem 1.7 one can prove a version of this result for random covers of cusped hyperbolic surfaces.

1.4. **Organisation of the paper.** Each section starts with a brief outline. Here we only give a broad overview of the paper. In Section 2 we state the result by Marković-Tošić [14] that covers admitting sufficiently large spectral gap have the Putman-Wieland property. This naturally leads us to the results by Magee-Naud-Puder [10], and Hide-Magee [6], that a random cover of a fixed hyperbolic surface has no new (sufficiently) small eigenvalues. The combination of these two results is the main idea behind the proof of Theorem 1.5.

The main difficulty we need to overcome is be able to compare the spectral gaps of a random cover and its compactification (obtained by filling in the punctures). We do this by showing that a random cover of a cusped hyperbolic surface has the L-horoball property, which by the work of Brooks [1] guarantees that the two spectral gaps are comparable. Assuming Theorem 3.5 we prove Theorem 1.5 in Section 3.

The remainder of the paper (after Section 3) is mostly devoted to proving Theorem 3.5. As explained in Section 5, showing that a random cover has the L-horoball property reduces to proving Theorem 1.7. The proof of Theorem 1.7 has combinatorial flavour and we explain the main steps in Sections 6 and 7 where we present the proof.

2. Spectral gap and the Putman-Wieland property

In this section we recall results which underpin the proof of Theorem 1.5. The first is that a subgroup $K \in \mathcal{K}_{g,p,n}$ has the Putman-Wieland property assuming that the spectral gap of the surface $\widehat{X}_K$ is uniformly bounded away from zero. Here $\widehat{X}_K$ denotes the compactification of the corresponding holomorphic covering surface X_K.

Next, we define the subset $\mathcal{K}_n^{\lambda_1}(X) \subset \mathcal{K}_{g,p,n}$ consisting of subgroups $K \in \mathcal{K}_{g,p,n}$ for which X_K has the same spectral gap as fixed hyperbolic surface X. We then state the results from [10], [6] saying that a random element of $\mathcal{K}_{g,p,n}$ is asymptotically almost surely in $\mathcal{K}_n^{\lambda_1}(X)$ (for suitable X).

2.1. **Pointed holomorphic covers.** Let $\mathcal{M}_{g,p}$ denote the moduli space of Riemann surfaces of genus g and with p cusps. A marked pointed Riemann surface is a triple (X, x, f), where $X \in \mathcal{M}_{g,p}$, $x \in X$, and $f : (\Sigma_g^p, \star) \to (X, x)$ is pointed homeomorphism.

Let $X \in \mathcal{M}_{g,p}$, and choose a marked pointed Riemann surface (X, x, f). Fix $K \in \mathcal{K}_{g,p,n}$, and denote by $\pi : (\Sigma', \star') \to (\Sigma_g^p, \star)$ a pointed cover corresponding to K. Then there exist a unique marked pointed Riemann surface

(X_K, x_K, f_K), and a holomorphic unbranched covering $\pi_K : X_K \to X$, such that the following diagram commutes

$$\begin{tikzcd}
(X_K, x_K) \arrow[d, "\pi_K"'] & (\Sigma', \star') \arrow[l, "f_K"'] \arrow[d, "\pi"] \\
(X, x) & (\Sigma_g^p, \star) \arrow[l, "f"]
\end{tikzcd}$$

Note that the Riemann surface X_K depends only on K and not on the choice of the point $x \in X$ or the marking f.

Remark 2. On the other hand, the point x_K and the map f_K depend on both x and f.

We close this subsection with the following definition.

Definition 2.1. *By $\widehat{X}_K$ we denote the closed Riemann surface obtained from X_K by filling in the punctures.*

2.2. **A random cover retains the spectral gap.** The starting point in the proof of Theorem 1.5 is the following result by Marković-Tošić [14] which states that covers admitting sufficiently large spectral gap must have the Putman-Wieland property (see Theorem 1.9 in [14]).

Theorem 2.2. *Let $g \geq 2$, $p \geq 0$, and $K \in \mathcal{K}_{g,p,n}$. Suppose that there exists $X \in \mathcal{M}_{g,p}$ such that*

$$\frac{1 + 2\lambda_1(\widehat{X}_K)}{2\lambda_1(\widehat{X}_K)} \leq g.$$

Then $K \in \mathcal{K}^{PW}_{g,p,n}$. Here λ_1 denotes the smallest non-zero eigenvalue of the (hyperbolic) laplacian.

This result indicates that in order to show $K \in \mathcal{K}^{PW}_{g,p,n}$ it suffices to bound from below $\lambda_1(\widehat{X}_K)$ for some $X \in \mathcal{M}_{g,p}$ (this bound needs to be uniform in n). This brings us to the second key ingredient in the proof Theorem 1.5 which is the result that a random cover of X does not have any new small eigenvalues.

Definition 2.3. *For $X \in \mathcal{M}_{g,p}$, we set*

$$\mathcal{K}_n^{\lambda_1}(X) = \big\{K \in \mathcal{K}_{g,p,n} : \ \lambda_1(X_K) = \lambda_1(X)\big\}.$$

Note that the inequality $\lambda_1(X_K) \leq \lambda_1(X)$ always holds because the pullback of an eigenfunction on X is an eigenfunction on X_K. Requiring $\lambda_1(X_K) = \lambda_1(X)$ means that the laplacian on the covering surface X_K has no new eigenvalues which are strictly smaller than $\lambda_1(X)$. The second key ingredient in the proof of Theorem 1.5 is the following:

Theorem 2.4. *Let $X \in \mathcal{M}_{g,p}$ be such that $\lambda_1(X) < \frac{3}{16}$. Then*

$$\lim_{n\to\infty} \frac{|\mathcal{K}_n^{\lambda_1}(X)|}{|\mathcal{K}_{g,p,n}|} = 1. \tag{2}$$

This theorem follows from deep results by Magee-Naud-Puder [10] in the case $p = 0$, and by Hide-Magee [6] when $p > 0$. However, they prove this with respect to the uniform measure on the degree n covers of X with a labelled fibre (which is denoted by $\mathcal{L}_{g,p,n}$ in Section 4), while Theorem 2.4 is the version of their results with respect to the uniform measure on the set $\mathcal{K}_{g,p,n}$ of index n subgroups of $\pi_1(\Sigma_g^p, \star)$. The two models are closely related which we explain in Section 4 (where we formally prove Theorem 2.4).

It is clear that combining Theorem 2.2 and Theorem 2.4 brings us closer to proving Theorem 1.5. However, the main obstacle is that the statement of Theorem 2.2 inputs $\lambda_1(\widehat{X}_K)$, while the statement of Theorem 2.4 outputs $\lambda_1(X_K)$. Therefore, we have to show that for a random cover of X these two geometric quantities are in some sense related to each other.

3. The L-horoball property

We explain the notion of the L-horoball property and its significance in relating the spectral gaps of a cusped hyperbolic surface and its compactification. Then we define the subset $\mathcal{K}_n^{\mathcal{H}}(X, L) \subset \mathcal{K}_{g,p,n}$ consisting of subgroups for which the induced holomorphic covering surface X_K has the L-horoball property. At the end of the section we prove Theorem 1.5 assuming Theorem 3.5 which states that a random subgroup belongs to $\mathcal{K}_n^{\mathcal{H}}(X, L)$ asymptotically almost surely.

Remark 3. In [11], Brooks and Makover developed a certain model of random surfaces and proved a random surface in this model has the L-horoball property for every L.

3.1. **The L-horoball property and the Cheeger constant.**

In this subsection we let S denote a hyperbolic surface. To control the behaviour of λ_1 under conformal compactification we use the L-horoball property devised by Brooks [1].

Definition 3.1. *Let S be a hyperbolic Riemann surface with at least one cusp. Given $L > 0$, we say that S has the L-horoball property if the horoballs of perimeter L around all punctures are pairwise disjoint and embedded.*

Remark 4. It is a known feature of hyperbolic geometry that every hyperbolic cusped surface has the 1-horoball property.

Let S^c denote the closed surface obtained from S by filling in the puncture. Brooks (see Theorem 4.1 in [1]) established a connection between the Cheeger constants of S and S^c which we denote by $h(S)$ and $h(S^c)$ respectively.

Theorem 3.2. *For every $C > 1$ there exists $L > 0$ such that if S is a finite area hyperbolic surface which has the L-horoball property then*

$$\frac{1}{C} h(S) \leq h(S^c) \leq C h(S). \tag{3}$$

3.2. **The L-horoball property and the spectral gap.** Let us define the set of covers with the L-horoball property.

Definition 3.3. *For $L > 0$, and $X \in \mathcal{M}_{g,p}$, we let*

$$\mathcal{K}_n^{\mathcal{H}}(X, L) = \big\{ K \in \mathcal{K}_{g,p,n} : \ X_K \ \text{ has } L\text{-horoball property} \big\}.$$

We prove the following lemma by combining Theorem 3.2 with the classical results by Cheeger and Buser.

Lemma 3.4. *There exist universal constants $q, L > 0$ such that for every $X \in \mathcal{M}_{g,p}$, and every $K \in \mathcal{K}_n^{\lambda_1}(X) \cap \mathcal{K}_n^{\mathcal{H}}(X, L)$, the inequality*

$$\lambda_1(\widehat{X}_K) \geq q \lambda_1^2(X) \tag{4}$$

holds.

Proof. Let L be the constant from Theorem 3.2 such that the inequalities (3) hold for $C = 2$. Consider any $K \in \mathcal{K}_n^{\lambda_1}(X) \cap \mathcal{K}_n^{\mathcal{H}}(X, L)$. We show that there exists a universal constant $q > 0$ such that (4) holds.

Firstly, the classical Cheeger's inequality gives

$$\lambda_1(\widehat{X}_K) \geq \frac{1}{4} h^2(\widehat{X}_K). \tag{5}$$

Moreover, from the choice of L, and the assumption $K \in \mathcal{K}_n^{\mathcal{H}}(X, L)$, we get

$$h(\widehat{X}_K) \geq \frac{1}{2} h(X_K). \tag{6}$$

Furthermore, by Theorem 7.1 of [2] there exists an absolute constant $c > 0$ such that

$$h(X_K) > c \lambda_1(X_K). \tag{7}$$

Putting together the last three inequalities shows that

$$\lambda_1(\widehat{X}_K) \geq \frac{c^2}{16} \lambda_1^2(X_K).$$

This inequality, combined with the assumption $K \in \mathcal{K}_n^{\lambda_1}(X)$, yields

$$\lambda_1(\widehat{X}_K) \geq \frac{c^2}{16} \lambda_1^2(X).$$

This proves (4) for $q = \frac{c^2}{16}$. $\square$

3.3. **A random cover has the L-horoball property.** The following theorem shows that a random cover of X has the L-horoball property. It is proved in Section 5.

Theorem 3.5. *For every $X \in \mathcal{M}_{g,p}$, and every $L > 0$, we have*

$$\lim_{n\to\infty} \frac{|\mathcal{K}_n^{\mathcal{H}}(X,L)|}{|\mathcal{K}_{g,p,n}|} = 1. \tag{8}$$

The equality (8) puts us in the position to put together Theorem 2.2 and Theorem 2.4 to prove the main result Theorem 1.5.

3.4. **Proof of Theorem 1.5.** We begin by stating the result which follows from the paper by Hide [5].

Proposition 3.6. *For every $\kappa < \frac{1}{2}$ there exist constants $g_1 > 0$, and $0 < \delta < \frac{3}{16}$, with the following property. If $g \geq g_1$ and $p \leq g^{\kappa}$, then there exists $X \in \mathcal{T}(\Sigma_{g,p})$ such that*

$$\delta \leq \lambda_1(X) < \frac{3}{16}. \tag{9}$$

Fix $\kappa < \frac{1}{2}$, and let q and L be the constant from Lemma 3.4. Set

$$g_0 = \max\left\{ g_1, \frac{1+2q\delta^2}{2q\delta^2} \right\}.$$

Claim 3.7. *Suppose $g \geq g_0$, $p \leq g^{\kappa}$. Then there exists $X \in \mathcal{M}_{g,p}$ such that $\mathcal{K}_n^{\lambda_1}(X) \cap \mathcal{K}_n^{\mathcal{H}}(X,L) \subset \mathcal{K}_{g,p,n}^{PW}$.*

Proof. Let $X \in \mathcal{M}_{g,p}$ be such that (9) holds, and suppose $K \in \mathcal{K}_n^{\lambda_1}(X) \cap \mathcal{K}_n^{\mathcal{H}}(X,L)$. Then by the inequality (4), and from (9) , we derive the inequality

$$\lambda_1(\widehat{X}_K) \geq q\delta^2.$$

Combining this with the lower bound

$$g \geq \frac{1+2q\delta^2}{2q\delta^2},$$

and applying Theorem 2.2, proves that $K \in \mathcal{K}_{g,p,n}^{PW}$. □

To finish the proof, we first observe the equality

$$\lim_{n\to\infty} \frac{|\mathcal{K}_n^{\lambda_1}(X) \cap \mathcal{K}_n^{\mathcal{H}}(X,L)|}{|\mathcal{K}_{g,p,n}|} = 1. \tag{10}$$

This follows by from Theorem 3.5 and Theorem 2.4. The proof of Theorem 1.5 now follows from the equality (10) and Claim 3.7.

4. Finite covers with labelled fibres

In this section we define the set $\mathcal{L}_{g,p,n}$ of fibre labelled covers of Σ_g^p which consists of monodromy homomorphisms of (fibre unlabelled) covers $\mathcal{K}_{g,p,n}$. We then observe that the natural projection $P_n : \mathcal{L}_{g,p,n} \to \mathcal{K}_{g,p,n}$ enables us to easily replace $\mathcal{K}_{g,p,n}$ by $\mathcal{L}_{g,p,n}$ in the statements of Theorem 2.4 and Theorem 3.5. At the end of the section we derive the proof of Theorem 2.4, and state Theorem 4.5 which is a version of Theorem 1.7 for transitive homomorphisms.

4.1. **Labelled fibres and the Symmetric group.** We say that a pair (π, ι) is a degree n cover with a labelled fibre if

(1) $\pi : \Sigma' \to \Sigma_g^p$ is a (connected) cover of degree n,
(2) $\iota : \pi^{-1}(\star) \to [n]$ a labelling

(recall the abbreviation $[n] = \{1, \dots, n\}$). Two such covers $\pi' : \Sigma' \to \Sigma_g^p$, and $\pi'' : \Sigma'' \to \Sigma_g^p$, are equivalent if there exists a homeomorphism $I : \big(\Sigma', (\pi')^{-1}(\star)\big) \to \big(\Sigma'', (\pi'')^{-1}(\star)\big)$ so that $\pi' = \pi'' \circ I$, and $I \circ (\iota'')^{-1} = (\iota')^{-1}$.

Definition 4.1. *The set of equivalence classes of degree n covers with a labelled fibre is denoted by $\mathcal{L}_{g,p,n}$.*

To each equivalence class $[\pi, \iota] \in \mathcal{L}_{g,p,n}$ we associate the monodromy homomorphism $\phi_{[\pi,\iota]} : \pi_1(\Sigma_g^p, \star) \to \mathbf{S}_n$ which describes how the fibre $\pi^{-1}(\star)$ is permuted when following lifts of a closed loop from Σ_g^p to Σ'. In fact, the equivalence class $[\pi, \iota]$ is uniquely determined by the monodromy homomorphism $\phi_{[\pi,\iota]}$ (see Section 1 in [10]).

Since the cover π is connected it follows that the homomorphism $\phi_{[\pi,\iota]}$ is transitive (i.e. the image group $\phi_{[\pi,\iota]}\big(\pi_1(\Sigma_g^p, \star)\big)$ acts transitively on the set $[n]$). Therefore, there is a natural bijection

$$\mathcal{L}_{g,p,n} \longleftrightarrow \{\text{transitive homomorphisms } \pi_1(\Sigma_g^p, \star) \to \mathbf{S}_n\}. \tag{11}$$

Convention 4.2. *Using bijection (11), we let $\mathcal{L}_{g,p,n} \subset \mathrm{Hom}_{m,n}$ denote the set of transitive homomorphisms $\pi_1(\Sigma_g^p, \star) \to \mathbf{S}_n$.*

Remark 5. It follows from the work by Liebeck-Shalev [9] (which generalises an old theorem of Dixon [3], see also the introduction in [15]) that non-transitive homomorphisms $\pi_1(\Sigma_g^p, \star) \to \mathbf{S}_n$ are statistically insignificant when n is large for any fixed g and p such that $3g + p - 3 > 0$. We have

$$\lim_{n\to\infty} \frac{|\mathcal{L}_{g,p,n}|}{|\mathrm{Hom}_{m,n}|} = 1. \tag{12}$$

4.2. **Subgroups and pointed covers.** Two pointed covers $\pi' : (\Sigma', \star') \to (\Sigma_g^p, \star)$, and $\pi'' : (\Sigma'', \star'') \to (\Sigma_g^p, \star)$, are equivalent if there exists a pointed homeomorphism $\mathrm{I} : (\Sigma', \star') \to (\Sigma'', \star'')$ such that $\pi' = \pi'' \circ I$. The equivalence class of a pointed cover $\pi : (\Sigma', \star') \to (\Sigma_g^p, \star)$ is uniquely determined by the subgroup $\pi_*\big(\pi_1(\Sigma', \star')\big)$, where $\pi_* : \pi_1(\Sigma', \star') \to \pi_1(\Sigma_g^p, \star)$ is the induced homomorphism. Therefore, the set $\mathcal{K}_{g,p,n}$ is in the bijection with the set of equivalence classes of degree n pointed covers of Σ_g^p.

Let (π, ι) be a fibre labelled cover $\pi : \Sigma' \to \Sigma_g^p$. Set $\star' = \iota^{-1}(1)$, and consider the pointed cover $\pi : (\Sigma', \star') \to (\Sigma_g^p, \star)$. It is elementary to check that to equivalent fibre labelled covers we associate equivalent pointed covers. Thus, we have constructed the map

$$P_n : \mathcal{L}_{g,p,n} \to \mathcal{K}_{g,p,n}. \tag{13}$$

Moreover, if $\phi, \psi \in \mathcal{L}_{g,p,n}$ then $P_n(\psi) = P_n(\phi)$ if and only if the homomorphism ϕ and ψ agree up to post-conjugation by a permutation of the set $\{2, \ldots, n\}$. There are exactly $(n-1)!$ such permutations. This enables us to conclude:

Lemma 4.3. *The pre-image (under the map P_n) of each element of $\mathcal{K}_{g,p,n}$ consists of exactly $(n-1)!$ different elements of $\mathcal{L}_{g,p,n}$.*

Thus, each index n subgroup $K \in \mathcal{K}_{g,p,n}$ corresponds to exactly $(n-1)!$ homomorphisms from $\mathcal{L}_{g,p,n}$.

Definition 4.4. *Let $\phi \in \mathcal{L}_{g,p,n}$. For $X \in \mathcal{M}_{g,p}$, we let $X_\phi = X_K$, where $K = P_n(\phi)$.*

It is important to observe that $X_\phi = X_\psi$ if $P_n(\phi) = P_n(\psi)$. Explicitly, the subgroup K is the stabiliser of 1 in the action $\phi\big(\pi_1(\Sigma_g^p, \star)\big)$ on $[n]$.

4.3. **Proof of Theorem 2.4.** Suppose $X \in \mathcal{M}_{g,p}$ is such that $\lambda_1(X) < \frac{3}{16}$. Let $\mathcal{L}_n^{\lambda_1}(X) \subset \mathcal{L}_{g,p,n}$ denote the set of homomorphisms ϕ such that $\lambda_1(X) = \lambda_1(X_\phi)$. It was shown in [10] for $p = 0$, and in [6] when $p > 0$, that

$$\lim_{n\to\infty} \frac{\left|\mathcal{L}_n^{\lambda_1}(X)\right|}{\left|\mathcal{L}_{g,p,n}\right|} = 1. \tag{14}$$

On the other hand, it follows from Lemma 4.3 that

$$\left|\mathcal{L}_n^{\lambda_1}(X)\right| = (n-1)! \left|\mathcal{K}_n^{\lambda_1}(X)\right|, \qquad \left|\mathcal{L}_{g,p,n}\right| = (n-1)! \left|\mathcal{K}_{g,p,n}\right|.$$

Together with (14) this yields the proof of Theorem 2.4.

4.4. **Random transitive permutations are tangle free.** The proof of Theorem 3.5 consists of a geometric and a combinatorial part. The combinatorial part reduces to the statement that a transitive random permutation is asymptotically almost surely tangle free.

Theorem 4.5. *Suppose $p > 0$, and let $w_1, w_2 \in \pi_1(\Sigma_g^p)$ denote two elements whose nontrivial powers are all distinct. Then for every $R > 0$, the equality*

$$\frac{\left|\{\phi \in \mathcal{L}_{g,p,n} : w_1, w_2 \text{ are } R\text{-tangled by } \phi\}\right|}{\left|\mathcal{L}_{g,p,n}\right|} \leq \frac{C}{n}$$

holds, where C depends only on $w_1, w_2, R, g,$ and p.

Proof. The combination of the equality (12) and Theorem 1.7 yields the proof of Theorem 4.5. □

5. Generic covers have L-horoball property

In this section we explain the geometric content of the proof of Theorem 3.5 and conclude its proof assuming Theorem 4.5.

Definition 5.1. *Given $X \in \mathcal{M}_{g,p}$, we let $\mathcal{L}_n^{\mathcal{H}}(X, L) \subset \mathcal{L}_{g,p,n}$ denote the set of homomorphisms ϕ such that X_ϕ has the L-horoball property.*

In view of Lemma 4.3, to prove Theorem 3.5 it suffices to prove the following:

Theorem 5.2. *For every $X \in \mathcal{M}_{g,p}$, and every $L > 0$, we have*

$$\lim_{n\to\infty} \frac{|\mathcal{L}_n^{\mathcal{H}}(X, L)|}{|\mathcal{L}_{g,p,n}|} = 1. \tag{15}$$

The remainder of the paper is devoted to proving Theorem 5.2. In this section we reformulate the L-horoball property in terms of the pairs of connected cusps on X which behave well under covers. This allows us to prove Theorem 5.2 assuming Theorem 4.5.

Below we prove Propositions 5.6 and 5.7 which relate the L-horoball property of a covering $\pi_\phi : X_\phi \to X$ to the branching degrees at pairs of connected cusps. Then, in Proposition 5.10 we express these branching degrees in terms of the combinatorial data of the homomorphism $\phi \in \mathcal{L}_{g,p,n}$.

5.1. Geometry of pairs of connected cusps.

In the remainder of this section we assume $X \in \mathcal{M}_{g,p}$ and $p > 0$.

Definition 5.3. *A pair of connected cusps on X is a triple (c_1, c_2, δ), where c_1, c_2 are cusps on X, and δ is a geodesic joining them.*

Note that we allow $c_1 = c_2$. We will need some more vocabulary to work with pairs of connected cusps. If c is a cusp on X, we let $\mathcal{H}_c(r)$ denote the horoball based at c such that that the horocycle $\partial\mathcal{H}_c(r)$ has perimeter equal to r.

Definition 5.4. *Let (c_1, c_2, δ) be a pair of connected cusps on X.*

- *The beam $\beta(c_1, c_2, \delta)$ is the segment of δ lying outside the horoballs $\mathcal{H}_{c_1}(1)$ and $\mathcal{H}_{c_2}(1)$.*
- *The length of (c_1, c_2, δ) is denoted by $|(c_1, c_2, \delta)|$. It is defined as the length of the beam $\beta(c_1, c_2, \delta)$.*

Having defined pairs of connected cusps we now state a few of their properties. We begin with the following elementary fact from hyperbolic geometry.

Lemma 5.5. *Let c be a cusp of X. Suppose $r \geq 1$. Then the distance between the horocycles $\partial\mathcal{H}_c(r)$ and $\partial\mathcal{H}_c(1)$ is $\log r$.*

The following proposition explains the relationship between pairs of connected cusps and the L-horoball property.

Proposition 5.6. *A hyperbolic surface X has the L-horoball property if and only if the length of every pair of connected cusps on X is at least $2\log L$.*

Proof. Suppose c_1 and c_2 are cusps on X (not necessarily distinct). From Lemma 5.5 we conclude that the horoballs $\mathcal{H}_{c_1}(L)$ and $\mathcal{H}_{c_2}(L)$ are disjoint if and only if for each geodesic δ connecting c_1 and c_2, the length $|(c_1, c_2, \delta)|$ is at least $2\log L$. This proves the proposition. $\square$

The reason why it is convenient to reformulate the L-horoball property in terms of pairs of connected cusps is that they behave well under covering maps.

Proposition 5.7. *Suppose $\pi : X' \to X$ is a holomorphic covering. Let (c_1', c_2', δ') be a pair of connected cusps on X', and (c_1, c_2, δ) a pair of connected cusps on X, such that $\pi(c_1', c_2', \delta') = (c_1, c_2, \delta)$. Denote the branching degrees of π at c_1', c_2', by d_1 and d_2 respectively. Then*

$$|(c_1', c_2', \delta')| = |(c_1, c_2, \delta)| + \log d_1 + \log d_2. \tag{16}$$

Proof. The preimage of $\mathcal{H}_{c_i}(1)$ is the horoball $\mathcal{H}_{c_i'}(d_i)$ on which π is a d_i-fold cyclic covering. From Lemma 5.5 we find that the length of the segment $\beta(c_1', c_2', \delta')$ is given by

$$|\beta(c_1', c_2', \delta')| = |\beta(c_1, c_2, \delta)| + \log d_1 + \log d_2,$$

which proves the proposition. $\square$

Finally, let us observe that the set of pairs of connected cusps of bounded length is finite.

Proposition 5.8. *Let $R > 0$, and fix $X \in \mathcal{M}_{g,p}$. Then the set of pairs of connected cusps whose length is at most R is finite.*

Remark 6. It can be shown that the number of such pairs of connected cusps is at most $2e^R \times$ (the number of cusps on X).

Proof. The set of pairs of connected cusps on X whose length is at most R is both compact and discrete, and thus finite. $\square$

5.2. **Lollipops.** To each pair of connected cusps (c_1, c_2, δ) on X we associate two elements of the fundamental group of X. By $x = x(c_1, c_2, \delta)$ we denote the midpoint of the beam $\beta = \beta(c_1, c_2, \delta)$. The point x divides the beam β into two segments which we denote by β_1 and β_2.

Definition 5.9. *By $b_i = b_i(c_1, c_2, \delta)$, we denote the loop based at the midpoint $x = x(c_1, c_2, \delta)$, obtained by following the half-beam β_i, then winding once along unit-length horocycle $\partial\mathcal{H}_{c_i}(1)$, and returning to x back via β_i. We refer to b_1 and b_2 as the lollipops.*

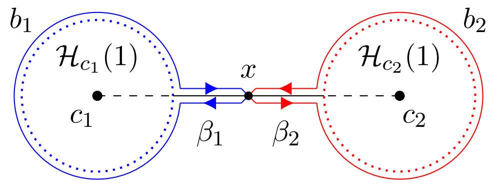

FIGURE 1. The lollipops b_1 (blue), and b_2 (red) represent elements of the fundamental group of $\pi_1(X, x)$

Choose a pointed marking $f : (\Sigma_g^p, \star) \to (X, x)$, and let $\phi \in \mathcal{L}_{g,p,n}$. Then there exist a unique marked pointed Riemann surface (X_ϕ, x_ϕ, f_ϕ), and a holomorphic unbranched covering $\pi_\phi : X_\phi \to X$, such that the following diagram commutes

$$\begin{tikzcd}
(X_\phi, x_\phi) \arrow[d, "\pi_\phi"'] & (\Sigma', \star') \arrow[l, "f_\phi"'] \arrow[d, "\pi"] \\
(X, x) & (\Sigma_g^p, \star) \arrow[l, "f"]
\end{tikzcd}$$

where $\pi : (\Sigma', \star') \to (\Sigma_g^p, \star)$ a pointed cover corresponding to the subgroup $P_n(\phi) \in \mathcal{K}_{g,p,n}$.

Recall that each $\phi \in \mathcal{L}_{g,p,n}$ correspond to the equivalence class of a fibre labelled cover (π, ι) where $\pi : \Sigma' \to \Sigma_g^p$, and $\iota : \pi^{-1}(\star) \to [n]$ is a labelling (that is, we have $\phi = \phi_{[\pi,\iota]}$). Let (π_ϕ, ι_ϕ) denote the fibre labelled cover where $\pi_\phi : X_\phi \to X$ is the aforementioned holomorphic covering, and $\iota_\phi : \pi_\phi^{-1}(x) \to [n]$ the labelling defined by $\iota_\phi = \iota \circ f_\phi^{-1}$. We set $x(k) = \iota_\phi^{-1}(k)$.

Proposition 5.10. *For each pair of connected cusps (c_1, c_2, δ) on $X \in \mathcal{M}_{g,p}$, there exists $a_1, a_2 \in \pi_1(\Sigma_g^p, \star)$ whose powers are mutually distinct, and with the following property. Let $\phi \in \mathcal{L}_{g,p,n}$, and let $(c_1(k), c_2(k), \delta(k))$ be the lift of (c_1, c_2, δ) (under the covering $\pi_\phi : X_\phi \to X$) whose midpoint is $x(k)$. Then*

$$d_1(k) + d_2(k) = |\mathrm{Orb}(\phi(a_1), k)| + |\mathrm{Orb}(\phi(a_2), k)| \tag{17}$$

where $d_i(k)$ denotes the branching degree of π_ϕ at $c_i(k)$.

Proof. Note that the branching degree $d_i(k)$ is the smallest positive integer m such that b_i^m lifts to a closed loop starting from $x(k)$. This is equivalent to saying that

$$d_i(k) = |\mathrm{Orb}(\psi(b_i), k)|, \qquad i = 1, 2, \tag{18}$$

where $\psi : \pi_1(X, x) \to \mathbf{S}_n$ is the monodromy homomorphism corresponding to the fibre labelled cover (π_ϕ, ι_ϕ). Observe that $\psi = \phi \circ (f_*)^{-1}$, where $f_* : \pi_1(\Sigma_g^p, \star) \to \pi_1(X, x)$ is the induced isomorphism. Replacing this in

(18) yields the equality

$$d_i(k) = |\mathrm{Orb}(\phi(a_i), k)|$$

where $a_i = f_*^{-1}(b_i)$. Clearly, a_1 and a_2 depend only on (c_1, c_2, δ), and not on ϕ. This implies the identity (17). The reader can verify that the nontrivial powers of a_1 and a_2 are distinct because the parabolic deck transforms of the universal cover induced by a_1 and a_2 have different fixed points.

□

5.3. **Proof of Theorem 5.2.**

Proposition 5.11. *For each $X \in \mathcal{M}_{g,p}$ there exists a finite collection of pairs $A \subset \pi_1(\Sigma_g^p, \star) \times \pi_1(\Sigma_g^p, \star)$ with the following properties. Firstly, if $(a_1, a_2) \in A$ then the powers of a_1 and a_2 are mutually distinct. Secondly, suppose $\phi \in \mathcal{L}_{g,p,n} \setminus \mathcal{L}_n^{\mathcal{H}}(X, L)$. Then there are $(a_1, a_2) \in A$ which are $2\log L$-tangled by ϕ.*

Proof. To each pair of connected cusps (c_1, c_2, δ) on X we associate the pair $(a_1, a_2) \in \pi_1(\Sigma_g^p, \star) \times \pi_1(\Sigma_g^p, \star)$ from Proposition 5.10. We define A as the collection of pairs (a_1, a_2) corresponding to pairs of connected cusps of length at most $2\log L$. Then, the collection A is finite by Proposition 5.8 (because the corresponding collection of pairs of connected cusps is finite).

The assumption $\phi \in \mathcal{L}_{g,p,n} \setminus \mathcal{L}_n^{\mathcal{H}}(X, L)$ means that X_ϕ does not have the L-horoball property. By Proposition 5.6 there exists a pair of connected cusps (c_1', c_2', δ') on X_ϕ whose length is at most $2\log L$. Let $(c_1, c_2, \delta) = \pi_\phi(c_1', c_2', \delta')$. Then $(c_1', c_2', \delta') = (c_1(k), c_2(k), \delta(k))$ for some $k \in [n]$. From Proposition 5.7 we conclude that

$$d_1(k) + d_2(k) + |(c_1, c_2, \delta)| = |(c_1(k), c_2(k), \delta(k))|.$$

Since we assume that $|(c_1(k), c_2(k), \delta(k))| \le 2\log L$, it follows that

$$|(c_1, c_2, \delta)| \le 2\log L, \tag{19}$$

and

$$d_1(k) + d_2(k) \le 2\log L. \tag{20}$$

From Proposition 5.10 we conclude that for certain $(a_1, a_2) \in A$, the following holds

$$d_1(k) + d_2(k) = |\mathrm{Orb}(\phi(a_1), k)| + |\mathrm{Orb}(\phi(a_2), k)|.$$

Combining this with (20) yields the inequality

$$|\mathrm{Orb}(\phi(a_1), k)| + |\mathrm{Orb}(\phi(a_2), k)| \le 2\log L.$$

But this means that a_1 and a_2 are $2\log L$-tangled by ϕ. This proves the proposition. □

We now complete the proof of Theorem 5.2. Fix $X \in \mathcal{M}_{g,p}$, and $L > 0$. We need to prove the equality (15). From Proposition 5.11 and Theorem 4.5 we conclude that

$$\begin{aligned} \frac{|\mathcal{L}_{g,p,n} \setminus \mathcal{L}^{\mathcal{H}}_n(X,L)|}{|\mathcal{L}_{g,p,n}|} &\leq \sum_{(a_1,a_2)\in A} \frac{|\{\phi \in \mathcal{L}_{g,p,n} :\, a_1, a_2 \text{ are } (2\log L)\text{-tangled by } \phi\}|}{|\mathcal{L}_{g,p,n}|} \\ &\leq \frac{C|A|}{n}. \end{aligned}$$

where C is the constant from Theorem 4.5. Letting $n \to \infty$ in the previous inequality implies the equality (15).

6. Random permutations are tangle free

It remains to prove Theorem 1.7. The key statement is that the set of homomorphisms from $\phi \in \mathrm{Hom}_{m,n}$ such that $\phi(w_1)$ and $\phi(w_2)$ have a common fixed point in the set $[n]$ is statistically insignificant compared to the size of the set $\mathrm{Hom}_{m,n}$ (here we must assume that the nontrivial powers of w_1 and w_2 are distinct.). We show that each such ϕ is carried by an edge labelled graph which enables us to effectively bound above the number of such homomorphisms ϕ.

6.1. **Carrier graphs.** Given a directed graph G we denote the vertex set of G by $V(G)$, and the set of oriented edges by $E(G)$. For $e \in E(G)$ we let $\iota(e)$ and $\tau(e)$ denote the initial and terminal vertices of e respectively. We write $\chi(G) = |V(G)| - |E(G)|$ to denote the Euler characteristic of G. Next, we introduce the key definitions of this section.

Definition 6.1. *We say that (G, h) is an edge labelled graph if:*

(1) G is a weakly connected directed graph,

(2) $h : E(G) \to [m]$ is an edge labelling such that if two edges $e_1, e_2 \in E(G)$ have the same initial and terminal vertices then $h(e_1) \neq h(e_2)$.

It turns out that edge labelled graphs are a convenient way of tracking fixed points of permutations. Let $\{s_1, \dots, s_m\}$ denote a generating set of the group $\mathbf{F}_m$.

Definition 6.2. *We say that an edge labelled graph (G, h) is f-compatible with $\phi \in \mathrm{Hom}_{m,n}$ if $f : V(G) \to [n]$ is vertex labelling such that*

$$\phi(s_{h(e)})\big(f(\iota(e))\big) = f(\tau(e))$$

for every $e \in E(G)$, where $s_{h(e)}$ is the corresponding generator of $\mathbf{F}_m$. Furthermore, we say that (G, h) carries ϕ if (G, h) is f-compatible with ϕ for some ***injective*** *vertex labelling f.*

The next proposition translates common fixed points of images of $\mathbf{F}_m$ in the symmetric group $\mathbf{S}_n$ into the language of carrier graphs with negative Euler characteristics. We postpone its proof until Section 7.

Proposition 6.3. *For every $w_1, w_2 \in \mathbf{F}_m$ there exists a constant $C = C(w_1, w_2)$ with the following properties. Suppose $\phi \in \mathrm{Hom}_{m,n}$ is such that $\phi(w_1), \phi(w_2)$ have a common fixed point in the set $[n]$. Then ϕ is carried by an edge labelled graph (G, h) which has at most C edges. If in addition we assume that all non-trivial powers of w_1 and w_2 are distinct then $\chi(G) < 0$.*

6.2. **The number of carried homomorphisms.** If w_1 and w_2 are R-tangled by some homomorphism ϕ then the permutations $\phi(w_1^{R!})$ and $\phi(w_2^{R!})$ have a common fixed point in the set $[n]$. Applying Proposition 6.3 to $w_1^{R!}, w_2^{R!} \in \mathbf{F}_m$, we find that such ϕ is carried by a suitable edge labelled graph (G, h).

The next step in the proof of Theorem 4.5 is to estimate the number of homomorphism from $\mathrm{Hom}_{m,n}$ which are carried by a fixed edge labelled graph.

Lemma 6.4. *Let (G, h) be an edge labelled graph. There exists a constant $C = C(G, h, m)$ such that for every integer $n > 0$ the following holds*

$$\frac{|\{\phi \in \mathrm{Hom}_{m,n} :\ (G,h) \text{ carries}\, \phi\}|}{|\,\mathrm{Hom}_{m,n}\,|} \leq C n^{\chi(G)} \tag{21}$$

Proof. Fix an edge labelled graph (G, h). The proof of the lemma is based on the following three claims. The first claim is elementary and its proof is left to the reader.

Claim 6.5. *The number of injective vertex labelings $f : V(G) \to [n]$ is equal to*

$$n(n-1)\cdots(n - |V(G)| + 1) = \frac{n!}{(n - |V(G)|)!}.$$

Claim 6.6. *Suppose that (G, h) carries some $\phi \in \mathrm{Hom}_{m,n}$. If $h(e_1) = h(e_2)$ then the implication*

$$\iota(e_1) = \iota(e_2) \implies e_1 = e_2 \tag{22}$$

holds.

Proof. Since (G, h) carries ϕ there exists an injective vertex labelling $f : V(G) \to [n]$ such that (G, h) is f-compatible with ϕ. Suppose $\iota(e_1) = \iota(e_2)$. It follows from the f-compatibility that $\phi(s_l)$ sends $f(\iota(e_j))$ to $f(\tau(e_j))$, for $j = 1, 2$, where $l = h(e_1) = h(e_2)$. This implies that $f(\tau(e_1)) = f(\tau(e_2))$. Since f is injective we derive the equality $\tau(e_1) = \tau(e_2)$. Thus, we have shown that e_1 and e_2 have the same initial and terminal vertices. Combining this with the condition (2) from Definition 6.1 shows that $e_1 = e_2$, and the implication (22) is proved. □

Next, define $E_l(G) = \{e \in E(G) :\ h(e) = l\}$.

Claim 6.7. *Let $f : V(G) \to [n]$ be an injective vertex labelling. Then*

$$|\{\phi \in \mathrm{Hom}_{m,n} : (G,h) \text{ is } f\text{-compatible with } \phi\}| \leq \prod_{l=1}^{m} (n - |E_l(G)|)! \tag{23}$$

Proof. We estimate the left hand side in (23) as

$$|\{\phi \in \mathrm{Hom}_{m,n} : (G,h) \text{ is } f\text{-compatible with } \phi\}| \leq \prod_{l=1}^{m} X_l \tag{24}$$

where X_l is the number of permutations in $\mathbf{S}_n$ which are realised as $\phi(s_l)$ for some ϕ which is f-compatible with (G,h). We now estimate X_l.

Define the subset $[n](G,h,f,l) \subset [n]$ by

$$[n](G,h,f,l) = \{f(\iota(e)) : e \in E_l(G)\}.$$

If (G,h) is f-compatible with some ϕ then the vertex labelling f specifies the values of the permutation $\phi(s_l)$ on the set $[n](G,h,f,l)$. Combining Claim 6.6 with the assumption that f is injective implies that the set $[n](G,h,f,l)$ has exactly $|E_l(G)|$ elements. Therefore, there are either zero or exactly $(n-|E_l(G)|)!$ possible permutations in $\mathbf{S}_n$ that can equal $\phi(s_l)$. That is, we established the estimate $X_l \leq (n-|E_l(G)|)!$. Replacing this in (24) proves the claim. ∎

We are ready to finish the proof of the lemma. The number of homomorphisms $\phi \in \mathrm{Hom}_{m,n}$ which are carried by (G,h) can be estimated above by the product of two numbers. The first is the number of injective vertex labelings f, and the second is the number of $\phi \in \mathrm{Hom}_{m,n}$ which are f-compatible with (G,h) for a fixed injective vertex labelling f. These two numbers we estimated in Claim 6.5 and Claim 6.7 respectively, and we derive the estimate

$$|\{\phi \in \mathrm{Hom}_{m,n} : \ (G,h) \text{ carries } \phi\}| \leq \frac{n!}{(n-|V(G)|)!} \prod_{l=1}^{m} (n-|E_l(G)|)!. \tag{25}$$

On the other hand, it is easy to derive (the well known) equality:

$$|\mathrm{Hom}_{m,n}| = (n!)^m. \tag{26}$$

Hence, dividing the left-hand side of (25) by the left-hand side of (26) yields the estimate

$$\begin{aligned}\frac{|\{\phi \in \mathrm{Hom}_{m,n} : \ (G,h)\, \text{carries}\, \phi\}|}{|\mathrm{Hom}_{m,n}|} &\leq \frac{n(n-1)\cdots(n-|V(G)|+1)}{\prod_{l=1}^{m} n(n-1)\cdots(n-|E_l(G)|+1)} \\ &\leq C\frac{n^{|V(G)|}}{n^{(|E_1(G)|+\cdots+|E_m(G)|)}} = Cn^{\chi(G)}\end{aligned}$$

for some C depending only on G, h and m. In the last step we used the equality $|E_1(G)| + \cdots + |E_m(G)| = |E(G)|$. ∎

6.3. **Proof of Theorem 4.5.** It remains to finish the proof of Theorem 4.5. Suppose that w_1 and w_2 are R-tangled by some $\phi \in \mathrm{Hom}_{m,n}$. Then the permutations $\phi(w_1^{R!}), \phi(w_2^{R!})$ have a common fixed point in the set $[n]$. Then Proposition 6.3 states that such ϕ is carried by an edge labelled graph (G,h) with at most C_1 edges, and with the negative Euler characteristic.

Here C_1 is the constant from Proposition 6.3 which depends only on $w_1^{R!}$ and $w_2^{R!}$. This implies the following estimate

$$|\{\phi \in \mathrm{Hom}_{m,n} : w_1, w_2 \text{ are R-tangled by } \phi\}| \leq AB \tag{27}$$

where A is the number of homomorphism ϕ carried by a fixed edge labelled graph (G,h) with $\chi(G) < 0$, and $|E(G)| \leq C_1$, and B is the number of edge labelled graphs (G,h) with $\chi(G) < 0$, and $|E(G)| \leq C_1$.

The number of graphs with at most C_1 edges is a finite number depending only on C_1. For each such graph G, the number of edge labelings $h : E(G) \to [m]$ depends only on $|E(G)|$ and m. Thus, the number B depends only C_1 and m. We conclude that B depends only on w_1, w_2, R, m.

On the other hand, for a fixed (G,h) we estimate A using Lemma 6.4 :

$$\begin{aligned}\frac{A}{|\mathrm{Hom}_{m,n}|} &= \frac{|\{\phi \in \mathrm{Hom}_{m,n} :\ (G,h)\,\text{carries}\,\phi\}|}{|\,\mathrm{Hom}_{m,n}\,|}\\ &\leq C_2 n^{\chi(G)} \leq \frac{C_2}{n},\end{aligned}$$

where C_2 is the constant from Lemma 6.4 depending only on G, h and m. In the last step we used that $\chi(G) < 0$. Returning this to (27) yields

$$\frac{|\{\phi \in \mathrm{Hom}_{m,n} : w_1, w_2 \text{ are R-tangled by } \phi\}|}{|\mathrm{Hom}_{m,n}|} \leq \frac{C}{n}$$

for $C = BC_2$. We have proved the theorem.

7. Proof of Proposition 6.3

The first step is to construct an edge labelled graph (G,h) which is f-compatible with ϕ without the requirement that f is injective. This is the content of Lemma 7.1 below. In the endgame we modify (G,h) to a new edge labelled graph (G_1,h_1) which carries ϕ.

Let Y be the directed graph which has a single vertex y, and m oriented loops (petals) which we denote by $\alpha_1, \ldots, \alpha_m$. There is an obvious isomorphism $\pi_1(Y,y) \to \mathbf{F}_m$ defined by $\alpha_j \to s_j$, $j \in [m]$.

Lemma 7.1. *Let $w_1, w_2 \in \mathbf{F}_m$, and suppose $\phi \in \mathrm{Hom}_{m,n}$ is such that the permutations $\phi(w_1), \phi(w_2)$ have a common fixed point in the set $[n]$. Then there exists an edge labelled graph (G,h) with the following properties:*

1. *the number of edges $|E(G)|$ is equal to the sum of the (reduced) word lengths of w_1 and w_2 in $\mathbf{F}_m$,*
2. *there exists a (non necessarily injective) vertex labelling $f : V(G) \to [n]$ such that (G,h) is f-compatible with ϕ,*
3. *there exists a graph morphism $\mu : G \to Y$ such that (for a suitable $v_0 \in V(G)$) the group $\mu_*(\pi_1(G,v_0))$ is equal to the subgroup of $\pi_1(Y,y)$ generated by w_1 and w_2,*

$$\phi\colon \begin{cases} s_1 & \mapsto (15)(687) \\ s_2 & \mapsto (172569) \\ s_3 & \mapsto (1934)(58) \end{cases}$$

$$w_1 = s_1^{-1}s_2^{-1}s_1s_2, \qquad \phi(w_1) = (29)(5678)$$
$$w_2 = s_1^{-1}s_3^{-1}s_1s_2^{-1}s_3, \qquad \phi(w_2) = (276458)$$

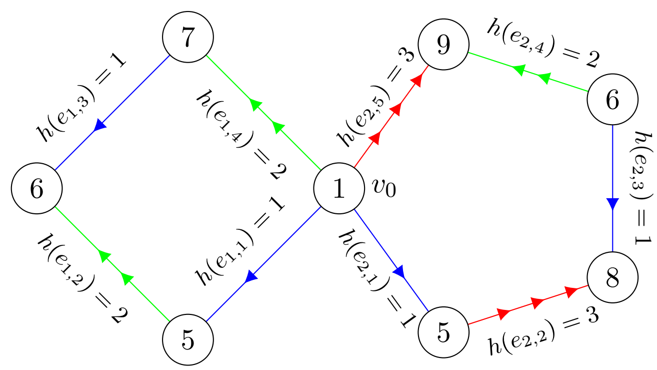

FIGURE 2. Both permutations $\phi(w_1), \phi(w_2)$ fix 1. Vertices represent the orbits of the common fixed point 1 under $\phi(w_1)$ and $\phi(w_2)$, and are labelled according the vertex labelling f.

(4) the equivalence

$$h(e_1) = h(e_2) \iff \mu(e_1) = \mu(e_2)$$

holds for every $e_1, e_2 \in E(G)$.

Remark 7. The significance of the condition (3) is that implies $\chi(G) < 0$ provided the group generated by w_1 and w_2 is not Abelian.

Remark 8. Figure 2 illustrates the construction of the edge labelled graph (G, h) and the corresponding vertex labelling.

Proof. Recall that $\{s_1, \dots, s_m\}$ is the generating set for $\mathbf{F}_m$. Therefore, each $w \in \mathbf{F}_m$ can be written as a (reduced) word in the corresponding alphabet. In particular, the elements w_1 and w_2 are spelled as:

$$w_i = s_{l_{i,1}}^{\sigma_{i,1}} s_{l_{i,2}}^{\sigma_{i,2}} \cdots s_{l_{i,k_i}}^{\sigma_{i,k_i}} \tag{28}$$

where $\sigma_{i,t} \in \{1, -1\}$.

Let Z be a directed graph which has a single vertex z, and 2 oriented loops (petals) which we denote by β_1 and β_2. The petal β_i is subdivided into oriented edges $e_{i,t}$, where $1 \le t \le k_i$, so that $e_{i,t}$ has the same orientation as β_i if and only of $\sigma_{i,t} = 1$. The resulting directed graph is denoted by G.

Then $|E(G)|$ equals the sum of word lengths of w_1 and w_2 which confirms (1).

Next, we define the morphism $\mu : G \to Y$ by letting

$$\mu(e_{i,t}) = \alpha_{l_{i,t}}. \tag{29}$$

Let $v_0 \in V(G)$ be the vertex arising from the vertex z of the rose Z. Then $\pi_1(G, v_0)$ is a free group of rank two generated by the loops β_1 and β_2. From (28) we conclude $\mu_*(\beta_i) = w_i$, where $\mu_* : \pi_1(G, v_0) \to \mathbf{F}_m$ is the induced homomorphism. This confirms (3).

Define the edge labelling $h : E(G) \to [m]$ by letting

$$h(e_{i,t}) = l_{i,t}. \tag{30}$$

Any two vertices in $V(G)$ are connected by at most one edge. Therefore, (G, h) is an edge labelled graph in the sense of Definition 6.1. Moreover, combining definitions (29) and (30) proves (4).

It remains to construct a vertex labelling $f : V(G) \to [n]$ such that (G, h) is f-compatible with ϕ. Denote by $k \in [n]$ the common fixed point of the permutations by $\phi(w_1)$ and $\phi(w_2)$. Set $f(v_0) = k$, and propagate the vertex labelling f to all other vertices so as to satisfy the condition

$$\phi(s_l)\big(f(\iota(e))\big) = f(\tau(e))$$

for each e, where $h(e) = l$. Since k is fixed by both permutations $\phi(w_1)$ and $\phi(w_2)$, such vertex labelling f is well defined and is in fact unique. Thus, we have shown that (G, h) is f-compatible with ϕ which confirms (2). $\square$

7.1. **The endgame.** Given $\phi \in \mathrm{Hom}_{m,n}$, in Lemma 7.1 we have constructed an edge labelled graph (G, h) which is f-compatible with ϕ, where $f : V(G) \to [n]$ is a labelling. We define a new graph G_1, together with a graph morphism $\rho : G \to G_1$, as follows (see Figure 3).

Define the equivalence relation $\sim_V$ on the vertex set $V(G)$ by letting $v_1 \sim v_2$ if and only if $f(v_1) = f(v_2)$. Denote by G' the corresponding quotient graphs such that $V(G') = V(G)/\sim$. Then, the graph G_1 is obtained from G' by a sequence of foldings where at each stage we identify two edges e_1 and e_2 if they have the same initial and terminal vertices, and the same label (that is, $h(e_1) = h(e_2)$). By $\rho : G \to G_1$ we denote the resulting morphism.

Given $e_1 \in E(G_1)$, there exists $l \in [m]$ such that $h(e) = l$ for every $e \in \rho^{-1}(e_1) \subset E(G)$. We define the new edge labelling $h_1 : E(G_1) \to [m]$ by letting $h_1(e_1) = l$. By construction (G_1, h_1) is an edge labelled graph in the sense of Definition 6.1.

Likewise, for each $v_1 \in V(G_1)$ there exists $k \in [n]$ such that $f(v) = k$ for every $v \in \rho^{-1}(v_1) \subset V(G)$. This enables us to define the vertex labelling $f_1 : V(G_1) \to [n]$ by letting $f_1(v_1) = k$. By construction f_1 is injective (since the vertex $v_1 \in V(G_1)$ is the equivalence class consisting of all vertices from $V(G)$ that are mapped to k). Moreover, since (G, h) is f-compatible with ϕ we conclude that (G_1, h_1) is f_1-compatible with ϕ.

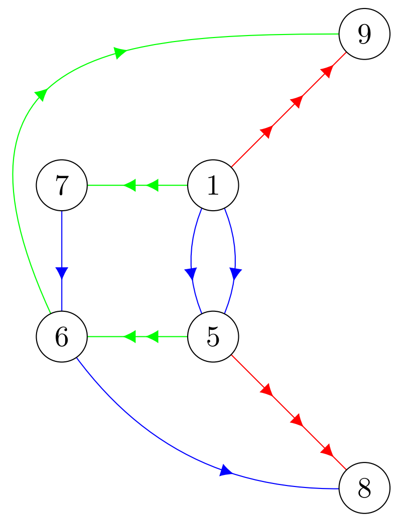

FIGURE 3. The graph G' from Section 7.1. The pairs of vertices with the labels 5 and 6 were identified by the relation $\sim_V$. The graph G_1 is then obtained from G' by replacing the two blue edges of label 1 between the vertices 1 and 5 with a single edge. Then $\chi(G_1) = -2$.

Thus, we have constructed an edge labelled graph (G_1, h_1) which carries ϕ. It remains to compute its Euler characteristic. It follows from the condition (4) in Lemma 7.1 that the morphism $\mu : G \to Y$ factors through G_1. That is, there exist a morphism $\mu_1 : G_1 \to Y$ so that $\mu = \mu_1 \circ \rho$. Then the condition (3) from Lemma 7.1 implies the inclusion

$$W = \mu_*\big(\pi_1(G, v_0)\big) \leq (\mu_1)_*\big(\pi_1(G_1, v_1)\big) \tag{31}$$

where W is the subgroup of $\pi_1(Y, y)$ generated by the words w_1 and w_2 (here $v_1 = \rho(v_0)$). If w_1 and w_2 have no non-trivial powers in common then W is a free group of rank 2. Combining this together with (31) yields the inequality $\chi(G) < 0$.

Therefore, we have proved that the labelled graph (G_1, h_1) carries ϕ. On the other hand, the obvious inequality $|E(G_1)| \leq |E(G)|$, and the condition (1) from Lemma 7.1, complete the proof of Proposition 6.3.

REFERENCES

[1] R. Brooks, *Platonic surfaces*. Commentarii Mathematici Helvetici, 74:156–170, (1999)
[2] P. Buser, *A note on the isoperimetric constant*. Annales Scientifiques Ecole Normale Superieure, 15:213-230, (1982)
[3] J. Dixon, *The probability of generating the symmetric group*. Math. Z., 110(3):199-205, (1969)
[4] F. Grunewald, M. Larsen, A. Lubotzky, J. Malestein, *Arithmetic quotients of the mapping class group*. Geometric and Functional Analysis volume 25, pages 1493-1542 (2015)

[5] W. Hide, *Spectral gap for Weil-Petersson random surfaces with cusps.* International Mathematics Research Notices, 2023, 10 (2022)
[6] W. Hide, M. Magee, *Near optimal spectral gaps for hyperbolic surfaces.* Annals of Mathematics, 198(2):791 – 824, (2023)
[7] A. Landesman, D. Litt, *Canonical representations of surface groups.* to appear in Annals of Mathematics
[8] E. Looijenga, *Prym representations of mapping class groups.* Geom. Dedicata, 64: 69-83, (1997)
[9] M. Liebeck, A. Shalev, *Fuchsian groups, coverings of Riemann surfaces, subgroup growth, random quotients and random walks.* J. Algebra, 276(2):552-601, (2004)
[10] M. Magee, F. Naud, D. Puder, *A random cover of a compact hyperbolic surface has relative spectral gap* $\left(\frac{3}{16}-\epsilon\right)$. Geometric and Functional Analysis, 32(3):595–661, (2022)
[11] R. Brooks, E. Makover, *Random construction of Riemann surfaces.* Journal Differential Geometry 68 , no. 1, 121-157 (2004)
[12] V. Marković, *Unramified correspondences and virtual properties of mapping class groups.* Bull. Lond. Math. Soc. 54, no. 6, 2324-2337 (2022)
[13] V. Marković, O. Tošić, *The second variation of the hodge norm and higher prym representations.* Journal of Topology, vol 17, March (2024)
[14] L. Monk, J. Thomas, *The tangle-free hypothesis on random hyperbolic surfaces.* Int. Math. Res. Not. IMRN, no.22, 18154-18185 (2022)
[15] D. Puder, T. Zimhoni, *Local Statistics of Random Permutations from Free Products.* arXiv:2203.12250v2
[16] A. Putman, B. Wieland, *Abelian quotients of subgroups of the mapping class group and higher Prym representations.* Journal of the London Mathematical Society 88 (2011)

Mathematical Institute
University of Oxford
United Kingdom

All Souls College
University of Oxford
United Kingdom