Topological Entropy and Diffeomorphisms of Surfaces with Wandering Domains*

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Abstract

Let M be a closed surface and f a diffeomorphism of M. A diffeomorphism is said to permute a dense collection of domains, if the union of the domains are dense and the iterates of any one domain are mutually disjoint. In this note, we show that if $f \in \mathrm{Diff}^{1+\alpha}(M)$, with $\alpha > 0$, and permutes a dense collection of domains with bounded geometry, then f has zero topological entropy.

1 Definitions and statement of results

A result of A. Norton and D. Sullivan [7] states that a diffeomorphism $f \in \mathrm{Diff}^3_0(\mathbb{T}^2)$ having *Denjoy-type* can not have a wandering disk whose iterates have the same *generic shape*. By diffeomorphisms of Denjoy-type are meant diffeomorphisms of the two-torus, isotopic to the identity, that are obtained as an extension of an irrational translation of the torus, for which the semi-conjugacy has countably many non-trivial fibers. If these fibers have non-empty interior, then the corresponding diffeomorphism has a wandering disk. Further, by generic shape is meant that the only elements of $\mathrm{SL}(2,\mathbb{Z})$ preserving the shape are elements of $\mathrm{SO}(2,\mathbb{Z})$, such as round disks and squares. In a similar spirit, C. Bonatti, J.M. Gambaudo, J.M. Lion and C. Tresser in [1] show that certain infinitely renormalizable diffeomorphisms of the two-disk that are sufficiently smooth, can not have wandering domains if these domains have a certain boundedness of geometry.

In this note, we study an analogous problem, namely the interplay between the geometry of iterates of domains under a diffeomorphism and its topological entropy. To state the precise result, we first need some definitions. Let (M, g) be a closed surface, that is, a smooth, closed, oriented Riemannian two-manifold, equipped with the canonical metric g induced from the standard conformal metric of the universal cover $\mathbb{P}^1, \mathbb{C}$ or $\mathbb{D}^2$. We denote by $d(\cdot,\cdot)$ the distance function relative to the metric g. Let $\mathrm{Diff}^r(M)$ be the group of diffeomorphisms of M, where for

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$r \geq 0$ finite, f is said to be of class C^r if f is continuously differentiable up to order $[r]$ and the $[r]$-th derivative is (r)-Hölder, with $[r]$ and (r) the integral and fractional part of r respectively. We identity $\mathrm{Diff}^0(M)$ with $\mathrm{Homeo}(M)$, the group of homeomorphisms of M.

Given $f \in \mathrm{Homeo}(M)$, for each $n \geq 1$, define the metric d_n on M given by $d_n(x,y) = \max_{1\leq i\leq n}\{d(f^i(x), f^i(y))\}$. Given $\epsilon > 0$, a subset $U \subset M$ is said to be (n,ϵ) separated if $d_n(x,y) \geq \epsilon$ for every $x, y \in U$ with $x \neq y$. Let $N(n,\epsilon)$ be the maximum cardinality of an (n,ϵ) separated set. The topological entropy is defined as

$$h_{\mathrm{top}}(f) = \lim_{\epsilon\to 0}\left(\lim_{n\to\infty}\sup \frac{1}{n}\log N(n,\epsilon)\right).$$

Next, we make precise the notion of a homeomorphism of a surface permuting a dense collection of domains.

Definition 1.1. Let $S \subset M$ be compact and $\mathcal{D} := \{D_k\}_{k\in\mathbb{Z}}$ the collection of connected components of the complement of S, with the property that $\mathrm{Int}(\mathrm{Cl}(D_k)) = D_k$, where $\mathrm{Cl}(D)$ is the closure of D in M. We say $f \in \mathrm{Homeo}(M)$ *permutes a dense collection of domains* if

(1) $f(S) = S$ and $\mathrm{Cl}(D_k) \cap \mathrm{Cl}(D_{k'}) = \emptyset$ if $k \neq k'$,

(2) for every $k \in \mathbb{Z}$, $f^n(D_k) \cap D_k = \emptyset$ for all $n \neq 0$, and

(3) $\bigcup_{k\in\mathbb{Z}} D_k$ is dense in M.

Note that we do not assume a domain to be recurrent, nor do we assume the orbit of a single domain to be dense. A *wandering domain* is a domain with mutually disjoint iterates under f such that the orbit of the domain is recurrent. Thus a diffeomorphism with a wandering domain with dense orbit is a special case of definition 1.1. Denote $\exp_p\colon T_pM\to M$ the exponential mapping at $p \in M$. The *injectivity radius* at a point $p \in M$ is defined as the largest radius for which $\exp_p$ is a diffeomorphism. The injectivity radius $\iota(M)$ of M is the infimum of the injectivity radii over all points $p \in M$. As M is compact, $\iota(M)$ is positive.

Definition 1.2 (Bounded geometry)**.** A collection of domains $\{D_k\}_{n\in\mathbb{Z}}$ on a surface M is said to have *bounded geometry* if the following holds: $\mathrm{Cl}(D_k)$ is contractible in M and there exists a constant $\beta \geq 1$ such that for every domain D_k in the collection, there exist $p_k \in D_k$ and $0 < r_k \leq R_k$ such that

$$B(p_k, r_k \subseteq D_k \subseteq B(p_k, R_k), \text{ with } R_k/r_k \leq \beta, \tag{1}$$

where $B(p,r) \subset M$ is the ball centered at $p \in M$ with radius $r > 0$. If no such β exists, then the collection is said to have *unbounded geometry*.

By $\mathrm{Cl}(D_k)$ being contractible in M we mean that $\mathrm{Cl}(D_k)$ is contained in an embedded topological disk in M. Our definition of bounded geometry is equivalent to the notion of bounded geometry in the theory of Kleinian groups and complex dynamics. It is not difficult, given a surface of any genus, to construct homeomorphisms of that surface with positive entropy that permute a dense collection of domains. We show that producing examples that have a certain amount of smoothness is possible only to a limited degree.

Theorem A (Topological entropy versus bounded geometry)**.** *Let M be a closed surface and $f \in \mathrm{Diff}^{1+\alpha}(M)$, with $\alpha > 0$. If f permutes a dense collection of domains with bounded geometry, then f has zero topological entropy.*

The outline of the proof of Theorem A is as follows. First we show that the bounded geometry of the permuted domains, combined with their density in the surface, give bounds on the dilatation of f on the complement of the union of the permuted domains. The differentiability assumptions on f allow us to estimate the rate of growth of the dilatation on the whole surface M. Using a result by Przytycki [8], we show this rate of growth is slow enough so as to ensure the topological entropy of f is zero.

2 Entropy and diffeomorphisms with wandering domains

First, we study the relation between geometry of domains and the complex dilatation of a diffeomorphism.

2.1 Geometry of domains and complex dilatation

We denote λ the measure associated to g and $d\lambda$ the Riemannian volume form. By compactness of M, there exists a constant $\kappa > 0$ such that

$$\lambda(B(p,r)) = \int_{B(p,r)} d\lambda \geq \kappa r^2. \tag{2}$$

where $B(p,r) \subset M$ is the ball centered at p with radius $r < \iota(M)/2$. A sequence of positive real numbers x_k is called a *null-sequence*, if for every given $\epsilon > 0$ there exist only finitely elements of the sequence for which $x_k \geq \epsilon$. Henceforth, we denote $\ell_k := \mathrm{diam}(D_k)$, the diameter of D_k measured in g, with $D_k \in \mathcal{D}$.

Lemma 2.1. *Let (M,g) be a closed surface and let $\{D_k\}_{k\in\mathbb{Z}}$ be a collection of mutually disjoint domains with bounded geometry. Then the sequence ℓ_k is a null-sequence.*

Proof. Suppose, to the contrary, that $\{D_k\}_{k\in\mathbb{Z}}$ is not a null-sequence. Then there exist an $\epsilon > 0$ and an infinite subsequence k_t such that $\mathrm{diam}(D_{k_t}) \geq \epsilon$. By the bounded geometry property, we have that $\operatorname{diam}(D_{k_t}) \leq \beta r_{k_t}$ and therefore $r_{k_t} \geq \epsilon/\beta$. Therefore, by (2),

$$\lambda(D_{k_t}) \geq \kappa r_{k_t}^2 \geq \frac{\kappa \epsilon^2}{\beta^2},$$

for every $t \in \mathbb{Z}$. But this yields that

$$\sum_{t \in \mathbb{Z}} \lambda(D_{k_t}) = \infty,$$

contradicting the fact that $\lambda(M) < \infty$ as M is compact. □

Recall that S is the complement of the union of the permuted domains, i.e. $S = M \setminus \bigcup_{k \in \mathbb{Z}} D_k$.

Lemma 2.2. *Let $f \in \operatorname{Homeo}(M)$ permute a dense collection $\mathcal{D}$ of domains with bounded geometry. For every $p \in S$, there exists a sequence of domains D_{k_t} with $\operatorname{diam}(D_{k_t}) \to 0$ for $t \to \infty$ such that $D_{k_t} \to p$.*

Proof. Fix $p \in S$ and let $U \subset M$ be an open (connected) neighbourhood of p. First assume that $p \in S \setminus \bigcup_{k \in \mathbb{Z}} \partial D_k$. This set in non-empty, as otherwise the surface M is a union of countably many mutually disjoint continua; but this contradicts Sierpiński's Theorem, which states that no countable union of disjoint continua is connected. We claim that U intersects infinitely many different elements of $\mathcal{D}$. Indeed, if U intersects only finitely many elements $D_{k_1}, \ldots, D_{k_m}$, then $\Omega := \bigcup_{i=1}^m \operatorname{Cl}(D_{k_i})$ is closed. This implies that $U \setminus \Omega$ is open and non-empty, as otherwise M would be a finite union of disjoint continua, which is impossible. However, as the union of the elements of $\mathcal{D}$ is dense, $U \setminus \Omega$ can not be open. Thus, there are infinitely many distinct elements $D_{k_1}, D_{k_2}, \ldots$ of $\mathcal{D}$ that intersect U. Taking a sequence of nested open connected neighbourhoods U_t containing p, we can find elements $D_{k_t} \subset U_t \setminus U_{t+1}$ for every $t \geq 1$. By Lemma 2.1, $\operatorname{diam}(D_{k_t})$ is a null-sequence and thus we obtain a sequence of domains D_{k_t} with $\operatorname{diam}(D_{k_t}) \to 0$ for $t \to \infty$ such that $D_{k_t} \to p$.

As $\operatorname{Int}(\operatorname{Cl}(D_k)) = D_k$, given $p \in \partial D_k$ and given any neighbourhood $U \ni p$, U has non-empty intersection with $M \setminus \operatorname{Cl}(D_k)$. By the same reasoning as above, p is again is a limit point of arbitrarily small domains in the collection $\mathcal{D}$. Thus we have proved the claim for all points $p \in S$ and this concludes the proof. □

Next, we turn to the *complex dilatation* of a diffeomorphism $f \in \operatorname{Diff}(M)$ and its behaviour under compositions of diffeomorphisms, see e.g. [4]. We first consider the case where $f \in \operatorname{Diff}(\mathbb{C})$. The complex dilatation μ_f of f is defined by

$$\mu_f \colon \mathbb{C} \to \mathbb{D}^2, \ \mu_f(p) = \frac{f_{\bar{z}}}{f_z}(p), \tag{3}$$

and the corresponding differential

$$\mu_f(p) \frac{d\bar{z}}{dz}, \tag{4}$$

is the *Beltrami differential* of f. The *dilatation* of f is defined by

$$K_f(p) = \frac{1 + |\mu_f(p)|}{1 - |\mu_f(p)|}, \tag{5}$$

which equals

$$K_f(p) = \frac{\max_v |Df_p(v)|}{\min_v |Df_p(v)|}, \tag{6}$$

where v ranges over the unit circle in $T_p\mathbb{C}$ and the norm $|\cdot|$ is induced by the standard (conformal) Euclidean metric g on $\mathbb{C}$. Denote $[\cdot,\cdot]$ be the hyperbolic distance in $\mathbb{D}^2$, i.e. the distance induced by the Poincaré metric on $\mathbb{D}^2$. When one composes two diffeomorphisms $f, g\colon \mathbb{C}\to\mathbb{C}$, then

$$\mu_{g\circ f}(p) = \frac{\mu_f(p) + \theta_f(p)\mu_g(f(p))}{1 + \overline{\mu_f(p)}\theta_f(p)\mu_g(f(p))}, \tag{7}$$

where $\theta_f(p) = \frac{\overline{f_z}}{f_z}(p)$. It follows that

$$\mu_{f^{n+1}}(p) = \frac{\mu_f(p) + \theta_f(p)\mu_{f^n}(f(p))}{1 + \overline{\mu_f(p)}\theta_f(p)\mu_{f^n}(f(p))}. \tag{8}$$

We can rewrite (7) as

$$\mu_{g\circ f}(p) = T_{\mu_f(p)}(\theta_f(p)\mu_g(f(p))) \tag{9}$$

where

$$T_a(z) = \frac{a + z}{1 + \bar{a}z} \in \text{Möb}(\mathbb{D}^2) \tag{10}$$

is an isometry relative to the Poincaré metric, for a given $a \in \mathbb{D}^2$. Further, the following relation holds

$$\log(K_{g\circ f^{-1}}(f(p))) = [\mu_g(p), \mu_f(p)]\,. \tag{11}$$

To define the complex (and maximal) dilatation of a diffeomorphism of a surface M, we first lift $f\colon M\to M$ to the universal cover $\widetilde{f}\colon \widetilde{M}\to\widetilde{M}$ and denote $\pi\colon \widetilde{M}\to M$ be the corresponding canonical projection mapping, where $M = \widetilde{M}/\Gamma$, with Γ a Fuchsian group. We assume here that $\widetilde{M}$ is either $\mathbb{C}$ or $\mathbb{D}^2$, the trivial case of the sphere $\mathbb{P}^1$ is excluded here. As π is an analytic local diffeomorphism, $\widetilde{f}$ is a diffeomorphism. Further, as M is compact, f is K-quasiconformal on M for some $K \geq 1$ and thus $\widetilde{f}$ is K-quasiconformal on $\widetilde{M}$. Since $\widetilde{f}\circ h\circ\widetilde{f}^{-1}$ is conformal for every $h \in \Gamma$, it follows from (7) that

$$\mu_{\widetilde{f}}(p) = \mu_{\widetilde{f}}(h(p))\frac{\overline{h_z}}{h_z}(p). \tag{12}$$

In other words, $\mu_{\widetilde{f}}$ defines a Beltrami differential on $\widetilde{M}$ for the group Γ, or equivalently, it defines a Beltrami differential for f on the surface M. Furthermore, the same formulas (5) and (6), defined relative to the canonical (conformal) metric defined on M, hold for the dilatation K_f of f on M.

The following lemma shows that the bounded geometry assumption of the domains has a strong effect on the dilatation of iterates of f on S. We say f has *uniformly bounded dilatation* on $S \subset M$, if $K_{f^n}(p)$ is bounded by a constant independent of $n \in \mathbb{Z}$ and $p \in S$.

Lemma 2.3 (Bounded dilatation)**.** *Let $f \in \mathrm{Diff}^1(M)$ permute a dense collection of domains $\mathcal{D}$. If the collection $\mathcal{D}$ has bounded geometry, then f has uniformly bounded dilatation on S.*

Proof. Suppose the collection of domains $\mathcal{D} = \{D_k\}_{k\in\mathbb{Z}}$ has β-bounded geometry for some $\beta \geq 1$. Fix $N \in \mathbb{Z}$ and $p \in S$ and take a small open neigbhourhood $U \subset M$ containing p. By Lemma 2.2, there exists a subsequence of domains D_{k_t}, where $|k_t| \to \infty$ and $\mathrm{diam}(D_{k_t}) \to 0$ for $t \to \infty$ and such that $D_{k_t} \to p$. Denote $q = f^N(p) \in S$. We may as well assume that for all $t \geq 1$ the domains D_{k_t} are contained in U. Define $D'_{k_t} := f^N(D_{k_t})$. If we denote $U' = f^N(U)$, then the sequence D'_{k_t} converges to q and $D'_{k_t} \subset U'$. By the bounded geometry assumption, for every $t \geq 1$, there exists $p_t \in D_{k_t}$ and $0 < r_t \leq R_t$ such that

$$B(p_t, r_t) \subseteq D_{k_t} \subseteq B(p_t, R_t)$$

with $R_k/r_k \leq \beta$. As $f \in \mathrm{Diff}^1(M)$, the local behaviour of f^N around q converges to the behaviour of the linear map Df_q^N. In particular, if we take $p_t \in D_{k_t}$, then $p_t \to p$ and thus $q_t := f^N(q_t) \to q$, and in order for all D'_{k_t} to have β-bounded geometry, it is required that

$$K_{f^N}(p) \leq \frac{R\beta}{r}.$$

Indeed, this is easily seen to hold if the map acts locally by a linear map and is thus sufficient as $f \in \mathrm{Diff}^1(M)$ and the increasingly smaller domains approach q. As $R/r \leq \beta$, we must therefore have $K_{f^N}(p) \leq \beta^2$. As this argument holds for every (fixed) $N \in \mathbb{Z}$ and every $p \in S$, we find β^2 the uniform bound on the dilatation on S. ☐

Our smoothness assumptions on f allow us to give bounds on the (complex) dilatation of iterates of f on M in terms of the diameters of the permuted domains.

Lemma 2.4 (Sum of diameters)**.** *Let $f \in \mathrm{Diff}^{1+\alpha}(M)$, with $\alpha > 0$, which permutes a collection of domains $\mathcal{D} = \{D_k\}_{k\in\mathbb{Z}}$ with β-bounded geometry. Then there exists a constant $C = C(\beta) > 0$ such that, if $p \in D_t$ (for some $t \in \mathbb{Z}$) and $q \in \partial D_t$, then*

$$\left[\mu_{f^{n+1}}(p), \mu_{f^{n+1}}(q)\right] \leq C \cdot \sum_{s=t}^{t+n} \ell_s^\alpha, \tag{13}$$

where the domains are labeled such that $f^s(D_t) = D_{t+s}$.

To prove Lemma 2.4, we use the following.

Lemma 2.5. *Let $f \in \mathrm{Diff}^1(M)$ and $p_0, q_0 \in M$. Then*

$$\left[\mu_{f^{n+1}}(p_0), \mu_{f^{n+1}}(q_0)\right] \leq \sum_{s=0}^{n} \left[T_{\mu_f(p_s)}(\theta_f(p_s)\mu_{f^{n-s}}(q_{s+1})), T_{\mu_f(q_s)}(\theta_f(q_s)\mu_{f^{n-s}}(q_{s+1}))\right], \tag{14}$$

where $p_s = f^s(p_0)$ and $q_s = f^s(q_0)$.

Proof. Using (9), we write

$$\left[\mu_{f^{n+1}}(p_0), \mu_{f^{n+1}}(q_0)\right] = \left[T_{\mu_f(p_0)}(\theta_f(p_0)\mu_{f^n}(p_1)), T_{\mu_f(q_0)}(\theta_f(q_0)\mu_{f^n}(q_1))\right].$$

By the triangle inequality, we thus have the following inequality

$$\begin{aligned}\left[\mu_{f^{n+1}}(p_0), \mu_{f^{n+1}}(q_0)\right] &\leq \left[T_{\mu_f(p_0)}(\theta_f(p_0)\mu_{f^n}(p_1)), T_{\mu_f(p_0)}(\theta_f(p_0)\mu_{f^n}(q_1))\right] \\ &+ \left[T_{\mu_f(p_0)}(\theta_f(p_0)\mu_{f^n}(q_1)), T_{\mu_f(q_0)}(\theta_f(q_0)\mu_{f^n}(q_1))\right].\end{aligned}$$

As both T_a (as defined by (10)) and rotations are isometries in the Poincaré disk, we have that

$$\left[T_{\mu_f(p_0)}(\theta_f(p_0)\mu_{f^n}(p_1)), T_{\mu_f(p_0)}(\theta_f(p_0)\mu_{f^n}(q_1))\right] = \left[\mu_{f^n}(p_1), \mu_{f^n}(q_1)\right].$$

Inequality (14) now follows by induction. □

As $\partial D_t \subset S$, by Lemma 2.3, $\mu_{f^{n-s}}(q_{s+1}) \in B_\delta$, with $B_\delta \subset \mathbb{D}^2$ the compact hyperbolic disk centered at $0 \in \mathbb{D}^2$ with radius

$$\delta = \frac{\beta^2 - 1}{\beta^2 + 1}. \tag{15}$$

Further, define

$$\delta' = \sup_{p \in M} |\mu_f(p)| < 1, \tag{16}$$

and let $B_{\delta'} \subset \mathbb{D}^2$ be the compact hyperbolic disk centered at $0 \in \mathbb{D}^2$ and radius δ'.

Lemma 2.6. *There exists a constant $C_1(\delta, \delta')$ such that*

$$\left[T_a(z), T_b(z)\right] \leq C_1 \left[a, b\right], \tag{17}$$

for given $a, b \in B_{\delta'}$ and $z \in B_\delta$.

Proof. First we observe that there exists a constant $0 < \delta'' < 1$ (depending only on δ and δ'), such that $[T_a(z), 0] \leq \delta''$, for every $a \in B_{\delta'}$ and every $z \in B_\delta$, as the disks $B_\delta, B_{\delta'} \subset \mathbb{D}^2$ are compact. Define $\bar{\delta} = \max\{\delta, \delta', \delta''\}$ and $B_{\bar{\delta}} \subset \mathbb{D}^2$ the compact disk with center $0 \in \mathbb{D}^2$ and radius $\bar{\delta}$.

As the Euclidean metric and the hyperbolic metric are equivalent on the compact disk $B_{\bar{\delta}}$, it suffices to show that there exists a constant $C_1'(\bar{\delta})$ such that

$$|T_a(z) - T_b(z)| \leq C_1' \, |a - b| \, , \tag{18}$$

where $|z - w|$ denotes the Euclidean distance between two points $z, w \in \mathbb{D}^2$. Indeed, if this is shown then (17) follows for a constant C_1 which differs from C_1' by a uniform constant depending only on $\bar{\delta}$. To prove (18), we compute that

$$|T_a(z) - T_b(z)| = \left| \frac{(a-b) + (a\bar{b} - \bar{a}b)z + (\bar{b} - \bar{a})z^2}{(1 + \bar{a}z)(1 + \bar{b}z)} \right| . \tag{19}$$

As $a, b \in B_{\delta'}$ and $z \in B_\delta$, there exists a constant $Q_1(\delta, \delta') > 0$ so that

$$|(1 + \bar{a}z)(1 + \bar{b}z)| \geq Q_1.$$

Therefore, it holds that

$$|T_a(z) - T_b(z)| \leq Q_1 \left(|a - b| + \delta' |a\bar{b} - \bar{a}b| + (\delta')^2 |a - b| \right) . \tag{20}$$

In order to prove (18), we show there exists a constant $Q_2(\delta') > 0$ such that

$$|a\bar{b} - \bar{a}b| \leq Q_2 |a - b|. \tag{21}$$

To this end, write $a = re^{i\phi}$ and $b = r'e^{i\phi'}$ and $x = a\bar{b}$, so that $x = rr'e^{i\nu}$ with $\nu = \phi - \phi'$. We may assume that $\nu \in [0, \pi)$. It follows that $a\bar{b} - \bar{a}b = x - \bar{x} = 2irr' \sin(\nu)$. Therefore,

$$|a\bar{b} - \bar{a}b| = |x - \bar{x}| = 2rr' \sin(\nu) \leq 2\delta' r \sin(\nu), \tag{22}$$

as $r' \leq \delta'$. As the angle between the vectors $a, b \in B_{\delta'}$ is ν, it is easily seen that $|a - b| \geq r \sin(\nu)$. Combining this estimate with (22), we obtain that

$$|a\bar{b} - \bar{a}b| \leq 2\delta' r \sin(\nu) \leq 2\delta' |a - b|. \tag{23}$$

Setting $Q_2 = 2\delta'$ yields (21). If we now combine (23) in turn with (20), we obtain a uniform constant

$$C_1'(\delta, \delta') = Q_1(1 + \delta' Q_2 + (\delta')^2) = Q_1(1 + 3(\delta')^2)$$

for which (18) holds, as required. □

Proof of Lemma 2.4. As $f \in \mathrm{Diff}^{1+\alpha}(M)$, we have that $\mu_f(p) \in C^\alpha(M, \mathbb{D}^2)$ and $\theta_f \in C^\alpha(M, \mathbb{C})$, are uniformly Hölder continuous by compactness of M. By the triangle inequality, we can estimate the summand in the right-hand side of (14) of Lemma 2.5 as

$$\left[T_{\mu_f(p_s)}(\theta_f(p_s)\mu_{f^{n-s}}(q_{s+1})), T_{\mu_f(q_s)}(\theta_f(q_s)\mu_{f^{n-s}}(q_{s+1}))\right] \leq \tag{24}$$

$$\left[T_{\mu_f(p_s)}(\theta_f(p_s)\mu_{f^{n-s}}(q_{s+1})), T_{\mu_f(q_s)}(\theta_f(p_s)\mu_{f^{n-s}}(q_{s+1}))\right] + \tag{25}$$

$$\left[T_{\mu_f(q_s)}(\theta_f(p_s)\mu_{f^{n-s}}(q_{s+1})), T_{\mu_f(q_s)}(\theta_f(q_s)\mu_{f^{n-s}}(q_{s+1}))\right]. \tag{26}$$

To estimate (25), define

$$z_s := \theta_f(p_s)\mu_{f^{n-s}}(q_{s+1}) \in B_\delta \text{ and } a_s = \mu_f(p_s), b_s = \mu_f(q_s) \in B_{\delta'} \subset \mathbb{D}^2.$$

Then (25) reads

$$\left[T_{\mu_f(p_s)}(\theta_f(p_s)\mu_{f^{n-s}}(q_{s+1})), T_{\mu_f(q_s)}(\theta_f(p_s)\mu_{f^{n-s}}(q_{s+1}))\right] = \left[T_{a_s}(z_s), T_{b_s}(z_s)\right]. \tag{27}$$

By Lemma 2.6, there exists a constant $C_1 > 0$ such that

$$[T_{a_s}(z_s), T_{b_s}(z_s)] \leq C_1[a_s, b_s]. \tag{28}$$

By Hölder continuity of μ_f, there exists a constant $\widehat{C}_1$ such that

$$[a_s, b_s] \leq \widehat{C}_1(d(p_s, q_s))^\alpha. \tag{29}$$

Therefore, combining equations (27), (28) and (29), we obtain that

$$\left[T_{\mu_f(p_s)}(\theta_f(p_s)\mu_{f^{n-s}}(q_{s+1})), T_{\mu_f(q_s)}(\theta_f(p_s)\mu_{f^{n-s}}(q_{s+1}))\right] \leq \widetilde{C}_1 \ell_{t+s}^\alpha, \tag{30}$$

as $d(p_s, q_s) \leq \ell_{t+s}$, with $\widetilde{C}_1 := C_1 \widehat{C}_1$.

To estimate (26), we note that the hyperbolic distance and the Euclidean distance are equivalent on the compact disk B_δ. Therefore, as the (Euclidean) distance between a point $z \in B_\delta$ and $e^{i\phi}z$ is bounded from above by a constant (depending only on δ) multiplied by the angle $|\phi|$, by Hölder continuity of θ_f there exists a constant $\widetilde{C}_2(\delta)$, such that

$$\left[\theta_f(p)z, \theta_f(p')z\right] \leq \widetilde{C}_2(d(p, p'))^\alpha,$$

for all $z \in B_\delta$ and $p, p' \in M$, using the local equivalence of the hyperbolic and Euclidean metric. Hence, (26) reduces to

$$\left[\theta_f(p_s)\mu_{f^{n-s}}(q_{s+1}), \theta_f(q_s)\mu_{f^{n-s}}(q_{s+1})\right] \leq \widetilde{C}_2 d(p_s, q_s))^\alpha \leq \widetilde{C}_2 \ell_{t+s}^\alpha, \tag{31}$$

as $d(p_s, q_s) \leq \ell_{t+s}$. Therefore, if we set $C := \widetilde{C}_1 + \widetilde{C}_2$, then (13) follows. □

2.2 Upper bounds on the entropy of a surface diffeomorphism

Next, we relate the topological entropy of a diffeomorphism to its dilatation.

Lemma 2.7 (Entropy and dilatation). *Let* $f \in \mathrm{Diff}^{1+\alpha}(M)$ *with* $\alpha > 0$. *Then*

$$h_{\rm top}(f) \leq \lim_{n\to\infty} \sup \frac{1}{2n} \log \int_M K_{f^n}(p) d\lambda(p), \tag{32}$$

with K_f *the dilatation of* f.

To prove this we use a result of F. Przytycki [8]. We need the following notation. Let $L : \mathbb{R}^m \to \mathbb{R}^m$ be a linear map and $L^{k\wedge} : \mathbb{R}^{m\wedge k} \to \mathbb{R}^{m\wedge k}$ the induced map on the k-th exterior algebra of $\mathbb{R}^m$. $L^\wedge$ denotes the induced map on the full exterior algebra. The norm $\|L^{k\wedge}\|$ of L^k has the following geometrical meaning. Let $\mathrm{Vol}_k(v_1, ..., v_k)$ be the k-dimensional volume of a parallelepiped spanned by the vectors $v_1, ..., v_k$, where $v_i \in \mathbb{R}^m$ with $1 \leq i \leq k$. Then

$$\|L^{k\wedge}\| = \sup_{v_i \in \mathbb{R}^m} \frac{\mathrm{Vol}_k(L(v_1), ..., L(v_k))}{\mathrm{Vol}_k(v_1, ..., v_k)}, \tag{33}$$

$$\|L^\wedge\| = \max_{1\leq k\leq m} \|L^{k\wedge}\|. \tag{34}$$

Further, let

$$\|L\| = \sup_{|v|=1} |L(v)|, \tag{35}$$

the standard norm on operators, with $v \in \mathbb{R}^m$ and $|\cdot|$ induced by the corresponding inner product on $\mathbb{R}^m$. The following result is due to F. Przytycki [8] (see also [3]).

Theorem 2.8. *Given a smooth, closed Riemannian manifold* M *and* $f \in \mathrm{Diff}^{1+\alpha}(M)$ *with* $\alpha > 0$. *Then*

$$h_{\rm top}(f) \leq \lim_{n\to\infty} \sup \frac{1}{n} \log \int_M \|(Df^n)^\wedge\| d\lambda(p). \tag{36}$$

where $h_{\rm top}(f)$ *is the topological entropy of* f, λ *is a Riemannian measure on* M *induced by a given Riemannian metric,* $Df^{n\wedge}$ *is a mapping between exterior algebras of the tangent spaces* T_pM *and* $T_{f^n(p)}M$, *induced by the* Df^n_p *and* $\|\cdot\|$ *is the norm on operators, induced from the Riemannian metric.*

Proof of Lemma 2.7. Fix $p \in M$ and let $Df^n_p : T_pM \to T_{f^n(p)}M$. Then

$$\|Df^n_p\|^2 = K_{f^n}(p) J_{f^n}(p).$$

Thus

$$\|(Df^n_p)^{1\wedge}\| = \sqrt{K_{f^n}(p) J_{f^n}(p)}, \text{ and } \|(Df^n_p)^{2\wedge}\| = J_{f^n}(p). \tag{37}$$

It follows that

$$\|(Df_p^n)^\wedge\| = \max\left\{\sqrt{K_{f^n}(p)J_{f^n}(p)}, J_{f^n}(p)\right\}. \tag{38}$$

As

$$\max\left\{\sqrt{K_{f^n}(p)J_{f^n}(p)}, J_{f^n}(p)\right\} \leq \sqrt{K_{f^n}(p)J_{f^n}(p)} + J_{f^n}(p),$$

we have that

$$\begin{aligned}\int_M \|(Df_p^n)^\wedge\| d\lambda(p) &\leq \int_M \left(\sqrt{K_{f^n}J_{f^n}} + J_{f^n}\right) d\lambda \\ &= \lambda(M) + \int_M \sqrt{K_{f^n}J_{f^n}} d\lambda\end{aligned}$$

as $\lambda(M) = \int_M J_{f^n} d\lambda$, for every $n \in \mathbb{Z}$. Either $\int_M \sqrt{K_{f^n}J_{f^n}} d\lambda$ is bounded as a sequence in n, in which case (32) holds trivially, or the sequence is unbounded in n, in which case it is readily verified that

$$\lim_{n\to\infty}\sup \frac{1}{n}\log\left(\lambda(M) + \int_M \sqrt{K_{f^n}J_{f^n}} d\lambda\right) = \lim_{n\to\infty}\sup \frac{1}{n}\log\int_M \sqrt{K_{f^n}J_{f^n}} d\lambda.$$

By the Cauchy-Schwartz inequality, we have that

$$\int_M \sqrt{K_{f^n}J_{f^n}} d\lambda \leq \sqrt{\lambda(M)} \cdot \sqrt{\int_M K_{f^n} d\lambda}.$$

and thus,

$$\log\int_M \sqrt{K_{f^n}J_{f^n}} d\lambda \leq \frac{1}{2}\log\lambda(M) + \frac{1}{2}\log\int_M K_{f^n} d\lambda.$$

It now follows that

$$\lim_{n\to\infty}\sup \frac{1}{n}\log\int_M \|(Df^n)^\wedge\| d\lambda \leq \lim_{n\to\infty}\sup \frac{1}{2n}\log\int_M K_{f^n} d\lambda.$$

and this proves (32). □

2.3 Proof of Theorem A

Let us now complete the proof. Let $f \in \mathrm{Diff}_A^{1+\alpha}(M)$, with $\alpha > 0$, and suppose that f permutes a dense collection of domains $\{D_k\}_{k\in\mathbb{Z}}$ with bounded geometry. By Lemma 2.1, the sequence ℓ_k is a null-sequence. Therefore, ℓ_k^α is a null-sequence as well, for every $\alpha > 0$. Let $p \in D_t$ for some $t \in \mathbb{Z}$ and $q \in \partial D_t$ and label the domains such that $f^s(D_t) = D_{t+s}$. By (11),

$$\log K_{f^n}(f(p)) = [\mu_{f^{n+1}}(p), \mu_f(p)]$$

and thus, by the triangle inequality,

$$\log K_{f^n}(f(p)) \leq \left[\mu_{f^{n+1}}(p), \mu_{f^{n+1}}(q)\right] + \left[\mu_{f^{n+1}}(q), \mu_f(p)\right] \tag{39}$$

As the second term in the right hand side of (39) stays uniformly bounded, we have that

$$\log K_{f^n}(f(p)) \leq \left[\mu_{f^{n+1}}(p), \mu_{f^{n+1}}(q)\right] + C' \tag{40}$$

for some constant $C' > 0$, independent of $p \in M$ and $n \in \mathbb{Z}$. Define

$$\xi(n) = \max \sum_{i=0}^{n} \ell_{k_i}^{\alpha}$$

where the maximum is taken over all collections of $n+1$ distinct elements $\{D_{k_0}, ..., D_{k_n}\}$ of $\mathcal{D}$. As ℓ_k^α is a null-sequence, we have that

$$\lim_{n\to\infty} \sup \frac{\xi(n)}{n} = 0. \tag{41}$$

By Lemma 2.4, we have that

$$\left[\mu_{f^{n+1}}(p), \mu_{f^{n+1}}(q)\right] \leq C \cdot \sum_{s=t}^{t+n} \ell_s^\alpha,$$

for some constant $C > 0$. Combined with (40), we obtain the following uniform estimate

$$\log K_{f^n}(f(p)) \leq C\xi(n) + C', \tag{42}$$

for every $p \in M$ and $n \in \mathbb{Z}$. Therefore

$$\begin{aligned}
\log \int_M K_{f^n} d\lambda &\leq \log \int_M \exp(C\xi(n) + C')d\lambda && (43)\\
&= \log\big((\exp(C\xi(n) + C')\lambda(M)\big) && (44)\\
&= C\xi(n) + C' + \log(\lambda(M)). && (45)
\end{aligned}$$

Combining (45) in turn with Lemma 2.7 yields

$$h_{\mathrm{top}}(f) \leq \lim_{n\to\infty} \sup \frac{1}{2n} \log \int_M K_{f^n} d\lambda \leq C \lim_{n\to\infty} \sup \frac{\xi(n)}{2n} = 0, \tag{46}$$

by (41). This proves Theorem A.

3 Concluding remarks

The proof of Theorem A, more precisely condition (41) in section 2.3, fails in the case where the Hölder constant $\alpha = 0$. This leads to the following natural

Question 1 (Differentiable counterexamples)**.** *Do there exist diffeomorphisms $f \in \mathrm{Diff}^1(M)$ with positive entropy that permute a dense collection of domains with bounded geometry?*

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