CONSTRUCTION OF SUBSURFACES VIA GOOD PANTS

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Abstract. This is a survey on the good pants construction and its applications.

The good pants technology is a systematic method that has been developed during the past few years to produce surface subgroups in cocompact lattices of $\mathrm{PSL}_2(\mathbf{C})$ or $\mathrm{PSL}_2(\mathbf{R})$ using frame flow. The $\mathrm{PSL}_2(\mathbf{C})$ case leads to a proof of the Surface Subgroup Conjecture, which plays a fundamental role in the resolution of the Virtual Haken Conjecture about 3-manifold topology, and the $\mathrm{PSL}_2(\mathbf{R})$ case leads to a proof of the Ehrenpreis Conjecture on Riemann surfaces. Prior to the work of Jeremy Kahn and Vladimir Markovic, Lewis Bowen first attempted to build surface subgroups by assembling pants subgroups using horocycle flow. This article is intended to guide the readers to these constructions due to Jeremy Kahn and Vladimir Markovic, and survey on some applications of their construction afterwards due to Yi Liu, Hongbin Sun, Ursula Hamenstädt and others.

1. Good pants and nice gluing

For any cocompact lattice of $\mathrm{PSL}_2(\mathbf{C})$ or $\mathrm{PSL}_2(\mathbf{R})$, any surface subgroup resulting from the Kahn–Markovic construction can be thought of as the subgroup $\pi_1(S)$ corresponding to a π_1-injectively immersed subsurface S of the quotient orbifold M of the underlying symmetric spaces $\mathbb{H}^3$ or $\mathbb{H}^2$ by the lattice. This allows us to discuss the construction from the perspective of hyperbolic geometry. In this section, we set up the general framework of the construction.

Without loss of generality, we may pass to a sublattice of finite index, and assume hereafter that M is a closed orientable hyperbolic manifold of dimension 3 or 2. The construction can be performed for any positive constant ϵ. It gives rise to a closed orientable subsurface S of the closed hyperbolic orbifold M that is $(1+\epsilon)$-bilipschitz equivalent to a finite cover of a model surface $S_0(R)$, as we describe in the following. The positive parameter R can be chosen arbitrarily as long as R is sufficiently large depending on ϵ and the lattice, and $S_0(R)$ has a defining pants decomposition which lifts to be a pants decomposition of the finite cover.

The model surface $S_0(R)$ is an orientable closed surface of genus 2, equipped with a hyperbolic structure described in term of a pants decomposition and its Fenchel–Nielsen parameters. For any positive constant R, a disk with two cone points of order 3 can be endowed with a unique hyperbolic structure to become a hyperbolic 2-orbifold with boundary length R. The unique planar surface cover of minimal degree of this 2-orbifold is a hyperbolic pair of pants $\Pi(R)$ with cuff length R which is symmetric under an isometric action of the 3-cyclic group. Take two

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oppositely oriented copies of the hyperbolic pair of pants, $\Pi(R)$ and $\bar{\Pi}(R)$. We glue $\Pi(R)$ and $\bar{\Pi}(R)$ by identifying their cuffs accordingly, but modify the gluing by a twist of length 1 along each identified curve in the direction induced from boundary of the pants. Note that there is no ambiguity for the twisting direction since the induced direction from $\Pi(R)$ and $\bar{\Pi}(R)$ are opposite to each other. The resulting hyperbolic surface is denoted as $S_0(R)$.

For universally large R, the injectivity radius of $S_0(R)$ is bounded uniformly from 0. It follows that the hyperbolic structure of $S_0(R)$ stays in a compact subset of the moduli space of hyperbolic structures on the underlying topological surface. For any finite cover $\tilde{S}_0(R)$ of $S_0(R)$ to which the pants decomposition lifts, $\tilde{S}_0(R)$ can be constructed by taking copies of the model pants and assembling according to the model gluing. The constant 1 twist in the gluing guarantees that the hyperbolic structure of $\tilde{S}_0(R)$ does not vary significantly under slight perturbation of the cuff length and the twist parameter of the gluing. The intuition leads to the following notion of good pants and nice gluing.

Let M be a closed orientable hyperbolic manifold of dimension 3 or 2. A π_1-injectively immersed pair of pants Π in M can be homotoped so that the cuffs are mutually distinct geodesically immersed curves. Fix a seam decomposition of Π into two hexagons along three arcs, called seams, which join mutually distinct pairs of cuffs. We may further homotope the immersion so that the seams are geodesic of the shortest length. Assuming that none of the seams degenerate to a point, the seams are perpendicular to the cuffs at all the endpoints, and the two hexagons are in fact isometric to each other. The inward unit tangent vectors of the seams at their endpoints are called *feets* of the seams on the cuffs. Therefore, the immersed pair of pants Π has six feets, each cuff carrying a pair of feets as unit normal vectors at an antipodal pair of points. We say that Π is (R,ϵ)-*good* if the length of each cuff of Π is approximately R up to error ϵ, and if the parallel transportation of a foot on each cuff to the antipodal point along the cuff is approximately the other foot up to error $(\epsilon\,/\,2)$, measured in the canonical metric on the unit vector bundle of M. Note that the second condition is automatically satisfied if M has dimension 2.

An immersed oriented subsurface S of M is said to be (R,ϵ)-*panted* if it has a decomposition along simple closed curves into pairs of pants, and the immersion restricted to each pair of pants is (R,ϵ)-good. In general, S is not necessarily π_1-injective since we have not controlled the gluing. We say that S is constructed from the (R,ϵ)-good pants by an (R,ϵ)-*nice* gluing, if for each decomposition curve c shared by pairs of pants P and P', the parallel transportation of distance 1 along c of any foot of P on c, in the direction induced from P, is approximately opposite to a foot of P' up to error $(\epsilon\,/\,R)$.

Theorem 1.1. *Let M be a closed orientable hyperbolic manifold of dimension 3 or 2. The following statement holds for any small positive constant ϵ depending on M and sufficiently large positive constant R depending on ϵ and M.*

Suppose that S is an immersed closed (R,ϵ)-panted subsurface of M constructed by an (R,ϵ)-nice gluing. Then S is π_1-injective and geometrically finite. Furthermore, S can be homotoped so that with respect to the path-induced metric, S is $K(\epsilon)$-bilipshitz equivalent to a finite cover of the model surface $S_0(R)$, where $K(t)$ is a positive function depending on M such that $K(t)\to 1$ as $t\to 0$.

See [Kahn–Markovic 2014, Theorem 2.1] and [Kahn–Markovic 2012, Theorem 2.1].

The construction of Kahn and Markovic produces closed subsurfaces of M by gluing a finite collection of good pants in a nice fashion. Theorem 1.1 describes the geometry of the resulting subsurfaces. To find a finite collection of good pants that admits a nice gluing is the difficult part of their construction, which can be reformulated as a problem of linear programing regarding measures of good pants.

Denote by $\mathbf{\Pi}_{R,\epsilon}$ the collection of the homotopy classes of oriented (R,ϵ)-good pants of M. Any finite collection of (R,ϵ)-good pants can be recorded by a counting measure over the set $\mathbf{\Pi}_{R,\epsilon}$. Denote by $\mathbf{\Gamma}_{R,\epsilon}$ the collection of the homotopy classes of (R,ϵ)-good curves of M, namely, geodesically immersed oriented loops of M of length approximately R and monodromy approximately trivial, up to error ϵ. There is a natural boundary operator between Borel measures over the sets of good pants and good curves:

$$\partial\colon \mathcal{M}(\mathbf{\Pi}_{R,\epsilon}) \to \mathcal{M}(\mathbf{\Gamma}_{R,\epsilon}),$$

which takes the atomic measure supported over any good pants Π to the sum of atomic measures supported over the cuffs of Π. The boundary operator ∂ can be lifted to an operator ranged in Borel measures over unit normal vectors on good curves:

$$\partial^\sharp\colon \mathcal{M}(\mathbf{\Pi}_{R,\epsilon}) \to \mathcal{M}(\mathscr{N}(\mathbf{\Gamma}_{R,\epsilon})),$$

which takes the atomic measure supported over Π to the sum of atomic measures supported over its feet. The space $\mathscr{N}(\mathbf{\Gamma}_{R,\epsilon})$ is the disjoint union of unit normal vector bundles $\mathscr{N}(\gamma)$ over good curves γ. To encode the model gluing, denote by

$$\overline{A_1}\colon \mathscr{N}(\mathbf{\Gamma}_{R,\epsilon}) \to \mathscr{N}(\mathbf{\Gamma}_{R,\epsilon})$$

the map that takes any unit normal vector n on a good curve γ to the parallel transportation of $-n$ along the orientation-reversal $\bar{\gamma}$ of distance 1.

Recall that over a metric space X, for any positive constant δ, two Borel measures μ, μ' are said to be δ-*equivalent* if $\mu(E)\leq \mu'(\mathrm{Nhd}_\delta(E))$ holds for all Borel subsets E of X and if $\mu(X)=\mu'(X)$. Being δ-equivalent is a reflexive and symmetric relation which is invariant under rescaling the measures by the same factor. We speak of approximately equivalent measures over $\mathscr{N}(\mathbf{\Gamma}_{R,\epsilon})$ with respect to its canonically induced metric.

Problem 1.2. *Find a probability measure $\mu\in\mathcal{M}(\mathbf{\Pi}_{R,\epsilon})$ such that $\partial^\sharp\mu$ is $(\epsilon\,/\,R)$-equivalent to $(\overline{A_1})_*\partial^\sharp\mu$ on $\mathscr{N}(\mathbf{\Gamma}_{R,\epsilon})$.*

A solution to Problem 1.2 implies a nontrivial integral measure in $\mathcal{M}(\mathbf{\Pi}_{R,\epsilon})$ with the same property, which gives rise to the finite collection of oriented (R,ϵ)-good pants. It turns out that these (R,ϵ)-good pants can be glued along common cuffs in an (R,ϵ)-nice fashion to create an immersed oriented closed subsurface of M, which meets the requirement of Theorem 1.1. To solve Problem 1.2, one needs to understand the statistics of good pants and the distribution of feet on good curves. The solutions to Problem 1.2 for cocompact lattices of $\mathrm{PSL}_2(\mathbf{C})$ and $\mathrm{PSL}_2(\mathbf{R})$ lead to affirmative answers to the Surface Subgroup Conjecture and the Ehrenpreis Conjecture respectively.

2. The Ehrenpreis Conjecture

For cocompact lattices of $\mathrm{PSL}_2(\mathbf{R})$, the problem of good pants construction can be resolved through dynamics of hyperbolic surfaces. In this section, we give an introduction to the proof of the Ehrenpreis Conjecture [Kahn–Markovic 2014], as an illustration how the exponential mixing property of the geodesic flow makes the construction possible.

Let S_1 and S_2 be two closed Riemann surfaces of the same genus. The Ehrenpreis Conjecture asserts that for any positive constant ϵ, there exists a $(1+\epsilon)$-quasiconformal map $f: S_1' \to S_2'$ between some finite covers of S_1 and S_2 accordingly. Leon Ehrenpreis verified the case of genus 1 and proposed the conjecture for genus at least 2, [Ehrenpreis 1970]. In hyperbolic geometry, the conjecture can be equivalently stated as two closed hyperbolic surfaces are virtually $(1+\epsilon)$-bilipschitz equivalent for any positive constant ϵ. There is no direct analog of the conjecture in higher dimensions because two closed hyperbolic manifolds of dimension at least 3 are necessarily commensurable with each other if they are mutually virtually quasi-isometric.

By Theorem 1.1, it suffices to find, for some large R, a finite cover S_i' of S_i that can be constructed by (R,ϵ)-nicely gluing (R,ϵ)-good pants. Therefore, we need to solve Problem 1.2 for cocompact lattices of $\mathrm{PSL}_2(\mathbf{R})$. In this case, a solution can be achieved through dynamics of the geodesic flow.

To illustrate the idea, let us first explain how to construct good curves and good pants in an oriented closed hyperbolic surface M using the mixing property of the geodesic flow.

Recall that the geodesic flow over an oriented closed hyperbolic surface M is a one-parameter family of morphisms of the unit tangent vector bundle

$$g_t : \mathrm{UT}(M) \to \mathrm{UT}(M)$$

which takes any unit tangent vector v at a point p of M to $g_t(v)$, the parallel transportation of v along the unit-speed geodesic ray emanating from p in the direction v for time t. The geodesic flow preserves the Liouville measure m of $\mathrm{UT}(M)$, induced from the Haar measure of $\mathrm{PSL}_2(\mathbf{R})$ by identifying $\mathrm{UT}(M)$ with the left quotient $\pi_1(M)\backslash\mathrm{PSL}_2(\mathbf{R})$. For any functions $\phi, \psi \in C^\infty(\mathrm{UT}(M))$ with integral 1 over $\mathrm{UT}(M)$, the mixing property of the geodesic flow implies:

$$\int_{\mathrm{UT}(M)} (g_t^*\phi)(v)\psi(v)\mathrm{d}m(v) \to \frac{1}{2\pi^2|\chi(M)|},$$

as t tends to $+\infty$.

For any positive constant δ, we claim that the following Connection Principle holds for sufficiently large L depending only on δ and the injectivity radius of M. For any two unit vectors v_p, v_q at points $p, q \in M$ respectively, there exists a geodesic segment connecting p and q such that the initial and terminal directions are approximately v_p and v_q respectively up to error δ. To this end, we may take a C^∞ function $\phi : \mathrm{PSL}_2(\mathbf{R}) \to [0,+\infty)$ with integral 1 over $\mathrm{PSL}_2(\mathbf{R})$, supported in a δ'-neighborhood U of the identity. For sufficiently small δ', the neighborhood $v_p \cdot U$ in $\mathrm{UT}(M)$ is isometric to U so there is an induced function ϕ_p over $\mathrm{UT}(M)$ supported on $v_p \cdot U$ defined by $\phi_p(v_p \cdot u) = \phi(u)$. In a similar fashion we define ϕ_q supported on $v_q \cdot U$. By the mixing property of the geodesic flow, the intersection of $(g_L^*\phi_p)(v_p \cdot U)$ and $v_q \cdot U$ is nonempty for sufficiently large L. In other words, there is a geodesic segment of which the initial and terminal direction vectors are δ'-close to v_p and v_q, respectively. Thus the claim follows since we can choose δ' sufficiently small, for instance, at most $10^{-2}\delta$.

In order to construct an (R,ϵ)-good curve in M for a given ϵ and any sufficiently large R, we apply the Connection Principle by taking p equal to q and v_p equal to v_q. By choosing δ to be $10^{-1}\epsilon$ and a sufficiently large L to be R, we obtain a geodesic segment with coincident endpoints p, which gives rise to a geodesic loop γ by identifying the endpoint and free homotopy. It can be verified by elementary estimation of hyperbolic geometry that γ is an (R,ϵ)-good curve. In order to construct an (R,ϵ)-good pair of pants, we may attach a nearly perpendicular bisecting arc α to γ as follows. Take a pair of antipodal points p,q on γ, and take unit normal vectors n_p, n_q at p,q by rotating the direction vectors of γ counterclockwise. By the Connecting Principle, construct a geodesic segment α connecting p and q with endpoint directions approximately n_p and $-n_q$ of length approximately $\frac{R}{2}+\log(2)$, up to error $10^{-1}\epsilon$. Attaching α to γ gives rise to a θ-shape graph, which is the spine of a unique immersed pair of pants Π up to homotopy. It can be verified, again, that Π is an (R,ϵ)-good pair of pants.

Now let us return to the original Problem 1.2 for cocompact lattices in $\mathrm{PSL}_2(\mathbf{R})$. The strategy is to solve the problem in two steps:

(1) Find a probability measure $\mu_0 \in \mathcal{M}(\mathbf{\Pi}_{R,\epsilon})$ so that $\partial^\sharp\mu_0$ is nontrivial and nearly evenly distributed over any (R,ϵ)-good curve γ, which means that $\partial^\sharp\mu_0$ is close to the Lebesgue measure on the unit normal bundle to γ.
(2) Resolve the small imbalance of boundary measures between oppositely oriented (R,ϵ)-good curves by a small correction $\mu_\phi \in \mathcal{M}(\mathbf{\Pi}_{R,\epsilon})$.

Then a solution of Problem 1.2 is given by the normalization of $\mu_0+\mu_\phi$. In the example constructions above, we have not estimated the amount of good curves and good pants in the hyperbolic surface M. The quantitative version can be achieved by knowing the exponential mixing rate of the geodesic flow. Specifically, the following theorems resolve the two steps.

To capture the imbalance of boundary measures, we introduce another boundary operator:

$$\partial_\Delta : \mathcal{M}(\mathbf{\Pi}_{R,\epsilon}) \to \mathcal{M}(\mathbf{\Gamma}_{R,\epsilon}),$$

by defining $(\partial_\Delta\mu)(\gamma) = \max\{0, (\partial\mu)_+(\gamma) - (\partial\mu)_-(\bar\gamma)\}$. Note that the unit normal vector bundle $\mathscr{N}(\gamma)$ for any (R,ϵ)-good curve γ has two components. Denote by $\mathscr{N}_+(\gamma)$ the unit normal vectors that form a special orthonormal frame with the unit direction vectors of γ. For any measure $\mu \in \mathcal{M}(\mathbf{\Pi}_{R,\epsilon})$, the restriction of the footed boundary $\partial^\sharp\mu$ to any $\mathscr{N}(\gamma)$ is hence supported on $\mathscr{N}_+(\gamma)$ (because $\mathbf{\Pi}_{R,\epsilon}$ consists of oriented good pants which have compatible orientation with the surface).

Theorem 2.1. *For some universal constants q, C and polynomial P, the following statement holds for any small ϵ depending on M and sufficiently large R depending on ϵ and M.*

There exists a probability measure $\mu_0 \in \mathcal{M}(\mathbf{\Pi}_{R,\epsilon})$ with $\partial_\Delta\mu_0$ supported on $\mathbf{\Gamma}_{R,\,\epsilon/2}$ (in fact, one may take μ_0 to be the normalized counting measure on $\mathbf{\Pi}_{R,\epsilon}$). Moreover,

- *for all $\gamma \in \mathbf{\Gamma}_{R,\epsilon}$, the restriction of $\partial^\sharp\mu_0$ to $\mathscr{N}_+(\gamma)$ is $(P(R)e^{-qR})$-equivalent to $(\partial\mu_0)(\gamma)$ times the Lebesgue measure;*

- *for all* $\gamma \in \mathbf{\Gamma}_{R,\epsilon/2}$*, the value* $(\partial\mu_0)(\gamma)$ *is at least* Ce^{-2R}*, and* $(\partial_\Delta\mu_0)(\gamma)$ *is at most* $P(R)e^{-(2+q)R}$.

Theorem 2.2. *For any measure* $\nu \in \mathcal{M}(\mathbf{\Gamma}_{R,\epsilon})$ *which is contained in the subspace* $\partial_\Delta\mathcal{M}(\mathbf{\Pi}_{R,\epsilon})$*, there exists a measure* $\phi(\nu) \in \mathcal{M}(\mathbf{\Pi}_{R,\epsilon})$ *such that* $\partial_\Delta\phi(\nu)$ *equals* ν*, and moreover,*

$$\|\phi(\nu)\| \leq P(R)e^{-R}\|\nu\|,$$

where $\|\cdot\|$ *is understood as the maximum value at individual* (R,ϵ)*-pants or* (R,ϵ)*-curves.*

See [Kahn–Markovic 2014, Theorems 3.1, 3.4, Lemma 3.3].

Starting with the measure $\mu_0 \in \mathcal{M}(\mathbf{\Pi}_{R,\epsilon})$ of Theorem 2.1, we may construct a correction term $\mu_\phi \in \mathcal{M}(\mathbf{\Pi}_{R,\epsilon})$ using Theorem 2.2, by defining μ_ϕ to be $-\phi(\partial_\Delta\mu_0)$ where $\bar{\mu}_0$ is the measure induced by the orientation reversion on $\mathbf{\Pi}_{R,\epsilon}$. It can be verified that $\mu_0+\mu_\phi$ is $(P(R)e^{-qR})$-equivalent to the Lebesgue measure restricted to any unit normal vector bundle $\mathscr{N}(\gamma)$, and that $\partial_\Delta(\mu_0+\mu_\phi)$ vanishes. By taking R sufficiently large, the normalization of $\mu_0+\mu_\phi$ provides a solution to Problem 1.2.

To close the present section, we remark that from Theorem 2.2 one can develop a quantitative good correction theory. The qualitative part of the theory is the so-called Good Pants Homology, which has been generalized to the $\mathrm{PSL}_2(\mathbf{C})$ case as we discuss in the next section. The quantitative part of the theory refers to the estimation of $\phi(\nu)$ in Theorem 2.2. The estimation can be achieved through a procedure called randomization that strengthens arguments involved the study of Good Pants Homology.

3. Surface Subgroup Conjecture

Most techniques of the good pants construction in $\mathrm{PSL}_2(\mathbf{R})$ find their counterparts in $\mathrm{PSL}_2(\mathbf{C})$. In this section, we introduce the proof of the Surface Subgroup Conjecture, which asserts that any cocompact lattice of $\mathrm{PSL}_2(\mathbf{C})$ contains a surface subgroup, [Kahn–Markovic 2012]. Then we discuss the generalization of the qualitative good correction theory for $\mathrm{PSL}_2(\mathbf{C})$, [Liu–Markovic 2014].

The Surface Subgroup Conjecture is related to the Virtual Haken Conjecture in 3-manifold topology. Essentially embedded subsurfaces of closed 3-manifolds are important objects in 3-manifold topology. Such subsurfaces can be constructed via topological methods when the fundamental group of the 3-manifold admits a nontrivial splitting. As a potential approach to the Poincaré Conjecture, Friedhelm Waldhausen conjectured in the 1960s that every closed 3-manifold with an infinite fundamental group has a finite cover which contains an essentially embedded subsurface, or in other words, is virtually Haken [Waldhausen 1968]. By the Geometrization Conjecture of William P. Thurston, confirmed by Grigori Perelman in 2003, Waldhausen's Virtual Haken Conjecture can be reduced to the case of closed hyperbolic 3-manifolds. Hence the Surface Subgroup Conjecture becomes a natural first step towards Waldhausen's conjecture. In general, a π_1-injectively immersed subsurface in a closed hyperbolic 3-manifold can be either geometrically infinite or quasi-Fuchsian. The only currently known way to construct geometrically infinite subsurfaces invokes the Sutured Manifold Hierarchy as well as the Virtual Special Cubulation, [Agol 2008, Gabai 1983, Wise 2011]. In particular, it requires passage to finite covers and offers no effective control on the geometry of the constructed subsurface. On the other hand, many arithmetic hyperbolic 3-manifolds are known to contain immersed Fuchsian subsurfaces, and hyperbolic surface bundles are known to contain immersed quasi-Fuchsian subsurfaces with arbitrarily thick hull, [Cooper–Long–Reid 1997, Masters 2006]. The Virtual Haken Conjecture has been proved by Ian Agol [Agol 2013] relying on [Kahn–Markovic 2012, Wise 2011].

The subsurfaces constructed through good pants techniques are π_1-injectively immersed and geometrically finite, as one can imply from Theorem 1.1. In fact, one can produce these subsurfaces to be arbitrarily nearly totally geodesic, in the sense that the limit set of a corresponding quasi-Fuchsian subgroup can be required to be contained in any given neighborhood of a round circle in the ideal boundary of $\mathbb{H}^3$. Once again, the task reduces to addressing Problem 1.2 for an oriented closed hyperbolic 3-manifold M. However, this time the solution is even easier since we can pass around the good correction theory. The $\mathrm{PSL}_2(\mathbf{C})$ case was actually resolved earlier than the $\mathrm{PSL}_2(\mathbf{R})$ case.

The first modification to the previous argument for $\mathrm{PSL}_2(\mathbf{R})$ is to employ the frame flow on the special orthonormal frame bundle $\mathrm{SO}(M)$ instead of the geodesic flow on the unit vector bundle $\mathrm{UT}(M)$. The reason is that good curves in closed hyperbolic 3-manifolds must be prescribed to have approximately trivial monodromy besides approximately given length, and similarly, good pants must have small bending in the normal direction. The mixing rate of the frame flow is known to be exponential. A weak analog of Theorem 2.1 is the following:

Theorem 3.1. *For some universal constants q and polynomial P, the following statement holds for any positive constant ϵ and sufficiently large R depending only on ϵ and M.*

There exists a probability measure $\mu_0 \in \mathcal{M}(\mathbf{\Pi}_{R,\epsilon})$ such that for all $\gamma \in \mathbf{\Gamma}_{R,\epsilon}$, the restriction of $\partial^\sharp\mu_0$ to $\mathscr{N}(\gamma)$ is $(P(R)e^{-qR})$-equivalent to $(\partial\mu_0)(\gamma)$ times the Lebesgue measure.

See [Kahn–Markovic 2012, Theorem 3.4].

Denote by $\bar{\mu}_0$ the measure induced by the orientation reversion on $\mathbf{\Pi}_{R,\epsilon}$. Then the probability measure $(\mu_0 + \bar{\mu}_0)\,/\,2$ yields a solution to Problem 1.2, which completes the proof of the Surface Subgroup Conjecture for cocompact lattices of $\mathrm{PSL}_2(\mathbf{C})$.

Before moving on to the correction theory, we make two technical remarks regarding Theorems 2.1 and 3.1:

(1) Current techniques of good pants construction do not apply to non-cocompact lattices in $\mathrm{PSL}_2(\mathbf{R})$ or $\mathrm{PSL}_2(\mathbf{C})$. The argument in the previous section to construct good curves or good pants would fail without the assumption of positive injectivity radius, so the existence of μ_0 is no longer guaranteed;
(2) The averaging trick in the $\mathrm{PSL}_2(\mathbf{C})$ case fails in the $\mathrm{PSL}_2(\mathbf{R})$ case essentially because the flip transformation of $\mathscr{N}(\mathbf{\Gamma}_{R,\epsilon})$ that takes any unit normal vector n to $-n$ is orientation reversing. The fact that the flip transformation is central but not contained in the identity component of the isometry group of $\mathscr{N}(\mathbf{\Gamma}_{R,\epsilon})$ obstructs μ_0 from being nearly symmetric under flipping. Similar technical obstructions can be observed for cocompact lattices in $\mathrm{SO}(2m, 1)$.

A qualitative correction theory has been developed for cocompact lattices in $\mathrm{PSL}_2(\mathbf{C})$. We state the result in terms of the good panted cobordism group.

Let M be an oriented closed hyperbolic 3-manifold. A possibly disconnect oriented immersed 1-submanifold L of M is called an (R,ϵ)-multicurve, if each component of L is homotopic to an (R,ϵ)-good curve. Two (R,ϵ)-multicurves L, L' are said to be (R,ϵ)-*panted cobordant* if there exists a possibly disconnected (R,ϵ)-panted subsurface bounded by $L \sqcup \bar{L}'$ (a (R,ϵ)-panted subsurface is a subsurface that comes with a decomposition into (R,ϵ) good pants). For any universally small ϵ and sufficiently large R, being (R,ϵ)-panted cobordant is an equivalence relation. Then we define the (R,ϵ)-*panted cobordism group* of M to be the set of (R,ϵ)-panted cobordism classes $[L]_{R,\epsilon}$ of (R,ϵ)-multicurves L, denoted as $\mathbf{\Omega}_{R,\epsilon}(M)$. This is a finitely generated Abelian group with the addition induced by the disjoint union operation and the inverse induced by the orientation reversion.

Theorem 3.2. *Let M be an oriented closed hyperbolic 3-manifold. For any universally small positive ϵ, and any sufficiently large positive R depending only on M and ϵ, there is a canonical isomorphism*

$$\Phi:\ \mathbf{\Omega}_{R,\epsilon}(M) \longrightarrow H_1(\mathrm{SO}(M);\mathbf{Z}),$$

where $\mathrm{SO}(M)$ denotes the bundle over M of special orthonormal frames with respect to the orientation of M. Moreover, for all $[L]_{R,\epsilon} \in \mathbf{\Omega}_{R,\epsilon}(M)$, the image of $\Phi([L]_{R,\epsilon})$ under the bundle projection is the homology class $[L] \in H_1(M;\mathbf{Z})$.

See [Liu–Markovic 2014, Theorem 5.2].

Since $H_1(\mathrm{SO}(M);\,\mathbf{Z})$ is a split extension of $H_1(M;\,\mathbf{Z})$ by $\mathbf{Z}_2$, the only obstruction for a null-homologous good multicurve to bound a good panted subsurface lies in the center $\mathbf{Z}_2$.

For an oriented closed hyperbolic surface M, the Good Pants Homology of M introduced in [Kahn–Markovic 2014] may be equivalently defined as $\mathbf{\Omega}_{R,\epsilon}(M)\otimes\mathbf{Q}$. It has been shown there that $\mathbf{\Omega}_{R,\epsilon}(M)\otimes\mathbf{Q}$ is canonically isomorphic to $H_1(M;\,\mathbf{Q})$, (see also [Calegari 2009]).

4. Further applications

There have been a number of proceedings of good pants constructions since [Kahn–Markovic 2012, Kahn–Markovic 2014]. Ursula Hamenstädt has considered good pants constructions for cocompact lattices of broader families of Lie groups.

Theorem 4.1 ([Hamenstädt 2014])**.** *Every cocompact lattice in a rank one simple Lie group other than $\mathrm{SO}(2m,1)$ for positive integers m contains a surface subgroup.*

The complete list of Lie groups in Hamenstädt's result consists of $\mathrm{SO}(2m+1,1)$, $\mathrm{SU}(n,1)$, $\mathrm{Sp}(n,1)$, and F_4^{-20}. It is also conjectured in [Hamenstädt 2014] the existence of surface subgroups for cocompact irreducible lattices in semisimple Lie groups with finite center, without compact factors and without factors locally isomorphic to $\mathrm{SL}_2(\mathbf{R})$.

The good correction theory for $\mathrm{PSL}_2(\mathbf{C})$ can be applied to construct bounded π_1-injectively immersed subsurface in closed hyperbolic 3-manifolds. The following theorem is the generalization of a result of Danny Calegari in the case of hyperbolic surfaces [Calegari 2009].

Theorem 4.2 ([Liu–Markovic 2014])**.** *Every π_1-injectively immersed oriented closed 1-submanifold in a closed hyperbolic 3-manifold which is rationally null-homologous admits an equidegree finite cover which bounds an oriented connected compact π_1-injective immersed quasi-Fuchsian subsurface.*

The fact that the good correction theory yields a finite cover of a closed hyperbolic surface can be generalized to a procedure called homological substitution as introduced in [Liu–Markovic 2014]. It replaces any π_1-injectively immersed one-vertexed 2-complex with a good panted complex so that the 1-cells correspond to a collection of good multicurves, and the 2-cells correspond to a collection of good panted subsurfaces. Homological substitution enables us to produce nicely glued good panted subsurfaces with control on the homology class.

Theorem 4.3 ([Liu–Markovic 2014])**.** *Every rational second homology class of a closed hyperbolic* 3*-manifold has a positve integral multiple represented by an oriented connected closed* π_1*-injectively immersed quasi-Fuchsian subsurface.*

Compare the result of Danny Calegari and Alden Walker that in a random group at any positive density, many second homology classes can be rationally represented by quasiconvex (closed) surface subgroups [Calegari–Walker 2013].

Some similar idea can be found in Hongbin Sun's work on virtual properties of 3-manifolds. The following results are based on good pants constructions and invoke the separability of quasiconvex subgroups for lattices of $\mathrm{PSL}_2(\mathbf{C})$, [Agol 2013].

Theorem 4.4 ([Sun 2014a])**.** *For any finite Abelian group* A*, every closed orientable hyperbolic* 3*-manifold has a finite cover of which the first integral homology contains* A *as a direct sum component.*

Theorem 4.5 ([Sun 2014b])**.** *For any closed orientable* 3*-manifold* N*, every closed orientable hyperbolic* 3*-manifold has a finite cover that maps onto* N *of degree* 2*.*

In [Sun 2014a], the core construction is a 2-complex which is good panted and nicely glued with certain mild singularity near a 1-submanifold locus. In [Sun 2014b], the core construction involves homological substitution of a carefully chosen one-vertexed 2-complex. In particular, the mapping degree 2 is essentially due to the $\mathbf{Z}_2$ obstruction discovered in Theorem 3.2.

In prospect, there are several directions rather interesting to explore. The first challenge is to generalize good pants constructions to non-cocompact lattices and one-ended word-hyperbolic groups. The next step is to develop qualitative and quantitative versions of the good correction theory. Finally, it remains widely open at this point how to construct higher dimensional submanifolds in locally symmetric spaces through their geometry and dynamics of various flow.

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