HOMOLOGY OF CURVES AND SURFACES IN CLOSED HYPERBOLIC 3-MANIFOLDS

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Abstract. Among other things, we prove the following two topologcal statements about closed hyperbolic 3-manifolds. First, every rational second homology class of a closed hyperbolic 3-manifold has a positve integral multiple represented by an oriented connected closed π_1-injectively immersed quasi-Fuchsian subsurface. Second, every rationally null-homologous, π_1-injectively immersed oriented closed 1-submanifold in a closed hyperbolic 3-manifold has an equidegree finite cover which bounds an oriented connected compact π_1-injective immersed quasi-Fuchsian subsurface. In part, we exploit techniques developed by Kahn and Markovic in [KM1, KM2], but we only distill geometric and topological ingredients from those papers so no hard analysis is involved in this paper.

1. Introduction

In this paper, we are concerned about the construction problem of homologically interesting connected quasi-Fuchsian subsurfaces in closed hyperbolic 3-manifolds. We show that in a closed hyperbolic 3-manifold, it is always possible to construct an oriented compact connected π_1-injectively immersed quasi-Fuchsian subsurface which is virtually bounded by prescribed multicurves and which virtually represents a prescribed rational relative second homology class (Theorem 1.3).

The following two results are motivational special cases of Theorem 1.3. For simplicity we state them first.

Corollary 1.1. *Every rational second homology class of a closed hyperbolic 3-manifold has a positve integral multiple represented by an oriented connected closed π_1-injectively immersed quasi-Fuchsian subsurface.*

Corollary 1.2. *Every rationally null-homologous, π_1-injectively immersed oriented closed 1-submanifold in a closed hyperbolic 3-manifold has an equidegree finite cover which bounds an oriented connected compact π_1-injective immersed quasi-Fuchsian subsurface.*

Here the closed 1-submanifold being π_1-injectively immersed means that all components are homotopically nontrivial, and a finite cover being equidegree means that the covering degree does not vary over different components of the 1-submanifold. However, we do not require the finite cover to be connected restricted to any component of the closed 1-submanifold.

Corollary 1.1 was a question that was recently (and informally) raised by William Thurston. Note that if not requiring the subsurface to be connected, one may easily

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obtain a componentwise quasi-Fuchsian embedded incompressible subsurface representing a second homology class that is nontrivial and non-fibered, or obtain a componentwise π_1-injectively immersed quasi-Fuchsian representative subsurface, using the Cooper–Long–Reid construction [CLR] in the fibered case or the Kahn–Markovic construction [KM1] in the trivial case. In the paper [CW], Danny Calegari and Alden Walker show that in a random group at any positive density, many second homology classes can be rationally represented by quasiconvex (closed) surface subgroups (cf. Remark 6.4.2 of [CW]). Corollary 1.2 answers a question of Calegari in the case of closed hyperbolic 3-manifold groups. Calegari proved the surface group case [Ca] but his question remains widely open for hyperbolic groups in general.

Next, we state our main result Theorem 1.3. A compact immersed subsurface F of a closed hyperbolic 3-manifold M is *quasi-Fuchsian* if it is an essential subsurface of a closed π_1-injectively immersed quasi-Fuchsian subsurface of M. Perhaps it would be better to call F 'quasi-Schottky' if it is quasi-Fuchsian with nonempty boundary.

Theorem 1.3. *Let M be a closed hyperbolic 3-manifold, and $L \subset M$ be the (possibly empty) union of finitely many mutually disjoint, π_1-injectively embedded loops. Then for any relative homology class $\alpha \in H_2(M, L; \mathbf{Q})$, there exists an oriented connected compact surface F, and an immersion of the pair*

$$j : (F, \partial F) \looparrowright (M, L),$$

such that j is π_1-injective and quasi-Fuchsian, and that F represents a positive integral multiple of α.

The reader is referred to Subsection 8.1 for more explanation about the formulation. In fact, the proof also implies that the claimed immersed subsurface is nearly geodesic and nearly regularly panted (cf. Section 2).

In the course of proving Theorem 1.3, we revisit the techniques developed in the work of Kahn–Markovic in [KM1, KM2], with an attempt to distill the topological ingredients from those papers. In particular, we recall the gluing construction of [KM1] and the topological part of the good correction theory of [KM2]. We do not touch any details of dynamics and randomization part of the good correction theory, so no hard analysis will be involved in the treatment of this paper.

The connectedness of the surface F in the conclusion of Theorem 1.3 comes from improving the gluing construction of [KM1]. The idea of the construction of [KM1] is to build a closed π_1-injectively immersed quasi-Fuchsian subsurface in a closed hyperbolic 3-manifold by gluing a sufficiently large finite collection of nearly regular pairs of pants with nearly evenly distributed feet. A crucial criterion was proved in [KM1, Theorem 2.1], asserting that a nearly unit shearing gluing yields the π_1-injectiveness and the quasi-Fuchsian property. In Section 2, we will review the program in more details with emphasis on the boundary operator on measures of nearly regular pairs of pants. However, the criterion of [KM1] does not necessarily produce a connected surface, so we provide a slightly stronger criterion (Theorem 2.9) which ensures connectedness of the output. The new criterion will be proved in Section 3 by applying a trick called *hybriding*. On the other hand, the assumptions of the new criterion are not hard to be satisfied, for instance, cf. Theorem 2.10.

The control of the homology class of the surface F in the conclusion of Theorem 1.3 comes from extending and strengthening the non-random good correction theory of [KM2] in the 3-dimensional case. For an oriented closed hyperbolic 3-manifold M, we will reformulate the Good Pants Homology introduced in [KM2] as the *nearly regularly panted cobordism group* $\mathbf{\Omega}_{R,\epsilon}(M)$ (Definition 5.1). In Section 5, we will find a canonical isomorphism Φ between $\mathbf{\Omega}_{R,\epsilon}(M)$ and the first integral homology of the special orthonormal frame bundle $\mathrm{SO}(M)$ over M (Theorem 5.2). This isomorphism fully characterizes the structure of $\mathbf{\Omega}_{R,\epsilon}(M)$, and improves the treatment of non-random good correction theory of [KM2] in that it accounts for the torsion part which was previously ignored. In Section 6, we will further show that any second integral homology class of M can be represented by an oriented closed nearly regularly panted subsurface (Theorem 6.1). With an extra property called *nearly regularly panted connectedness* introduced in Section 7, our study of nearly regularly panted cobordisms can be summarized by the following Theorem 1.4, stated in a form analogous to Theorem 1.3 (cf. Section 2 for the notations). Note that Corollaries 1.2 and 1.1 are also parallel to Theorems 5.2 and 6.1 in their statements respectively. These results are all based on geometric constructions using ∂-framed segments as we will study in Section 4.

Theorem 1.4. *Let M be closed hyperbolic 3-manifold. For any sufficiently small positive constant ϵ depending on the injectivity radius of M and any sufficiently large positive constant R depending only on M and ϵ, the following holds. There exists a nontrivial invariant $\sigma(L)$ valued in $\mathbf{Z}_2$, defined for all null-homologous oriented (R,ϵ)-multicurve L in M, satisfying the following.*

1. *For any null-homologous oriented (R,ϵ)-multicurve L_1, L_2,*
$$\sigma(L_1 \sqcup L_2) = \sigma(L_1) + \sigma(L_2).$$
2. *The invariant $\sigma(L)$ vanishes if and only if L bounds a connected compact oriented (R,ϵ)-panted subsurface F immersed in M.*
3. *When $\sigma(L)$ vanishes, every relative homology class $\alpha \in H_2(M,L;\mathbf{Z})$ with $\partial\alpha$ equal to the fundamental class $[L] \in H_1(L;\mathbf{Z})$ is represented by a connected compact oriented (R,ϵ)-panted immersed subsurface F bounded by L.*

In fact, $\sigma(L)$ is defined as $\Phi([L]_{R,\epsilon})$, where $[L]_{R,\epsilon}$ is the (R,ϵ)-panted cobordism class of L, so $\sigma(L)$ lies in a canonical submodule of $H_1(\mathrm{SO}(M);\mathbf{Z})$ isomorphic to $\mathbf{Z}_2$. The proofs of Theorems 1.3 and 1.4 will be completed in Section 8. A few further questions will be proposed in Section 9.

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2. Methodology

For a typical construction problem of quasi-Fuchsian subsurfaces in a closed hyperbolic 3-manifold, such as addressed in Theorem 1.3, one may generally follow two steps: first, decide a suitable finite collection of (oriented) nearly geodesic pairs of pants whose cuff lengths are nearly equal; secondly, glue these pairs of pants up along boundary in a well controlled fashion to output a connected quasi-Fuchsian subsurface. The second step is supposed to be automatic once we have fed in the collection of pairs of pants as initial data, so the real task is to provide such a collection. Regarding the collection as a finite measure over the set of pairs of pants, we will translate the compatibility condition for the gluing into a linear system of equations of that measure, and we will introduce properties on solutions to ensure a suitable gluing. In other words, we will be interested in certain solutions of the linear system of equations associated to a *boundary operator* on measures of pants. The purpose of this section is to set up the framework, and to divide the discussion into several aspects that can be treated separatedly in the rest of this paper.

2.1. **Measures of pants.** Throughout this subsection, M will be a closed hyperbolic 3-manifold. Identifying the universal cover $\widehat{M}$ of M as the 3-dimensional hyperbolic space $\mathbb{H}^3$, we will regard the deck transformation group $\pi_1(M)$ as a torsion-free cocompact discrete subgroup of the group of isometries $\mathrm{Isom}(\mathbb{H}^3)$.

2.1.1. *Curves and pants.* Let S^1 be the topological circle with a fixed orientation. An *oriented curve* in M, or simply a *curve*, is the free homotopy class of a π_1-injective immersion $\gamma : S^1 \looparrowright M$. We often abuse the notations for curves and their representatives, and write a curve as

$$\gamma \looparrowright M.$$

Every curve can be homotoped to a unique oriented closed geodesic in M with the length parametrization up to a rotation, so we define the *visual torus* $\mathscr{N}_\gamma$ of γ to be the unit normal vector bundle of the geodesic representative. We think of the visual torus to be a holomorphic torus, and the name comes from the fact that we may alternatively define $\mathscr{N}_\gamma$ as follows. Let $\hat{\gamma}$ be any elevation of a curve γ in $\mathbb{H}^3$. As $\hat{\gamma}$ is a quasi-geodesic with endpoints p, q on the sphere at infinity $\hat{\mathbf{C}}$, we may define $\mathscr{N}_\gamma$ to be the holomorphic cylinder $\hat{\mathbf{C}} \setminus \{p, q\}$ quotiented by the stabilizer $\mathrm{Stab}_{\pi_1(M)}(\hat{\gamma})$. The two definitions of $\mathscr{N}_\gamma$ are certainly equivalent, but the latter might be more natural from a perspective of geometric group theory.

The collection of curves in M will be denoted as $\mathbf{\Gamma}(M)$, or simply $\mathbf{\Gamma}$. The quotient of $\mathbf{\Gamma}$ under the free involution induced by orientation reversion of curves is the collection of *unoriented curves* in M, and we will denote it as $|\mathbf{\Gamma}|$.

Let $\Sigma_{0,3}$ be a topological pair of pants, namely, a compact three-holed sphere. For convenience, we will fix an orientation of $\Sigma_{0,3}$. An *unmarked oriented pair of pants* in M, or simply a *pair of pants*, is the homotopy class of a π_1-injective immersion $\Pi : \Sigma_{0,3} \looparrowright M$, up to orientation-preserving self-homeomorphisms of $\Sigma_{0,3}$. We often abuse the notations for homotopy classes and their representatives, and write a pair of pants as

$$\Pi \looparrowright M.$$

The *cuffs* of $\Sigma_{0,3}$ are the three boundary curves of $\Sigma_{0,3}$, and the *seams* of $\Sigma_{0,3}$ are three mutually disjoint, properly embedded arcs connecting the three pairs of cuffs, which are unique up to orientation-preserving self-homeomorphisms of $\Sigma_{0,3}$. Every pair of pants can be homotoped so that the cuffs are the unique geodesic closed curves, and that the seams are the unique geodesic arcs orthogonal to the adjacent cuffs, or possibly points in the degenerate case. We say a pair of pants in M is *nonsingular* if no seam degenerates to a point under the straightening as above.

The collection of nonsingular pants in M will be denoted as $\mathbf{\Pi}(M)$, or simply $\mathbf{\Pi}$. The quotient of $\mathbf{\Pi}$ under the free involution induced by orientation reversion of pants is the collection of *unoriented nonsingular pants* in M, and we will denote it as $|\mathbf{\Pi}|$.

Suppose $\Pi \looparrowright M$ is a nonsingular pair of pants, straightened so that the cuffs are geodesic and the seams are geodesic and orthogonal to the cuffs. For every pair of cuffs γ and γ', the seam η from γ to γ' defines a unit normal vector v at γ, pointing along η towards γ'. We call $v \in \mathscr{N}_\gamma$ the *foot* of Π at γ toward γ', and it is the 'visual direction of the nearest point' as we observe γ' from γ. There are exactly six feet of Π, two at each cuff toward the other two cuffs respectively.

2.1.2. *Boundary operators.* Throughout this paper, a measure is always considered to be nonnegative. Let $\mathcal{M}(\mathbf{\Pi})$ denote all finitely-supported finite measures on the set of nonsingular pants $\mathbf{\Pi}$ in M. We usually write a nontrivial element of $\mathcal{M}(\mathbf{\Pi})$ as a finite formal sum of elements of $\mathbf{\Pi}$ with positive coefficients. Similarly, let $\mathcal{M}(\mathbf{\Gamma})$ denote all finitely-supported finite measures on the set of curves $\mathbf{\Gamma}$ in M. There is a natural *boundary operator*

$$\partial : \mathcal{M}(\mathbf{\Pi}) \to \mathcal{M}(\mathbf{\Gamma}),$$

defined by assigning $\partial\Pi$ to be the sum of the cuffs of Π. We will consider two related notions: the footed boundary $\partial^\sharp$, which is a geometric refinement of ∂ remembering the feet; and the net boundary $\partial^\flat$, which is an algebraic reduction of ∂ forgetting the orientation.

Definition 2.1. Let $\mathcal{M}(\mathscr{N}_\gamma)$ denote all Borel measures on the visual torus of any curve γ in M, and let $\mathcal{M}(\mathscr{N}_{\mathbf{\Gamma}})$ denote the direct sum of $\mathcal{M}(\mathscr{N}_\gamma)$ as γ runs over all curves $\mathbf{\Gamma}$. The *footed boundary operator* is the homomorphism:

$$\partial^\sharp : \mathcal{M}(\mathbf{\Pi}) \to \mathcal{M}(\mathscr{N}_{\mathbf{\Gamma}}),$$

defined by assigning $\partial^\sharp\Pi$ to be one half of the sum of the atomic measures supported at the six feet of Π, where $\Pi \in \mathbf{\Pi}$ is any nonsingular pants.

Remark 2.2. The normalization coefficient $\frac{1}{2}$ has been chosen so that the total measure on $\mathscr{N}_\gamma$ of each cuff γ is equal to 1.

Definition 2.3. Let $\mathcal{M}(|\mathbf{\Gamma}|)$ denote all finite measures on the set of unoriented curves $|\mathbf{\Gamma}|$ in M. We identify $\mathcal{M}(|\mathbf{\Gamma}|)$ as the subspace of $\mathcal{M}(\mathbf{\Gamma})$ fixed under the free involution induced by the orientation reversion $\gamma \mapsto \bar{\gamma}$, in other words, regard the atomic measure supported on the unoriented class $\{\gamma, \bar{\gamma}\}$ as the measure $\frac{1}{2}(\gamma + \bar{\gamma})$. The *net boundary operator* is the homomorphism:

$$\partial^\flat : \mathcal{M}(\mathbf{\Pi}) \to \mathcal{M}(|\mathbf{\Gamma}|),$$

defined by

$$\partial^\flat \mu = \frac{1}{2}\left|\partial\mu - \overline{\partial\mu}\right|.$$

Remark 2.4. If we regard $\mathcal{M}(|\mathbf{\Pi}|)$ as the subspace of $\mathcal{M}(\mathbf{\Pi})$ fixed under the orientation reverion, then $\mathcal{M}(|\mathbf{\Pi}|)$ lies in the kernel of $\partial^\flat$.

We have the following commutative diagram relating various operators:

$$\begin{tikzcd} & \mathcal{M}(\mathscr{N}_{\mathbf{\Gamma}}) \arrow[d, "\mathrm{Tot}"] \\ \mathcal{M}(\mathbf{\Pi}) \arrow[ur, "\partial^\sharp"] \arrow[r, "\partial"] \arrow[dr, "\partial^\flat"'] & \mathcal{M}(\mathbf{\Gamma}) \arrow[d, "\mathrm{Net}"] \\ & \mathcal{M}(|\mathbf{\Gamma}|) \end{tikzcd}$$

Here Tot is the componentwise total $\mathrm{Tot}(\mu) = \sum_{\gamma\in\mathbf{\Gamma}} \mu(\mathscr{N}_\gamma)\,\gamma$, and Net is the unorientation reduction defined by linearly extending $\mathrm{Net}(\gamma) = \frac{1}{2}|\gamma - \bar{\gamma}|$ for all $\gamma \in \mathbf{\Gamma}$.

2.1.3. *Shape controlling.* For most of our treatment we will focus on pairs of pants in M that are nearly geodesic with cuffs of nearly equal length, or *nearly regular pants* as we will introduce below.

First recall that for a boundary-framed segment in $\mathbb{H}^3$ (with the canonical orientation), the (oriented geometric) *complex length* of it can be defined as a complex value in

$$(0, +\infty) + (-\pi, \pi]\,\mathrm{i}.$$

More precisely, an oriented *∂-framed segment* is an oriented geodesic arc with a unit normal vector at each endpoint, so the real part of the complex length is the usual length of the geodesic arc, and the imaginary part is the signed angle from the initial normal vector to the parallel transportation of the terminal normal vector to the initial point of the geodesic arc, with respect to the initial tangent vector. The complex length does not change if we reverse the orientation of the ∂-framed segment. It is clear that the complex length of ∂-framed segments also makes sense in any oriented hyperbolic 3-manifold M. For a geodesic loop in M, we may pick a normal vector at a point, and define the *complex length* of the geodesic loop as the complex length of the boundary-framed segment obtained from cutting the geodesic loop along the chosen point, and endowing both the end-points with the same chosen normal vector. The definition is clearly independent of the choices of the point or the normal vector.

Let M be a closed hyperbolic 3-manifold. Suppose $\Pi \looparrowright M$ be a nonsingular pair of pants, straightened by homotopy so that the cuffs and seams are geodesic and orthogonal as before. Observe that each cuff γ of Π is bisected into two boundary-framed segments with the boundary framing given by the two feet. In fact, these two boundary-framed segments, called *half cuffs*, have the same complex length. We define the *complex half length* of the cuff γ of Π to be the complex length of either of the half cuffs, denoted as $\mathbf{hl}_\Pi(\gamma)$. We denote the complex length of γ as $\mathbf{l}(\gamma)$. When the imaginary part of $\mathbf{hl}_\Pi(\gamma)$ is at most the right angle, $\mathbf{l}(\gamma)$ is equal to twice $\mathbf{hl}_\Pi(\gamma)$.

Definition 2.5. Let M be a closed hyperbolic 3-manifold. Suppose $R \in (0, +\infty)$ and $\epsilon \in [0, \pi]$.

(1) We say that a curve $\gamma \looparrowright M$ is (R, ϵ)-*nearly hyperbolic*, if

$$|\mathbf{l}(\gamma) - R| < \epsilon.$$

The subcollection of (R,ϵ)-nearly hyperbolic curves in M will be denoted as $\mathbf{\Gamma}_{R,\epsilon} \subset \mathbf{\Gamma}$.

(2) We say that a nondegenerate pair of pants $\Pi \looparrowright M$ is (R,ϵ)-*nearly regular*, if for each cuff γ of Π,

$$\left|\mathbf{hl}_{\Pi}(\gamma) - \frac{R}{2}\right| < \frac{\epsilon}{2}.$$

The subcollection of (R,ϵ)-nearly regular pants in M will be denoted as $\mathbf{\Pi}_{R,\epsilon} \subset \mathbf{\Pi}$.

We often simply say *nearly hyperbolic* or *nearly regular* with the usage explained in the following Convention 2.6.

Convention 2.6. When ambiguously saying *nearly* instead of (R,ϵ)-*nearly*, we suppose that $R \in (0,+\infty)$ and $\epsilon \in [0,\pi]$ are understood from the context. Presumably, ϵ will be universally small, and R will be sufficiently large, depending on M and ϵ. This precisely means that for some universal constant $\hat{\epsilon} > 0$ to be determined, ϵ is assumed to satisfy $0 < \epsilon < \hat{\epsilon}$, and that for any given closed hyperbolic 3-manifold M, and for some constant $\hat{R} = \hat{R}(M,\epsilon) > 0$ to be determined, R is assumed to satisfy $R > \hat{R}$.

From Definition 2.5, it follows that the restriction of the boundary operator yields:

$$\partial : \mathcal{M}(\mathbf{\Pi}_{R,\epsilon}) \to \mathcal{M}(\mathbf{\Gamma}_{R,\epsilon}),$$

and similarly for $\partial^\sharp$ and $\partial^\flat$.

2.2. **From pants measures to panted subsurfaces.** For any finite collection of pairs of pants in a closed hyperbolic 3-manifold M, we can try to glue them along common cuffs with opposite induced orientations, and this will give rise to a panted subsurface in M, precisely as follows.

Definition 2.7. Suppose $R \in (0,+\infty)$ and $\epsilon \in [0,\pi]$. An (R,ϵ)-*nearly regularly panted subsurface* of M, or simply an (R,ϵ)-*panted subsurface*, is a (possibly disconnected) compact oriented surface F with a pants decomposition, and with an immersion $j : F \looparrowright M$ into M such that the restriction of j to each component pair of pants is (R,ϵ)-nearly regular. Let $\mu \in \mathcal{M}(\mathbf{\Pi}_{R,\epsilon})$ be the integral measure such that for each $\Pi \in \mathbf{\Pi}_{R,\epsilon}$, there are exactly $\mu(\{\Pi\})$ copies of Π in all component pairs of pants of F immersed via j. Then we say that the (R,ϵ)-panted subsurface is *subordinate to* μ.

In general, the panted subsurface would be neither π_1-injective quasi-Fuchsian nor connected. However, we wish to introduce conditions on μ to ensure that some quasi-Fuchsian connected panted subsurface therefore exists and is subordinate to μ.

Recall that for a metric space (X,d), and for a positive number δ, two Borel probability measures μ,μ' are said to be δ-*equivalent*, if for every Borel subset A of X, $\mu(A) \leq \mu'(\mathcal{N}_\delta(A))$, where $\mathcal{N}_\delta(A) \subset X$ is the δ-neighborhood of A. Note that δ-equivalence is a symmetric relation. For any nonvanishing finite Borel measure μ on X, we may speak of δ-equivalence after normalization, namely, after dividing μ by $\mu(X)$.

Definition 2.8. Let M be a closed hyperbolic 3-manifold. Suppose $R \in (0,+\infty)$ and $\epsilon \in [0,\pi]$. Let $\mu \in \mathcal{M}(\mathbf{\Pi}_{R,\epsilon})$ be a measure of nearly regular pants.

(1) We say that μ is *ubiquitous*, if μ is positive at every $\Pi \in \mathbf{\Pi}_{R,\epsilon}$, and if $\partial\mu$ is positive at every $\gamma \in \mathbf{\Gamma}_{R,\epsilon}$.
(2) We say that μ is *irreducible*, if for any nontrivial decomposition $\mu = \mu' + \mu''$, μ' and μ'' have adjacent supports, namely, that there is a curve $\gamma \in \mathbf{\Gamma}_{R,\epsilon}$ which lies in the support of $\partial\mu'$ and the orientation-reversal of which lies in the support of $\partial\mu''$.
(3) We say that μ is (R,ϵ)*-nearly evenly footed*, if for every curve $\gamma \in \mathbf{\Gamma}_{R,\epsilon}$ on which $\partial\mu$ is nonvanishing, the normalization of $(\partial^\sharp\mu)|_{\mathscr{N}_\gamma}$ is $(\frac{\epsilon}{R})$-equivalent to the normalization of the Lebesgue measure, with respect to the Euclidean metric on the visual torus $\mathscr{N}_\gamma$ induced from the unit normal vector bundle of the geodesic representative of γ. We often simply say *nearly evenly footed* following Convention 2.6.
(4) We say that μ is *rich*, if the net boundary of μ at any unoriented curve is a relatively small portion compared to the cancelled part, or specifically for our application, that $\partial^\flat\mu(|\gamma|) \leq \frac{1}{5}\,\partial\mu(\{\gamma,\bar{\gamma}\})$. Here $|\gamma|$ means the unoriented class $\{\gamma,\bar{\gamma}\}$ for any curve $\gamma \subset \mathbf{\Gamma}_{R,\epsilon}$.

The following criterion about connected quasi-Fuchsian gluing will be proved in Section 3.

Theorem 2.9. *Let M be a closed hyperbolic 3-manifold. For a universally small positive ϵ, and for all sufficiently large positive R, depending on M and ϵ, the following statement holds. For any nontrivial rational measure $\mu \in \mathcal{M}(\mathbf{\Pi}_{R,\epsilon})$, if μ is irreducible, (R,ϵ)-nearly evenly footed, and rich, then there exists an oriented, connected, compact, π_1-injectively immersed quasi-Fuchsian subsurface:*

$$j : F \looparrowright M,$$

which is (R,ϵ)-nearly regularly panted subordinate to a positive integral multiple of μ.

2.3. **Homology via pants measures.** For a closed hyperbolic 3-manifold M, we wish to understand the structure of the boundary operator $\partial : \mathcal{M}(\mathbf{\Pi}_{R,\epsilon}) \to \mathcal{M}(\mathbf{\Gamma}_{R,\epsilon})$. More specifically, the following Theorem 2.10 should be viewed from this perspective.

Let $\mathcal{L} \subset \mathbf{\Gamma}_{R,\epsilon}$ be a collection of distinct curves, invariant under orientation reversion. We write $|\mathcal{L}| \subset |\mathbf{\Gamma}_{R,\epsilon}|$ for the corresponding unoriented curves, namely, the quotient of $\mathcal{L}$ by orientation reversion. Let

$$\mathcal{ZM}(\mathbf{\Pi}_{R,\epsilon}, |\mathcal{L}|)$$

denote the subset of $\mathcal{M}(\mathbf{\Pi}_{R,\epsilon})$ consisting of measures μ with the net boundary $\partial^\flat\mu$ supported on (possibly a proper subset of) the unoriented curves $|\mathcal{L}|$. Choosing a collection of mutually disjoint, embedded unoriented loops $k_1, \cdots, k_r$ representing elements of $|\mathcal{L}|$, we write $H_2(M, |\mathcal{L}|; \mathbf{R})$ for $H_2(M, k_1 \cup \cdots \cup k_r; \mathbf{R})$. Note that $H_2(M, |\mathcal{L}|; \mathbf{R})$ is well defined up to natural isomorphisms for different choices of the loops k_i. Thus there is a natural homomorphism between semimodules over the semiring of nonnegative real numbers:

$$[\cdot] : \mathcal{ZM}(\mathbf{\Pi}_{R,\epsilon}, |\mathcal{L}|) \to H_2(M, |\mathcal{L}|; \mathbf{R}).$$

In particular, when $\mathcal{L}$ is empty, we denote the kernel of the homomorphism $[\cdot]$ above as

$$\mathcal{BM}(\mathbf{\Pi}_{R,\epsilon}) \subset \mathcal{ZM}(\mathbf{\Pi}_{R,\epsilon}, \emptyset).$$

Nevertheless, $\mathcal{BM}(\mathbf{\Pi}_{R,\epsilon})$ is naturally contained in $\mathcal{ZM}(\mathbf{\Pi}_{R,\epsilon},|\mathcal{L}|)$ for any $\mathcal{L}$ as well.

Theorem 2.10. *Let M be a closed hyperbolic 3-manifold. For a universally small positive ϵ, and for all sufficiently large positive R, depending on M and ϵ, the following statements hold. Suppose $\mathcal{L}\subset\mathbf{\Gamma}_{R,\epsilon}$ is a collection of distinct curves invariant under orientation reversion.*

(1) There is a short exact sequence of semimodules over the semiring of non-negative real numbers:
$$0\longrightarrow\mathcal{BM}(\mathbf{\Pi}_{R,\epsilon})\longrightarrow\mathcal{ZM}(\mathbf{\Pi}_{R,\epsilon},|\mathcal{L}|)\longrightarrow H_2(M,|\mathcal{L}|;\mathbf{R})\longrightarrow 0.$$

(2) There exists a nontrivial measure $\mu_0\in\mathcal{BM}(\mathbf{\Pi}_{R,\epsilon})$ which is ubiquitous, irreducible, (R,ϵ)-nearly evenly footed, and rich. Moreover, every measure in $\mathcal{ZM}(\mathbf{\Pi}_{R,\epsilon},|\mathcal{L}|)$ can be adjusted to satisfy the same properties, by adding some measure in $\mathcal{BM}(\mathbf{\Pi}_{R,\epsilon})$.

Furthermore, the same statements hold for rational coefficients instead of real coefficients as well.

Theorem 2.10 will be proved in Section 8. As Theorem 2.10 feeds Theorem 2.9 with workable input, homologically interesting connected quasi-Fuchsian subsurfaces can be produced in closed hyperbolic 3-manifolds under fairly general conditions.

3. Quasi-Fuchsian connected gluing

In this section, we prove Theorem 2.9, restated as Proposition 3.2 in terms of gluing. Let M be a closed hyperbolic 3-manifold, and let (R,ϵ) be a pair of undetermined constants, assuming that ϵ is a universally small positive number, and that R is a sufficiently large positive number depending on M and ϵ.

Given a panted surface F of which the pants structure is given by a union of disjoint simple closed curves $C\subset F$, we may cut F along C to obtain a disconnected surface $\mathcal{F}$ whose components are all pairs of pants. Denote the union of all the new boundary components of $\mathcal{F}$ coming from the cutting as $\mathcal{C}\subset\partial\mathcal{F}$. Then the panted surface F can be recovered as the quotient of $\mathcal{F}$ by an orientation-reversing involution $\phi:\mathcal{C}\to\mathcal{C}$, which sends any preimage component of C to its opposite boundary component. With this in mind, we introduce the notion of gluing as follows.

Definition 3.1. For any integral measure $\mu\in\mathcal{M}(\mathbf{\Pi}_{R,\epsilon})$, let $\mathcal{F}$ be the finite disjoint union of copies of nearly regular pants prescribed by μ, namely, such that for any $\Pi\in\mathbf{\Pi}_{R,\epsilon}$, there are exactly $\mu(\{\Pi\})$ copies of Π in $\mathcal{F}$. By a *gluing* of $\mathcal{F}$, we mean a pair $(\mathcal{C},\phi)$, such that
$$\mathcal{C}\subset\partial\mathcal{F}$$
is a subunion of cuffs, and that
$$\phi:\mathcal{C}\to\mathcal{C}$$
is a free involution which sends each cuff $c\subset\mathcal{C}$ to its orientation-reversal, regarded as in $\mathbf{\Gamma}_{R,\epsilon}$. We say that $(\mathcal{C},\phi)$ is *maximal* if ϕ cannot be extended to any subunion of cuffs $\mathcal{C}'\subset\partial\mathcal{F}$ larger than $\mathcal{C}$. Since the quotient of $\mathcal{F}$ by ϕ yields a compact oriented (R,ϵ)-panted subsurface $j:F\looparrowright M$, the quotient image of any cuff $c\subset\mathcal{C}$ in F will be called a *glued cuff*.

Proposition 3.2. *Let M be a closed hyperbolic 3-manifold. For all universally small positive ϵ, and for all sufficiently large positive R depending on M and ϵ, the following holds. If a rational nontrivial pants measure $\mu \in \mathcal{M}(\mathbf{\Pi}_{R,\epsilon})$ is irreducible, (R,ϵ)-nearly evenly footed, and rich, then possibly after passing to a positive integral multiple of μ, the prescribed oriented compact surface $\mathcal{F}$ admits a maximal gluing $(\mathcal{C},\phi)$, which yields a π_1-injectively immersed, quasi-Fuchsian, and connected subsurface $j : F \looparrowright M$.*

The key technique to ensure the connectedness of the resulting surface is a trick called *hybriding*. To illustrate the idea, the reader may assume for simplicity that $\partial\mu$ is zero, so that any panted surface F resulted from a maximal gluing is closed. We say that a gluing is *nearly unit shearing*, if for any glued cuff c on the resulting (R,ϵ)-panted surface F, the feet of the pair of pants on one side of c is almost exactly opposite to the feet of the pair of pants on the other side of c after a parallel transportation along c of distance 1 (Definition 3.5). By the construction of [KM1], the assumption that μ is (R,ϵ)-nearly evenly footed implies that such a maximal gluing always exists, resulting in a π_1-injectively immersed surface which is quasi-Fuchsian. Since F might be disconnected, we wish to slightly modify the gluing without affecting the nearly unit shearing property, nevertheless the number of components of F can be decreased in that case. Denote the components of F as $F_1, \cdots, F_r$, where r is at least two. If two components of F, say F_1 and F_2, has glued cuffs $c_1 \subset F_1$ and $c_2 \subset F_2$ that are homotopic to each other, supposing that c_i is nonseparating on F_i, we may modify the gluing by cutting F_i along c_i, and regluing in a cross fashion. Then the new resulting surface F' has a connected component F_{12} instead of the previous two components F_1 and F_2. We say that F_{12} is obtained by *hybriding* F_1 and F_2 along c_1 and c_2. To preserve the nearly unit shearing property, we need to require that the feet of pants on one side of c_1 is almost the same as the feet of pants on the same side of c_2. Such F_i and c_i can be found by the following argument. First, the irreducibility of μ implies that there is some curve class $\gamma \in \mathbf{\Gamma}_{R,\epsilon}$, such that there are at least two distinct components of F that have glued cuffs homotopic to γ. Because the footed boundary $\partial^\sharp\mu$ restricted to the unit normal vector bundle $\mathscr{N}_\gamma$ over γ is nearly evenly distributed, the connectedness of $\mathscr{N}_\gamma$ implies that at least distinct two components F_1 and F_2 of F (not necessarily the components that we started with) have glued cuffs c_1 and c_2 homotopic to γ with their feet on the same side very close to those of each other. Therefore, performing the hybriding on these F_i along c_i will decrease the number of components of the resulting surface, preserving the nearly unit shearing property. Iterating the process until the resulting surface become connected, then we are done. A minor point here is that the hybriding trick also require that c_i be nonseparating on F_i. In fact, with the somewhat technical assumption that μ is rich, the nonseparating property of glued cuffs can be satisfied if we pass to a cyclic finite cover of F, and hence pass to positive multiple of μ. Note also that the arguments above certainly works as well when F has boundary. In practice, one need to be slightly careful to control the error so that the new resulting surface remains (R,ϵ)-panted, but the general idea of hybriding follows the outline above.

Roughly speaking, the assumption that μ is nearly evenly footed allows us to control the shape of F along each glued cuff, which ensures the π_1-injectivity and the quasi-Fuchsian property; the assumption that μ is rich allows us to construct F so that glued cuff is nonseparating in F, so combined with the assumption that μ

is irreducible, we may perform a hybriding trick to obtain a connected F, possibly after passing to a further positive integral multiple of μ.

In the rest of this section, we prove Proposition 3.2. In Subsection 3.1 we explain how to control the gluing so that the glued cuffs are nonseparating; in Subsection 3.2, we review the nearly unit shearing condition that is used in [KM1]; Subsection 3.3 is the hybriding argument; Subsection 3.4 summarizes the proof of Proposition 3.2.

It will be convenient to introduce a measure

$$\nu^\sharp_{\mathcal{C}} \in \mathcal{M}(\mathscr{N}_{\mathbf{\Gamma}_{R,\epsilon}}),$$

naturally associated to any subunion of cuffs $\mathcal{C} \subset \partial\mathcal{F}$. This measure records the contribution to the footed boundary $\partial^\sharp\mu$ from those pairs of pants which contain components of $\mathcal{C}$. More concretely, each component $c \subset \mathcal{C}$ lies in a unique pair of pants $P_c \subset \mathcal{F}$. If $c \subset \mathcal{C}$ is a copy of $\gamma \in \mathbf{\Gamma}_{R,\epsilon}$, and if P_c is a copy of $\Pi \in \mathbf{\Pi}_{R,\epsilon}$, we define the *marked footed boundary* $\partial^\sharp_c(P_c) \in \mathcal{M}(\mathscr{N}_\gamma)$ to be sum of the two feet (as atomic measures) of Π at the cuff corresponding to $c \subset \partial P_c$. Note that potentially P_c could have other cuffs which are copies of γ but which might not come from $\mathcal{C}$, so we need to specify c rather than just mentioning γ. We define

$$\nu^\sharp_{\mathcal{C}} = \sum_{c\subset\mathcal{C}} \partial^\sharp_c(P_c).$$

In particular, we also write

$$\nu^\sharp_c = \partial^\sharp_c(P_c).$$

3.1. **Nonseparating glued cuffs.** The lemma below essentially follows from the condition that μ is rich.

Lemma 3.3. *With the notations above, possibly after passing to a positive integral multiple of μ, we may assume that the prescribed disjoint union of pants $\mathcal{F}$ admits a subunion of cuffs $\mathcal{C} \subset \partial\mathcal{F}$, satisfying the following:*

- *Any gluing $(\mathcal{C}, \phi)$ of $\mathcal{F}$ along $\mathcal{C}$ is maximal;*
- *Restricted to any $\mathscr{N}_\gamma$, the measure $\nu^\sharp_{\mathcal{C}}$ is a positive rational multiple of $\partial^\sharp\mu$;*
- *Any pair of pants $P \subset \mathcal{F}$ contains at least two cuffs from $\mathcal{C}$.*

Proof. For simplicity, we write m_γ for $(\partial\mu)(\{\gamma\})$, and n_Π for $\mu(\{\Pi\})$. Let $k_{\gamma,\Pi} \in \{0, 1, 2, 3\}$ be the number of times that a curve γ occurs as the cuff of a pair of pants Π. For any curve $\gamma \in \mathbf{\Gamma}_{R,\epsilon}$, let

$$\mu_\gamma = \sum_{\Pi\in\mathbf{\Pi}_{R,\epsilon}} n_\Pi\, k_{\gamma,\Pi} \cdot \Pi.$$

Let

$$\tilde{\mu}_\gamma = \frac{m_\gamma - m_{\bar{\gamma}}}{m_\gamma} \cdot \mu_\gamma,$$

if $m_\gamma > m_{\bar{\gamma}}$; otherwise, let $\tilde{\mu}_\gamma = 0$. Since μ is rich, it is clear by Definition 2.8 that

$$\frac{m_\gamma - m_{\bar{\gamma}}}{m_\gamma} \leq \frac{1}{3},$$

whenever $m_\gamma > m_{\bar{\gamma}}$. It follows that

$$\sum_{\gamma\in\mathbf{\Gamma}_{R,\epsilon}} \tilde{\mu}_\gamma \leq \mu,$$

because every pair of pants has only three cuffs. Furthermore, possibly after passing to a positive integral multiple of μ, we may assume that μ and all $\tilde{\mu}_\gamma$ are integral. Therefore, in the disjoint union of pairs of pants $\mathcal{F}$ prescribed by μ, we may find mutually disjoint subunions $\mathcal{F}_\gamma$ prescribed by $\tilde{\mu}_\gamma$, and for each component $P \subset F_\gamma$, we may mark one cuff $c \subset P$ which is a copy of γ. Let

$$\mathcal{C} \subset \partial \mathcal{F}$$

be the union of all the unmarked cuffs.

It is straightforward to check that the three listed properties about $\mathcal{C}$ are satisfied by our construction. In fact, for any $\gamma, \bar{\gamma} \in \mathbf{\Gamma}_{R,\epsilon}$, suppose without loss of generality that $m_\gamma \geq m_{\bar{\gamma}}$. Because $\mathcal{C}$ has exactly $m_{\bar{\gamma}}$ components homotopic to γ and exactly $m_{\bar{\gamma}}$ components homotopic to $\bar{\gamma}$, any gluing $(\mathcal{C}, \phi)$ is maximal. The measure $\nu^\sharp_{\mathcal{C}}$ restricted to $\mathscr{N}_\gamma$ equals $\frac{m_{\bar{\gamma}}}{m_\gamma}$ times $\partial^\sharp \mu$ and restricted to $\mathscr{N}_{\bar{\gamma}}$ equals $\partial^\sharp \mu$, both proportional to $\partial^\sharp \mu$. For any $P \in \mathcal{F}$, we marked at most one cuff $c \subset P$ in the construction above, so it contains at least two cuffs from $\mathcal{C}$. $\square$

Suppose $(\mathcal{C}, \phi)$ is a gluing of $\mathcal{F}$ prescribed by μ. For a disjoint union of pants $\tilde{\mathcal{F}}$ prescribed by a positive integral multiple of μ, we say that a gluing $(\tilde{\mathcal{C}}, \tilde{\phi})$ *covers* $(\mathcal{C}, \phi)$, if $\tilde{\mathcal{C}}$ is the preimage of $\mathcal{C}$ under the natural covering $\kappa : \tilde{\mathcal{F}} \to \mathcal{F}$, and if the following diagram commutes:

$$\begin{array}{ccc} \tilde{\mathcal{C}} & \xrightarrow{\tilde{\phi}} & \tilde{\mathcal{C}} \\ \downarrow{\scriptstyle \kappa} & & \downarrow{\scriptstyle \kappa} \\ \mathcal{C} & \xrightarrow{\phi} & \mathcal{C} \end{array}$$

In this case, the associated surface $\tilde{F}$ naturally covers F as well.

Lemma 3.4. *Let $\mathcal{C} \subset \mathcal{F}$ be a subunion of cuffs satisfying the conclusion of Lemma 3.3. Suppose that $(\mathcal{C}, \phi)$ is a gluing of $\mathcal{F}$ prescribed by μ. Then the disjoint union of pants $\tilde{\mathcal{F}}$ prescribed by 2μ admits a gluing $(\tilde{\mathcal{C}}, \tilde{\phi})$ covering $(\mathcal{C}, \phi)$, such that every glued cuff in the resulted surface $\tilde{F}$ is nonseparating.*

Proof. The glued cuffs induces a decomposition of F into pairs of pants, and let Λ be the (possibly disconnected) dual graph. By Lemma 3.3, the valence of any vertex of Λ is at least two. It follows from an easy construction that Λ admits a double cover $\tilde{\Lambda}$ in which every edge is nonseparating. In other words, there is a double cover $\tilde{F}$ of F induced from a cover $(\tilde{\mathcal{C}}, \tilde{\phi})$ of the gluing $(\mathcal{C}, \phi)$, in which every glued cuff is nonseparating. Note also that $\tilde{\mathcal{F}}$ can be identified with $\tilde{F}$ cut along the glued cuffs, so it is prescribed by 2μ. $\square$

3.2. **Gluing with nearly unit shearing.** To control the shape of F along a glued cuff, we will require the gluing ϕ to be *nearly unit shearing*, which can be described on the visual tori as follows.

Observe that for any nearly purely hyperbolic curve $\gamma \in \mathbf{\Gamma}_{R,\epsilon}$ (or indeed for any curve), there is a natural action of the additive group of complex numbers on $\mathscr{N}_\gamma$. More precisely, for any $\zeta \in \mathbf{C}$, there is an isomorphism between holomorphic tori:

$$A_\zeta : \mathscr{N}_\gamma \to \mathscr{N}_\gamma,$$

satisfying that for any $r \in \mathbf{R}$, A_r parallel transports any normal vector along γ by signed distance r, and that for any $\theta \in \mathbf{R}$, $A_{\theta \mathrm{i}}$ rotates the direction of any normal vector by a signed angle θ. It is clear that the kernel of the action is the lattice in $\mathbf{C}$ generated by $2\pi\mathrm{i}$ and the complex length $\mathbf{l}(\gamma)$ of γ. There is also a canonical anti-isomorphism

$$\bar{\ }: \mathscr{N}_\gamma \to \mathscr{N}_{\bar{\gamma}},$$

taking any unit normal vector (p, v) to the opposite vector $\overline{(p, v)} = (p, -v)$ at the same point $p \in |\gamma|$, where $|\gamma|$ is regarded as the unoriented geodesic representative. Note that an anti-isomorphism is orientation-reversing. We will think of the composition of the bar anti-isomorphism with A_ζ as the model of a ζ-shearing gluing along γ, denoted as:

$$\overline{A_\zeta} : \mathscr{N}_\gamma \to \mathscr{N}_{\bar{\gamma}}.$$

In other words, a nearly unit shearing gluing $(\mathcal{C}, \phi)$ should behave very much like $\overline{A_1}$ along each glued cuff.

Definition 3.5. A gluing $(\mathcal{C}, \phi)$ of $\mathcal{F}$ is said to be (R, ϵ)-*nearly unit shearing*, if for every pair of cuffs $c, c' \subset \mathcal{C}$ with c' equal to $\phi(c)$, the feet measure $\nu^\sharp_{c'}$ is $(\frac{\epsilon}{R})$-equivalent to $(\overline{A_1})_*(\nu^\sharp_c)$ on $\mathscr{N}_{\bar{\gamma}}$, where $\gamma \in \mathbf{\Gamma}_{R,\epsilon}$ is the curve class of c.

Remark 3.6. A reader familiar with the Kahn–Markovic construction should recognize the definition above as equivalent to the condition

$$|s(c) - 1| < \frac{\epsilon}{R}$$

in [KM1, Theorem 2.1].

The lemma below is a consequence of the condition that $\mu \in \mathcal{M}(\mathbf{\Pi}_{R,\epsilon})$ is (R, ϵ)-nearly evenly footed.

Lemma 3.7. *With the notations above, there is a gluing $(\mathcal{C}, \phi)$ of $\mathcal{F}$, which is (R, ϵ)-nearly unit shearing.*

Proof. Let $\mathcal{C} \subset \partial\mathcal{F}$ be a subunion of cuffs as ensured by the conclusion of Lemma 3.3. The lemma follows from the Hall Marriage argument, cf. [KM1, Theorem 3.2 and Subsection 3.5]. □

Lemma 3.8. *There exists $\hat{\epsilon} > 0$, and for any $0 < \epsilon < \hat{\epsilon}$, there exists $\hat{R} > 0$ depending on M and ϵ, such that for any $R > \hat{R}$, the following holds. If a gluing $(\mathcal{C}, \phi)$ of $\mathcal{F}$ is (R, ϵ)-nearly unit shearing, then the resulting surface $j : F \looparrowright M$ is π_1-injectively immersed and quasi-Fuchsian.*

Proof. This is exactly [KM1, Theorem 2.1] if F is closed. In the general case, recall that a compact immersed subsurface F of M is *quasi-Fuchsian* in our sense if it is an essential subsurface of a closed immersed quasi-Fuchsian subsurface F' of M (Section 1). We may take a ubiquitous (R, ϵ)-nearly evenly footed measure $\mu_0 \in \mathcal{M}(\mathbf{\Pi}_{R,\epsilon})$, for instance, as guaranteed by Theorem 2.10 (2). Then for a sufficiently large integer N, we may assume that $N\mu - \mu_0$ is still ubiquitous and (R, ϵ)-nearly evenly footed. Let $\mathcal{F}'$ be the disjoint union of (R, ϵ)-nearly regular pants prescribed by $N\mu_0$. We may identify $\mathcal{F}$ as a subunion of components of $\mathcal{F}'$. It is not hard to see that the gluing $(\mathcal{C}, \phi)$ can be extended to be a gluing $(\partial\mathcal{F}', \phi')$ which is still (R, ϵ)-nearly unit shearing, provided N sufficiently large. Then the gluing $(\partial\mathcal{F}', \phi')$ yields a possibly disconnected, componentwise π_1-injectively immersed quasi-Fuchsian closed subsurface $F' \looparrowright M$ by [KM1, Theorem 2.1]. The subsurface F obtained via the gluing $(\mathcal{C}, \phi)$ of $\mathcal{F}$ is an essential subsurface of F', so it is π_1-injectively immersed and quasi-Fuchsian. □

3.3. **Hybriding disconnected components.** The following lemma uses the condition that μ is irreducible.

Lemma 3.9. *Suppose that $(\mathcal{C}, \phi)$ is a gluing of $\mathcal{F}$ prescribed by a positive integral multiple of μ which is (R, ϵ)-nearly unit shearing with all glued cuffs nonseparating on the resulting surface F. Then possibly after passing to a further positive multiple of μ, there is a gluing $(\mathcal{C}, \phi')$, which is $(R, 2\epsilon)$-nearly unit shearing with all glued cuffs nonseparating on the resulting surface F', and moreover, F' is connected.*

Proof. For simplicity, we rewrite the positive integral multiple of μ prescribing $\mathcal{F}$ as μ. Note that we may still assume μ to be irreducible, (R, ϵ)-nearly evenly footed, and rich. We also observe that if F has r components $F_1, \cdots, F_r$, then for any positive integral multiple $m\mu$, there is a gluing $(\tilde{\mathcal{C}}, \tilde{\phi})$ covering $(\mathcal{C}, \phi)$ such that the resulting surface $\tilde{F}$ is an m-fold cover of F with r components as well. Indeed, each component $\tilde{F}_i$ of $\tilde{F}$ can be chosen as the m-fold cyclic cover of F_i dual to a glued cuff $c_i \subset F_i$, and $\tilde{F}$ has an induced pants decomposition that describes $(\tilde{\mathcal{C}}, \tilde{\phi})$. Moreover, $(\tilde{\mathcal{C}}, \tilde{\phi})$ is clearly (R, ϵ)-nearly unit shearing with all glued cuffs nonseparating as well. Therefore, possibly after passing to a positive integral multiple of μ, and considering the gluing $(\tilde{\mathcal{C}}, \tilde{\phi})$ instead of $(\mathcal{C}, \phi)$, we may further assume that

$$\partial\mu(\{\gamma\}) > r$$

for any curve $\gamma \in \mathbf{\Gamma}_{R,\epsilon}$, unless $\partial\mu(\{\gamma\}) = 0$.

Let

$$F = F_1 \sqcup \cdots \sqcup F_r$$

be the decomposition of F into connected components. Then there is an induced decomposition of $\mathcal{F}$ into subunion of pairs of pants $\mathcal{F}_1, \cdots, \mathcal{F}_r$, such that components of each $\mathcal{F}_i$ is projected to be F_i under the gluing. It follows that μ equals $\mu_1 + \cdots + \mu_r$, where the measure μ_i prescribes $\mathcal{F}_i$. Similarly, there is an induced decomposition of

$$\mathcal{C} = \mathcal{C}_1 \sqcup \cdots \sqcup \mathcal{C}_r$$

such that each $\mathcal{C}_i$ is the subunion of cuffs of $\mathcal{F}_i$, which is invariant under ϕ. It follows that

$$\nu^\sharp_{\mathcal{C}} = \nu^\sharp_{\mathcal{C}_1} + \cdots + \nu^\sharp_{\mathcal{C}_r}.$$

Consider a simplicial graph X as follows of r vertices $v_1, \cdots, v_r$. For any $1 \leq i < j \leq r$, the vertices v_i and v_j are connected by an edge if and only if there is a pair of cuffs $c \subset \mathcal{C}_i$ and $c' \subset \mathcal{C}_j$ representing the same curve class $\gamma \in \mathbf{\Gamma}_{R,\epsilon}$, such that $\nu^\sharp_{\phi(c)}$ and $\nu^\sharp_{\phi(c')}$ are $(\frac{\epsilon}{R})$-equivalent on $\mathscr{N}_{\bar{\gamma}}$. We hence fix a choice of c, c' as above, rewriting as c_{ij}, c'_{ij}. Since we have assumed that $\partial\mu(\{\gamma\})$ is either 0 or at least r, we may assume that c_{ij} are mutually distinct components of $\mathcal{C}$, and similarly for c'_{ij}.

Observe that X is connected. In fact, let $X_1, \cdots, X_s$ be the components of X, and let $I_k \subset \{1, \cdots, r\}$ be the subset of indices so that $i \in I_k$ if and only if $v_i \in X_k$. Suppose on the contrary that $s > 1$. We write the $(\frac{\epsilon}{R})$-neighborhood of the support of $\nu^\sharp_{\mathcal{C}_i}$ as U_i, and write the union of $U_1, \cdots, U_r$ as U. Because μ is (R, ϵ)-nearly evenly distributed and rich (Definition 2.8), $U \cap \mathscr{N}_\gamma$ is either the emptyset or $\mathscr{N}_\gamma$, for any curve $\gamma \in \mathbf{\Gamma}_{R,\epsilon}$. If $U \cap \mathscr{N}_\gamma$ equals $\mathscr{N}_\gamma$, then for all the U_i that meets $\mathscr{N}_\gamma$ in a nonempty set, the connectedness of $\mathscr{N}_\gamma$ implies that the corresponding vertices v_i must lie on the same component of X. Therefore, writing μ_{I_k} for the sum of μ_i for $i \in I_k$, it follows that the supports of $\partial\mu_{I_k}$ are mutually disjoint subsets of

$\mathbf{\Gamma}_{R,\epsilon}$. However, μ equals $\mu_{I_1} + \cdots + \mu_{I_s}$. This is contrary to the assumption that μ is irreducible (Definition 2.8).

Guided by the simplicial graph X together with the decorating data c_{ij}, c'_{ij}, we perform the hybriding construction to obtain a new gluing

$$(\mathcal{C}, \phi').$$

More precisely,

$$\phi' : \mathcal{C} \to \mathcal{C}$$

is the gluing such that $\phi'(c_{ij})$ equals $\phi(c'_{ij})$, and that $\phi'(c'_{ij})$ equals $\phi(c_{ij})$, and that $\phi'(c)$ equals $\phi(c)$ for all $c \subset \mathcal{C}$ other than any c_{ij}, c'_{ij} above. Because c_{ij}, c'_{ij} are projected to nonseparating glued cuffs of F_i, F_j respectively, and because X is connected, the new gluing $(\mathcal{C}, \phi')$ of $\mathcal{F}$ results in a connected surface F'.

Because $\nu^\sharp_{\phi(c_{ij})}$ and $\nu^\sharp_{\phi(c'_{ij})}$ are $(\frac{\epsilon}{R})$-equivalent on $\mathscr{N}_{\vec{\gamma}_{ij}}$, where $\gamma_{ij} \in \mathbf{\Gamma}_{R,\epsilon}$ denotes the homotopy class represented by both c_{ij} and c'_{ij}, and because $(\overline{A_1})_*(\nu^\sharp_{c_{ij}})$ and $\nu^\sharp_{\phi(c_{ij})}$ as $(\mathcal{C}, \phi)$ is (R, ϵ)-nearly unit shearing, the construction of ϕ' implies that $(\mathcal{C}, \phi')$ is $(R, 2\epsilon)$-nearly unit shearing. This completes the proof. □

3.4. **Proof of Proposition 3.2.** We summarize the proof of Proposition 3.2 as follows. As μ is irreducible, (R, ϵ)-nearly evenly footed, and rich, possibly after passing to a positive integral multiple of μ, we may construct an $(R, 2\epsilon)$-nearly unit-shearing gluing $(\mathcal{C}, \phi)$ of $\mathcal{F}$ prescribed by μ, so that the resulting surface F is connected (Lemmas 3.4, 3.5, 3.9). If ϵ is sufficiently small so that $2\epsilon < \hat{\epsilon}$ as in Lemma 3.8, and if R is sufficiently large depending only on ϵ, then the induced immersion $j : F \looparrowright M$ is π_1-injective and quasi-Fuchsian. This completes the proof of Proposition 3.2.

4. Hyperbolic geometry of segments with framed endpoints

In this section, we study the techniques of constructing (R, ϵ)-panted surfaces via ∂-framed bigons and tripods in oriented closed hyperbolic 3-manifolds, which generalizes the constructions of [KM2] in the 2-dimension case. In fact, our attempt is to develop a theory of geometry of ∂-framed segments in a closed oriented hyperbolic 3-manifold, which seems to be generalizable to any closed oriented hyperbolic manifold.

Following the spirit of Euclid, objects of the hyperbolic ∂-framed segment geometry are shapes that can be constructed via ∂-framed segments, so the theory of the geometry naturally contains two parts, about *shapes* and about *constructions*. In the first part, for our purpose of application, we will provide an approximate formula that calculates the length and phase of sufficiently tame reduced concatenations of approximately consecutive chains and cycles, which should be compared to the Cosine Law in elementary Euclidean geometry. In the second part, we will define a list of basic constructions, and discuss several more efficient constructions, which can be derived from composing the basic ones. These basic constructions should be regarded as axioms that can be implemented in an oriented closed hyperbolic 3-manifold. The axiomatic approach to constructions brings at least two benefits: first, it highlights the Connection Principle (Lemma 4.13) as a featuring axiom (Definition 4.10 (4)) in the theory; secondly, it allows us to analyze limits of constructions, for instance, as the Spine Principle (Lemma 4.14) implies, that any construction provides no extra information about the second homology of the

3-manifold M. The second point will be of particular importance to the treatment of our paper. It suggests that in order to construct any homologically interesting (R,ϵ)-panted surface, certain *a priori* knowledge about the fundamental group of M should be necessary. In our case, this piece of information will be supplemented by a finite presentation of $\pi_1(M)$, topologically realized by the associated presentation complex, (cf. Sections 5, 6).

In Subsection 4.1, we introduce some basic concepts in the geometry of ∂-framed segments. In Subsection 4.2, we state and prove the Length and Phase Formula (Lemma 4.7). In Subsection 4.3, we introduce the basic constructions in terms of the constructible classes, and show the Connection Principle (Lemma 4.13) and the Spine Principle (Lemma 4.14). In Subsection 4.4, we develop several useful derived constructions, namely, *splitting*, *swapping*, *rotation*, and *antirotation*.

4.1. **Terminology.** Suppose M is an oriented hyperbolic 3-manifold. We introduce several basic concepts in the geometry of ∂-framed segments.

4.1.1. *Segments with framed endpoints.*

Definition 4.1. An *oriented ∂-framed segment* in M is a triple

$$\mathfrak{s}=(s,\vec{n}_{\mathtt{ini}},\vec{n}_{\mathtt{ter}}),$$

such that s is an immersed oriented compact geodesic segment, and that $\vec{n}_{\mathtt{ini}}$ and $\vec{n}_{\mathtt{ter}}$ are two unit normal vectors at the initial endpoint and the terminal endpoint, respectively.

- The *carrier segment* is the oriented segment s;
- The *initial endpoint* $p_{\mathtt{ini}}(\mathfrak{s})$ and the *terminal endpoint* $p_{\mathtt{ter}}(\mathfrak{s})$ are the initial endpoint and the terminal endpoint of s, respectively;
- The *initial framing* $\vec{n}_{\mathtt{ini}}(\mathfrak{s})$ and the *terminal framing* $\vec{n}_{\mathtt{ter}}(\mathfrak{s})$ are the unit normal vectors $\vec{n}_{\mathtt{ini}}$ and $\vec{n}_{\mathtt{ter}}$,
- The *initial direction* $\vec{t}_{\mathtt{ini}}(\mathfrak{s})$ and the *terminal direction* $\vec{t}_{\mathtt{ter}}(\mathfrak{s})$ are the unit tangent vectors in the direction of s at the initial point and the terminal point, respectively.

The *orientation reversal* of $\mathfrak{s}$ is defined to be

$$\bar{\mathfrak{s}}=(\bar{s},\vec{n}_{\mathtt{ter}},\vec{n}_{\mathtt{ini}}),$$

where $\bar{s}$ is the orientation reversal of s. The *framing rotation* of $\mathfrak{s}$ by an angle $\phi\in\mathbf{R}\,/\,2\pi\mathbf{Z}$ is defined to be

$$\mathfrak{s}(\phi)=\left(s,\,\vec{n}_{\mathtt{ini}}\cos\phi+(\vec{t}_{\mathtt{ini}}\times\vec{n}_{\mathtt{ini}})\sin\phi,\,\vec{n}_{\mathtt{ter}}\cos\phi+(\vec{t}_{\mathtt{ter}}\times\vec{n}_{\mathtt{ter}})\sin\phi\right),$$

where $\times$ means the cross product in the tangent space. In particular, the *framing flipping* of $\mathfrak{s}$ is defined to be framing rotation by π, denoted as

$$\mathfrak{s}^*=(s,-\vec{n}_{\mathtt{ini}},-\vec{n}_{\mathtt{ter}}).$$

It follows from the definition that

$$\overline{\mathfrak{s}(\phi)}=\bar{\mathfrak{s}}(-\phi),$$

and in particular, framing flipping commutes with orientation reversion.

Definition 4.2. For an oriented ∂-framed segment $\mathfrak{s}$ in M, the *length* of $\mathfrak{s}$, denoted as

$$\ell(\mathfrak{s})\in(0,+\infty),$$

is the length of the unframed segment s carrying $\mathfrak{s}$, and the *phase* of $\mathfrak{s}$, denoted as

$$\varphi(\mathfrak{s}) \in \mathbf{R} \,/\, 2\pi\mathbf{Z},$$

is the angle from the initial framing $\vec{n}_{\mathtt{ini}}$ to the transportation of $\vec{n}_{\mathtt{ter}}$ to the initial point of s via s, signed with respect to the normal orientation induced from $\vec{t}_{\mathtt{ini}}$ and the orientation of M. We may combine the length and phase into a complex value known as the *phasor* of $\mathfrak{s}$, defined as

$$\boldsymbol{\lambda}(\mathfrak{s}) \;=\; e^{\ell(\mathfrak{s})+\mathrm{i}\varphi(\mathfrak{s})}.$$

The value of a phasor always lies outside the unit circle of $\mathbf{C}$. For an oriented closed geodesic curve c in M, we will also speak of its *length*, *phase*, or *phasor*, by taking an arbitrary unit normal vector $\vec{n}$ at a point $p \in c$, and regarding c as a ∂-framed segment obtained from cutting at p and endowed with framing $\vec{n}$ at both endpoints. Note that the geometric complex length $\mathbf{l}(c)$ of c satisfies

$$\mathbf{l}(c) = \ell(c) + \mathrm{i}|\varphi(c)|,$$

where $|.|$ on $\mathbf{R}/2\pi\mathbf{Z}$ is understood as the distance from zero valued in $[0, \pi]$.

It follows from the definition that length and phase are invariant under orientation reversal and under framing rotation.

4.1.2. *Consecutiveness and fellow travelling.*

Definition 4.3. Let $0 \leq \delta < \frac{\pi}{3}$, and $L > 0$, and $0 < \theta < \pi$ be constants. Let M be an oriented hyperbolic 3-manifold of injectivity radius at least 2δ.

(1) Two oriented ∂-framed segments $\mathfrak{s}$ and $\mathfrak{s}'$ are said to be *δ-consecutive* if terminal endpoint of $\mathfrak{s}'$ is the initial endpoint of $\mathfrak{s}'$, and if the the terminal framing of $\mathfrak{s}$ is δ-close to transportation of the initial framing of $\mathfrak{s}'$ to the terminal endpoint of $\mathfrak{s}$ via the unique δ-short geodesic path in M. We simply say *consecutive* if δ equals zero. The *bending angle* between $\mathfrak{s}$ and $\mathfrak{s}'$ is the angle between the the terminal direction of $\mathfrak{s}$ and the initial direction of $\mathfrak{s}'$, which is valued in $[0, \pi]$.

(2) A *δ-consecutive chain* of oriented ∂-framed segments is a finite sequence $\mathfrak{s}_1, \cdots, \mathfrak{s}_m$ such that each $\mathfrak{s}_i$ is δ-consecutive to $\mathfrak{s}_{i+1}$. It is a *δ-consecutive cycle* if furthermore $\mathfrak{s}_m$ is δ-consecutive to $\mathfrak{s}_1$. A δ-consecutive chain or cycle is said to be *(L, θ)-tame*, if each $\mathfrak{s}_i$ has length at least $2L$, and the bending angle at each joint point is at most θ.

(3) For an (L, θ)-tame δ-consecutive chain $\mathfrak{s}_1, \cdots, \mathfrak{s}_m$, the *reduced concatenation*, denoted as

$$\mathfrak{s}_1 \cdots \mathfrak{s}_m,$$

is the oriented ∂-framed segment as follows. The unframed oriented segment of $\mathfrak{s}_1 \cdots \mathfrak{s}_m$ is homotopic to the piecewise geodesic path obtained from concatenating the unframed oriented segments carrying $\mathfrak{s}_i$, relative to the initial point of $\mathfrak{s}_1$ and the terminal point of $\mathfrak{s}_m$; the initial framing of $\mathfrak{s}_1 \cdots \mathfrak{s}_m$ is the unit normal vector at the initial endpoint that is the closest to the initial framing of $\mathfrak{s}_1$; the terminal framing of $\mathfrak{s}_1 \cdots \mathfrak{s}_m$ is the unit normal vector at the terminal endpoint that is the closest to the terminal framing of $\mathfrak{s}_m$.

(4) In the case of (L,θ)-tame δ-consecutive cycles, the *reduced cyclic concatenation*, denoted as
$$[\mathfrak{s}_1 \cdots \mathfrak{s}_m],$$
is the oriented closed geodesic curve free-homotopic to the concatenation of the unframed oriented segments defined similarly as above, assuming the result not contractible to a point.

Definition 4.4. Let $0 \leq \delta < \frac{\pi}{3}$ be a constant. Let M be an oriented closed hyperbolic 3-manifold of injectivity radius at least 2δ. For any two oriented ∂-framed segments $\mathfrak{s}$ and $\mathfrak{s}'$ in M, we say that $\mathfrak{s}$ and $\mathfrak{s}'$ form a *δ-fellow-travel* pair, if there exist lifts $\tilde{\mathfrak{s}}, \tilde{\mathfrak{s}}'$ of $\mathfrak{s}, \mathfrak{s}'$ in $\mathbb{H}^3$ respectively, such that the initial point of $\tilde{\mathfrak{s}}$ is δ-close to the initial point of $\tilde{\mathfrak{s}}'$, and that the initial framing of $\tilde{\mathfrak{s}}$ is δ-close to the transportation of the initial framing of $\tilde{\mathfrak{s}}'$ to the initial point of $\tilde{\mathfrak{s}}$, and that the same holds for the terminal points and the terminal framings.

4.1.3. *Bigons and tripods.* We introduce (L,δ)-tame bigons and tripods. These objects should be thought of as nearly hyperbolic curves and geodesic 2-simplices in the context of ∂-framed segment geometry.

Definition 4.5. An *(L,δ)-tame bigon* is an (L,δ)-tame δ-consecutive cycle of two oriented ∂-framed segments $\mathfrak{a}, \mathfrak{b}$ of phase δ-close to 0. We usually say that the reduced cyclic concatenation $[\mathfrak{a}\mathfrak{b}]$ is a *(L,δ)-tame bigon* with the cycle understood. Furthermore, it is said to be *(l,δ)-nearly regular* if the edges $\mathfrak{a}$ and $\mathfrak{b}$ have length δ-close to l.

Note that framing flipping does not change the reduced cyclic concatenation, namely, $[\mathfrak{a}^*\mathfrak{b}^*]$ is the same as $[\mathfrak{a}\mathfrak{b}]$. However, the orientation of $[\bar{\mathfrak{a}}\bar{\mathfrak{b}}]$ is exactly opposite to that of $[\mathfrak{a}\mathfrak{b}]$.

Definition 4.6. An *(L,δ)-tame tripod*, denoted as
$$\mathfrak{a}_0 \vee \mathfrak{a}_1 \vee \mathfrak{a}_2,$$
is a triple $(\mathfrak{a}_0, \mathfrak{a}_1, \mathfrak{a}_2)$ of oriented ∂-framed segments of length at least $2L$ and of phase δ-close to 0, such that $\bar{\mathfrak{a}}_i$ is δ-consecutive to $\mathfrak{a}_{i+1}$ with bending angle δ-close to $\frac{\pi}{3}$, for $i \in \mathbf{Z}_3$. Furthermore, it is said to be *(l,δ)-nearly regular* if the legs $\mathfrak{a}_i$ have length δ-close to l, for $i \in \mathbf{Z}_3$. For each $i \in \mathbf{Z}_3$, $\mathfrak{a}_i$ will be referred to as a *leg* of $\mathfrak{a}_0 \vee \mathfrak{a}_1 \vee \mathfrak{a}_2$, and
$$\mathfrak{a}_{i,i+1} = \bar{\mathfrak{a}}_i \mathfrak{a}_{i+1}$$
will be referred to as a *side* of $\mathfrak{a}_0 \vee \mathfrak{a}_1 \vee \mathfrak{a}_2$. Note that the initial framings of $\mathfrak{a}_i$ are δ-close to each other, so approximately the ordered initial directions rotates either couterclockwise or clockwise around any of them. We say that $\mathfrak{a}_0 \vee \mathfrak{a}_1 \vee \mathfrak{a}_2$, is *right-hand* if it is the former case, or *left-hand* if the latter.

Note that framing flipping switches the chirality and the side order of a nearly regular tripod, namely, the chirality of $\mathfrak{a}_0^* \vee \mathfrak{a}_1^* \vee \mathfrak{a}_2^*$ is exactly opposite to that of $\mathfrak{a}_0 \vee \mathfrak{a}_1 \vee \mathfrak{a}_2$. However, the chirality of $\mathfrak{a}_0^* \vee \mathfrak{a}_{-1}^* \vee \mathfrak{a}_{-2}^*$ is the same as that of $\mathfrak{a}_0 \vee \mathfrak{a}_1 \vee \mathfrak{a}_2$, where the indices are considered to be in $\mathbf{Z}_3$.

4.2. **The Length and Phase Formula.** For sufficiently tame concatenations of approximately consecutive chains and cycles, the change of length and phase under reduction of the concatenation can be approximately calculated.

Recall that for any bending angle $0 \leq \vartheta < \pi$, the *limit inefficiency* associated to ϑ is defined as

$$I(\vartheta) \,=\, 2\log(\sec(\vartheta/2)).$$

The function $I(\vartheta)$ is strictly convex and increasing on $[0,\pi)$, and the geometric meaning is explained by Lemma 4.8 (2).

Lemma 4.7 (Length and Phase Formula)**.** *Given any constants $0 < \delta < \frac{1}{100}$, and $0 < \theta < \pi - 100\delta$, and $L > I(\theta) + \frac{1}{10}$, the following statements hold in any oriented hyperbolic 3-manifold of injectivity radius at least 2δ.*

(1) If $\mathfrak{s}_1, \cdots, \mathfrak{s}_m$ is an (L,θ)-tame δ-consecutive chain of oriented ∂-framed segments, denoting the bending angle between $\mathfrak{s}_i$ and $\mathfrak{s}_{i+1}$ as $\theta_i \in [0,\theta]$, then

$$\left|\ell(\mathfrak{s}_1\cdots\mathfrak{s}_m) - \sum_{i=1}^{m}\ell(\mathfrak{s}_i) + \sum_{i=1}^{m-1} I(\theta_i)\right| \,<\, (90+\tan(\theta/2))(m-1)\delta,$$

and

$$\left|\varphi(\mathfrak{s}_1\cdots\mathfrak{s}_m) - \sum_{i=1}^{m}\varphi(\mathfrak{s}_i)\right| \,<\, 100(m-1)\delta,$$

where $|.|$ on $\mathbf{R}/2\pi\mathbf{Z}$ is understood as the distance from zero valued in $[0,\pi]$.

(2) If $\mathfrak{s}_1, \cdots, \mathfrak{s}_m$ is an (L,θ)-tame δ-consecutive cycle of oriented ∂-framed segments, denoting the bending angle between $\mathfrak{s}_i$ and $\mathfrak{s}_{i+1}$ as $\theta_i \in [0,\pi-\theta]$ with $\mathfrak{s}_{m+1}$ equal to $\mathfrak{s}_0$ by convention, then

$$\left|\ell([\mathfrak{s}_1\cdots\mathfrak{s}_m]) - \sum_{i=1}^{m}\ell(\mathfrak{s}_i) + \sum_{i=1}^{m} I(\theta_i)\right| \,<\, (90+\tan(\theta/2))m\delta,$$

and

$$\left|\varphi([\mathfrak{s}_1\cdots\mathfrak{s}_m]) - \sum_{i=0}^{m}\varphi(\mathfrak{s}_i)\right| \,<\, 100m\delta,$$

where $|.|$ on $\mathbf{R}/2\pi\mathbf{Z}$ is understood as the distance from zero valued in $[0,\pi]$.

The proof relies on a lemma in elementary hyperbolic geometry, which provides some key estimation for tame concatenation of segments.

Lemma 4.8. *Given any constants $0 < \delta < \frac{\pi}{2}$, and $0 < \theta < \pi - 2\delta$, and $L > I(\theta)$, suppose that $\triangle ABC$ is a geodesic triangle in hyperbolic space, where $|CA|$, $|CB| \geq L$, and $\angle C = \pi - \theta$, then*

(1) $\angle A + \angle B \,<\, 4\delta\sin(\theta/2) \,<\, 4\delta$.
(2) $I(\theta) + 4\delta/\log(2\delta) \,<\, |CA| + |CB| - |AB| \,<\, I(\theta)$.

Proof. To prove the inequality (1), it suffices to assume that $|CA| = |CB| = L$ since in this case $\angle A + \angle B$ achieves its unique maximum. Let M be the midpoint of AB. In the right triangle $\triangle ACM$, it follows from the Dual Law of Cosines that

$$-\cos\frac{\angle C}{2}\cos\angle A + \sin\frac{\angle C}{2}\sin\angle A\cosh|AC| \,=\, 0.$$

Therefore,

$$\angle A \,<\, \tan\angle A \,=\, \frac{\tan(\theta/2)}{\cosh L}.$$

Since $0 < \theta < \pi - 2\delta$,

$$\cosh L \;>\; \cosh I(\theta) \;>\; e^{I(\theta)}/2 \;>\; \sec^2(\theta/2)/2 \;>\; \sec(\theta/2)\csc(\delta)/2,$$

thus,

$$\angle A \;<\; \frac{\tan(\theta/2)}{\sec(\theta/2)\csc(\delta)/2} \;=\; 2\sin\delta\sin(\theta/2) \;<\; 2\delta\sin(\theta/2).$$

The same estimation holds for $\angle B$, so the inequality (1) follows.

To prove the inequality (2), consider the inscribed circle $\odot J$ of $\triangle ABC$, denoting the tangent point of $\odot J$ with AB, BC, and CA as T_c, T_b, and T_a, respectively. Then

$$|CA| + |CB| - |AB| \;=\; |CT_a| + |CT_b|,$$

which approaches the supremum as $|CA|$ and $|CB|$ tend to $+\infty$, and achieves the unique minimum when $|CA| = |CB| = L$. A direct computation shows that the supremum is exactly $I(\theta)$, so the upper bound of inequality (2) holds.

For the lower bound, it hence suffices to assume that $|CA| = |CB| = L$. We write $\triangle A^*B^*C$ for the triangle with ideal points A^* and B^*, and let $\odot J^*$ be the inscribed circle which is tangent to A^*B^*, B^*C and CA^* at T_c, T_a^* and T_b^*, respectively. Note that now T_c is the midpoint of AB, and similarly for T_c^*. It is also clear that

$$|CT_b^*| - |CT_b| \;=\; |T_b^*T_b| \;<\; |J^*J| \;<\; |T_cT_c^*|.$$

In the right triangle $\triangle CT_cA$, it follows from the Dual Law of Cosines that

$$\sin\angle A\cosh|AT_c| \;=\; \cos\frac{\angle C}{2}.$$

Therefore,

$$\cosh|AT_c| \;=\; \frac{\cos((\pi-\theta)/2)}{\sin\angle A} \;>\; \frac{\sin(\theta/2)}{2\delta\sin(\theta/2)} \;=\; \frac{1}{2\delta}.$$

Since $e^{|AT_c|} \;>\; \cosh|AT_c|$, we obtain

$$|AT_c| > -\log(2\delta).$$

On the other hand, the difference between the area of $\triangle A^*B^*C$ and $\triangle ABC$ is the area of the quadrilateral $AT_cT_c^*A^*$, which clearly equals the value of $\angle A$ in $\triangle ABC$. Because AT_c and $A^*T_c^*$ are perpendicular to $T_cT_c^*$, there is the comparison of area:

$$|T_cT_c^*|\cdot|AT_c| \;<\; \mathrm{Area}(AT_cT_c^*A^*) \;=\; \angle A.$$

Therefore,

$$|T_cT_c^*| \;<\; \frac{\angle A}{|AT_c|} \;<\; \frac{2\delta}{-\log(2\delta)}.$$

We obtain

$$\begin{aligned} |CA| + |CB| - |AB| \;&=\; |CT_a| + |CT_b| \\ &>\; |CT_a^*| + |CT^*| - 2|T_cT_c^*| \\ &=\; I(\theta) + \frac{4\delta}{\log(2\delta)}, \end{aligned}$$

which verifies the lower bound in the inequality (2). $\square$

Proof of Lemma 4.7. Since it follows from certain standard estimation argument provided Lemma 4.8, we only sketch the proof.

To see the statement (1), we write

$$\Delta\ell(\mathfrak{s}_1\cdots\mathfrak{s}_m) = \ell(\mathfrak{s}_1\cdots\mathfrak{s}_m) - \sum_{i=1}^{m}\ell(\mathfrak{s}_i) + \sum_{i=1}^{m-1} I(\theta_i)$$

and

$$\Delta\varphi(\mathfrak{s}_1\cdots\mathfrak{s}_m) = \varphi(\mathfrak{s}_1\cdots\mathfrak{s}_m) - \sum_{i=1}^{m}\varphi(\mathfrak{s}_i)$$

for the error terms that we will estimate.

By perturbing $\vec{n}_{\mathtt{ter}}(\mathfrak{s}_i)$ and $\vec{n}_{\mathtt{ini}}(\mathfrak{s}_{i+1})$ appropriately for each $1 \leq i < m$, we may obtain a consecutive $(\tilde{L},\tilde{\theta})$-chain $\tilde{\mathfrak{s}}_1,\cdots,\tilde{\mathfrak{s}}_m$, where $(\tilde{L},\tilde{\theta})$ equals $(L,\theta+\delta)$. Moreover, we may assume $|\ell(\tilde{\mathfrak{s}}_i)-\ell(\mathfrak{s}_i)|$, $|\varphi(\tilde{\mathfrak{s}})-\varphi(\mathfrak{s}_i)|$, and $|\tilde{\theta}_i-\theta_i|$ are all bounded by δ. Hence $|I(\tilde{\theta}_i)-I(\theta_i)|$ is bounded by $|\tilde{\theta}_i-\theta_i|\cdot\tan((\theta+\delta)/2) < \delta+\delta\tan(\theta/2)$. Therefore, the new errors of length and phase differ from the old by

$$|\Delta\ell(\tilde{\mathfrak{s}}_1\cdots\tilde{\mathfrak{s}}_m)-\Delta\ell(\mathfrak{s}_1\cdots\mathfrak{s}_m)| < (3+\tan(\theta/2))(m-1)\delta,$$

and

$$|\Delta\varphi(\tilde{\mathfrak{s}}_1\cdots\tilde{\mathfrak{s}}_m)-\Delta\varphi(\mathfrak{s}_1\cdots\mathfrak{s}_m)| < (m-1)\delta.$$

Note that

$$L > I(\theta+\delta)-\delta(1+\tan(\theta/2))+\frac{1}{10} > I(\theta+\delta)+\frac{1}{10}-\delta(1+\cot(50\delta)),$$

so with $\delta < \frac{1}{100}$,

$$\tilde{L} > I(\tilde{\theta}).$$

It is also obvious that $\tilde{\theta} < \pi - 2\delta$. We will be safe to apply Lemma 4.8 in the following with respect to δ, $\tilde{\theta}$ and $\tilde{L}$.

If m equals 1, we have already done, as $\Delta\ell(\tilde{\mathfrak{s}}_1)$ and $\Delta\varphi(\tilde{\mathfrak{s}}_1)$ are both 0 by definition. If m is greater than 1, we may consider a chain $\mathfrak{s}'_1,\cdots,\mathfrak{s}'_{m-1}$ as follows. For each $1 < i < m$, write $\tilde{\mathfrak{s}}_i$ as the concatenation of two consecutive oriented ∂-framed segments $\tilde{\mathfrak{s}}_{i-}$ and $\tilde{\mathfrak{s}}_{i+}$ of equal length and phase. For $1 < i < m-1$, let $\mathfrak{s}'_i$ be $\tilde{\mathfrak{s}}_{i+}\tilde{\mathfrak{s}}_{(i+1)-}$. Let $\mathfrak{s}'_1$ be $\tilde{\mathfrak{s}}_1\tilde{\mathfrak{s}}_{2-}$, and $\mathfrak{s}'_m$ be $\tilde{\mathfrak{s}}_{(m-1)+}\tilde{\mathfrak{s}}_m$, or in the case that m equals 2, let $\mathfrak{s}'_1$ be $\tilde{\mathfrak{s}}_1\tilde{\mathfrak{s}}_2$. It follows immediately from Lemma 4.8 that $\mathfrak{s}'_1,\cdots,\mathfrak{s}'_{m-1}$ is (10δ)-consecutive and $(\tilde{L},10\delta)$-tame, and that $\Delta\ell(\mathfrak{s}'_1\cdots\mathfrak{s}'_{m-1})$ is $(-4(m-1)\delta/\log(2\delta))$-close, and hence $(10(m-1)\delta)$-close, to $\Delta\ell(\tilde{\mathfrak{s}}_1\cdots\tilde{\mathfrak{s}}_m)$. It is also clear that $\Delta\varphi(\mathfrak{s}'_1\cdots\mathfrak{s}'_{m-1})$ is $(10(m-1)\delta)$-close to $\Delta\varphi(\tilde{\mathfrak{s}}_1\cdots\tilde{\mathfrak{s}}_m)$.

If m equals 2, we have done since the chain $\mathfrak{s}'_1,\cdots,\mathfrak{s}'_{m-1}$ has only one term. If m is greater than 2, we may further obtain a chain $\mathfrak{s}''_1,\cdots,\mathfrak{s}''_{m-2}$ from $\mathfrak{s}'_1,\cdots,\mathfrak{s}'_{m-1}$, in a similar way as we obtain the latter from $\tilde{\mathfrak{s}}_1,\cdots,\tilde{\mathfrak{s}}_m$. However, because $\mathfrak{s}'_1,\cdots,\mathfrak{s}'_{m-1}$ is (10δ)-consecutive and $(\tilde{L},10\delta)$-tame, applying the finer bound of Lemma 4.8 (1) will imply that $\mathfrak{s}''_1,\cdots,\mathfrak{s}''_{m-2}$ is much more closely consecutive and tamer, for example, as sufficiently for our estimation, $(10\delta)^2$-consecutive and $(\tilde{L},(10\delta)^2)$-tame. It follows that $\Delta\ell(\mathfrak{s}''_1\cdots\mathfrak{s}''_{m-2})$ is $((m-1)(10\delta)^2)$-close, to $\Delta\ell(\mathfrak{s}'_1\cdots\mathfrak{s}'_{m-1})$, and that $\Delta\varphi(\mathfrak{s}''_1\cdots\mathfrak{s}''_{m-2})$ is $((m-1)(10\delta)^2)$-close to $\Delta\varphi(\mathfrak{s}'_1\cdots\mathfrak{s}'_{m-1})$.

Proceed iteratively to obtain new chains $\mathfrak{s}_1^r, \cdots, \mathfrak{s}_{m-r}^r$ from $\mathfrak{s}_1^{r-1}, \cdots, \mathfrak{s}_{m-r-1}^{r-1}$ until $m-r$ equals 1. Summing up the error of length in each step yields that

$$|\Delta\ell(\tilde{\mathfrak{s}}_1 \cdots \tilde{\mathfrak{s}}_m)| \,<\, \sum_{r=1}^{m-1}(m-1)(10\delta)^r \,<\, 20(m-1)\delta,$$

and that

$$|\Delta\varphi(\tilde{\mathfrak{s}}_1 \cdots \tilde{\mathfrak{s}}_m)| \,<\, \sum_{r=1}^{m-1}(m-1)(10\delta)^r \,<\, 20(m-1)\delta.$$

Therefore, we have the estimations of the statement (1).

The statment (2) can be proved similarly. We first obtain a consecutive cycle $\tilde{\mathfrak{s}}_1, \cdots, \tilde{\mathfrak{s}}_m$ from $\mathfrak{s}_1, \cdots, \mathfrak{s}_m$, then construct iteratively the cycles $\mathfrak{s}_1^{r+1}, \cdots, \mathfrak{s}_m^{r+1}$ from $\mathfrak{s}_1^r, \cdots, \mathfrak{s}_m^r$ by joining consequential midpoints, starting with $\mathfrak{s}_1^0, \cdots, \mathfrak{s}_m^0$ which is $\tilde{\mathfrak{s}}_1, \cdots, \tilde{\mathfrak{s}}_m$. Similar estimations as before hold in this case, and as r tends to infinity, the (non-reduced) cyclic concatenation $\mathfrak{s}_1^r, \cdots, \mathfrak{s}_m^r$ converges to $[\mathfrak{s}_1 \cdots \mathfrak{s}_m]$ geometrically. Note that the summation of errors in this case will be a geometric series, but the upper bound $20m\delta$ will stay unchanged. Combining the estimations as before yields the estimations in the statement (2). □

4.3. **Principles of construction.** Before we discuss how to construct various (R, ϵ)-panted surfaces with ∂-framed segments (Subsection 4.4), in this subsection, we wish to formally discuss what we mean by a *construction* (Definition 4.12). We will enumerate our basic constructions as axioms (Definition 4.10). These constructions are realizable in an oriented closed hyperbolic 3-manifold M essentially because of the Connection Principle (Lemma 4.13, cf. Lemma 4.11). Then we will state the Spine Principle (Lemma 4.14), which morally says that since we are only drawing auxiliary ∂-framed segments in any such construction, we will not gain any new knowledge about the second homology of M. This observation will be of fundamental importance when we pantify a second homology class (Section 6). In practice, it will be convenient to describe constructions more naturally in terms of ∂-framed segments, and at the end of this subsection, we will explain how to translate between the natural description and the formal description in terms of constructible extensions. However, the reader may safely skip the discussion of this subsection until Section 6.

4.3.1. *Constructible classes.* We provide a formal definition of constructible objects in terms of partially-Δ spaces over an oriented closed hyperbolic 3-manifold.

Definition 4.9. Let M be an oriented hyperbolic 3-manifold. A *partially-Δ space over M* is a triple (X, X_Δ, f_X) as follows. The space X_Δ is a CW subspace of a CW space X, enriched with a Δ-complex structure; the map $f_X : X \to M$ is geodesic restricted to the 1-skeleton of X_Δ. We often simply mention a partially-Δ space X with X_Δ and f_X implicitly assumed. A *partially combinatorial* map between two partially-Δ space X and Y is a CW map $\phi : (X, X_\Delta) \to (Y, Y_\Delta)$, combinatorial with respect to the Δ-complex structure of X_Δ and Y_Δ, such that $f_Z = f_X \circ \phi$. For a partially-Δ space Z over M, an *extension* of Z is a partially-Δ space X together with a partially combinatorial embedding $\phi : (Z, Z_\Delta) \to (X, X_\Delta)$.

We list our basic constructions by the following axioms, and verify that they are all possible in the situation that we will be concerned with.

Definition 4.10. For an oriented hyperbolic 3-manifold M and a pair (L,δ) of positive constants, the *axioms of constructions* are the following statements:

(1) *Vertex Creation.* Suppose that $p \in M$ is a point. If X is a partially-Δ space over M, then there exists a partially-Δ space X' as follows. The space pair (X', X'_Δ) is $(X \sqcup v,\, X_\Delta \sqcup v)$ where v is a new vertex; the map $f_{X'}$ is an extension of f_X such that $f_{X'}(v) = p$.

(2) *Vertex Insertion.* Suppose that $p \in M$ is a point. If X is a partially-Δ space over M with an edge e of X_Δ, such that $f(e)$ passes through p, then there exists a partially-Δ space X' as follows. The space pair (X', X'_Δ) is $(X \cup e_- \cup e_+ \cup D,\, X_\Delta \cup e_- \cup e_+)$, where e_- and e_+ are new edges consecutive at a new vertex v, with the initial vertex of e_- the initial vertex of e, and the terminal vertex of e_+ the terminal vertex of e, and where D is a new disk with the boundary 1-cycle $e_-, e_+, \bar{e}$; the map $f_{X'}$ is an extension of f_X so that $f_{X'}(v) = p$, and that $f_{X'}|_{e_-}$ and $f_{X'}|_{e_+}$ are geodesic subsegments of $f_{X'}|_e$ in the sense that $f_{X'}|_{e_- \cup e_+}$ homotopic to $f_{X'}|_e$ relative to the endpoints.

(3) *Edge Extension.* Suppose that l is a positive number. If X is a partially-Δ space over M with an edge e of X_Δ such that $f(e)$ is a nondegenerate geodesic segment, then there exists a partially-Δ space X' over M as follows. The space pair (X', X'_Δ) is $(X \cup e',\, X_\Delta \cup e')$, where e is a new edge the terminal vertex of e to a new vertex v; the map $f_{X'}$ is an extension of f_X such that $f_{X'}|_{e'}$ is a geodesic segment of length l that extends the geodesic segment $f_X|_e$.

(4) *Edge Connection.* Suppose that $\vec{t}_p$, $\vec{n}_p \in T_pM$ and $\vec{t}_q$, $\vec{n}_q \in T_qM$ are pairs of orthogonal unit vectors at points $p, q \in M$ respectively, and that λ is a complex number of modulus at least L. If X is a partially-Δ space over M with (not necessarily distinct) vertices v, w of X_Δ such that $f_X(v) = p$ and $f_X(w) = q$, then there exists a partially-Δ space X' over M as follows. The space pair (X', X'_Δ) is $(X \cup e,\, X_\Delta \cup e)$, where e is a new edge from v to w; the map $f_{X'}$ is an extension of f_X such that $f_{X'}|_e$ carries an oriented ∂-framed segment $\mathfrak{s}$ from p to q satisfying the following.
   - The oriented ∂-framed segment $\mathfrak{s}$ has length and phase δ-close to $\log|\lambda|$ and $\arg(\lambda)$, respectively. The initial direction and framing of $\mathfrak{s}$ are δ-close to $\vec{t}_p$ and $\vec{n}_p$, respectively. The terminal direction and framing of $\mathfrak{s}$ are δ-close to $\vec{t}_q$ and $\vec{n}_q$, respectively.

(5) *Reduction of Concatenation.* Suppose that $\mathfrak{s}_1, \cdots, \mathfrak{s}_m$ is an (L,θ)-tame δ-consecutive chain of oriented ∂-framed segments in M. If X is a partially-Δ space over M with a 1-chain $e_1, \cdots, e_m$ of X_Δ such that $f_X|_{e_i}$ carries $\mathfrak{s}_i$, then there exists a partially-Δ space X' over M as follows. The space pair (X', X'_Δ) is $(X \cup e \cup D,\, X_\Delta \cup e)$, where e is a new edge from the initial vertex of e_1 to the terminal vertex of e_1, and D is a new disk with the boundary 1-cycle of D the 1-cycle $e_1, \cdots, e_m, \bar{e}$; the map $f_{X'}$ is an extension of f_X such that $f_{X'}|_e$ carries the reduced concatenation $\mathfrak{s}_1 \cdots \mathfrak{s}_m$.

(6) *Reduction of Cyclic Concatenation.* Suppose that $\mathfrak{s}_1, \cdots, \mathfrak{s}_m$ is an (L,θ)-tame δ-consecutive cycle of oriented ∂-framed segments. If X is a partially-Δ space over M with a 1-cycle $e_1, \cdots, e_m$ of X_Δ such that $f_X|_{e_i}$ carries $\mathfrak{s}_i$, then there exists a partially-Δ space X' over M as follows. The space pair (X', X'_Δ) is $(X \cup v \cup e \cup A,\, X_\Delta \cup v \cup e)$, where v is a new vertex, e

is a new edge with both vertices attached to v, and A is a new oriented annulus with the boundary 1-cycle of A the sum of the 1-cycles $e_1, \cdots, e_m$ and the orientation reversal of e; the map $f_{X'}$ is an extension of f_X such that $f_{X'}|_{v\cup e}$ carries the reduced cyclic concatenation $[\mathfrak{s}_1 \cdots \mathfrak{s}_m]$.

(7) *Tripod Zipping.* If X is a partially-Δ space over M with a 2-simplex σ of X_Δ such that the immersed oriented geodesic triangle $f_X|_{\partial\sigma}$ has all the internal angles δ-close to 0, then there exists a partially-Δ space X' over M as follows. The space pair (X', X'_Δ) is $(X \cup C(\sigma\cup\partial\sigma),\, X'_\Delta \cup C(\sigma\cup\partial\sigma))$, where $C(\sigma\cup\partial\sigma)$ is the cone over the Δ-complex closure of σ in X_Δ; the map $f_{X'}$ is an extension of f_X such that $f_{X'}$ restricted to the cone over vertices of σ carries a $(10,\delta)$-tame tripod $\mathfrak{a}_0 \vee \mathfrak{a}_1 \vee \mathfrak{a}_2$ satisfying the following.
- For each $i \in \mathbf{Z}_3$, the length of $\mathfrak{a}_i$ is at least 10, and the phase of $\mathfrak{a}_i$ is 0. The initial endpoints and initial framings of all $\mathfrak{a}_i$ are the same. The initial directions of all $\mathfrak{a}_i$ form 120° angle between each other.
- The carrier segment of $\mathfrak{a}_i$ lies on the unique geodesic 2-simplex homotopic to $f_X|_\sigma$ relative to the boundary, for $i \in \mathbf{Z}_3$. The carrier segment of $\mathfrak{a}_{i,i+1}$ is the oriented sides of $f_X|_{\partial\sigma}$ for $i \in \mathbf{Z}_3$.

(8) *Fellow Traveller Zipping.* Suppose that $\mathfrak{a}_1, \cdots, \mathfrak{a}_n$ are oriented ∂-framed segments of length at least $4L+1$, such that the terminal endpoints of $\mathfrak{a}_i$ are the same, and the terminal directions and framings of $\mathfrak{a}_i$ are $(\delta/(2\cosh(2L)))$-close to each other respectively for all i. If X is a partially-Δ space over M with edges $e_1, \cdots, e_m$ of X_Δ having the same terminal vertex v, such that $f|_{e_i}$ carries $\mathfrak{a}_i$, then there exists a partially-Δ space X' over M as follows. The partially-Δ space (X', X'_Δ) is $(X \cup w \cup \tilde{e}_0 \cup \cdots \cup \tilde{e}_n \cup s \cup D_1 \cup \cdots \cup D_n,\, X'_\Delta \cup w \cup \tilde{e}_0 \cup \cdots \cup \tilde{e}_n \cup s)$, where w is a new vertex, and $\tilde{e}_i$ are new edges from the initial vertices of e_i to w, and s is a new edge from w to v, and D_i are a new oriented CW disk with boundary 1-cycles $\tilde{e}_i, s, \bar{e}_i$; the map $f_{X'}$ is an extension of f_X such that $f|_s$ and $f|_{\tilde{e}_i}$ carry oriented ∂-framed segments $\mathfrak{s}$ and $\tilde{\mathfrak{a}}_i$ respectively, satisfying the following.
- The chains $\tilde{\mathfrak{a}}_i, \mathfrak{s}$ are δ-consecutive and (L,δ)-tame for all i.
- The reduced concatenation $\tilde{\mathfrak{a}}_i\mathfrak{s}$ is the same as $\mathfrak{a}_i$ up to $(\delta/(2\cosh(2L)))$-small change of framings for all i.

Lemma 4.11. *Suppose that M is an oriented closed hyperbolic 3-manifold. If δ is universally small positive, and if L is sufficiently large depending only on δ and M, the axioms of constructions are true statements.*

Proof. Axioms (1), (2), (3) are clearly true for any hyperbolic 3-manifold, since they are true for the universal cover $\mathbb{H}^3$ by straightforward constructions.

Axiom (4) holds for any oriented closed hyperbolic 3-manifold M. In fact, it suffices to prove the existence of $f_{X'}|_e$, and this follows immediately from the Connection Principle (Lemma 4.13), which we prefer to state separately later because of its importance.

Axioms (5) and (6) occur only if the hyperbolic 3-manifold has nontrivial fundamental group. The constructions are straightforward from the statement.

Axiom (7) holds for any oriented hyperbolic 3-manifold. In fact, it suffices to prove the existence of the tripod $\mathfrak{a}_0 \vee \mathfrak{a}_1 \vee \mathfrak{a}_2$ for $\mathbb{H}^3$. Moreover, the following argument also implies that $\mathfrak{a}_0 \vee \mathfrak{a}_1 \vee \mathfrak{a}_2$ is unique up to rotations of the indices of legs subject to either prescribed chirality. Let p be the Fermat point on the geodesic 2-simplex $f_X(\sigma)$ that minimizes sum of the distances to the three vertices. Then the oriented geodesic segments s_0, s_1, s_2 from p to the three vertices of $f_X(\sigma)$ form $120°$ angle to each other if the internal angles of σ are all less than $120°$. An easy argument of elementary hyperbolic geometry shows that if the internal angles are all universally small, the length of each s_i will be at least 20. Possibly after switching the order of s_0, s_1, s_2, we may assume that they are rotating counterclockwise around the unit normal vector $\vec{n}_p$ at p of σ in M. For each $i \in \mathbf{Z}_3$, let $\mathfrak{a}_i$ be the unique oriented ∂-framed segment carried by s_i of phase 0 and with the initial framing $\vec{n}_p$. This gives rise to the right-hand tripod $\mathfrak{a}_0 \vee \mathfrak{a}_1 \vee \mathfrak{a}_2$ as desired, which is unique up to rotation of the indices in $\mathbf{Z}_3$. Taking $\mathfrak{a}_0^* \vee \mathfrak{a}_1^* \vee \mathfrak{a}_2^*$ instead of $\mathfrak{a}_0 \vee \mathfrak{a}_1 \vee \mathfrak{a}_2$, we obtain the unique left-hand tripod as desired.

Axiom (8) holds for any oriented hyperbolic 3-manifold. Again, it suffices to prove the existence of $\mathfrak{s}$ and $\tilde{\mathfrak{a}}_i$ for $\mathbb{H}^3$. We may decompose $\mathfrak{a}_1$ into the concatenation of two consecutive oriented ∂-framed segments $\tilde{\mathfrak{a}}_1\mathfrak{s}$, where $\mathfrak{s}$ has length $2L + \frac{1}{10}$. Let $\tilde{\mathfrak{a}}_i$ be an oriented ∂-orientable segment carried by the geodesic segment from $p_{\mathtt{ter}}(\mathfrak{s})$ to $p_{\mathtt{ter}}(\mathfrak{a}_i)$, and with initial and terminal framings as close as possible to $\vec{n}_{\mathtt{ter}}(\mathfrak{s})$ and $\vec{n}_{\mathtt{ter}}(\mathfrak{a}_i)$, respectively. Using elementary hyperbolic geometry, a direct estimation will verify that $\tilde{\mathfrak{a}}_1, \cdots, \tilde{\mathfrak{a}}_n$ and $\mathfrak{s}$ are as claimed. □

Let M be an oriented close hyperbolic 3-manifold and let (L, δ) be a pair of positive constants satisfying the assumption of Lemma 4.11.

Definition 4.12. Suppose that Z is a partially-Δ space over M. We define the *class of constructible extensions of* Z with respect to (L, δ) to be the smallest class $\mathscr{C}$ of extensions of Z, satisfying that $Z \in \mathscr{C}$, and that $X \in \mathscr{C}$ implies $X' \in \mathscr{C}$ for any X' obtained from X by one of the axioms of constructions above. A partially-Δ space U over M is said to be *constructible from* Z if there exists a constructible extension $X \in \mathscr{C}$ of Z and a partially combinatorial map $\psi : U \to X$, such that $f_U = f_X \circ \psi$.

In practice, Z will serve as the object that a construction starts with, where Z_Δ records the piece of information that are available for the construction; X will serve as the recipe of the construction; and U will serve as the resulting object of the construction, where U_Δ is meant to record certain properties that result should satisfy.

4.3.2. *The Connection Principle.* We emphasize the following Connection Principle as it is the fundamental reason for all our constructions of nearly regular pants to work. For example, it implies that $\mathbf{\Gamma}_{R,\epsilon}$ and $\mathbf{\Pi}_{R,\epsilon}$ are nonempty for an oriented closed hyperbolic 3-manifold M, provided that ϵ is universally small positive and that R is sufficiently large depending only on M and ϵ.

Lemma 4.13 (Connection Principle)**.** *For any universally small positive δ, and for any sufficiently large positive L depending only on δ and M, the following statement holds. If $\vec{t}_p$, $\vec{n}_p \in T_pM$ and $\vec{t}_q$, $\vec{n}_q \in T_qM$ are pairs of orthogonal unit vectors at points $p, q \in M$ respectively, and if λ is a complex number of modulus at least L, then there exists an oriented ∂-framed segment $\mathfrak{s}$ from p to q satisfying the following.*

- *The oriented ∂-framed segment $\mathfrak{s}$ has length and phase δ-close to* $\log|\lambda|$ *and* $\arg(\lambda)$*, respectively. The initial direction and framing of $\mathfrak{s}$ are δ-close to $\vec{t}_p$ and $\vec{n}_p$, respectively. The terminal direction and framing of $\mathfrak{s}$ are δ-close to $\vec{t}_q$ and $\vec{n}_q$, respectively.*

Proof. This follows from the fact that the frame flow is mixing [KM1, Theorem 4.2]. In fact, it suffices to prove the existence of $f_{X'}|_e$ and the argument is the same as that of [KM1, Lemma 4.4]. □

4.3.3. *The Spine Principle.* The Spine Principle says that any constructible extension of a partially-Δ space over an oriented hyperbolic 3-manifold is *relatively* 1-*spined*, or precisely as follows.

Lemma 4.14 (Spine Principle)**.** *If (X, X_Δ, f_X) is a constructible extension of a partially-Δ space (Z, Z_Δ, f_Z) over an oriented hyperbolic 3-manifold M, then the defining inclusion $\phi : Z \to X$ can be extended to be a homotopy equivalence $Z' \simeq X$ where Z' is obtained from Z by attaching cells of dimension at most 1.*

Proof. This follows immediately from inspecting the construction axioms listed in Definition 4.12. □

4.3.4. *Describing a construction.* In the rest of this paper, we will often describe a construction without explicitly writing down the associated partially-Δ spaces. Instead, the hypothesis of a construction will be stated in terms of ∂-framed segments. For our applications, the result of a construction is often an (R, ϵ)-panted surface $j : F \to M$ with the boundary prescribed in terms of the ∂-framed segments from the hypothesis. See any construction of Subsection 4.4 for an example. To translate such a description into one with partially-Δ spaces, one may take the partially-Δ space (Z, Z_Δ, f_Z) to be as follows. Take Z_Δ to be a 1-complex such that each ∂-framed segment in the hypothesis corresponds to an oriented 1-cell, and any two 1-cells have identified endpoints if and only if the corresponding ∂-framed segments are declared to be consecutive; take Z to be Z_Δ; and take f_Z to be the obvious map that send any 1-cell to the carrier segment of its defining ∂-framed segment. In the same fashion, the recipe of the construction applies the axioms of the construction (Definition 4.10) step by step, by indicating at each step that an auxiliary point, segment, or ∂-framed segment should be drawn, so the procedure gives rise to an extension X of Z. The result of the construction can then be translated in terms of a partially-Δ space $(F, \partial F, j)$ constructible from Z, where ∂F has a preferred 1-complex structure since it is prescribed by the ∂-framed segments from the hypothesis. In fact, one can always write down the partially combinatorial map $\psi : F \to X$ explicitly, where X is the extension of Z from the recipe of the construction.

4.4. **Derived constructions.** In this subsection, we exhibit several constructions of (R, ϵ)-panted surfaces using ∂-framed segments that will be applied in the rest of this paper. Throughout this subsection, we assume M to be an oriented closed hyperbolic 3-manifold. We will assume that ϵ is at most 1, and δ positive and less than the minimum among $\frac{1}{100}$, $\frac{\epsilon}{10000}$ and half the injectivity radius of M, and L at least 100 satisfying the conclusion of the Connection Principle (Lemma 4.13) with respect to δ and M, and R at least $100L$. The constructions below are all *derived* from the basic constructions listed in Definition 4.10, and are all *definite* in the sense that the number of ∂-framed segments involved are universally bounded.

4.4.1. *Splitting.* The splitting construction below gives rise to a nearly regular pair of pants by adding a bisecting segment to a nearly purely hyperbolic curve. The reader should compare it with [KM2, Lemma 3.2 and Remark].

Construction 4.15 (Splitting)**.** *Let $\mathfrak{s},\mathfrak{s}'$ be two cyclically consecutive oriented ∂-framed segments of the same length, $(\frac{\epsilon}{2})$-close to $\frac{R}{2}$, and the same phase, $(\frac{\epsilon}{2})$-close to 0. Then a pair of pants $\Pi\in\mathbf{\Pi}_{R,\epsilon}$ can be constructed, with one cuff $[\mathfrak{s}\mathfrak{s}']\in\mathbf{\Gamma}_{R,\epsilon}$, and with the other two cuffs in $\mathbf{\Gamma}_{R,\delta}$.*

Proof. By the Connection Principle (Lemma 4.13), draw an oriented ∂-framed segment $\mathfrak{m}$ from $p_{\mathtt{ter}}(\mathfrak{s})$ to $p_{\mathtt{ini}}(\mathfrak{s})$ as follows:

- The initial and terminal directions are δ-close to $\vec{t}_{\mathtt{ter}}(\mathfrak{s})\times\vec{n}_{\mathtt{ter}}(\mathfrak{s})$ and $-\vec{t}_{\mathtt{ini}}(\mathfrak{s})\times\vec{n}_{\mathtt{ini}}(\mathfrak{s})$ respectively, and the initial and terminal framings are δ-close to $\vec{n}_{\mathtt{ter}}(\mathfrak{s})$ and $\vec{n}_{\mathtt{ini}}(\mathfrak{s})$ respectively, and the length and phase are δ-close to $R-\ell(\mathfrak{s})+I(\frac{\pi}{2})$ and $-\varphi(\mathfrak{s})$ respectively.

Then a pair of pants $\Pi\in\mathbf{\Pi}_{R,\epsilon}$ can be constructed, such that one cuff of Π is $[\mathfrak{s}\mathfrak{s}']\in\mathbf{\Gamma}_{R,\epsilon}$, and that the other two cuffs are $\overline{[\mathfrak{s}\mathfrak{m}]}$, $\overline{[\bar{\mathfrak{m}}\mathfrak{s}']}\in\mathbf{\Gamma}_{R,\delta}$. The verification follows from the Length and Phase Formula (Lemma 4.7). □

4.4.2. *Swapping.* The swapping construction below allows us to exchange the arcs of two nearly purely hyperbolic curves if they fellow travel near a pair of common points, as long as the result are still purely hyperbolic curves. The reader should compare it with the Geometric Square Lemma [KM2, Lemma 5.4]. In fact, our construction largely follows the idea there.

Definition 4.16. A *δ-swap pair* of bigons is a pair of $(1,\delta)$-tame bigons $[\mathfrak{a}\mathfrak{b}]$ and $[\mathfrak{a}'\mathfrak{b}']$, such that the cycles $\mathfrak{a},\mathfrak{b}'$ and $\mathfrak{a}',\mathfrak{b}$ are also δ-consecutive, and that $\mathfrak{a}$ and $\mathfrak{a}'$ have length and phase δ-close to each other respectively, and that the same holds for $\mathfrak{b}$ and $\mathfrak{b}'$. In this case, we say that the new pair of $(1,\delta)$-tame bigons $[\mathfrak{a}\mathfrak{b}']$ and $[\mathfrak{a}'\mathfrak{b}]$ is the δ-swap pair resulted from *swapping* $[\mathfrak{a}\mathfrak{b}]$ and $[\mathfrak{a}'\mathfrak{b}']$, and vice versa.

Construction 4.17 (Swapping)**.** *Let $[\mathfrak{a}\mathfrak{b}]$ and $[\mathfrak{a}'\mathfrak{b}']$ a δ-swap pair of $(100,\delta)$-tame bigons. Suppose that the sum of length $\ell(\mathfrak{a})+\ell(\mathfrak{b})$ is δ-close to R. Then an oriented connected compact (R,ϵ)-panted surface F can be constructed, with exactly four boundary components $[\mathfrak{a}\mathfrak{b}]$, $[\mathfrak{a}'\mathfrak{b}']$, $\overline{[\mathfrak{a}\mathfrak{b}']}$, $\overline{[\mathfrak{a}'\mathfrak{b}]}\in\mathbf{\Gamma}_{R,\epsilon}$.*

Proof. We will first construct swap surfaces for pairs of squares, which can be regarded as a special case of swapping pairs of bigons. Then a simple reduction from the bigon case to the square case using fellow traveller replacement will complete the proof.

Parallel to the definition for bigons, we will refer to an (L,δ)-tame cycle of four oriented ∂-framed segments $\mathfrak{a},\mathfrak{s},\mathfrak{b},\mathfrak{t}$, or ambiguously their reduced cyclic concatenation $[\mathfrak{a}\mathfrak{s}\mathfrak{b}\mathfrak{t}]$, as an *$(L,\delta)$-tame square*. A *$\delta$-swap pair* of squares with respect to the common segments $\mathfrak{s}$ and $\mathfrak{t}$ is a pair of $(1,\delta)$-tame squares $[\mathfrak{a}\mathfrak{s}\mathfrak{b}\mathfrak{t}]$ and $[\mathfrak{a}'\mathfrak{s}\mathfrak{b}'\mathfrak{t}]$ such that $\mathfrak{a}$ and $\mathfrak{a}'$ has length and phase δ-close to each other respectively, and that the same holds for $\mathfrak{b}$ and $\mathfrak{b}'$. In this case, we say that the new pair of $(1,\delta)$-tame squares $[\mathfrak{a}\mathfrak{s}\mathfrak{b}'\mathfrak{t}]$ and $[\mathfrak{a}'\mathfrak{s}\mathfrak{b}\mathfrak{t}]$ is the δ-swap pair resulted from *swapping* $[\mathfrak{a}\mathfrak{s}\mathfrak{b}\mathfrak{t}]$ and $[\mathfrak{a}'\mathfrak{s}\mathfrak{b}'\mathfrak{t}]$, and vice versa.

Step 1. Let $[\mathfrak{a}\mathfrak{s}\mathfrak{b}\mathfrak{t}]$ and $[\mathfrak{a}'\mathfrak{s}\mathfrak{b}'\mathfrak{t}]$ be a δ-swap pair of two $(2L,\delta)$-tame squares. Suppose that the sum of length $\ell(\mathfrak{a})+\ell(\mathfrak{b})$ is δ-close to $R-\ell(\mathfrak{s})-\ell(\mathfrak{t})$, and that the sum of phase $\varphi(\mathfrak{a})+\varphi(\mathfrak{b})$ is δ-close to $-\varphi(\mathfrak{s})-\varphi(\mathfrak{t})$. In addition, suppose for this step that both $\ell(\mathfrak{a})$ and $\ell(\mathfrak{b})$ are less than $\frac{R}{2}-4L$. We construct an oriented connected compact (R,ϵ)-panted surface F with exactly four boundary components $[\mathfrak{a}\mathfrak{s}\mathfrak{b}\mathfrak{t}]$, $[\mathfrak{a}'\mathfrak{s}\mathfrak{b}'\mathfrak{t}]$, $\overline{[\mathfrak{a}\mathfrak{s}\mathfrak{b}'\mathfrak{t}]}$, $\overline{[\mathfrak{a}'\mathfrak{s}\mathfrak{b}\mathfrak{t}]}\in\mathbf{\Gamma}_{R,\epsilon}$, using four pairs of pants from $\mathbf{\Pi}_{R,\epsilon}$ as follows.

Observe that the additional assumption allows one can find a segment that approximately tamely bisects all the squares. More precisely, let p be the point on the carrier segment of $\mathfrak{s}$ of distance $2L$ from the terminal endpoint of $\mathfrak{s}$, and the assumption implies that there is a point q on the carrier segment of $\mathfrak{t}$ of distance $\frac{R}{2}-2L-\ell(\mathfrak{b})>2L$ from the initial endpoint of $\mathfrak{t}$. Suitably assigning framings at p and q, we decompose $\mathfrak{s}$ and $\mathfrak{t}$ as the concatenation of consecutive oriented ∂-framed segments $\mathfrak{s}_-\mathfrak{s}_+$ and $\mathfrak{t}_-\mathfrak{t}_+$, respectively, such that the reduced concatenation $\mathfrak{t}_+\mathfrak{a}\mathfrak{s}_-$ is (L,δ)-tame with the length and phase δ-close to $\frac{R}{2}$ and 0 respectively, and that the same holds for the reduced concatenations $\mathfrak{t}_+\mathfrak{a}'\mathfrak{s}_-$, $\mathfrak{s}_+\mathfrak{b}\mathfrak{t}_-$, and $\mathfrak{s}_+\mathfrak{b}'\mathfrak{t}_-$ as well. Draw an oriented ∂-framed segment $\mathfrak{m}$ from p to q satisfying the following:

- $\mathfrak{m}$ has length and phase δ-close to $\frac{R}{2}+2I(\frac{\pi}{2})$ and 0 respectively. The initial and terminal directions of $\mathfrak{m}$ are δ-close to the cross product $-\vec{t}_{\mathtt{ter}}(\mathfrak{s}_-)\times\vec{n}_{\mathtt{ter}}(\mathfrak{s}_-)$ and $\vec{t}_{\mathtt{ter}}(\mathfrak{t}_-)\times\vec{n}_{\mathtt{ter}}(\mathfrak{t}_-)$ respectively. The initial and terminal framings of $\mathfrak{m}$ are δ-close to $\vec{n}_{\mathtt{ter}}(\mathfrak{s}_-)$ and $\vec{n}_{\mathtt{ter}}(\mathfrak{t}_-)$ respectively.

With $\mathfrak{u}$ standing for $\mathfrak{a}$ or $\mathfrak{a}'$, and $\mathfrak{v}$ standing for $\mathfrak{b}$ or $\mathfrak{b}'$, there is a unique pair of pants $\Pi_{\mathfrak{u},\mathfrak{v}}\in\mathbf{\Pi}_{R,\epsilon}$ determined by its cuffs $[\mathfrak{u}\mathfrak{s}\mathfrak{v}\mathfrak{t}]$, $\overline{[\mathfrak{u}\mathfrak{s}_-\mathfrak{m}\mathfrak{t}_+]}$, $\overline{[\mathfrak{s}_+\mathfrak{v}\mathfrak{t}_-\bar{\mathfrak{m}}]}\in\mathbf{\Gamma}_{R,\epsilon}$. Note that the curve $\overline{[\mathfrak{u}\mathfrak{s}_-\mathfrak{m}\mathfrak{t}_+]}$ appears in exactly two pairs of pants as a cuff, and that the same holds for $\overline{[\mathfrak{s}_+\mathfrak{v}\mathfrak{t}_-\bar{\mathfrak{m}}]}$. Thus the four pairs of pants $\Pi_{\mathfrak{a},\mathfrak{b}},\Pi_{\mathfrak{a},\mathfrak{b}'},\overline{\Pi_{\mathfrak{a}',\mathfrak{b}}},\overline{\Pi_{\mathfrak{a}',\mathfrak{b}'}}\in\mathbf{\Pi}_{R,\epsilon}$ can be glued along these cuffs, yielding the desired (R,ϵ)-panted surface F, which is a torus with four holes.

Step 2. Let $[\mathfrak{a}\mathfrak{s}\mathfrak{b}\mathfrak{t}]$ and $[\mathfrak{a}'\mathfrak{s}\mathfrak{b}'\mathfrak{t}]$ be a δ-swap pair of two $(10L,\delta)$-tame squares. Suppose that the sum of length $\ell(\mathfrak{a})+\ell(\mathfrak{b})$ is δ-close to $R-\ell(\mathfrak{s})-\ell(\mathfrak{t})$, and that the sum of phase $\varphi(\mathfrak{a})+\varphi(\mathfrak{b})$ is δ-close to $-\varphi(\mathfrak{s})-\varphi(\mathfrak{t})$. In addition, suppose for this step that both $\ell(\mathfrak{a})$ and $\ell(\mathfrak{b})$ are less than $\frac{3R}{4}-30L$. We construct an oriented connected compact (R,ϵ)-panted surface F with exactly four boundary components $[\mathfrak{a}\mathfrak{s}\mathfrak{b}\mathfrak{t}]$, $[\mathfrak{a}'\mathfrak{s}\mathfrak{b}'\mathfrak{t}]$, $\overline{[\mathfrak{a}\mathfrak{s}\mathfrak{b}'\mathfrak{t}]}$, $\overline{[\mathfrak{a}'\mathfrak{s}\mathfrak{b}\mathfrak{t}]}\in\mathbf{\Gamma}_{R,\epsilon}$, using at most eight pairs of pants from $\mathbf{\Pi}_{R,\epsilon}$ as follows.

Because of Step 1, it suffices to find F assuming that the length of either $\mathfrak{a}$ or $\mathfrak{b}$ is at least $\frac{R}{2}-4L$. Without loss of generality, we may suppose that $\ell(\mathfrak{a})$ and $\ell(\mathfrak{a}')$ are greater than $\frac{R}{2}-6L$. Let p be the point on the carrier segment of $\mathfrak{s}$ of distance $2L$ from the terminal endpoint of $\mathfrak{s}$, and now there is a point q on the carrier segment of $\mathfrak{a}$ of distance $\ell(\mathfrak{a})+\ell(\mathfrak{s})-\frac{R}{2}-2L$, which is between $12L$ and $\frac{R}{4}-12L$, from the initial endpoint of $\mathfrak{a}$, and similarly a point q' on the carrier segment of $\mathfrak{a}'$. One can suitably assign framings at the points p, q, q' to decompose the oriented ∂-framed segments $\mathfrak{s}$, $\mathfrak{a}$, $\mathfrak{a}'$ as the concatenations $\mathfrak{s}_-\mathfrak{s}_+$, $\mathfrak{a}_-\mathfrak{a}_+$, $\mathfrak{a}'_-\mathfrak{a}'_+$, respectively, such that the reduced concatenation $\mathfrak{a}_+\mathfrak{s}_-$ is (L,δ)-tame with the length and phase δ-close to $\frac{R}{2}$ and 0 respectively, and that the same holds for the reduced concatenations $\mathfrak{a}'_+\mathfrak{s}_-$, $\mathfrak{s}_+\mathfrak{b}\mathfrak{t}\mathfrak{a}_-$, and $\mathfrak{s}_+\mathfrak{b}\mathfrak{t}\mathfrak{a}'_-$ as well. Choose an auxiliary point $x\in M$. Draw an oriented ∂-framed segment $\mathfrak{r}$ from p to x, and two oriented ∂-framed segments $\mathfrak{c}$ and $\mathfrak{c}'$ from x to q and to q' respectively, satisfying the following:

- $\mathfrak{r}$ has length and phase δ-close to $\frac{R}{4}+I(\frac{\pi}{2})-2L$ and 0 respectively. The initial direction of $\mathfrak{r}$ is δ-close to the cross product $-\vec{t}_{\mathtt{ter}}(\mathfrak{s}_-)\times\vec{n}_{\mathtt{ter}}(\mathfrak{s}_-)$. The initial framing of $\mathfrak{r}$ are δ-close to $\vec{n}_{\mathtt{ter}}(\mathfrak{s}_-)$.
- $\mathfrak{c}$ and $\mathfrak{c}'$ have length and phase δ-close to $\frac{R}{4}+I(\frac{\pi}{2})+2L$ and 0, respectively. The terminal directions of $\mathfrak{c}$ and $\mathfrak{c}'$ are δ-close to $\vec{t}_{\mathtt{ter}}(\mathfrak{a}_-)\times\vec{n}_{\mathtt{ter}}(\mathfrak{a}_-)$ and $\vec{t}_{\mathtt{ter}}(\mathfrak{a}'_-)\times\vec{n}_{\mathtt{ter}}(\mathfrak{a}'_-)$, respectively. The terminal framings of $\mathfrak{c}$ and $\mathfrak{c}'$ are δ-close to $\vec{n}_{\mathtt{ter}}(\mathfrak{a}_-)$ and $\vec{n}_{\mathtt{ter}}(\mathfrak{a}'_-)$, respectively.

- The chains $\mathfrak{r},\mathfrak{c}$ and $\mathfrak{r},\mathfrak{c}'$ are δ-consecutive. Note that they are also $(1,\delta)$-tame.

With $\mathfrak{v}$ standing for $\mathfrak{b}$ or $\mathfrak{b}'$, there is a unique pair of pants $\Pi_{\mathfrak{a},\mathfrak{v}} \in \mathbf{\Pi}_{R,\epsilon}$ determined by its cuffs $[\mathfrak{a}\mathfrak{s}\mathfrak{b}\mathfrak{t}]$, $\overline{[\mathfrak{a}_+\mathfrak{s}_-\mathfrak{r}\mathfrak{c}]}$, $\overline{[\mathfrak{s}_+\mathfrak{v}\mathfrak{t}\mathfrak{a}_-\bar{\mathfrak{c}}\bar{\mathfrak{r}}]} \in \mathbf{\Gamma}_{R,\epsilon}$, and similarly, $\Pi_{\mathfrak{a}',\mathfrak{v}} \in \mathbf{\Pi}_{R,\epsilon}$ with cuffs $[\mathfrak{a}'\mathfrak{s}\mathfrak{b}\mathfrak{t}]$, $\overline{[\mathfrak{a}'_+\mathfrak{s}_-\mathfrak{r}\mathfrak{c}']}$, $\overline{[\mathfrak{s}_+\mathfrak{v}\mathfrak{t}\mathfrak{a}'_-\bar{\mathfrak{c}}'\bar{\mathfrak{r}}]} \in \mathbf{\Gamma}_{R,\epsilon}$. Note that the curve $\overline{[\mathfrak{a}_+\mathfrak{s}_-\mathfrak{r}\mathfrak{c}]}$ appears in exactly two pairs of pants as a cuff, and that the same holds for $\overline{[\mathfrak{a}'_+\mathfrak{s}_-\mathfrak{r}\mathfrak{c}']}$. Another four cuffs form two δ-swap pairs of squares, namely, the pair $\overline{[\mathfrak{s}_+\mathfrak{b}\mathfrak{t}\mathfrak{a}_-\bar{\mathfrak{c}}\bar{\mathfrak{r}}]}$ and $\overline{[\mathfrak{s}_+\mathfrak{b}'\mathfrak{t}\mathfrak{a}'_-\bar{\mathfrak{c}}'\bar{\mathfrak{r}}]}$, and the other pair $\overline{[\mathfrak{s}_+\mathfrak{b}\mathfrak{t}\mathfrak{a}'_-\bar{\mathfrak{c}}'\bar{\mathfrak{r}}]}$ and $\overline{[\mathfrak{s}_+\mathfrak{b}'\mathfrak{t}\mathfrak{a}_-\bar{\mathfrak{c}}\bar{\mathfrak{r}}]}$, with respect to the common ∂-framed segments $\bar{\mathfrak{r}}\mathfrak{s}_+$ and $\mathfrak{t}$. Moreover, swapping each pair results in the other. By Lemma 4.7, the lengths of $\mathfrak{b}$ and $\mathfrak{b}'$ are less than $\frac{R}{2} - 32L$, and that the lengths of $\mathfrak{a}_-\bar{\mathfrak{c}}$ and $\mathfrak{a}'_-\bar{\mathfrak{c}}'$ are less than $\frac{R}{2} - 8L$, where $2L$ has been deducted in each bound to buffer the error. By Step 1, there is an oriented (R,ϵ)-panted four-hold torus E with boundary exactly four curves $[\mathfrak{s}_+\mathfrak{b}\mathfrak{t}\mathfrak{a}_-\bar{\mathfrak{c}}\bar{\mathfrak{r}}]$, $[\mathfrak{s}_+\mathfrak{b}'\mathfrak{t}\mathfrak{a}'_-\bar{\mathfrak{c}}\bar{\mathfrak{r}}]$, $\overline{[\mathfrak{s}_+\mathfrak{b}\mathfrak{t}\mathfrak{a}'_-\bar{\mathfrak{c}}\bar{\mathfrak{r}}]}$, and $\overline{[\mathfrak{s}_+\mathfrak{b}'\mathfrak{t}\mathfrak{a}_-\bar{\mathfrak{c}}\bar{\mathfrak{r}}]}$. Thus the two pairs of pants $\Pi_{\mathfrak{a},\mathfrak{b}}$ and $\overline{\Pi_{\mathfrak{a},\mathfrak{b}'}}$ can be glued along the oppositely oriented common cuffs $\overline{[\mathfrak{a}_+\mathfrak{s}_-\mathfrak{r}\mathfrak{c}]}$ and $[\mathfrak{a}_+\mathfrak{s}_-\mathfrak{r}\mathfrak{c}]$, and similarly, $\overline{\Pi_{\mathfrak{a}',\mathfrak{b}}}$ and $\Pi_{\mathfrak{a}',\mathfrak{b}'}$ can be glued their oppositely oriented common cuffs. Another four cuffs then match up with the boundary components of E with respect to orientation. Gluing up along these curve results in a connected oriented compact (R,ϵ) panted surface F with exactly four boundary components $[\mathfrak{a}\mathfrak{s}\mathfrak{b}\mathfrak{t}]$, $[\mathfrak{a}'\mathfrak{s}\mathfrak{b}'\mathfrak{t}]$, $\overline{[\mathfrak{a}\mathfrak{s}\mathfrak{b}'\mathfrak{t}]}$, $\overline{[\mathfrak{a}'\mathfrak{s}\mathfrak{b}\mathfrak{t}]} \in \mathbf{\Gamma}_{R,\epsilon}$ as desired.

Step 3. Let $[\mathfrak{a}\mathfrak{s}\mathfrak{b}\mathfrak{t}]$ and $[\mathfrak{a}'\mathfrak{s}\mathfrak{b}'\mathfrak{t}]$ be a δ-swap pair of two $(20L,\delta)$-tame squares. Suppose that the sum of length $\ell(\mathfrak{a}) + \ell(\mathfrak{b})$ is δ-close to $R - \ell(\mathfrak{s}) - \ell(\mathfrak{t})$, and that the sum of phase $\varphi(\mathfrak{a}) + \varphi(\mathfrak{b})$ is δ-close to $-\varphi(\mathfrak{s}) - \varphi(\mathfrak{t})$. We construct an oriented connected compact (R,ϵ)-panted surface F with exactly four boundary components $[\mathfrak{a}\mathfrak{s}\mathfrak{b}\mathfrak{t}]$, $[\mathfrak{a}'\mathfrak{s}\mathfrak{b}'\mathfrak{t}]$, $\overline{[\mathfrak{a}\mathfrak{s}\mathfrak{b}'\mathfrak{t}]}$, $\overline{[\mathfrak{a}'\mathfrak{s}\mathfrak{b}\mathfrak{t}]} \in \mathbf{\Gamma}_{R,\epsilon}$, using at most twelve pairs of pants from $\mathbf{\Pi}_{R,\epsilon}$ as follows.

Because of Step 2, it suffices to find F assuming that the length of either $\mathfrak{a}$ or $\mathfrak{b}$ is at least $\frac{3R}{4} - 30L$. Without loss of generality, we may suppose that $\ell(\mathfrak{a})$ and $\ell(\mathfrak{a}')$ are greater than $\frac{3R}{4} - 32L$. We perform a similar construction as in Step 2. Now the point p is chosen on the carrier segment of $\mathfrak{s}$ of distance $2L$ from the terminal endpoint of $\mathfrak{s}$. Construct q, q', $\mathfrak{a}_\pm$, $\mathfrak{a}'_\pm$, and $\mathfrak{s}_\pm$ as before. The length of $\mathfrak{a}_-$ is between $\frac{R}{4} + 6L$ and $\frac{R}{2} - 42L$. Construct $\mathfrak{c}$, $\mathfrak{c}'$, and $\mathfrak{r}$ as before with the same parameters. It follows that the lengths of $\mathfrak{b}$ and $\mathfrak{b}'$ are less than $\frac{R}{4} - 46L$, and that the lengths of $\mathfrak{a}_-\bar{\mathfrak{c}}$ and $\mathfrak{a}'_-\bar{\mathfrak{c}}'$ are less than $\frac{3R}{4} - 38L$, where $2L$ has been deducted in each bound to buffer the error. We may construct the surfaces as before except applying Step 2 instead of Step 1 for the swap surface E. This yields an oriented compact (R,ϵ)-panted surface with exactly four boundary components $[\mathfrak{a}\mathfrak{s}\mathfrak{b}\mathfrak{t}]$, $[\mathfrak{a}'\mathfrak{s}\mathfrak{b}'\mathfrak{t}]$, $\overline{[\mathfrak{a}\mathfrak{s}\mathfrak{b}'\mathfrak{t}]}$, $\overline{[\mathfrak{a}'\mathfrak{s}\mathfrak{b}\mathfrak{t}]} \in \mathbf{\Gamma}_{R,\epsilon}$ as desired.

Step 4. We construct an oriented connected compact (R,ϵ)-panted surface F in the bigon case. Let $[\mathfrak{a}\mathfrak{b}]$ and $[\mathfrak{a}'\mathfrak{b}']$ be the δ-swap pair of $(100,\delta)$-tame bigons as assumed.

The trick is that a very consecutive and tame swap pair of bigons can be regarded as as a swap pair of square, and vice versa. On one hand, a δ-swap pair of (L,δ)-tame squares $[\mathfrak{a}\mathfrak{s}\mathfrak{b}\mathfrak{t}]$ and $[\mathfrak{a}'\mathfrak{s}\mathfrak{b}'\mathfrak{t}]$ can be regarded as a (100δ)-swap pair of $(2L,100\delta)$-tame bigons $[(\mathfrak{a}\mathfrak{s})(\mathfrak{b}\mathfrak{t})]$ and $[(\mathfrak{a}'\mathfrak{s})(\mathfrak{b}'\mathfrak{t})]$, by the length and phase formula (Lemma 4.7). On the other hand, a δ-swap pair of $(2K+1,\delta)$-tame bigons $[\mathfrak{a}\mathfrak{b}]$ and $[\mathfrak{a}'\mathfrak{b}']$

can also be regarded as a $(2(\cosh(2K)+1)\delta)$-swap pair of $(K, 2\cosh(2K)\delta))$-tame squares $[\tilde{\mathfrak{a}}\mathfrak{s}\tilde{\mathfrak{b}}\mathfrak{t}]$ and $[\tilde{\mathfrak{a}}'\mathfrak{s}\tilde{\mathfrak{b}}'\mathfrak{t}]$, by the fellow traveller zipping (Definition 4.10 (8)), provided $K > 1$ and $2\delta\cosh(2K) < \frac{1}{100}$.

Therefore, we can regard $[\mathfrak{a}\mathfrak{b}]$ and $[\mathfrak{a}'\mathfrak{b}']$ as $\tilde{\delta}$-swap pair of $(20\tilde{L}, \tilde{\delta})$-tame squares, where $(\tilde{L}, \tilde{\delta})$ can be chosen as $(2, 2\cosh(80)\delta)$, respectively. Now the bigon case simply reduces to Step 3 assuming δ to be so small that Step 3 can be applied with respect to the pair $(\tilde{L}, \tilde{\delta})$ instead of (L, δ). This completes the proof. □

4.4.3. *Rotation.* When two nearly regular tripods of opposite chiralities have legs almost opposite to the ones of each other, one can naturally build a nearly regular pair of pants spined on the union of the two tripods. This will be the first statement of the rotation construction below. However, there is another case when the tripods have identical chirality. Thus we have two rotation constructions. Note that in the identical chirality case, we take two copies of the curves arising from the construction. This turns out to be necessary due to Theorem 5.2.

Definition 4.18. A *δ-rotation pair* of tripods is a pair of $(10, \delta)$-tame tripods $\mathfrak{a}_0 \vee \mathfrak{a}_1 \vee \mathfrak{a}_2$ and $\mathfrak{b}_0 \vee \mathfrak{b}_1 \vee \mathfrak{b}_2$, where $\mathfrak{a}_0 \vee \mathfrak{a}_1 \vee \mathfrak{a}_2$ is (l_a, δ)-nearly regular, and $\mathfrak{b}_0 \vee \mathfrak{b}_1 \vee \mathfrak{b}_2$ is (l_b, δ)-nearly regular. Moreover, the chains $\mathfrak{a}_i, \bar{\mathfrak{b}}_j$ are δ-consecutive for all $i, j \in \mathbf{Z}_3$.

Construction 4.19 (Rotation)**.** *Let $\mathfrak{a}_0 \vee \mathfrak{a}_1 \vee \mathfrak{a}_2$ and $\mathfrak{b}_0 \vee \mathfrak{b}_1 \vee \mathfrak{b}_2$ be a δ-rotation pair of tripods. Suppose $\ell(\mathfrak{a}_i) + \ell(\mathfrak{b}_j)$ is δ-close to $\frac{R}{2} + I(\frac{\pi}{3})$, for $i, j \in \mathbf{Z}_3$. Then an oriented connected compact (R, ϵ)-panted surface F can be constructed satisfying the following.*

1. *If $\mathfrak{a}_0 \vee \mathfrak{a}_1 \vee \mathfrak{a}_2$ and $\mathfrak{b}_0 \vee \mathfrak{b}_1 \vee \mathfrak{b}_2$ are of opposite charalities, then F is a pair of pants $\Pi \in \mathbf{\Pi}_{R,\epsilon}$ with cuffs $[\mathfrak{a}_{i,i+1}\bar{\mathfrak{b}}_{i,i+1}]$ in $\mathbf{\Gamma}_{R,\epsilon}$, for $i \in \mathbf{Z}_3$.*
2. *If $\mathfrak{a}_0 \vee \mathfrak{a}_1 \vee \mathfrak{a}_2$ and $\mathfrak{b}_0 \vee \mathfrak{b}_1 \vee \mathfrak{b}_2$ are of identical charality, then F has exactly six boundary components, namely, two copies of each $[\mathfrak{a}_{i,i+1}\bar{\mathfrak{b}}_{i,i+1}]$ in $\mathbf{\Gamma}_{R,\epsilon}$, for $i \in \mathbf{Z}_3$.*

Proof. To prove the statement (1), suppose that $\mathfrak{a}_0 \vee \mathfrak{a}_1 \vee \mathfrak{a}_2$ and $\mathfrak{b}_0 \vee \mathfrak{b}_1 \vee \mathfrak{b}_2$ are of opposite chiralities. This is the simple case because F can be naturally chosen as the pair of pants $\Pi \in \mathbf{\Pi}_{R,\epsilon}$ with cuffs $[\mathfrak{a}_{i,i+1}\bar{\mathfrak{b}}_{i,i+1}]$ for $i \in \mathbf{Z}_3$. The spine of Π is the figure-θ graph which is approximately the union of the carrier segments of $\mathfrak{a}_i\bar{\mathfrak{b}}_i$ for $i \in \mathbf{Z}_3$.

To prove the statment (2), suppose that $\mathfrak{a}_0 \vee \mathfrak{a}_1 \vee \mathfrak{a}_2$ and $\mathfrak{b}_0 \vee \mathfrak{b}_1 \vee \mathfrak{b}_2$ are of identical chirality. Considering the framing flipping $\mathfrak{a}_0^* \vee \mathfrak{a}_1^* \vee \mathfrak{a}_2^*$ and $\mathfrak{b}_0^* \vee \mathfrak{b}_1^* \vee \mathfrak{b}_2^*$ instead if necessary, we may assume that the tripods are both right-hand without loss of generality.

Step 1. We construct an oriented connected compact (R, ϵ)-panted surface F assuming that $\mathfrak{b}_0 \vee \mathfrak{b}_1 \vee \mathfrak{b}_2$ can be written as $\mathfrak{c}_0\mathfrak{r} \vee \mathfrak{c}_1\mathfrak{r} \vee \mathfrak{c}_2\mathfrak{r}$ where $\mathfrak{c}_i, \mathfrak{r}$ is a δ-consecutive (L, δ)-tame chain for each $i \in \mathbf{Z}_3$. It follows that $\mathfrak{c}_0 \vee \mathfrak{c}_1 \vee \mathfrak{c}_2$ is an $(l_b - \ell(\mathfrak{r}), \delta)$-nearly regular right-hand tripod. Draw an auxiliary oriented ∂-oriented segment $\mathfrak{s}$ satisfying the following.

- The length and phase of $\mathfrak{s}$ is δ-close to $\ell(\mathfrak{r})$ and 0 respectively. The initial and terminal endpoints of $\mathfrak{s}$ coincide with $p_{\mathtt{ini}}(\mathfrak{r})$ and $p_{\mathtt{ter}}(\mathfrak{r})$ respectively. The initial and terminal directions of $\mathfrak{s}$ are δ-close to $\vec{t}_{\mathtt{ini}}(\mathfrak{r})$ and $\vec{t}_{\mathtt{ter}}(\mathfrak{r})$ respectively. The initial framing of $\mathfrak{s}$ is δ-close to $-\vec{n}_{\mathtt{ini}}(\mathfrak{r})$, and the terminal framing of $\mathfrak{s}$ is δ-close to $\vec{n}_{\mathtt{ter}}(\mathfrak{r})$.

It follows that $\mathfrak{c}_0^*\mathfrak{s} \vee \mathfrak{c}_1^*\mathfrak{s} \vee \mathfrak{c}_2^*\mathfrak{s}$ is an $(l_b - \ell(\mathfrak{r}), \delta)$-nearly regular left-hand tripod, where $\mathfrak{c}_i^*, \mathfrak{s}$ is a δ-consecutive (L, δ)-tame chain for each $i \in \mathbf{Z}_3$.

Observe the following (R,ϵ)-panted surfaces. By swapping (Construction 4.17), for each $i \in \mathbf{Z}_3$, there is an (R,ϵ)-panted surface E_i with boundary components the curves $[\mathfrak{a}_{i,i+1}\bar{\mathfrak{r}}\,\bar{\mathfrak{c}}_{i,i+1}\mathfrak{r}]$, $[\bar{\mathfrak{a}}_{i,i+1}\,\bar{\mathfrak{s}}\mathfrak{c}^*_{i,i+1}\mathfrak{s}]$, $\overline{[\bar{\mathfrak{a}}_{i,i+1}\bar{\mathfrak{r}}\,\bar{\mathfrak{c}}_{i,i+1}\mathfrak{r}]}$, $\overline{[\mathfrak{a}_{i,i+1}\bar{\mathfrak{s}}\mathfrak{c}^*_{i,i+1}\mathfrak{s}]}$ in $\mathbf{\Gamma}_{R,\epsilon}$; also by swapping (Construction 4.17), for each $i \in \mathbf{Z}_3$, there is an (R,ϵ)-panted surface E_i' with boundary components the curves $[\mathfrak{a}_{i,i+1}\bar{\mathfrak{r}}\,\bar{\mathfrak{c}}_{i,i+1}\mathfrak{r}]$, $[\bar{\mathfrak{a}}^*_{i,i+1}\,\bar{\mathfrak{s}}^*\mathfrak{c}_{i,i+1}\mathfrak{s}^*]$, $\overline{[\mathfrak{a}_{i,i+1}\bar{\mathfrak{r}}\mathfrak{c}_{i,i+1}\mathfrak{r}]}$, $\overline{[\bar{\mathfrak{a}}^*_{i,i+1}\,\bar{\mathfrak{s}}^*\bar{\mathfrak{c}}_{i,i+1}\mathfrak{s}^*]}$ in $\mathbf{\Gamma}_{R,\epsilon}$; by rotation in the opposite chirality case (Statement (1)) applied to the δ-rotation pair $\mathfrak{c}_0^*\mathfrak{s} \vee \mathfrak{c}_1^*\mathfrak{s} \vee \mathfrak{c}_2^*\mathfrak{s}$ and $\mathfrak{a}_0 \vee \mathfrak{a}_1 \vee \mathfrak{a}_2$, there is a pair of pants $\Pi \in \mathbf{\Pi}_{R,\epsilon}$ with boundary components the curves $\overline{[\bar{\mathfrak{a}}^*_{i,i+1}\bar{\mathfrak{s}}^*\mathfrak{c}_{i,i+1}\mathfrak{s}^*]}$ in $\mathbf{\Gamma}_{R,\epsilon}$, where i runs over $\mathbf{Z}_3$; for another copy Π' of Π, we may rewrite the boundary components of Π' as $\overline{[\bar{\mathfrak{a}}_{i,i+1}\bar{\mathfrak{s}}\mathfrak{c}^*_{i,i+1}\mathfrak{s}]}$ in $\mathbf{\Gamma}_{R,\epsilon}$, where i runs over $\mathbf{Z}_3$. Furthermore, in the above, observe the following common boundary components of opposite orientations. The third curve of ∂E_i is the orientation reversal of the third curve of $\partial E_i'$; the fourth curve of ∂E can be rewritten as $\overline{[\mathfrak{a}^*_{i,i+1}\bar{\mathfrak{s}}^*\mathfrak{c}_{i,i+1}\mathfrak{s}^*]}$, which is clearly the orientation reversal of the fourth curve of $\partial E'$; the orientation reversal of each curve of $\partial\Pi$ appears exactly once as the second curve in ∂E_i, and similarly for $\partial\Pi'$ and $\partial E_i'$. Gluing the (R,ϵ)-panted surfaces E_i, E_i', Π, and Π' along these oppositely oriented common boundary components, the result is an oriented connected compact surface F with exactly six boundary components, namely, two copies for each $[\mathfrak{a}_{i,i+1}\bar{\mathfrak{r}}\bar{\mathfrak{c}}_{i,i+1}\mathfrak{r}]$, where i runs over $\mathbf{Z}_3$. Since $[\mathfrak{a}_{i,i+1}\bar{\mathfrak{r}}\,\bar{\mathfrak{c}}_{i,i+1}\mathfrak{r}]$ equals $[\mathfrak{a}\bar{\mathfrak{b}}_{i,i+1}]$ under the additional assumption of the current step, the (R,ϵ)-panted surface F is as desired.

Step 2. We will finish the proof of the statement (2) in the general case. Possibly after switching the roles of $\mathfrak{a}_0 \vee \mathfrak{a}_1 \vee \mathfrak{a}_2$ and $\mathfrak{b}_0 \vee \mathfrak{b}_1 \vee \mathfrak{b}_2$, we may assume without loss of generality that l_b is greater than or equal to l_a. Then an auxiliary (l_b, δ)-nearly regular right-hand tripod $\mathfrak{b}_0' \vee \mathfrak{b}_1' \vee \mathfrak{b}_2'$ can be drawn so that $\mathfrak{a}_0 \vee \mathfrak{a}_1 \vee \mathfrak{a}_2$ and $\mathfrak{b}_0' \vee \mathfrak{b}_1' \vee \mathfrak{b}_2'$ form a δ-rotation pair, and that $\mathfrak{b}_0' \vee \mathfrak{b}_1' \vee \mathfrak{b}_2'$ satisfies the additional assumption of Step 1.

Observe the following (R,ϵ)-panted surfaces. By Step 1, there is an (R,ϵ)-panted surface F' with six boundary components, namely, two copies of each curve $[\mathfrak{a}_{i,i+1}\bar{\mathfrak{b}}'_{i,i+1}]$ in $\mathbf{\Gamma}_{R,\epsilon}$ for $i \in \mathbf{Z}_3$; by rotation in the opposite chirality case (Statement (1)), applied to the δ-rotation pair $\mathfrak{b}_0 \vee \mathfrak{b}_1 \vee \mathfrak{b}_2$ and $\mathfrak{a}_0 \vee \mathfrak{a}_{-1} \vee \mathfrak{a}_{-2}$, there is a pair of pants $\Pi \in \mathbf{\Pi}_{R,\epsilon}$ with boundary components $[\mathfrak{a}_{-i,-i-1}\bar{\mathfrak{b}}_{i,i+1}]$ in $\mathbf{\Gamma}_{R,\epsilon}$ for $i \in \mathbf{Z}_3$; also by rotation in the opposite chirality case (Statement (1)), applied to the δ-rotation pair $\mathfrak{b}_0' \vee \mathfrak{b}_1' \vee \mathfrak{b}_2'$ and $\mathfrak{a}_0 \vee \mathfrak{a}_{-1} \vee \mathfrak{a}_{-2}$, there is a pair of pants $\Pi' \in \mathbf{\Pi}_{R,\epsilon}$ with boundary components $\overline{[\mathfrak{a}_{-i,-i-1}\bar{\mathfrak{b}}'_{i,i+1}]}$ in $\mathbf{\Gamma}_{R,\epsilon}$ for $i \in \mathbf{Z}_3$; by swapping (Construction 4.17), for each $i \in \mathbf{Z}_3$, there is an (R,ϵ)-panted surface E_i with boundary components the curves $[\mathfrak{a}_{i,i+1}\bar{\mathfrak{b}}_{i,i+1}]$ $[\mathfrak{a}_{-i,-i-1}\bar{\mathfrak{b}}'_{i,i+1}]$ $\overline{[\mathfrak{a}_{i,i+1}\bar{\mathfrak{b}}'_{i,i+1}]}$ $\overline{[\mathfrak{a}_{-i,-i-1}\bar{\mathfrak{b}}_{i,i+1}]}$ in $\mathbf{\Gamma}_{R,\epsilon}$. Gluing the (R,ϵ)-panted surfaces F', and two copies of the (R,ϵ)-panted surfaces Π, Π', E_0, E_1, and E_2, along these oppositely oriented common boundary components, the result is an oriented connected compact surface F with exactly six boundary components, namely, two copies for each $[\mathfrak{a}_{i,i+1}\bar{\mathfrak{b}}_{i,i+1}]$ for $i \in \mathbf{Z}_3$ as desired. This completes the proof of the statement (2). $\square$

4.4.4. *Antirotation.* The antirotation construction is a variation of rotation when we join the legs in an 'unnatural' way. As before, we have two cases depending on the chiralities of the pair of tripods in consideration.

Construction 4.20 (Antirotation)**.** *Let $\mathfrak{a}_0 \vee \mathfrak{a}_1 \vee \mathfrak{a}_2$ and $\mathfrak{b}_0 \vee \mathfrak{b}_1 \vee \mathfrak{b}_2$ be a δ-rotation pair of tripods. Suppose $\ell(\mathfrak{a}_i) + \ell(\mathfrak{b}_j)$ is δ-close to $\frac{R}{2} + I(\frac{\pi}{3})$, for $i, j \in \mathbf{Z}_3$. Then an oriented connected compact (R, ϵ)-panted surface F can be constructed satisfying the following.*

(1) If $\mathfrak{a}_0 \vee \mathfrak{a}_1 \vee \mathfrak{a}_2$ and $\mathfrak{b}_0 \vee \mathfrak{b}_1 \vee \mathfrak{b}_2$ are of opposite charalities, then F has exactly six boundary components, namely, two copies of each $[\mathfrak{a}_{i,i+1}\bar{\mathfrak{b}}_{i+1,i}]$ in $\mathbf{\Gamma}_{R,\epsilon}$, for $i \in \mathbf{Z}_3$.
(2) If $\mathfrak{a}_0 \vee \mathfrak{a}_1 \vee \mathfrak{a}_2$ and $\mathfrak{b}_0 \vee \mathfrak{b}_1 \vee \mathfrak{b}_2$ are of identical charality, then F has exactly three boundary components namely, $[\mathfrak{a}_{i,i+1}\bar{\mathfrak{b}}_{i+1,i}]$ in $\mathbf{\Gamma}_{R,\epsilon}$, for $i \in \mathbf{Z}_3$.

Proof. Because $[\mathfrak{a}_{01}\bar{\mathfrak{b}}_{10}]$ and $[\mathfrak{a}_{20}\bar{\mathfrak{b}}_{20}]$ form a δ-swap pair, by swapping (Construction 4.17), there is an oriented connected compact (R, ϵ)-panted surface E with exactly four boundary components $[\mathfrak{a}_{01}\bar{\mathfrak{b}}_{10}]$, $[\mathfrak{a}_{20}\bar{\mathfrak{b}}_{02}]$, $\overline{[\mathfrak{a}_{01}\bar{\mathfrak{b}}_{02}]}$, $\overline{[\mathfrak{a}_{20}\bar{\mathfrak{b}}_{10}]}$. Thus, it suffices an oriented connected compact (R, ϵ)-panted surface F' satisfying the following.

(1) If $\mathfrak{a}_0 \vee \mathfrak{a}_1 \vee \mathfrak{a}_2$ and $\mathfrak{b}_0 \vee \mathfrak{b}_1 \vee \mathfrak{b}_2$ are of opposite charalities, then F' has exactly six boundary components, namely, two copies of $[\mathfrak{a}_{01}\bar{\mathfrak{b}}_{02}]$, $[\mathfrak{a}_{12}\bar{\mathfrak{b}}_{21}]$, and $[\mathfrak{a}_{20}\bar{\mathfrak{b}}_{10}]$.
(2) If $\mathfrak{a}_0 \vee \mathfrak{a}_1 \vee \mathfrak{a}_2$ and $\mathfrak{b}_0 \vee \mathfrak{b}_1 \vee \mathfrak{b}_2$ are of identical charality, then F' has exactly three boundary components namely, $[\mathfrak{a}_{01}\bar{\mathfrak{b}}_{02}]$, $[\mathfrak{a}_{12}\bar{\mathfrak{b}}_{21}]$, and $[\mathfrak{a}_{20}\bar{\mathfrak{b}}_{10}]$.

In fact, this follows immediately from rotation (Construction 4.19) with respect to the δ-rotation pair $\mathfrak{a}_0 \vee \mathfrak{a}_1 \vee \mathfrak{a}_2$ and $\mathfrak{b}_0 \vee \mathfrak{b}_2 \vee \mathfrak{b}_1$. Note that the chirality of $\mathfrak{b}_0 \vee \mathfrak{b}_2 \vee \mathfrak{b}_1$ is exactly opposite to that of $\mathfrak{b}_0 \vee \mathfrak{b}_1 \vee \mathfrak{b}_2$. This completes the proof. □

5. Panted cobordism group

In this section, we introduce the (R, ϵ)-panted cobordism group of oriented (R, ϵ)-multicurves in an oriented closed hyperbolic 3-manifold. This will serve as a correction theory which will reduce the relative case of Theorem 2.10 (1) to the absolute case.

Let M be an oriented closed hyperbolic 3-manifold, and let (R, ϵ) be a pair of undetermined constants, assuming that ϵ is a universally small positive number, and that R is a sufficiently large positive number depending on M and ϵ. By an (R, ϵ)-*nearly hyperbolic multicurve*, or simply an (R, ϵ)-*multicurve*, we mean a (possibly disconnected) nonempty oriented closed 1-submanifold immersed in M, of which all components are (R, ϵ)-nearly hyperbolic curves.

Definition 5.1. An (R, ϵ)-*nearly regularly panted cobordism*, or simply an (R, ϵ)-*panted cobordism*, between two (R, ϵ)-multicurves L, L' in M is an (R, ϵ)-panted subsurface F such that ∂F is the disjoint union $L \sqcup \bar{L}'$, where $\bar{L}'$ denotes the orientation-reversal of L'. We say that L, L' are (R, ϵ)-*panted cobordant* if there exists an (R, ϵ)-panted cobordism between them. Hence being (R, ϵ)-panted cobordant is an equivalence relation on the set of (R, ϵ)-multicurves. The set of all (R, ϵ)-panted cobordism classes of (R, ϵ)-multicurves will be denoted as $\mathbf{\Omega}_{R,\epsilon}(M)$, or simply $\mathbf{\Omega}_{R,\epsilon}$, the cobordism class of any (R, ϵ)-multicurve L will be denoted as $[L]_{R,\epsilon}$.

It will be verified in Subsubsection 5.1.1 that $\mathbf{\Omega}_{R,\epsilon}$ is an abelian group with the addition induced by the disjoint union operation between (R, ϵ)-multicurves.

Theorem 5.2. *Let M be an oriented closed hyperbolic 3-manifold. For all sufficiently small positive ϵ depending on the injectivity radius of M, and for all sufficiently large positive R depending on M and ϵ, there is a canonical isomorphism*

$$\Phi:\,\mathbf{\Omega}_{R,\epsilon}(M)\longrightarrow H_1(\mathrm{SO}(M);\mathbf{Z}),$$

where $\mathrm{SO}(M)$ *denotes the bundle over M of special orthonormal frames with respect to the orientation of M. Moreover, for all $[L]_{R,\epsilon}\in\mathbf{\Omega}_{R,\epsilon}(M)$, the image of $\Phi([L]_{R,\epsilon})$ under the bundle projection is the homology class $[L]\in H_1(M;\mathbf{Z})$.*

Remark 5.3. The last part of the statement implies that the integral module $\mathbf{\Omega}_{R,\epsilon}(M)$ is equivalent to $H_1(\mathrm{SO}(M);\,\mathbf{Z})$ as a splitting extesion of $H_1(M;\,\mathbf{Z})$ by $\mathbf{Z}_2$ (Subsubsection 5.1.2). Therefore, all the isomorphisms between $\mathbf{\Omega}_{R,\epsilon}(M)$ and $H_1(\mathrm{SO}(M);\,\mathbf{Z})$ that are extension equivalences are in natural bijection to $H^1(M;\,\mathbf{Z}_2)$, where the canonical isomorphism Φ corresponds to 0.

The idea of Theorem 5.2 is developed from the (non-random) correction theory part of [KM2]. In that paper, the notion *Good Pants Homology* of an oriented closed hyperbolic surface S is informally introduced, and with our notations above, the Good Pants Homology of S there means precisely the rational (R,ϵ)-panted cobordism group $\mathbf{\Omega}_{R,\epsilon}(S)\otimes\mathbf{Q}$, cf. [KM2, Definition 3.2]. The proof of the Good Correction Theorem [KM2, Theorem 3.2] essentially implies that there is an isomorphism $\phi:\,\mathbf{\Omega}_{R,\epsilon}(S)\otimes\mathbf{Q}\to H_1(S;\mathbf{Q})$, such that $\phi([\gamma]_{R,\epsilon})$ equals $[\gamma]$. In fact, most part of the proof of [KM2, Theorem 3.2] can be extended directly to the 3-dimensional case, yielding an isomorphism $\phi:\,\mathbf{\Omega}_{R,\epsilon}(M)\otimes\mathbf{Q}\to H_1(M;\mathbf{Q})$ as above. Motivated by pushing the result to the integral coefficient case, the main innovation of Theorem 5.2 lies in the observation that in many senses, it should be more natural to replace $H_1(M;\mathbf{Z})$ with $H_1(\mathrm{SO}(M);\mathbf{Z})$. One reason, for instance, is that (R,ϵ)-multicurves and (R,ϵ)-pants admit certain canonical lifts into $\mathrm{SO}(M)$ because of their geometry; another reason is that technically, passing to $\mathrm{SO}(M)$ resolves certain ambiguity in the definition of the inverse of Φ; the reader may also observe a vague analogy between the statement of Theorem 5.2 and the Thom–Pontrjagin correspondence in cobordism theory, thinking of $H_1(\mathrm{SO}(M);\mathbf{Z})$ as $\pi_1(\mathrm{SO}(M))$ modulo the action of conjugations.

The proof of Theorem 5.2 is organized slightly differently from the treatment of [KM2]. Fix a basepoint $*$ of M and a special orthonormal frame $\mathbf{e}\in\mathrm{SO}(M)|_*$ as a basepoint of $\mathrm{SO}(M)$. We will construct the homorphism Φ and a homomorphism $\Psi:\,\pi_1(\mathrm{SO}(M),\mathbf{e})\to\mathbf{\Omega}_{R,\epsilon}$, which descends to be a homomorphism $\Psi^{\mathsf{ab}}:H_1(\mathrm{SO}(M);\mathbf{Z})\to\mathbf{\Omega}_{R,\epsilon}$ by abelianization. We will verify that $\Phi\circ\Psi^{\mathsf{ab}}=\mathrm{id}$ and that Ψ surjects $\mathbf{\Omega}_{R,\epsilon}$. This will imply that Φ is an isomorphism with the inverse Ψ^{ab}. To briefly describe Φ, note that any curve $\gamma\in\mathbf{\Gamma}_{R,\epsilon}$ has a framing over its geodesic representative given by (nearly) parallel transporting a frame around γ. The *canonical lift* $\hat{\gamma}:S^1\to\mathrm{SO}(M)$ of γ, well defined up to homotopy, is then a framing that differs from the the parallel-transportation framing by a loop of matrices $S^1\to\mathrm{SO}(3)$ that represents the nontrivial element of $\pi_1(\mathrm{SO}(3))\cong\mathbf{Z}_2$ (Definition 5.11). The homomorphism $\Phi:\,\mathbf{\Omega}_{R,\epsilon}\to H_1(\mathrm{SO}(M);\mathbf{Z})$ will hence be uniquely defined so that $\Phi([\gamma]_{R,\epsilon})$ equals $[\hat{\gamma}]$. The definition of Ψ will depend on a choice of a finite triangular generating set $\hat{g}_1,\cdots,\hat{g}_s$ of $\pi_1(\mathrm{SO}(M),\mathbf{e})$, together with some other setup data. Here *triangular* means that all the relators of length at most 3 in the generating set gives rise to a finite presentation of $\pi_1(\mathrm{SO}(M),\mathbf{e})$.

We will define $\Psi(\hat{g}_i)$ (or decorated with some setup data, $\Psi^{h}_{D'}(\hat{g}_i)$) to be represented by some (R,ϵ)-multicurve constructed from $\hat{g}_i$, so that $[\Psi(\hat{g}_i)]$ is obviously equal to $[\hat{g}_i]$ in $H_1(\mathrm{SO}(M);\mathbf{Z})$. With the derived constructions of Subsection 4.4, we will verify that $\Psi(\hat{g}_i)+\Psi(\hat{g}_j)+\Psi(\hat{g}_k)=0$ in $\mathbf{\Omega}_{R,\epsilon}$ whenever there is a triangular relation $\hat{g}_i\hat{g}_j\hat{g}_k=\mathrm{id}$. It will follow that $\Psi:\pi_1(\mathrm{SO}(M),\mathbf{e})\to\mathbf{\Omega}_{R,\epsilon}$ is a well defined homomorphism. We point out that the triangular generating set of $\pi_1(\mathrm{SO}(M),\mathbf{e})$, or more essentially, a finite triangular presentation of $\pi_1(M,*)$, is the source of topological information which makes the construction of Ψ possible. This idea will be further investigated in Section 6 when we pantify second homology classes.

The rest of this section is devoted to the proof of Theorem 5.2. In Subsection 5.1, we define the homomorphism Φ. In Subsection 5.2, we define the homomorphism Ψ and verify that Ψ^{ab} is the inverse of Φ. In Subsection 5.3, we summarize the proof of Theorem 5.2.

Throughout this section, after fixing a basepoint $*$ of M, we will no longer distinguish a nontrivial element of $\pi_1(M,*)$ from its pointed geodesic loop representative, so it makes sense to speak of the length, or the initial or terminal direction of a nontrivial element in $\pi_1(M,*)$.

5.1. The homomorphism Φ.

In this subsection, we define the homomorphism Φ. We also need to mention some basic facts about the (R,ϵ)-panted cobordism gorup $\mathbf{\Omega}_{R,\epsilon}$, and about the special orthonomal framing bundle $\mathrm{SO}(M)$.

5.1.1. *The panted cobordism group $\mathbf{\Omega}_{R,\epsilon}$.*

Lemma 5.4. *For all universally small positive ϵ, and for all sufficiently large positive R depending on M and ϵ, $\mathbf{\Omega}_{R,\epsilon}$ is a finitely generated abelian group. Here the addition is induced by the disjoint union operation between (R,ϵ)-multicurves.*

Proof. If R is sufficiently large with respect to M and universally small ϵ, the Connection Principle (Lemma 4.13) and splitting (Construction 4.15) implies that $\mathbf{\Gamma}_{R,\epsilon}$ and $\mathbf{\Pi}_{R,\epsilon}$ are both nonempty, but finite. The zero of $\mathbf{\Omega}_{R,\epsilon}$ is represented by the sum of the three cuffs for any $\Pi\in\mathbf{\Pi}_{R,\epsilon}$. It is then straightforward to see that $\mathbf{\Omega}_{R,\epsilon}$ is a finitely generated abelian group with the addition induced by disjoint union. In fact, we have a natural presentation of $\mathbf{\Omega}_{R,\epsilon}$ given by the exact sequence

$$\mathbf{Z\Gamma}_{R,\epsilon}\xrightarrow{\partial}\mathbf{Z\Pi}_{R,\epsilon}\longrightarrow\mathbf{\Omega}_{R,\epsilon}\longrightarrow 0.$$

□

Lemma 5.5. *If ϵ is universally small and R is sufficiently large, then for all (R,ϵ)-multicurve L in M,*

$$[L]_{R,\epsilon}=-[\bar{L}]_{R,\epsilon}.$$

Proof. It suffices to assume that L has only one component. Then $[c]_{R,\epsilon}=-[\bar{c}]_{R,\epsilon}$ if there is a pair of pants $\Pi_c\in\mathbf{\Pi}_{R,\epsilon}$ with a cuff c, for each component c of L. The condition certainly holds for universally small ϵ if R is sufficiently large (Construction 4.15). In fact, one may take two oppositely oriented copies of $\Pi^{\pm}_c$ of Π_c, and glue them along their two cuffs other than c or $\bar{c}$. The resulting (R,ϵ)-panted surface has exactly two boundary component c and $\bar{c}$. □

Lemma 5.6. *For all universally small positive ϵ and ϵ' such that $\epsilon'\leq\epsilon$, and for all sufficiently large positive R depending on M, ϵ and ϵ', $\mathbf{\Omega}_{R,\epsilon}$ is generated by (R,ϵ)-panted cobordism classes of (R,ϵ')-multicurves.*

Proof. This is the restatement of splitting (Construction 4.15). $\square$

5.1.2. *The special orthonormal frame bundle* $\mathrm{SO}(M)$. Fix an orthonormal frame

$$\mathbf{e} = (\vec{t}, \vec{n}, \vec{t} \times \vec{n})$$

at a fixed basepoint $*$ of M, and regard $\mathbf{e}$ as a basepoint of the total space $\mathrm{SO}(M)$ of the bundle over M of special orthonormal frames with respect to the orientation of M. One may naturally identify $\mathrm{SO}(M)$ as $\mathrm{Isom}_0(\mathbb{H}^3) / \pi_1(M)$. Since the tangent bundle of a closed orientable 3-manifold is always trivializable, $\mathrm{SO}(M)$ is a trivial $\mathrm{SO}(3)$-principal bundle over M. There are canonical short exact sequences

$$1 \longrightarrow \pi_1(\mathrm{SO}(3), I) \longrightarrow \pi_1(\mathrm{SO}(M), \mathbf{e}) \longrightarrow \pi_1(M, *) \longrightarrow 1,$$

and

$$0 \longrightarrow \mathbf{Z}_2 \longrightarrow H_1(\mathrm{SO}(M); \mathbf{Z}) \longrightarrow H_1(M; \mathbf{Z}) \longrightarrow 0,$$

both of which are splitting but not naturally. Note that $\pi_1(\mathrm{SO}(3), I) \cong \mathbf{Z}_2$ is the center of $\pi_1(\mathrm{SO}(M), \mathbf{e})$. We will usually write

$$\hat{c} \in \pi_1(\mathrm{SO}(M), \mathbf{e})$$

for the nontrivial central element.

Certain noncentral elements of $\pi_1(\mathrm{SO}(M), \mathbf{e})$, namely, the δ-sharp elements in the sense of Definition 5.7 below, can be naturally represented by their associated oriented ∂-framed segments. This provides a convenient way to understand such elements, which will be especially useful when we construct the inverse of Φ.

Definition 5.7. Let δ be a positive constant at most $\frac{1}{100}$. A noncentral element $\hat{g} \in \pi_1(\mathrm{SO}(M), \mathbf{e})$ is said to be *δ-sharp* if its image g in $\pi_1(M, *)$ has the initial and terminal directions δ-close to $\vec{t}$ and $-\vec{t}$, respectively. For a δ-sharp $\hat{g}$, we will say that an oriented ∂-framed segment $\mathfrak{g}$ is *associated to* $\hat{g}$, and vice versa, if $\mathfrak{g}$ satisfies the following.

- The carrier segment of $\mathfrak{g}$ is g. The phase of $\mathfrak{g}$ is δ-close to 0. The initial and terminal framings of $\mathfrak{g}$ are δ-close to each other.
- The element $\hat{g}$ is represented by a loop of frames based at $\mathbf{e} \in \mathrm{SO}(M)$ as follows. The loop first flows $\mathbf{e}$ along g by parallel transportation, and then rotates 180° counterclockwise about $\vec{n}_{\mathtt{ter}}(\mathfrak{g})$, and then rotates back to $\mathbf{e}$ along a δ-short path within $\mathrm{SO}(M)|_*$.

We point out that our biased choice of the 180° *counterclockwise* rotation in Definition 5.7 determines our choices of chirality for tripods in the rest of this section. The essential difference between distinct chiralities is revealed by Lemma 5.9 and Remark 5.10.

Lemma 5.8. *For any δ-sharp element $\hat{g} \in \pi_1(\mathrm{SO}(M), \mathbf{e})$, there is an oriented ∂-framed segment $\mathfrak{g}$ associated to $\hat{g}$, unique up to δ-small change of framings at endpoints. Moreover, $\mathfrak{g}^*$ is associated to $\hat{c}\hat{g}$, and $\bar{\mathfrak{g}}^*$ is associated to $\hat{g}^{-1}$.*

Proof. Let $\vec{m}$ be a unit vector orthogonal to $\vec{t}$ such that the parallel transportation of $\vec{m}$ along g to the other end is δ-close to $\vec{m}$. Up to δ-small change, there are only two possible such vectors, namely, $\pm\vec{m}$. Enriching g with initial and terminal framings both δ-close to $\pm\vec{m}$ yield oriented ∂-framed segments $\mathfrak{g}_\pm$ satisfying the first part of the listed properties in Definition 5.7. It is clear that exactly one of $\mathfrak{g}_\pm$ fulfills the second part of the listed properties, so we pick it as $\mathfrak{g}$. The 'moreover' part is straightforward from the construction above as well. $\square$

Lemma 5.9. *Let δ be a positve constant at most $\frac{1}{100}$. Suppose that $\mathfrak{t}_0 \vee \mathfrak{t}_1 \vee \mathfrak{t}_2$ is a $(10, \delta)$-tame left-hand tripod with the terminal directions of legs δ-close to $\vec{t}$. Then there exist angles $\phi_0, \phi_1, \phi_2 \in \mathbf{R}/2\pi\mathbf{Z}$ satisfying*

$$\phi_0 + \phi_1 + \phi_2 = 0,$$

such that $2\phi_{i+2} \in \mathbf{R}/2\pi\mathbf{Z}$ is δ-close to the directed angle from $\vec{n}_{\mathtt{ter}}(\mathfrak{t}_i)$ to $\vec{n}_{\mathtt{ter}}(\mathfrak{t}_{i+1})$ with respect to the common orthogonal vector at $*$ which is δ-close to $\vec{t}$, and that each $\mathfrak{t}_{i,i+1}(\phi_{i+2})$ (Definition 4.1) is associated to a δ-sharp element $\hat{g}_{i+2} \in \pi_1(\mathrm{SO}(M), \mathbf{e})$ for $i \in \mathbf{Z}_3$. For any such ϕ_i as above, the triangular relation

$$\hat{g}_0\hat{g}_1\hat{g}_2 = \mathrm{id}$$

is satisfied. Moreover, adding two of the three ϕ_i by π yields another triple of angles satisfying the conditions above, with two corresponding $\hat{g}_i$ changed into $\hat{c}\hat{g}_i$.

Remark 5.10. If $\mathfrak{t}_0 \vee \mathfrak{t}_1 \vee \mathfrak{t}_2$ is right-hand, we must either replace the triangular relation in the conclusion with the twisted triangular relation $\hat{g}_0\hat{g}_1\hat{g}_2 = \hat{c}$, or alternatively, replace the equation for the anlges with $\phi_0 + \phi_1 + \phi_2 = \pi$.

Proof. For each $i \in \mathbf{Z}_3$, pick a unit vector $\vec{n}_i$ at $*$ orthogonal to $\vec{e}$, such that $\vec{n}_i$ is δ-close to $\vec{n}_{\mathtt{ini}}(\mathfrak{t}_i)$. Let $\psi_{i,i+1} \in \mathbf{R}/2\pi\mathbf{Z}$ be the angle from $\vec{n}_i$ to $\vec{n}_{i+1}$ with respect to $\vec{t}$, and let ϕ'_{i+2} be half of $\psi_{i,i+1}$, valued in $\mathbf{R}/\pi\mathbf{Z}$. Choose a lift $\phi_{i+2} \in \mathbf{R}/2\pi\mathbf{Z}$ for each ϕ'_{i+2}, so that $\phi_0 + \phi_1 + \phi_2 = 0$. Note that any other lift can be obtained from changing two ϕ_i by adding π. It is clear that $\mathfrak{t}_{i,i+1}(\phi_{i+2})$ is associated to a δ-sharp $\hat{g}_{i+2} \in \pi_1(\mathrm{SO}(M), \mathbf{e})$, and $\hat{g}_0\hat{g}_1\hat{g}_2$ equals either id or $\hat{c}$. We claim that it is the former case.

It suffices to verify that $\hat{g}_0\hat{g}_1\hat{g}_2$ is trivial in $H_1(\mathrm{SO}(M), \mathbf{e})$. The argument is routine and easy so we only include an outline below. Let $\hat{\beta}_{i+2}$ be a path of frames from $(\vec{t}, \vec{n}_i, \vec{t} \times \vec{n}_{i+1})|_*$ to $(\vec{t}, \vec{n}_{i+1}, \vec{t} \times \vec{n}_{i+1})|_*$ that first flowing by parallel transportation along g_{i+2}, and then rotates 180° counterclockwise about $\vec{n}_{i+1}$, and then rotates to $(\vec{t}, \vec{n}_{i+1}, \vec{t} \times \vec{n}_{i+1})|_*$ via a δ-short path in $\mathrm{SO}(M)|_*$. Here $g_{i+2} \in \pi_1(M, *)$ is the image of $\hat{g}_{i+2}$, also regarded as a pointed geodesic loop. Since $\vec{n}_i$ are all δ-close to the normal vector of the 2-simplex σ spanned by the concatenation of g_0, g_1, g_2, the loop of frames $\hat{\beta}_0\hat{\beta}_1\hat{\beta}_2$ obtained by concatenation is homotopic to the constant loop $(\vec{t}, \vec{n}_1, \vec{t} \times \vec{n}_1)|_*$ in $\mathrm{SO}(M)|_*$ as $\mathfrak{t}_0 \vee \mathfrak{t}_1 \vee \mathfrak{t}_2$ is left-hand. (Compare with the right-hand case where the resulting loop would be the 360° counterclockwise rotation of the frame $(\vec{t}, \vec{n}_1, \vec{t} \times \vec{n}_1)|_*$ about $\vec{n}_1$.) In particular,

$$[\hat{\beta}_0\hat{\beta}_1\hat{\beta}_2] = 0$$

in $H_1(\mathrm{SO}(M), \mathbf{e})$. Let $\vec{m}_{i+2}$ be a unit vector at $*$ orthogonal to $\vec{t}$, such that $\vec{m}_{i+2}$ is δ-close to both the initial and terminal framings of $\mathfrak{t}_{i,i+1}(\phi_{i+2})$. Let $\hat{\xi}_{i+2}$ be the path of frames in $\mathrm{SO}(M)|_*$ from $(\vec{t}, \vec{m}_{i+2}, \vec{t} \times \vec{m}_{i+2})$ to $(\vec{t}, \vec{n}_i, \vec{t} \times \vec{n}_i)$ by a rotation of angle δ-close to ϕ_{i+2}, and let $\hat{\eta}_{i+2}$ be the path of frames in $\mathrm{SO}(M)|_*$ from $(\vec{t}, \vec{n}_{i+1}, \vec{t} \times \vec{n}_{i+1})$ to $(\vec{t}, \vec{m}_{i+2}, \vec{t} \times \vec{m}_{i+2})$ by a rotation of angle δ-close to ϕ_{i+2}. Then the loop of frames $\hat{\xi}_{i+2}\hat{\beta}_{i+2}\hat{\eta}_{i+2}$ based at $(\vec{t}, \vec{m}_{i+2}, \vec{t} \times \vec{m}_{i+2})$ can be conjugated to the loop based at $\mathbf{e}$ representing $\hat{g}_{i+2}$ described in Definition 5.7. Thus

$$[\hat{g}_{i+2}] = [\hat{\xi}_{i+2}\hat{\beta}_{i+2}\hat{\eta}_{i+2}]$$

in $H_1(\mathrm{SO}(M), \mathbf{e})$. Note that $\hat{\xi}_0\hat{\eta}_0\hat{\xi}_1\hat{\eta}_1\hat{\xi}_2\hat{\eta}_2$ is a loop of frames in $\mathrm{SO}(M)|_*$ that rotates counterclockwise about $\vec{t}$ of angle $2(\phi_0 + \phi_1 + \phi_2)$. As $\phi_0 + \phi_1 + \phi_2 = 0$

modulo 2π, the winding number of this loop around $\vec{t}$ is even. Hence

$$[\hat{\xi}_0\hat{\eta}_0\hat{\xi}_1\hat{\eta}_1\hat{\xi}_2\hat{\eta}_2] = 0$$

in $H_1(\mathrm{SO}(M);\mathbf{Z})$. Therefore, in $H_1(\mathrm{SO}(M);\mathbf{Z})$,

$$\begin{aligned}[\hat{g}_0\hat{g}_1\hat{g}_2] &= [\hat{\xi}_0\hat{\beta}_0\hat{\eta}_0] + [\hat{\xi}_1\hat{\beta}_1\hat{\eta}_1] + [\hat{\xi}_2\hat{\beta}_2\hat{\eta}_2] \\ &= [\hat{\xi}_0\hat{\beta}_0\hat{\eta}_0\hat{\xi}_1\hat{\beta}_1\hat{\eta}_1\hat{\xi}_2\hat{\beta}_2\hat{\eta}_2] \\ &= [\hat{\beta}_0\hat{\beta}_1\hat{\beta}_2] + [\hat{\xi}_0\hat{\eta}_0\hat{\xi}_1\hat{\eta}_1\hat{\xi}_2\hat{\eta}_2] \\ &= 0.\end{aligned}$$

This implies that $\hat{g}_0\hat{g}_1\hat{g}_2 = \mathrm{id}$ in $\pi_1(\mathrm{SO}(M),\mathbf{e})$. □

There is a canonical way to lift (R,ϵ)-curves and (R,ϵ)-pants into $\mathrm{SO}(M)$, up to homotopy.

Definition 5.11. For any curve $\gamma \in \mathbf{\Gamma}_{R,\epsilon}$, a *canonical lift* of γ is a loop of frames

$$\hat{\gamma} : S^1 \to \mathrm{SO}(M)$$

as follows. Choose a point p on the geodesic representative of γ, and a normal vector $\vec{n}_p$ of γ at p. Let $\vec{t}_p$ be the direction vector of γ at p. The frame $\mathbf{e}_{\gamma,\vec{n}_p} = (\vec{t}_p, \vec{n}_p, \vec{t}_p \times \vec{n}_p)$ is an element of $\mathrm{SO}(M)|_p$. With these notations, a base-point free loop of frames $\hat{\gamma}$ starts from $\mathbf{e}_{\gamma,\vec{n}_p}$, and then flows once around γ by parallel transportation, and then rotates $360°$ counterclockwise about $\vec{n}_p$, and then rotates back to $\mathbf{e}_{\gamma,\vec{n}_p}$ along an ϵ-short path within $\mathrm{SO}(M)|_p$. For any pair of pants $\Pi \in \mathbf{\Pi}_{R,\epsilon}$, a *canonical lift* of Π is a lift

$$\hat{\Pi} :\ \Sigma_{0,3} \to \mathrm{SO}(M)$$

of Π, such that the three cuffs are canonically lifted.

Lemma 5.12. *For any positive constant ϵ at most $\frac{1}{100}$, suppose $\gamma \in \mathbf{\Gamma}_{R,\epsilon}$ and $\Pi \in \mathbf{\Pi}_{R,\epsilon}$. The canonical lifts $\hat{\gamma}$ and $\hat{\Pi}$ as described in Definition 5.11 exist and are unique up to homotopy.*

Remark 5.13. However, if the canonical lifts of $\hat{\Pi}$ on the cuffs have been chosen, $\hat{\Pi}$ is unique only up to homotopy relative to cuffs together with $\mathbf{Z}_2$-Dehn twists in the fiber $\mathrm{SO}(3)$ near the boundary. In other words, the relative homotopy class of $\hat{\Pi}$ is determined by any class of $H_2(\mathrm{SO}(\Pi),\mathrm{SO}(\partial\Pi);\mathbf{Z})$ that projects to the fundamental class $[\Pi]$ in $H_2(\Pi,\partial\Pi;\mathbf{Z})$.

Proof. The existence of $\hat{\gamma}$ is by definition. The uniqueness follows from the fact that the set of homotopy classes of framings of $TM|_\gamma$ is bijective to $[S^1,\mathrm{SO}(3)] \cong \mathbf{Z}_2$. To see the existence of $\hat{\Pi}$, note that the pull-back tangent bundle $TM|_\Pi$, namely, $\Pi^*(TM)$, is isomorphic to $T\Sigma_{0,3}\oplus\epsilon^1$. Consider a trivialization of $T\Sigma_{0,3}$, for example, by embedding $\Sigma_{0,3}$ into the plane and endowing with the standard framing of $\mathbf{R}^2$. By direct summing with the trivialization induced by the orientation of Π, the trivialization of $T\Sigma_{0,3}$ naturally induces a framing of $TM|_\Pi$ up to homotopy. The restriction of this framing on any cuff γ of Π is the canonical lift of γ. Thus this framing of $TM|_\Pi$ is a canonical lift $\hat{\Pi} : \Sigma_{0,3} \to \mathrm{SO}(M)$ of Π by definition. To see the uniqueness of $\hat{\Pi}$, note that the set of homotopy classes of framings of $TM|_{\Sigma_{0,3}}$ is bijective to $[\Sigma_{0,3},\mathrm{SO}(3)] \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2$. Thus the homotopy class of a framing of $TM|_{\Sigma_{0,3}}$ is uniquely determined by its restriction to the cuffs. This completes the proof. □

5.1.3. *Construction of* Φ. We construct

$$\Phi : \mathbf{\Omega}_{R,\epsilon} \to H_1(\mathrm{SO}(M);\, \mathbf{Z})$$

as follows. Suppose γ is a geodesic representative of a curve in $\mathbf{\Gamma}_{R,\epsilon}$. We define $\Phi([\gamma]_{R,\epsilon})$ in $H_1(\mathrm{SO}(M); \mathbf{Z})$ to be represented by the canonical lift of γ (Definition 5.11). For an (R,ϵ)-multicurve L, we define

$$\Phi([L]_{R,\epsilon}) \in H_1(\mathrm{SO}(M);\, \mathbf{Z})$$

to be the sum of Φ defined for each of its components. We verify that Φ is well defined.

Lemma 5.14. *The homology class $\Phi([L]_{R,\epsilon})$ in $H_1(\mathrm{SO}(M);\, \mathbf{Z})$ constructed above depends only on the (R,ϵ)-panted cobordism class $[L]_{R,\epsilon} \in \mathbf{\Omega}_{R,\epsilon}$ of L. Moreover, the induced map Φ from $\mathbf{\Omega}_{R,\epsilon}$ to $H_1(\mathrm{SO}(M);\, \mathbf{Z})$ is a homomorphism.*

Proof. If $[L]_{R,\epsilon}$ vanishes in $\mathbf{\Omega}_{R,\epsilon}$, there exists an (R,ϵ)-panted surface $F \looparrowright M$ bounded by L. The canonical lifts of pairs of pants of F (Definition 5.11) yield a canonical lift $F \to \mathrm{SO}(M)$, whose restriction to the boundary are the canonical lifts of components of L. This implies that (R,ϵ)-panted cobordant multicurves yield homologous canonical lifts, or in other words, that $\Phi([L]_{R,\epsilon})$ in $H_1(\mathrm{SO}(M);\, \mathbf{Z})$ depends only $[L]_{R,\epsilon}$. The 'moreover' part is straightforward from the definition. □

Lemma 5.15. *For any (R,ϵ)-panted cobordism class $[L]_{R,\epsilon}$, the image of $\Phi([L]_{R,\epsilon})$ under the natural projection from $H_1(\mathrm{SO}(M);\, \mathbf{Z})$ to $H_1(M; \mathbf{Z})$ is the homology class $[L]$.*

Proof. This follows immediately from the construction of Φ. □

5.2. **The inverse of Φ.** Fix an orthonormal frame $\mathbf{e} = (\vec{t}, \vec{n}, \vec{t} \times \vec{n})$ at a fixed basepoint $*$ of M as before. In this subsection, we construct a homomorphism

$$\Psi :\, \pi_1(\mathrm{SO}(M), \mathbf{e}) \to \mathbf{\Omega}_{R,\epsilon}$$

which, descending to the abelianization,

$$\Psi^{\mathsf{ab}} :\, H_1(\mathrm{SO}(M); \mathbf{Z}) \to \mathbf{\Omega}_{R,\epsilon},$$

yields the inverse of Φ. We need to choose some setup data including a triangular finite generating set of $\pi_1(M,*)$, and define Ψ on a subset of $\pi_1(M,*)$ that contains the triangular generating set. We verify that Ψ extends as a homomorphism by showing that it vanishes on words corresponding to the triangular relations. We also verify that Ψ^{ab} is the inverse of Φ by showing that it is surjective and is the pre-inverse of Φ.

5.2.1. *Setup.* Given any positive number ϵ, we need to fix some setup data

$$(\mathcal{C}_h, \mathcal{B}_{D'}, \tau_h)$$

in terms of $\pi_1(M,*)$, restrained by some environmental data (δ', L') and (δ, L), which are constant pairs fulfilling the conditions of the Connection Principle (Lemma 4.13). Such a collection of data is provided by Lemma 5.16 below, together with a positive constant

$$R(\epsilon, M)$$

such that the homomorphism Ψ from $\pi_1(\mathrm{SO}(M), \mathbf{e})$ to $\mathbf{\Omega}_{R,\epsilon}$ can be constructed for any constant R greater than $R(\epsilon, M)$.

In the following, a *triangular generating set* of $\pi_1(M,*)$ means a generating set in which all the relations of word length at most 3 yield a presentation of $\pi_1(M,*)$ (cf. Subsection 6.1 for discussion in more details); the conjugation τ_h induced by an element $h \in \pi_1(M,*)$ acts on $\pi_1(M,*)$ by $\tau_h(g) = h^{-1}gh$; a δ-*fellow-travel pair* of geodesic segments in a hyperbolic 3-manifold is understood in the same sense as in the ∂-framed case (Definition 4.4), except with the framing conditions disregarded.

Lemma 5.16. *Let M be an oriented closed hyperbolic 3-manifold. Given any positive number ϵ, there is a collection of data depending only on M and ϵ as follows.*

1. *There exist positive constants δ' and L'. The constant δ' is less than the minimum among $\frac{1}{100}$, $\frac{\epsilon}{10000}$ and half the injectivity radius of M; the constant L' is at least 100, and L' satisfies the conclusion of the Connection Principle (Lemma 4.13) with respect to δ' and M.*
2. *There exist finite subsets $\mathcal{B}_{D'}$ and $\mathcal{B}_D$ of $\pi_1(M,*)$, where $\mathcal{B}_d$ denotes the elements of $\pi_1(M,*)$ with length at most d. The constants D and D' are both greater than $10L'$; the subset $\mathcal{B}_{D'}$ contains a generating set of $\pi_1(M,*)$; the subset $\mathcal{B}_D$ contains a triangular generating set of $\pi_1(M,*)$ which further contains $\mathcal{B}_{D'}$.*
3. *There exist positive constants δ and L. The constant δ is less than δ' and $\frac{1}{1000\sqrt{\mathrm{Card}(\mathcal{B}_D)}}$, where* Card *denotes the cardinality; the constant L is greater than $10L'$, and L satisfies the conclusion of Connection Principle (Lemma 4.13) with respect to δ and M, and the condition for the Length and Phase Formula (Lemma 4.7) with respect to δ and M.*
4. *There exists a conjugation τ_h of $\pi_1(M,*)$ by an element $h \in \pi_1(M,*)$ of length at least $10L$. For each $g \in \mathcal{B}_D$, $\tau_h(g)$ is δ-sharp (Definition 5.7) of length at least $10L$; for each pair $g, g' \in \mathcal{B}_D$, the initial subsegments of $\tau_h(g)$ and $\tau_h(g')$ of length $2L$ form a δ-fellow-travel pair, and hence the same holds for terminal subsegments as well.*
5. *There exists a positive constant*
$$R(\epsilon, M),$$
which is greater than $10L + K$, where K is maximal length of $\tau_h(g)$ for all $g \in \mathcal{B}_D$.
6. *There exists a finite subset $\mathcal{C}_h$ of $\pi_1(M,*)$ containing $\mathcal{B}_D$, consisting of elements u so that $\tau_h(u)$ has length between $4L$ and $R(\epsilon, M) - 10L$, and that for each $g \in \mathcal{B}_D$, the initial subsegments of $\tau_h(u)$ and $\tau_h(g)$ of length $2L$ form a δ-fellow-travel pair, and hence the same holds for terminal subsegments as well.*

Proof. It suffices to prove the statements (2) and (4), since the other ones are obvious from the statements themselves.

The statement (2) follows from the fact that any presentation $(\mathcal{S}, \mathcal{R})$ of a group G induces a triangular generating set $\tilde{\mathcal{S}}$ consisting of all the elements that are represented by subwords of all the relators from $\mathcal{R}$. Note that $\tilde{\mathcal{S}}$ is finite if $(\mathcal{S}, \mathcal{R})$ is a finite presentation. Since $\pi_1(M,*)$ is finitely presented, we may choose $D' > 10L'$ sufficiently large so that $\mathcal{B}_{D'}$ generates $\pi_1(M,*)$. Then for any finite presentation of $\pi_1(M,*)$ over $\mathcal{B}_{D'}$, the induced finite triangular generating set $\tilde{\mathcal{B}}_{D'}$ is finite, and hence contained in $\mathcal{B}_D$ for some sufficiently large $D \geq D'$.

To prove the statement (4), we construct h to be h_2h_1 as follows. Since $\mathcal{B}_D$ is a finite set, we may first find some $h_1 \in \pi_1(M,*)$ such that $\tau_{h_1}(g)$ has length $2L$ for all $g \in \mathcal{B}_D$. In fact, choosing a basepoint O of the universal cover $\mathbb{H}^3$ which covers $*$, we may regard g as acting on $\mathbb{H}^3$ by a deck transformation. Let V_g be the subset of $\mathbb{H}^3$ consisting of points which translates of distance at most $2L$ under the action of g. Note that V_g is a round tubular neighborhood of the axis of g which is strictly convex. It follows that there is a conjugate $h_1.O$ of O which lies outside all V_g for $g \in \mathcal{B}_D$. We may choose the h_1 above as claimed. Since $\mathrm{Card}(\mathcal{B}_D)\cdot(2000\delta)^2\pi$ is bounded by 4π, it is easy to see that there exists a unit vector $\vec{v}$ at $*$ to that $\vec{v}$ is (2000δ)-away from the initial direction of $\tau_{h_1}(g)$ for any $g \in \mathcal{B}_D$. By the Connection Principle (Lemma 4.13), there is an element $h_2 \in \pi_1(M,*)$ represented by a pointed geodesic loop of length at least $10L$, of which the initial direction is δ-close to $\vec{v}$ and the terminal direction is δ-close to $\vec{t}$. We may choose the h_2 above as claimed. Note that $\tan(1000\delta)$ is bounded by 1. With the Length and Phase Formula (Lemma 4.7) and the statement (3), it is straightforward to check that the conjugation τ_h satisfies the conclusion of the statement (4). □

5.2.2. *Construction of* Ψ. Given any positive constant ϵ, we fix a collection of setup data $(\mathcal{C}_h, \mathcal{B}_{D'}, \tau_h)$ subject to (δ', L') and (δ, L) as provided by Lemma 5.16, and obtain a positive constant $R(\epsilon, M)$ accordingly. For any constant R at least $R(\epsilon, M)$, we will construct the homomorphism

$$\Psi:\,\pi_1(\mathrm{SO}(M),\mathbf{e}) \to \mathbf{\Omega}_{R,\epsilon}$$

in the following. More precisely, let $\hat{\mathcal{C}}_h$ and $\hat{\mathcal{B}}_{D'}$ denote the preimages of $\mathcal{C}_h$ and $\mathcal{B}_{D'}$ in $\pi_1(\mathrm{SO}(M),\mathbf{e})$, respectively. We will construct a set-theoretic map

$$\Psi^h_D:\,\hat{\mathcal{C}}_h \to \mathbf{\Omega}_{R,\epsilon}.$$

The restriction of Ψ^h_D to $\hat{\mathcal{B}}_{D'}$, denoted as

$$\Psi^h_{D'}:\,\hat{\mathcal{B}}_{D'} \to \mathbf{\Omega}_{R,\epsilon},$$

will extend uniquely over $\pi_1(\mathrm{SO}(M),\mathbf{e})$ to be a homomorphism, which will still be denoted as $\Psi^h_{D'}$. However, it will be verified that $\Psi^h_{D'}$ descends to the abelianization, yielding a homomorphism

$$(\Psi^h_{D'})^{\mathsf{ab}}:\,H_1(\mathrm{SO}(M);\mathbf{Z}) \to \mathbf{\Omega}_{R,\epsilon},$$

which is exactly the inverse of Φ (Subsubsection 5.2.3), so eventually we may drop the scripts h and D' and simply write $\Psi^h_{D'}$ as Ψ.

Since $\pi_1(\mathrm{SO}(M),\mathbf{e})$ is a central extension of $\pi_1(M,*)$ by $\mathbf{Z}_2$, the conjugation τ_h of $\pi_1(M,*)$ naturally induces a conjugation of $\pi_1(\mathrm{SO}(M),\mathbf{e})$, which will be denoted as $\hat{\tau}_h$. For each noncentral $\hat{g} \in \hat{\mathcal{B}}_D$, by Lemma 5.7 and Lemma 5.16 (4), we may choose an oriented ∂-framed segment $\mathfrak{s}^h_{\hat{g}}$ associated to the δ-sharp element $\hat{\tau}_h(\hat{g})$, or simply written as

$$\mathfrak{s}_{\hat{g}}$$

with the fixed τ_h understood. Note that $\mathfrak{s}_{\hat{g}}$ is unique up to δ-small change of the initial and terminal framings, and $\mathfrak{s}_{\hat{c}\hat{g}}$ can be chosen as the framing flipping $\mathfrak{s}^*_{\hat{g}}$.

We define the claimed set-theoretic map Ψ^h_D as follows. For each noncentral element $\hat{g} \in \hat{\mathcal{C}}_h$, choose an oriented ∂-framed segment $\mathfrak{s}_{\hat{g}}$ which is associated to $\hat{\tau}_h(\hat{g})$ as above. For convenience, choose a unit vector

$$\vec{n}_{\hat{g}} \in T_*(M)$$

orthogonal to $\vec{t}$, such that $\vec{n}_{\mathtt{ini}}(\mathfrak{s}_{\hat{g}})$ and $\vec{n}_{\mathtt{ter}}(\mathfrak{s}_{\hat{g}})$ are both δ-close to $\vec{n}_{\hat{g}}$. Choose a right-hand nearly regular tripod $\mathfrak{a}_0 \vee \mathfrak{a}_1 \vee \mathfrak{a}_2$ and an oriented ∂-framed segment $\mathfrak{b}$ satisfying the following:

- The right-hand tripod $\mathfrak{a}_0 \vee \mathfrak{a}_1 \vee \mathfrak{a}_2$ is $(\frac{R}{2} - \frac{\ell(\mathfrak{s}_{\hat{g}})}{2} + \frac{1}{2} I(\frac{\pi}{3}), 10\delta)$-nearly regular. For each $i \in \mathbf{Z}_3$, the terminal endpoint $p_{\mathtt{ter}}(\mathfrak{a}_i)$ equals $*$, and the terminal direction $\vec{t}_{\mathtt{ter}}(\mathfrak{a}_i)$ is (10δ)-close to $-\vec{t}$, and the terminal framing $\vec{n}_{\mathtt{ter}}(\mathfrak{a}_i)$ is (10δ)-close to $\vec{n}_{\hat{g}}$.
- The oriented ∂-framed segment $\mathfrak{b}$ has length (100δ)-close to $\frac{R}{2} - \ell(\mathfrak{s}_{\hat{g}})$ and phase (100δ)-close to 0. The initial and terminal directions of $\mathfrak{b}$ are δ-close to $\vec{t}$ and $-\vec{t}$ respectively, and the initial and terminal framings of $\mathfrak{b}$ are both δ-close to $\vec{n}_{\hat{g}}$.

In $\mathbf{\Omega}_{R,\epsilon}$, define

$$\Psi^h_D(\hat{g}) \;=\; [\mathfrak{s}_{\hat{g}}\mathfrak{a}_{01}]_{R,\epsilon} + [\mathfrak{s}_{\hat{g}}\mathfrak{a}_{12}]_{R,\epsilon} + [\mathfrak{s}_{\hat{g}}\mathfrak{a}_{20}]_{R,\epsilon} - \;[\mathfrak{s}_{\hat{g}}\mathfrak{b}]_{R,\epsilon} - \;[\mathfrak{s}_{\hat{g}}\bar{\mathfrak{b}}]_{R,\epsilon},$$

and define

$$\Psi^h_D(\mathrm{id}) \;=\; 0,$$

and define

$$\Psi^h_D(\hat{c}) \;=\Psi^h_D(\hat{g}) - \Psi^h_D(\hat{c}\hat{g}),$$

using any noncentral $\hat{g} \in \hat{\mathcal{C}}_h$.

Remark 5.17. The reader should compare the definition of $\Psi^h_{D'}$ with the definition of the operator A_T in [KM2, Subsection 7.1]. Since A_T was defined with a coefficient $\frac{1}{2}$, it does not work in integral coefficients. In fact, the argument of Good Correction Theorem [KM2, Theorem 3.2] essentially implies that A_T induces an isomorphism $\psi^{\mathsf{ab}} : H_1(S;\mathbf{Q}) \to \mathbf{\Omega}_{R,\epsilon}(S)$ for any closed oriented hyperbolic surface S, which is the inverse of the homomorphism $\phi : \mathbf{\Omega}_{R,\epsilon}(S) \to H_1(S;\mathbf{Q})$ given by $\phi([\gamma]_{R,\epsilon}) = [\gamma]$. By introducing a right-hand tripod $\mathfrak{a}_0 \vee \mathfrak{a}_1 \vee \mathfrak{a}_2$ in addition to segment $\mathfrak{b}$, we may get rid of the coefficient $\frac{1}{2}$ and write down an expression of $\Psi^h_{D'}$ with integral coefficients. However, the ambiguity of the choice of $\vec{n}_{\hat{g}}$ makes it necessary to pass to $\mathrm{SO}(M)$ rather than to stay in M.

Lemma 5.18. *The set-theoretic map Ψ^h_D is well defined from $\hat{\mathcal{C}}_h$ to $\mathbf{\Omega}_{R,\epsilon}$ with respect to the fixed conjugation τ_h. In other words,*

(1) All the reduced cyclic concatenations involved are curves in $\mathbf{\Gamma}_{R,\epsilon}$;
(2) For any noncentral $\hat{g} \in \hat{\mathcal{C}}_h$, $\Psi^h_D(\hat{g})$ depends only on $\hat{g}$;
(3) Different choices of noncentral $\hat{g}$ defines the same $\Psi^h_D(\hat{c})$.

Proof. The statement (1) follows from straightforward verification using the Length and Phase Formula (Lemma 4.7) under our fixed choice of setup data (Lemma 5.16). It remains to prove the statements (2) and (3).

To prove the statement (2), observe that $\Psi^h_D(\hat{g})$ is clearly independent of the choice of $\mathfrak{s}_{\hat{g}}$ and $\vec{n}_{\hat{g}}$, since $\mathfrak{s}_{\hat{g}}$ is unique up to δ-fellow travelling and $\vec{n}_{\hat{g}}$ is unique up to δ-closeness. Suppose that $\mathfrak{a}'_0 \vee \mathfrak{a}'_1 \vee \mathfrak{a}'_2$ is another oriented ∂-framed segment satisfying the same conditions as of $\mathfrak{a}_0 \vee \mathfrak{a}_1 \vee \mathfrak{a}_2$, and similarly for $\mathfrak{b}'$. We write the new $\Psi^h_D(\hat{g})$ as $\Psi^{h'}_D(\hat{g})$ to distinguish. We must show that $\Psi^h_D(\hat{g})$ equals $\Psi^{h'}_D(\hat{g})$ in $\mathbf{\Omega}_{R,\epsilon}$.

Choose an auxiliary left-hand nearly regular tripod $\mathfrak{c}_0 \vee \mathfrak{c}_1 \vee \mathfrak{c}_2$ satisfying the following.

- The left-hand tripod $\mathfrak{c}_0 \vee \mathfrak{c}_1 \vee \mathfrak{c}_2$ is $(\frac{\ell(\mathfrak{s}_{\hat{g}})}{2} + \frac{1}{2}I(\frac{\pi}{3}), \delta)$-nearly regular. For each $i \in \mathbf{Z}_3$, the terminal endpoint $p_{\mathtt{ter}}(\mathfrak{c}_i)$ equals $*$, and the terminal direction $\vec{t}_{\mathtt{ter}}(\mathfrak{c}_i)$ is δ-close to $\vec{t}$, and the terminal framing $\vec{n}_{\mathtt{ter}}(\mathfrak{c}_i)$ is δ-close to $\vec{n}_{\hat{g}}$.

Since both $\mathfrak{a}_0 \vee \mathfrak{a}_1 \vee \mathfrak{a}_2$ and $\mathfrak{a}'_0 \vee \mathfrak{a}'_1 \vee \mathfrak{a}'_2$ have the opposite chirality to that of $\mathfrak{c}_0 \vee \mathfrak{c}_1 \vee \mathfrak{c}_2$, it follows from rotation (Construction 4.19 (1)) that

$$\sum_{i \in \mathbf{Z}_3} [\mathfrak{a}_{i,i+1} \bar{\mathfrak{c}}_{i,i+1}]_{R,\epsilon} = 0$$

and

$$\sum_{i \in \mathbf{Z}_3} [\mathfrak{a}'_{i,i+1} \bar{\mathfrak{c}}_{i,i+1}]_{R,\epsilon} = 0,$$

where $\mathfrak{a}_{i,i+1}$ means $\bar{\mathfrak{a}}_i \mathfrak{a}_{i+1}$ for $i \in \mathbf{Z}_3$, and similarly for the notations $\mathfrak{a}'_{i,i+1}$ and $\mathfrak{c}_{i,i+1}$. On the other hand, by swapping (Construction 4.17),

$$[\mathfrak{s}_{\hat{g}} \mathfrak{a}_{i,i+1}]_{R,\epsilon} - [\mathfrak{s}_{\hat{g}} \mathfrak{a}'_{i,i+1}]_{R,\epsilon} = [\bar{\mathfrak{c}}_{i,i+1} \mathfrak{a}_{i,i+1}]_{R,\epsilon} - [\bar{\mathfrak{c}}_{i,i+1} \mathfrak{a}'_{i,i+1}]_{R,\epsilon},$$

and

$$[\mathfrak{s}_{\hat{g}} \mathfrak{b}]_{R,\epsilon} - [\mathfrak{s}_{\hat{g}} \mathfrak{b}']_{R,\epsilon} = [\bar{\mathfrak{s}}_{\hat{g}} \mathfrak{b}]_{R,\epsilon} - [\bar{\mathfrak{s}}_{\hat{g}} \mathfrak{b}']_{R,\epsilon}.$$

For convenience, we write $\mathfrak{b}_{01}$ and $\mathfrak{b}_{10}$ for $\mathfrak{b}$ and $\bar{\mathfrak{b}}$, respectively, and similarly for $\mathfrak{b}'_{01}$ and $\mathfrak{b}'_{10}$. Then $\Psi^h_D(\hat{g}) - \Psi^{h'}_D(\hat{g})$ equals

$$\begin{aligned} & \sum_{i \in \mathbf{Z}_3} \left([\mathfrak{s}_{\hat{g}} \mathfrak{a}_{i,i+1}]_{R,\epsilon} - [\mathfrak{s}_{\hat{g}} \mathfrak{a}'_{i,i+1}]_{R,\epsilon}\right) - \sum_{j \in \mathbf{Z}_2} \left([\mathfrak{s}_{\hat{g}} \mathfrak{b}_{j,j+1}]_{R,\epsilon} - [\mathfrak{s}_{\hat{g}} \mathfrak{b}'_{j,j+1}]_{R,\epsilon}\right) \\ = & \sum_{i \in \mathbf{Z}_3} \left([\bar{\mathfrak{c}}_{i,i+1} \mathfrak{a}_{i,i+1}]_{R,\epsilon} - [\bar{\mathfrak{c}}_{i,i+1} \mathfrak{a}'_{i,i+1}]_{R,\epsilon}\right) - 0 \\ = & \ 0, \end{aligned}$$

or in other words, $\Psi^h_D(\hat{g})$ equals $\Psi^{h'}_D(\hat{g})$ in $\mathbf{\Omega}_{R,\epsilon}$. This proves the statement (2).

To prove the statement (3), we must show that for any noncentral elements $\hat{g}, \hat{g}'$ of $\hat{\mathcal{C}}_h$,

$$\Psi^h_D(\hat{g}) - \Psi^h_D(\hat{c}\hat{g}) = \Psi^h_D(\hat{g}') - \Psi^h_D(\hat{c}\hat{g}').$$

Step 1. We prove the equation above assuming that $\ell(\mathfrak{s}_{\hat{g}})$ is δ-close to $\ell(\mathfrak{s}_{\hat{g}'})$, and that $\vec{n}_{\hat{g}}$ is δ-close to $\vec{n}_{\hat{g}'}$.

Observe that the defining right-hand tripod $\mathfrak{a}_0 \vee \mathfrak{a}_1 \vee \mathfrak{a}_2$ and the defining ∂-framed segment $\mathfrak{b}$ can be chosen as the same for both $\hat{g}$ and $\hat{g}'$. Moreover, $\mathfrak{s}_{\hat{c}\hat{g}}$ can be chosen as the framing flipping $\mathfrak{s}^*_{\hat{g}}$, and the defining right-hand tripod and the defining ∂-framed segment can be chosen as $\mathfrak{a}^*_0 \vee \mathfrak{a}^*_{-1} \vee \mathfrak{a}^*_{-2}$ and $\mathfrak{b}^*$ respectively, and similarly for $\mathfrak{s}_{\hat{c}\hat{g}'}$. For convenience, we write $\mathfrak{b}_{01}$ and $\mathfrak{b}_{10}$ for $\mathfrak{b}$ and $\bar{\mathfrak{b}}$, respectively.

Thus,

$$
\begin{aligned}
& \Psi^h_D(\hat{g}) - \Psi^h_D(\hat{c}\hat{g}) \\
= & \sum_{i\in\mathbf{Z}_3} \left([\mathfrak{s}_{\hat{g}}\mathfrak{a}_{i,i+1}]_{R,\epsilon} - [\mathfrak{s}^*_{\hat{g}}\mathfrak{a}^*_{-i,-i-1}]_{R,\epsilon}\right) - \sum_{j\in\mathbf{Z}_2} \left([\mathfrak{s}_{\hat{g}}\mathfrak{b}_{j,j+1}]_{R,\epsilon} - [\mathfrak{s}^*_{\hat{g}}\mathfrak{b}^*_{j,j+1}]_{R,\epsilon}\right) \\
= & \sum_{i\in\mathbf{Z}_3} \left([\mathfrak{s}_{\hat{g}}\mathfrak{a}_{i,i+1}]_{R,\epsilon} - [\mathfrak{s}_{\hat{g}}\mathfrak{a}_{-i,-i-1}]_{R,\epsilon}\right) - \sum_{j\in\mathbf{Z}_2} \left([\mathfrak{s}_{\hat{g}}\mathfrak{b}_{j,j+1}]_{R,\epsilon} - [\mathfrak{s}_{\hat{g}}\mathfrak{b}_{j,j+1}]_{R,\epsilon}\right) \\
= & \sum_{i\in\mathbf{Z}_3} \left([\mathfrak{s}_{\hat{g}'}\mathfrak{a}_{i,i+1}]_{R,\epsilon} - [\mathfrak{s}_{\hat{g}'}\mathfrak{a}_{-i,-i-1}]_{R,\epsilon}\right) - \sum_{j\in\mathbf{Z}_2} \left([\mathfrak{s}_{\hat{g}'}\mathfrak{b}_{j,j+1}]_{R,\epsilon} - [\mathfrak{s}_{\hat{g}'}\mathfrak{b}_{j,j+1}]_{R,\epsilon}\right) \\
= & \sum_{i\in\mathbf{Z}_3} \left([\mathfrak{s}_{\hat{g}'}\mathfrak{a}_{i,i+1}]_{R,\epsilon} - [\mathfrak{s}^*_{\hat{g}'}\mathfrak{a}^*_{-i,-i-1}]_{R,\epsilon}\right) - \sum_{j\in\mathbf{Z}_2} \left([\mathfrak{s}_{\hat{g}'}\mathfrak{b}_{j,j+1}]_{R,\epsilon} - [\mathfrak{s}^*_{\hat{g}'}\mathfrak{b}^*_{j,j+1}]_{R,\epsilon}\right) \\
= & \Psi^h_D(\hat{g}') - \Psi^h_D(\hat{c}\hat{g}'),
\end{aligned}
$$

where the third equality following from swapping (Construction 4.17), applied to each bigon pair $[\mathfrak{s}_{\hat{g}}\mathfrak{a}_{i,i+1}]$, $[\mathfrak{s}_{\hat{g}'}\mathfrak{a}_{-i,-i-1}]$, for $i \in \mathbf{Z}_3$, and to each bigon pair $[\mathfrak{s}_{\hat{g}}\mathfrak{b}_{j,j+1}]$, $[\mathfrak{s}_{\hat{g}'}\mathfrak{a}_{j,j+1}]$, for $j \in \mathbf{Z}_2$.

Step 2. We prove the equation $\Psi^h_D(\hat{g}) - \Psi^h_D(\hat{c}\hat{g}) = \Psi^h_D(\hat{g}') - \Psi^h_D(\hat{c}\hat{g}')$ in the general case.

By the Connection Principle (Lemma 4.13), we may interpolate a sequence of elements $\hat{g}_0, \cdots, \hat{g}_N$ in $\hat{\mathcal{C}}_h$, where $\hat{g}_0 = \hat{g}$ and $\hat{g}_N = \hat{g}'$, such that $\ell(\mathfrak{s}_{\hat{g}_k})$ is δ-close to $\ell(\mathfrak{s}_{\hat{g}_{k+1}})$, and that $\vec{n}_{\hat{g}_k}$ is δ-close to $\vec{n}_{\hat{g}_{k+1}}$, for each $0 \leq k < N$. By Step 1,

$$\Psi^h_D(\hat{g}_k) - \Psi^h_D(\hat{c}\hat{g}_k) = \Psi^h_D(\hat{g}_{k+1}) - \Psi^h_D(\hat{c}\hat{g}_{k+1}),$$

for each $0 \leq k < N$. We conclude that $\Psi^h_D(\hat{g}) - \Psi^h_D(\hat{c}\hat{g})$ equals $\Psi^h_D(\hat{g}') - \Psi^h_D(\hat{c}\hat{g}')$. This completes the proof of the statement (3). □

Lemma 5.19. *For any triple of elements $\hat{g}_0, \hat{g}_1, \hat{g}_2$ in $\hat{\mathcal{C}}_h$ satisfying the triagular relation $\hat{g}_0\hat{g}_1\hat{g}_2 = \mathrm{id}$,*

$$\Psi^h_D(\hat{g}_0) + \Psi^h_D(\hat{g}_1) + \Psi^h_D(\hat{g}_2) \;=\; 0.$$

Hence the restriction of Ψ^h_D to $\hat{\mathcal{B}}_{D'}$ extends uniquely to be a homomorphism $\Psi^h_{D'}$ from $\pi_1(\mathrm{SO}(M), \mathbf{e})$ to $\mathbf{\Omega}_{R,\epsilon}$.

Remark 5.20. We point out that in the proof of Lemma 5.19, the last equality in Step 1 uses antirotation of tripod pairs with opposite charilities (Construction 4.20 (1)). The presence of the coefficient 2 there is not only indispensible but also crucial for Theorem 5.2 to work in the integral coefficient case. The corresponding fact is that in the paper [KM2], the conclusion of the Second Rotation Lemma (Lemma 8.2) should be

$$2\sum_{i=0}^{2}(R_i\bar{R}_{i+1})_T = 0$$

if one attempts to state with integral coefficients.

Proof. The 'hence' part follows from the facts that $\mathcal{B}_{D'}$ contains a generating set of $\pi_1(M, *)$, and that $\mathcal{C}_h$ contains $\mathcal{B}_D$ which further contains a triangular generating set of $\pi_1(M, *)$ (Lemma 5.16 (2)(6)). The former implies that the extension is unique, and the latter implies that the extension exists. It remains to prove the main statement.

First consider the case when at least one of $\hat{g}_i$ is central. If exactly one of $\hat{g}_i$ is central, and if it is nontrivial, the equation follows from the definition of $\hat{h}_D(\hat{c})$. If exactly one of $\hat{g}_i$ is central, and if it is the identity, we must prove that for any noncentral $\hat{g} \in \hat{\mathcal{C}}_h$,

$$\Psi^h_D(\hat{g}^{-1}) = -\Psi^h_D(\hat{g}).$$

Observe that $\mathfrak{s}_{\hat{g}^{-1}}$ can be chosen as the orientation reversed framing flipping $\bar{\mathfrak{s}}^*_{\hat{g}}$, and that the defining right-hand tripod and ∂-framed segment can be chosen as $\mathfrak{a}^*_0 \vee \mathfrak{a}^*_{-1} \vee \mathfrak{a}^*_{-2}$ and $\bar{\mathfrak{b}}^*$ respectively, provided that $\mathfrak{s}_{\hat{g}}$, $\mathfrak{a}_0 \vee \mathfrak{a}_1 \vee \mathfrak{a}_2$, and $\mathfrak{b}$ have been chosen to define $\Psi^h_D(\hat{g})$. This should be compared to the wrong choice $\bar{\mathfrak{s}}_{\hat{g}}$, $\mathfrak{a}_0 \vee \mathfrak{a}_1 \vee \mathfrak{a}_2$, and $\mathfrak{b}$, which is actually right for defining $\Psi^h_D(\hat{c}\hat{g}^{-1})$ by Lemma 5.16 (4). Choose an auxiliary left-hand $(\frac{\ell(\mathfrak{s}_{\hat{g}})}{2} + \frac{1}{2}I(\frac{\pi}{3}), \delta)$-nearly regular tripod $\mathfrak{c}_0 \vee \mathfrak{c}_1 \vee \mathfrak{c}_2$ as in the proof of Lemma 5.18 (2), namely, such that $\mathfrak{c}_0 \vee \mathfrak{c}_1 \vee \mathfrak{c}_2$ and $\mathfrak{a}_0 \vee \mathfrak{a}_1 \vee \mathfrak{a}_2$ form a δ-rotation pair. For convenience, we write $\mathfrak{b}_{01}$ and $\mathfrak{b}_{10}$ for $\mathfrak{b}$ and $\bar{\mathfrak{b}}$, respectively. Then $\Psi^h_D(\hat{g}) + \Psi^h_D(\hat{g}^{-1})$ equals

$$\begin{aligned}
& \sum_{i \in \mathbf{Z}_3} ([\mathfrak{s}_{\hat{g}}\mathfrak{a}_{i,i+1}]_{R,\epsilon} + [\bar{\mathfrak{s}}^*_{\hat{g}}\mathfrak{a}^*_{-i,-i-1}]_{R,\epsilon}) - \sum_{j \in \mathbf{Z}_2} ([\mathfrak{s}_{\hat{g}}\mathfrak{b}_{j,j+1}]_{R,\epsilon} + [\bar{\mathfrak{s}}^*_{\hat{g}}\bar{\mathfrak{b}}^*_{j,j+1}]_{R,\epsilon}) \\
= & \sum_{i \in \mathbf{Z}_3} ([\mathfrak{s}_{\hat{g}}\mathfrak{a}_{i,i+1}]_{R,\epsilon} + [\bar{\mathfrak{s}}_{\hat{g}}\mathfrak{a}_{-i,-i-1}]_{R,\epsilon}) - \sum_{j \in \mathbf{Z}_2} ([\mathfrak{s}_{\hat{g}}\mathfrak{b}_{j,j+1}]_{R,\epsilon} + [\bar{\mathfrak{s}}_{\hat{g}}\bar{\mathfrak{b}}_{j,j+1}]_{R,\epsilon}) \\
= & \sum_{i \in \mathbf{Z}_3} ([\mathfrak{s}_{\hat{g}}\mathfrak{a}_{i,i+1}]_{R,\epsilon} + [\bar{\mathfrak{s}}_{\hat{g}}\mathfrak{a}_{i+1,i}]_{R,\epsilon}) - 0 \\
= & \ 0.
\end{aligned}$$

This proves $\Psi^h_D(\hat{g}^{-1}) = -\Psi^h_D(\hat{g})$. Finally, if all $\hat{g}_i$ are central, it suffices to show that $2\Psi^h_D(\hat{c}) = 0$. However, this can be derived by the previous case, indeed, for any noncentral $\hat{g} \in \hat{\mathcal{C}}_h$,

$$\begin{aligned}
2\Psi^h_D(\hat{c}) &= (\Psi^h_D(\hat{g}\hat{c}) + \Psi^h_D(\hat{g}^{-1})) + (\Psi^h_D(\hat{g}^{-1}\hat{c}) + \Psi^h_D(\hat{g})) \\
&= (\Psi^h_D(\hat{g}\hat{c}) + \Psi^h_D(\hat{g}^{-1}\hat{c})) + (\Psi^h_D(\hat{g}^{-1}) + \Psi^h_D(\hat{g})) \\
&= 0.
\end{aligned}$$

It remains to consider the case when none of $\hat{g}_i$ is central. From the relation $\hat{g}_0\hat{g}_1\hat{g}_2 = \mathrm{id}$, the carrier segments of $\mathfrak{s}_{\hat{g}_i}$ form the boundary cycle of an oriented 2-simplex σ in M, so the tripod zipping (Definition 4.10 (7), and Lemma 4.11) provides us a $(2L + 10 + I(\frac{\pi}{3}), \delta)$-tame tripod $\mathfrak{t}_0 \vee \mathfrak{t}_1 \vee \mathfrak{t}_2$, such that $\mathfrak{t}_{i,i+1}$ and $\mathfrak{s}_{\hat{g}_{i+2}}$ are carried by the same segment for all i. Moreover, we may choose $\mathfrak{t}_0 \vee \mathfrak{t}_1 \vee \mathfrak{t}_2$ to be left-hand, then by Lemma 5.9, there are $\phi_0, \phi_1, \phi_2 \in \mathbf{R}/2\pi\mathbf{Z}$ with $\phi_0 + \phi_1 + \phi_2 = 0$ and $2\phi_{i+2}$ δ-close to the angle from $\mathfrak{t}_i$ to $\mathfrak{t}_{i+1}$ with respect to $\vec{t}$, such that $\mathfrak{t}_{i,i+1}(\phi_{i+2})$ is the same as $\mathfrak{s}_{\hat{g}_{i+2}}$ up to δ-small change of the initial and termal framings. Note that the central case allows us to alternatively prove the identity after switching two of the three $\hat{g}_i$ into $\hat{c}\hat{g}_i$.

Step 1. We prove the base step when $\mathfrak{t}_0 \vee \mathfrak{t}_1 \vee \mathfrak{t}_2$ is $(\frac{l}{2} + \frac{I(\frac{\pi}{3})}{2}, \delta)$-nearly regular for some constant l at least $2L + 10$, and when the terminal framings of $\mathfrak{t}_i$ are all δ-close to a unit vector $\vec{m}$ at $* \in M$ orthogonal to $\vec{t}$.

Possibly after switching two $\hat{g}_i$ to $\hat{c}\hat{g}_i$, we may assume that ϕ_i are all 0. Observe that in this case, for each $\hat{g}_{r+2}$ where $r \in \mathbf{Z}_3$, $\mathfrak{s}_{\hat{g}_{r+2}}$ may be chosen as $\mathfrak{t}_{r,r+1}$, and the defining tripod can be chosen as $\mathfrak{a}_0 \vee \mathfrak{a}_1 \vee \mathfrak{a}_2$, and the defining ∂-framed segment $\mathfrak{b}$

may be chosen as $\mathfrak{a}_{r,r+1}$. Then $\Psi^h_D(\hat{g}_0)+\Psi^h_D(\hat{g}_1)+\Psi^h_D(\hat{g}_2)$ equals

$$\begin{aligned}
& \sum_{r\in\mathbf{Z}_3}\left(\left(\sum_{i\in\mathbf{Z}_3}[\mathfrak{t}_{r,r+1}\mathfrak{a}_{i,i+1}]_{R,\epsilon}\right)-[\mathfrak{t}_{r,r+1}\mathfrak{a}_{r,r+1}]_{R,\epsilon}-[\mathfrak{t}_{r,r+1}\bar{\mathfrak{a}}_{r,r+1}]_{R,\epsilon}\right)\\
= & \sum_{r\in\mathbf{Z}_3}[\mathfrak{t}_{r,r+1}\mathfrak{a}_{r+1,r+2}]_{R,\epsilon}+\sum_{r\in\mathbf{Z}_3}[\mathfrak{t}_{r,r+1}\mathfrak{a}_{r+2,r}]_{R,\epsilon}-\sum_{r\in\mathbf{Z}_3}[\mathfrak{t}_{r,r+1}\bar{\mathfrak{a}}_{r,r+1}]_{R,\epsilon}\\
= & \sum_{r\in\mathbf{Z}_3}[\mathfrak{t}_{r,r+1}\mathfrak{a}_{r+1,r+2}]_{R,\epsilon}+\sum_{r\in\mathbf{Z}_3}[\mathfrak{t}_{r,r+1}\mathfrak{a}_{r+2,r}]_{R,\epsilon}-\sum_{r\in\mathbf{Z}_3}[\mathfrak{t}_{r,r+1}\bar{\mathfrak{a}}_{r,r+1}]_{R,\epsilon}\\
= & \ 2\sum_{r\in\mathbf{Z}_3}[\mathfrak{t}_{r,r+1}\mathfrak{a}_{r,r+1}]_{R,\epsilon}-0\\
= & \ 2\sum_{r\in\mathbf{Z}_3}[\mathfrak{t}_{r,r+1}\bar{\mathfrak{a}}_{r+1,r}]_{R,\epsilon}\\
= & \ 0.
\end{aligned}$$

In the third equality, the last summation equals zero by rotation (Construction 4.19 (1)); the first two summations are both equal to the summation of $[\mathfrak{t}_{r,r+1}\mathfrak{a}_{r,r+1}]_{R,\epsilon}$ over $r\in\mathbf{Z}_3$, since by swapping (Construction 4.17),

$$\begin{aligned}
& [\mathfrak{t}_{01}\mathfrak{a}_{12}]_{R,\epsilon}+[\mathfrak{t}_{12}\mathfrak{a}_{20}]_{R,\epsilon}+[\mathfrak{t}_{20}\mathfrak{a}_{01}]_{R,\epsilon}\\
= & [\mathfrak{t}_{01}\mathfrak{a}_{12}]_{R,\epsilon}+[\mathfrak{t}_{12}\mathfrak{a}_{01}]_{R,\epsilon}+[\mathfrak{t}_{20}\mathfrak{a}_{20}]_{R,\epsilon}\\
= & [\mathfrak{t}_{01}\mathfrak{a}_{01}]_{R,\epsilon}+[\mathfrak{t}_{12}\mathfrak{a}_{12}]_{R,\epsilon}+[\mathfrak{t}_{20}\mathfrak{a}_{20}]_{R,\epsilon},
\end{aligned}$$

and similarly for $[\mathfrak{t}_{01}\mathfrak{a}_{20}]_{R,\epsilon}+[\mathfrak{t}_{12}\mathfrak{a}_{01}]_{R,\epsilon}+[\mathfrak{t}_{20}\mathfrak{a}_{12}]_{R,\epsilon}$. The last equality follows from antirotation (Construction 4.20 (1)). This proves $\Psi^h_D(\hat{g}_0)+\Psi^h_D(\hat{g}_1)+\Psi^h_D(\hat{g}_2)=0$ for the base step.

Step 2. We prove a connecting step which is the following claim. Suppose that $\mathfrak{c}_0\mathfrak{r}_0\vee\mathfrak{c}_1\mathfrak{r}_1\vee\mathfrak{c}_2\mathfrak{r}_2$ and $\mathfrak{c}_0\mathfrak{r}'_0\vee\mathfrak{c}_1\mathfrak{r}'_1\vee\mathfrak{c}_2\mathfrak{r}'_2$ are (L,δ)-tame left-hand tripods satisfying the following:

- The left-hand tripod $\mathfrak{c}_0\vee\mathfrak{c}_1\vee\mathfrak{c}_2$ is $(10,\delta)$-nearly regular.
- For each $i\in\mathbf{Z}_3$, the chain $\mathfrak{c}_i,\mathfrak{r}_i$ is δ-consecutive and $(10,\delta)$-tame. The length of $\mathfrak{c}_i\mathfrak{r}_i$ is at most $\frac{1}{2}(R-2L+I(\frac{\pi}{3})-20)$. The terminal direction of $\mathfrak{c}_i\mathfrak{r}_i$ is δ-close to $\vec{t}$. The same holds for $\mathfrak{c}_i\mathfrak{r}'_i$.
- For each $i\in\mathbf{Z}_3$, $\ell(\mathfrak{r}_i)$ is δ-close to $\ell(\mathfrak{r}'_i)$, and $\vec{n}_{\mathtt{ter}}(\mathfrak{r}_i)$ is δ-close to $\vec{n}_{\mathtt{ter}}(\mathfrak{r}'_i)$.

Let $\phi_0,\phi_1,\phi_2\in\mathbf{R}/2\pi\mathbf{Z}$ be the angles guaranteed by Lemma 5.9 with respect to $\mathfrak{c}_0\mathfrak{r}_0\vee\mathfrak{c}_1\mathfrak{r}_1\vee\mathfrak{c}_2\mathfrak{r}_2$, which, hence, works for $\mathfrak{c}_0\mathfrak{r}'_0\vee\mathfrak{c}_1\mathfrak{r}'_1\vee\mathfrak{c}_2\mathfrak{r}'_2$ as well. Let $\hat{g}_{i+2},\hat{g}'_{i+2}\in\pi_1(\mathrm{SO}(M),\mathbf{e})$ be the δ-sharp element associated to $(\bar{\mathfrak{r}}_i\mathfrak{c}_{i,i+1}\mathfrak{r}_i)(\phi_{i+2})$, $(\bar{\mathfrak{r}}'_i\mathfrak{c}_{i,i+1}\mathfrak{r}'_i)(\phi_{i+2})$, respectively, which, in fact, lie in $\hat{\mathcal{C}}_h$. With the notations above, we claim

$$\Psi^h_D(\hat{g}_0)+\Psi^h_D(\hat{g}_1)+\Psi^h_D(\hat{g}_2)=\Psi^h_D(\hat{g}'_0)+\Psi^h_D(\hat{g}'_1)+\Psi^h_D(\hat{g}'_2).$$

To prove the claim, observe that it suffices to prove a simple case that $\mathfrak{r}_i$ equals $\mathfrak{r}'_i$ except for one $i\in\mathbf{Z}_3$. Then the claim follows by applying the simple case successively to each neighboring pair in the sequence of tripods $\mathfrak{c}_0\mathfrak{r}_0\vee\mathfrak{c}_1\mathfrak{r}_1\vee\mathfrak{c}_2\mathfrak{r}_2$, $\mathfrak{c}_0\mathfrak{r}_0\vee\mathfrak{c}_1\mathfrak{r}_1\vee\mathfrak{c}_2\mathfrak{r}'_2$, $\mathfrak{c}_0\mathfrak{r}_0\vee\mathfrak{c}_1\mathfrak{r}'_1\vee\mathfrak{c}_2\mathfrak{r}'_2$, $\mathfrak{c}_0\mathfrak{r}'_0\vee\mathfrak{c}_1\mathfrak{r}'_1\vee\mathfrak{c}_2\mathfrak{r}'_2$. Without loss of generality, we may assume that $\mathfrak{r}_i=\mathfrak{r}'_i$ except for i being 0. Then $\Psi^h_D(\hat{g}_0)=\Psi^h_D(\hat{g}'_0)$, and we must show $\Psi^h_D(\hat{g}_1)+\Psi^h_D(\hat{g}_2)=\Psi^h_D(\hat{g}'_1)+\Psi^h_D(\hat{g}'_2)$. Observe further that for both $\hat{g}_1$ and $\hat{g}'_1$, we may choose the same defining right-hand tripod $\mathfrak{a}^{(1)}_0\vee\mathfrak{a}^{(1)}_1\vee\mathfrak{a}^{(1)}_2$ and the same defining ∂-framed segment $\mathfrak{b}^{(1)}$. Similarly, for $\hat{g}_2$ and $\hat{g}'_2$ we choose the same $\mathfrak{a}_0^{(2)} \vee \mathfrak{a}_1^{(2)} \vee \mathfrak{a}_2^{(2)}$ and $\mathfrak{b}^{(2)}$ for both $\hat{g}_2$ and $\hat{g}_2'$. By swapping the bigon pair $[\mathfrak{r}_0\,(\mathfrak{a}_{01}^{(2)}(-\phi_1)\bar{\mathfrak{r}}_2\mathfrak{c}_{20})]$ and $[\mathfrak{r}_0'\,(\bar{\mathfrak{a}}_{01}^{(1)}(\phi_2)\bar{\mathfrak{r}}_1\bar{\mathfrak{c}}_{01})]$ into $[\mathfrak{r}_0'\,(\mathfrak{a}_{01}^{(2)}(-\phi_1)\bar{\mathfrak{r}}_2\mathfrak{c}_{20})]$ and $[\mathfrak{r}_0\,(\bar{\mathfrak{a}}_{01}^{(1)}(\phi_2)\bar{\mathfrak{r}}_1\bar{\mathfrak{c}}_{01})]$ (Construction 4.17), after rearrangement, we have

$$[\mathfrak{t}_{01}(\phi_2)\mathfrak{a}_{01}^{(1)}]_{R,\epsilon} + [\mathfrak{t}_{20}(\phi_1)\mathfrak{a}_{01}^{(2)}]_{R,\epsilon} = [\mathfrak{t}_{01}'(\phi_2)\mathfrak{a}_{01}^{(1)}]_{R,\epsilon} + [\mathfrak{t}_{20}'(\phi_1)\mathfrak{a}_{01}^{(2)}]_{R,\epsilon},$$

where $\mathfrak{t}_{i,i+1} = \bar{\mathfrak{r}}_i\mathfrak{c}_{i,i+1}\mathfrak{r}_{i+1}$ and $\mathfrak{t}_{i,i+1}' = \bar{\mathfrak{r}}_i'\mathfrak{c}_{i,i+1}\mathfrak{r}_{i+1}'$; similarly,

$$[\mathfrak{t}_{01}(\phi_2)\mathfrak{a}_{12}^{(1)}]_{R,\epsilon} + [\mathfrak{t}_{20}(\phi_1)\mathfrak{a}_{12}^{(2)}]_{R,\epsilon} = [\mathfrak{t}_{01}'(\phi_2)\mathfrak{a}_{12}^{(1)}]_{R,\epsilon} + [\mathfrak{t}_{20}'(\phi_1)\mathfrak{a}_{12}^{(2)}]_{R,\epsilon};$$

$$[\mathfrak{t}_{01}(\phi_2)\mathfrak{a}_{20}^{(1)}]_{R,\epsilon} + [\mathfrak{t}_{20}(\phi_1)\mathfrak{a}_{20}^{(2)}]_{R,\epsilon} = [\mathfrak{t}_{01}'(\phi_2)\mathfrak{a}_{20}^{(1)}]_{R,\epsilon} + [\mathfrak{t}_{20}'(\phi_1)\mathfrak{a}_{20}^{(2)}]_{R,\epsilon};$$

and

$$-[\mathfrak{t}_{01}(\phi_2)\mathfrak{b}^{(1)}]_{R,\epsilon} - [\mathfrak{t}_{20}(\phi_1)\mathfrak{b}^{(2)}]_{R,\epsilon} = -[\mathfrak{t}_{01}'(\phi_2)\mathfrak{b}^{(1)}]_{R,\epsilon} - [\mathfrak{t}_{20}'(\phi_1)\mathfrak{b}^{(2)}]_{R,\epsilon};$$

$$-[\mathfrak{t}_{01}(\phi_2)\bar{\mathfrak{b}}^{(1)}]_{R,\epsilon} - [\mathfrak{t}_{20}(\phi_1)\bar{\mathfrak{b}}^{(2)}]_{R,\epsilon} = -[\mathfrak{t}_{01}'(\phi_2)\bar{\mathfrak{b}}^{(1)}]_{R,\epsilon} - [\mathfrak{t}_{20}'(\phi_1)\bar{\mathfrak{b}}^{(2)}]_{R,\epsilon}.$$

Summing up the five equations above yields $\Psi_D^h(\hat{g}_1) + \Psi_D^h(\hat{g}_2) = \Psi_D^h(\hat{g}_1') + \Psi_D^h(\hat{g}_2')$. This finishes the proof the claim of the connecting step.

Step 3. We finish the proof of the general noncentral case. Let $\mathfrak{t}_0 \vee \mathfrak{t}_1 \vee \mathfrak{t}_2$ and ϕ_0, ϕ_1, ϕ_2 be as before so that $\hat{g}_{i+2}$ is associated to $\mathfrak{t}_{i,i+1}(\phi_{i+2})$. We may write $\mathfrak{t}_i$ as concatenation of consecutive ∂-framed segments $\mathfrak{c}_i\mathfrak{r}_i$, so that $\mathfrak{c}_0 \vee \mathfrak{c}_1 \vee \mathfrak{c}_2$ is $(10+I(\frac{\pi}{3}), \delta)$-nearly regular. Hence $\mathfrak{r}_i$ has at least $2L+10$ and phase δ-close to 0. By the Connection Principle (Lemma 4.13), we may interpolate a sequence of tripods $\mathfrak{t}_0^{(k)} \vee \mathfrak{t}_1^{(k)} \vee \mathfrak{t}_2^{(k)}$ where k runs over $0, \cdots, N$, such that $\mathfrak{t}_i^{(k)}$ is the δ-concatenation $\mathfrak{c}_i\mathfrak{r}_i^{(k)}$, and $\mathfrak{r}_i^{(k)}$ satisfies the following.

- For all $i \in \mathbf{Z}_3$ and $0 \le k \le N$, $\mathfrak{r}_i^{(k)}$ have length at least $2L$ and phase δ-close to 0.
- For $i \in \mathbf{Z}_3$ and $0 \le k < N$, $\mathfrak{r}_i^{(k)}$ and $\mathfrak{r}_i^{(k+1)}$ have length δ-close to each other, and terminal framings δ-close to each other.
- For $i \in \mathbf{Z}_3$, $\mathfrak{r}_i^{(0)}$ equals $\mathfrak{r}_i$.
- For $i \in \mathbf{Z}_3$, $\mathfrak{r}_i^{(N)}$ have length δ-close to each other, and terminal framings δ-close to each other.

For each $\mathfrak{t}_0^{(k)} \vee \mathfrak{t}_1^{(k)} \vee \mathfrak{t}_2^{(k)}$, let $\phi_0^{(k)}, \phi_1^{(k)}, \phi_2^{(k)} \in \mathbf{R}/2\pi\mathbf{Z}$ be a triple of angles guaranteed by Lemma 5.9, and let $\hat{g}_{i+2}^{(k)} \in \pi_1(\mathrm{SO}(M), \mathbf{e})$ be the δ-sharp element associated to $\mathfrak{t}_{i,i+1}^{(k)}(\phi_{i+2})$. It follows that $\hat{g}_0^{(k)}\hat{g}_1^{(k)}\hat{g}_2^{(k)} = \mathrm{id}$ for $0 \le k \le N$. Moreover, we may assume without loss of generality that $\phi_0^{(0)}, \phi_1^{(0)}, \phi_2^{(0)}$ are all ϕ_0, ϕ_1, ϕ_2 respectively, and that $\phi_0^{(N)}, \phi_1^{(N)}, \phi_2^{(N)}$ are all 0. Therefore, Step 1 implies that

$$\Psi_D^h(\hat{g}_0^{(N)}) + \Psi_D^h(\hat{g}_1^{(N)}) + \Psi_D^h(\hat{g}_2^{(N)}) = 0,$$

and Step 2 implies that

$$\Psi_D^h(\hat{g}_0^{(k)}) + \Psi_D^h(\hat{g}_1^{(k)}) + \Psi_D^h(\hat{g}_2^{(k)}) = \Psi_D^h(\hat{g}_0^{(k+1)}) + \Psi_D^h(\hat{g}_1^{(k+1)}) + \Psi_D^h(\hat{g}_2^{(k+1)}),$$

for $0 \le k < N$. It follows that when k equals 0, we have

$$\Psi_D^h(\hat{g}_0) + \Psi_D^h(\hat{g}_1) + \Psi_D^h(\hat{g}_2) = 0.$$

This completes the proof. □

By Lemma 5.19, we conclude that the restriction of Ψ^h_D to $\hat{\mathcal{B}}_{D'}$ extends uniquely to a homomorphism

$$\Psi^h_{D'}\colon \pi_1(\mathrm{SO}(M),\mathbf{e}) \to \mathbf{\Omega}_{R,\epsilon}.$$

Descending to the abelianization, we denote the induced homomorphism as

$$(\Psi^h_{D'})^{\mathsf{ab}}\colon H_1(\mathrm{SO}(M);\mathbf{Z}) \to \mathbf{\Omega}_{R,\epsilon}.$$

5.2.3. *Verifications.* It remains to verify that $(\Psi^h_{D'})^{\mathsf{ab}}$ is the inverse of Φ. We complete this by proving that $(\Psi^h_{D'})^{\mathsf{ab}}$ is the pre-inverse of Φ (Lemma 5.21), and that $\Psi^h_{D'}$ is onto (Lemma 5.22).

Lemma 5.21. *For any element $\hat{g}$ in $\hat{\mathcal{C}}_h$,*

$$\Phi(\Psi^h_D(\hat{g})) = [\hat{g}]$$

in $H_1(\mathrm{SO}(M);\mathbf{Z})$. Hence the composition $\Phi\circ(\Psi^h_{D'})^{\mathsf{ab}}$ is the identity transformation of $H_1(\mathrm{SO}(M);\mathbf{Z})$.

Proof. First consider the case when $\hat{g}$ is noncentral. Let $\hat{a}_{i,i+1}$ in $\pi_1(\mathrm{SO}(M),\mathbf{e})$ be the δ-sharp element associated to $\mathfrak{a}_{i,i+1}$ for $i\in\mathbf{Z}_3$, and $\hat{b}\in\pi_1(\mathrm{SO}(M),\mathbf{e})$ be associated to $\mathfrak{b}$ (Definition 5.7). Hence $\hat{c}\hat{b}^{-1}$ is associated to $\bar{\mathfrak{b}}$. It is clear from the construction of Φ that the the image of $[\mathfrak{s}_{\hat{g}}\mathfrak{a}_{i,i+1}]_{R,\epsilon}$ under Φ is equal to $[\hat{g}\hat{a}_{i,i+1}]$. Similarly, $\Phi([\mathfrak{s}_{\hat{g}}\mathfrak{b}]_{R,\epsilon}) = [\hat{g}\hat{b}]$, and $\Phi([\mathfrak{s}_{\hat{g}}\bar{\mathfrak{b}}]_{R,\epsilon}) = [\hat{g}\hat{c}\hat{b}^{-1}]$. Then in $H_1(\mathrm{SO}(M);\mathbf{Z})$, $\Phi(\Psi^h_D(\hat{g}))$ equals

$$\begin{aligned} & [\hat{g}\hat{a}_{01}]+[\hat{g}\hat{a}_{12}]+[\hat{g}\hat{a}_{20}]-[\hat{g}\hat{b}]-[\hat{g}\hat{c}\hat{b}^{-1}] \\ =\ & [\hat{g}]+[\hat{a}_{01}]+[\hat{a}_{12}]+[\hat{a}_{20}]-[\hat{c}] \\ =\ & [\hat{g}]+[\hat{a}_{01}\hat{a}_{12}\hat{a}_{20}\hat{c}] \\ =\ & [\hat{g}], \end{aligned}$$

where $[\hat{a}_{01}\hat{a}_{12}\hat{a}_{20}\hat{c}]=0$ because $\mathfrak{a}_0\vee\mathfrak{a}_1\vee\mathfrak{a}_2$ is right-hand, (Remark 5.10). This shows $\Phi(\Psi^h_D(\hat{g}))=[\hat{g}]$ in the noncentral case. The central case is an immediate consequence of the noncentral case, so we have completed the proof of the main statement. The 'hence' part follows immediately from the fact that $\hat{\mathcal{B}}_{D'}$ generates $\pi_1(\mathrm{SO}(M),\mathbf{e})$ (Lemma 5.16 (2)). □

Lemma 5.22. *For any curve $\gamma\in\mathbf{\Gamma}_{R,\epsilon}$, the (R,ϵ)-panted cobordism class $[\gamma]_{R,\epsilon}$ is equal to an integral linear combination of elements in the image of $\hat{\mathcal{B}}_{D'}$ under $\Psi^h_{D'}$. Hence the homomorphism $(\Psi^h_{D'})^{\mathsf{ab}}$ surjects $\mathbf{\Omega}_{R,\epsilon}$.*

Proof. The 'hence' part follows immediately from the main statement and the fact that $\hat{\mathcal{B}}_{D'}$ generates $\pi_1(\mathrm{SO}(M),\mathbf{e})$ (Lemma 5.16 (2)). It remains to prove the main statement.

Remember that (L',δ') are the pair of constants guaranteed by Lemma 5.16 (1). By Lemma 5.6, it suffices to assume $\gamma\in\mathbf{\Gamma}_{R,\delta'}$. Let $\mathcal{A}$ denote the subset of $\pi_1(M,*)$ consisting of elements u such that the length of u is at most $\frac{R}{2}+2L'+2$, and that the initial and terminal directions of u is $(10\delta')$-close to $\vec{t}_{\mathtt{ter}}(h)$ and $\vec{t}_{\mathtt{ini}}(h)$, respectively. Note that $\tau_h(\mathcal{A})$ is contained in $\mathcal{C}_h$. Let $\hat{\mathcal{A}}$ denote the preimage of $\mathcal{A}$ in $\pi_1(\mathrm{SO}(M),\mathbf{e})$.

Step 1. We find $\hat{x}_\pm\in\hat{\tau}_h(\hat{\mathcal{A}})$, such that

$$[\gamma]_{R,\epsilon}=\Phi^h_D(\hat{x}_-)+\Phi^h_D(\hat{x}_+).$$

Since $\gamma \in \mathbf{\Gamma}_{R,\delta'}$, we may bisect γ into an $(\frac{R}{2}, \delta')$-nearly regular bigon $[\mathfrak{s}_-\mathfrak{s}_+]$ by interpolating a pair of antipodal points with suitably chosen normal vectors. Enrich h with initial and terminal framings to obtain a ∂-framed segment $\mathfrak{h}$ of phase 0. By the Connection Principle (Lemma 4.13) there are oriented ∂-framed segments $\mathfrak{u}_\pm$ from $p_{\mathtt{ter}}(\mathfrak{u}_\pm)$ to $*$, satisfying the following.

- The length and phase of $\mathfrak{u}_\pm$ are δ'-close to $L' + I(\frac{\pi}{2}) + 1$ and 0 respectively. The initial direction $\mathfrak{u}_\pm$ is δ'-close to $\pm\vec{n}_{\mathtt{ter}}(\mathfrak{s}_\pm) \times \vec{t}_{\mathtt{ter}}(\mathfrak{s}_\pm)$, and the initial framing of $\mathfrak{u}_\pm$ is δ'-close to $\vec{n}_{\mathtt{ter}}(\mathfrak{s}_\pm)$. The terminal direction of $\mathfrak{u}_\pm$ is δ'-close to $\vec{t}_{\mathtt{ini}}(\mathfrak{h})$, and the terminal framings of $\mathfrak{u}_\pm$ and are δ'-close to $\vec{n}_{\mathtt{ini}}(\mathfrak{h}(\pm\pi/2))$.

Let $\mathfrak{x}_\pm = \bar{\mathfrak{w}}_\mp \mathfrak{s}_\pm \mathfrak{w}_\pm$, where $\mathfrak{w}_\pm = \mathfrak{u}_\pm \mathfrak{h}(\pm\pi/2)$, be the reduced concatenation. Then $\mathfrak{x}_\pm(\pm\pi/2)$ is associated to a δ'-sharp element $\hat{x}_\pm \in \hat{\tau}_h(\hat{\mathcal{A}})$.

Choose a right-hand tripod $\mathfrak{a}_0 \vee \mathfrak{a}_1 \vee \mathfrak{a}_2$ and a ∂-framed segment $\mathfrak{b}$ for defining $\Psi^h_D(\hat{x}_+)$, and choose $\mathfrak{a}^*_0 \vee \mathfrak{a}^*_{-1} \vee \mathfrak{a}^*_{-2}$ with the indices understood in $\mathbf{Z}_3$ and $\bar{\mathfrak{b}}^*$ for $\Psi^h_D(\hat{x}_-)$. Note that for $i \in \mathbf{Z}_3$,

$$\begin{aligned}
& [\mathfrak{x}_+(\pi/2)\mathfrak{a}_{i,i+1}]_{R,\epsilon} + [\mathfrak{x}_-(-\pi/2)\mathfrak{a}^*_{-i,-i-1}]_{R,\epsilon} \\
= \ & [\mathfrak{x}_+\mathfrak{a}_{i,i+1}(\pi/2)]_{R,\epsilon} + [\mathfrak{x}_-\overline{\mathfrak{a}_{i,i+1}(\pi/2)}]_{R,\epsilon} \\
= \ & [\mathfrak{s}_+(\mathfrak{w}_+\mathfrak{a}_{i,i+1}(\pi/2)\bar{\mathfrak{w}}_-)]_{R,\epsilon} + [\mathfrak{s}_-\overline{(\mathfrak{w}_+\mathfrak{a}_{i,i+1}(\pi/2)\bar{\mathfrak{w}}_-)}]_{R,\epsilon} \\
= \ & [\gamma]_{R,\epsilon},
\end{aligned}$$

where the last equality follows from splitting $\mathfrak{s}_+\mathfrak{s}_-$ (Construction 4.15). Thus

$$\begin{aligned}
[\mathfrak{x}_+(\pi/2)\mathfrak{a}_{01}]_{R,\epsilon} + [\mathfrak{x}_-(-\pi/2)\mathfrak{a}^*_{02}]_{R,\epsilon} &= [\gamma]_{R,\epsilon}; \\
[\mathfrak{x}_+(\pi/2)\mathfrak{a}_{12}]_{R,\epsilon} + [\mathfrak{x}_-(-\pi/2)\mathfrak{a}^*_{21}]_{R,\epsilon} &= [\gamma]_{R,\epsilon}; \\
[\mathfrak{x}_+(\pi/2)\mathfrak{a}_{20}]_{R,\epsilon} + [\mathfrak{x}_-(-\pi/2)\mathfrak{a}^*_{10}]_{R,\epsilon} &= [\gamma]_{R,\epsilon};
\end{aligned}$$

Similarly,

$$\begin{aligned}
-[\mathfrak{x}_+(\pi/2)\mathfrak{b}]_{R,\epsilon} + -[\mathfrak{x}_-(-\pi/2)\bar{\mathfrak{b}}^*]_{R,\epsilon} &= -[\gamma]_{R,\epsilon}; \\
-[\mathfrak{x}_+(\pi/2)\bar{\mathfrak{b}}]_{R,\epsilon} + -[\mathfrak{x}_-(-\pi/2)\mathfrak{b}^*]_{R,\epsilon} &= -[\gamma]_{R,\epsilon}.
\end{aligned}$$

Summing up the five equations above shows that $\Phi^h_D(\hat{x}_+) + \Phi^h_D(\hat{x}_-) = [\gamma]_{R,\epsilon}$.

Step 2. For any $\hat{z} \in \hat{\mathcal{A}}$ of length at least $10L'$ we find $\hat{y}_\pm \in \hat{\mathcal{A}}$, such that

$$\hat{z} = \hat{y}_-\hat{y}_+$$

and that for the images $y_\pm, z \in \pi_1(M,*)$ of $\hat{y}_\pm, \hat{z}$ respectively, both $\tau_h(y_\pm)$ have length less than $\frac{1}{2}\ell(z) + 3L'$.

In fact, we may write the pointed geodesic loop as the concatenation of to geodesic segments $\zeta_-\zeta_+$ joint at the midpoint of z. By the Connection Principle (Lemma 4.13), applied to the unframed case simply by ignoring the framings, there is a path υ from the midpoint of z to $*$, satisfying the following: the length of υ is δ'-close to $L' + I(\frac{\pi}{2})$; the initial direction of υ is δ'-closely perpendicular to z; and the terminal direction of υ is δ'-close to $\vec{t}$. Let y_- and y_+ in $\pi_1(M,*)$ be $\zeta_-\upsilon$ and $\bar{\upsilon}\zeta_+$, respectively. Since $z = y_-y_+$ in $\pi_1(M,*)$, we may choose lifts $\hat{y}_\pm$ of $y_\pm$ in $\pi_1(\mathrm{SO}(M),\mathbf{e})$ so that $\hat{z} = \hat{y}_-\hat{y}_+$. It is straightforward to see that $\hat{y}_\pm$ are as desired.

Step 3. We complete the proof of the main statement. As mentioned above, we may assume that $\gamma \in \mathbf{\Gamma}_{R,\delta'}$. By Step 1, $[\gamma]_{R,\epsilon}$ can be written as a sum of elements in the image of $\hat{\tau}_h(\hat{\mathcal{A}})$ under Ψ^h_D. By iterately applying Step 2, any element in $\Psi^h_D(\hat{\tau}_h(\hat{\mathcal{A}}))$ can be replaced with a sum of elements of the form $\Psi^h_D(\hat{\tau}_h(\hat{y}))$ where the image of $\hat{y} \in \hat{\mathcal{A}}$ in $\pi_1(M, *)$ has length at most $10L'$. In particular, $\hat{y} \in \hat{\mathcal{B}}_{D'}$ since D' is assumed to be at least $10L'$ (Lemma 5.16 (2)). Thus $[\gamma]_{R,\epsilon}$ is equal to an integral linear combination of elements in the image of $\hat{\mathcal{B}}_{D'}$ under $\Psi^h_{D'}$, as $\Psi^h_{D'}$ is the restriction of Ψ^h_D to $\hat{\mathcal{B}}_{D'}$. This completes the proof. □

5.3. **Proof of Theorem 5.2.** In summary, given any oriented closed hyperbolic 3-manifold M, suppose that ϵ is a positive constant smaller than the minimum between $\frac{1}{100}$ and half the injectivity radius of M. With the positive constant $R(\epsilon, M)$ guaranteed by Lemma 5.16 (5), suppose that R is a positive constant greater than $R(\epsilon, M)$. Then the homomorphism

$$\Phi : \mathbf{\Omega}_{R,\epsilon}(M) \to H_1(\mathrm{SO}(M), \mathbf{e})$$

constructed in Subsection 5.1 is a canonically defined isomorphism (Lemma 5.14 and Subsection 5.2). By Lemma 5.15, for all $[L]_{R,\epsilon} \in \mathbf{\Omega}_{R,\epsilon}(M)$, the image of $\Phi([L]_{R,\epsilon})$ under the bundle projection is the homology class $[L] \in H_1(M; \mathbf{Z})$. This completes the proof of Theorem 5.2.

6. Pantifying second homology classes

In this section, we show that second homology classes of an oriented closed hyperbolic 3-manifold M can be represented by (R, ϵ)-panted surfaces, as precisely stated in Theorem 6.1. This will imply the absolute case of Theorem 2.10 (1), namely, when the collection of curves $\mathcal{L} \subset \mathbf{\Gamma}_{R,\epsilon}$ is empty (Subsection 8.2). Roughly speaking, Theorem 6.1 follows from inspecting the homology classes of the (R, ϵ)-panted surfaces constructed in the proof of Theorem 5.2, so our argument and notations will heavily rely on Section 5. In particular, throughout this section, it will suffice to assume ϵ to be a positive constant smaller than the minimum between $\frac{1}{100}$ and half the injectivity radius of M, and R to be a positive constant greater than the constant $R(\epsilon, M)$ as guaranteed by Lemma 5.16 (5).

Theorem 6.1. *Let M be an oriented closed hyperbolic 3-manifold. For any small positive ϵ and sufficiently large positive R depending on M and ϵ, the following holds. For any homology class $\alpha \in H_2(M; \mathbf{Z})$, there exists an (oriented) closed (R, ϵ)-panted subsurface $j : F \looparrowright M$ so that $j_*[F]$ equals α.*

Remark 6.2. There is a canonical free resolution of the integral module $\mathbf{\Omega}_{R,\epsilon}$ given by

$$0 \longrightarrow N \longrightarrow \mathbf{Z}\,\mathbf{\Pi}_{R,\epsilon} \xrightarrow{\partial} \mathbf{Z}\,\mathbf{\Gamma}_{R,\epsilon} \longrightarrow \mathbf{\Omega}_{R,\epsilon} \longrightarrow 0,$$

where N denotes the kernel of of the boundary homomorphism. There is also a natural homomorphism $N \to H_2(M; \mathbf{Z})$ since the natural homomorphism $\mathbf{Z}\,\mathbf{\Pi}_{R,\epsilon} \to H_2(M, |\mathbf{\Gamma}_{R,\epsilon}|; \mathbf{Z})$ uniquely lifts to $H_2(M; \mathbf{Z})$ restricted to N. Therefore, Theorem 6.1 asserts that N surjects $H_2(M; \mathbf{Z})$. In this sense, it reveals certain finer structure of (R, ϵ)-panted cobordisms in addition to Theorem 5.2.

The key idea of the proof of Theorem 6.1 is to apply a process called *homologous substitution*. To illustrate how it works, suppose that S is a connected oriented closed surface and that $f : S \to M$ is a map so that $f_*[S]$ equals α. To replace $f : S \to M$ with a homologous (R, ϵ)-panted subsurface $j : F \looparrowright M$, we endow S with a triangulation with a single vertex $*$, and assume that f has been homotoped so that $*$ is sent to a chosen basepoint of M, and that the 1-simplices of S are long geodesic segments, and that the 2-simplices of S are totally geodesic in M.

Following the construction of Ψ in Subsection 5.2, we may replace any (oriented) 1-simplex e with an (R,ϵ)-multicurve $L(e)$ as in the definition of Ψ, without passing to the (R,ϵ)-panted cobordism class. By convention, we define $L(\bar{e})$ to be $\overline{L(e)}$. Moreover, we may replace any 2-simplex σ with an (R,ϵ)-panted surface $F(\sigma)$, so that if $\partial\sigma$ is a cycle e_0, e_1, e_2, $F(\sigma)$ will be bounded by $L(e_0) \sqcup L(e_1) \sqcup L(e_2)$. The (R,ϵ)-panted surface $F(\sigma)$ can be obtained explicitly by the constructions in Lemmas 5.18, 5.19. Thus the (R,ϵ)-panted surface F can be obtained by naturally gluing the (R,ϵ)-panted surfaces $F(\sigma)$ along the (R,ϵ)-multicurves $L(e)$ on their boundary, according to the triangulation structure of S. Intuitively, it should follow from the Spine Principle (Lemma 4.14) that there are natural isomorphisms $H_2(F(\sigma),\partial F(\sigma);\mathbf{Z}) \cong H_2(\sigma,\partial\sigma;\mathbf{Z})$. Then a Mayer–Vietoris argument will imply that there is a natural isomorphism $H_2(F;\mathbf{Z}) \cong H_2(S;\mathbf{Z})$ that commutes with the homomorphisms $f_* : H_2(S;\mathbf{Z}) \to H_2(M;\mathbf{Z})$ and $j_* : H_2(F;\mathbf{Z}) \to H_2(M;\mathbf{Z})$. In other words, the (R,ϵ)-panted surface F is homologous to S in M, and hence represents α as desired. In practice, it is actually more convenient not to specify the homology class $\alpha \in H_2(M;\mathbf{Z})$. Instead, we consider a triangular presentation complex $f : (K,*) \to (M,*)$ (cf. Subsection 6.1) of $\pi_1(M,*)$ rather than the triangulated $(S,*) \to (M,*)$. Then a similar process of homologous subsitution will yield an (R,ϵ)-panted complex $j : \mathcal{K} \looparrowright M$ (a 2-complex obtained by gluing (R,ϵ)-panted surfaces along (R,ϵ)-multicurves on the boundary, cf. Subsection 6.2). In general, there will be a natural epimorphism $H_2(\mathcal{K};\mathbf{Z}) \to H_2(K;\mathbf{Z})$ that commutes with f_* and j_*. Because of the easy observation that $f_* : H_2(K;\mathbf{Z}) \to H_2(M;\mathbf{Z})$ is onto, $j_* : H_2(\mathcal{K};\mathbf{Z}) \to H_2(M;\mathbf{Z})$ will also be onto. In other words, any homology class $\alpha \in H_2(M;\mathbf{Z})$ comes from some $\tilde{\alpha} \in H_2(\mathcal{K};\mathbf{Z})$, so it is represented by some (R,ϵ)-panted surface obtained by a composition $F \to \mathcal{K} \to M$.

The rest of this section is devoted to the proof of Theorem 6.1. In Subsections 6.1, 6.2, we introduce some notations that we will adopt, namely, triangular presentation complexes and (R,ϵ)-panted complexes; Subsection 6.3 is the homologous substitution argument, which is core of the proof; Subsection 6.4 completes the proof of Theorem 6.1 by a brief summary.

Let M be an oriented closed 3-manifold. The bundle of special orthonormal frames of M will be denoted as $\mathrm{SO}(M)$. Fix an orthonormal frame $\mathbf{e} = (\vec{t}, \vec{n}, \vec{t} \times \vec{n})$ at a fixed basepoint $*$ of M, so the special orthonormal bundle $\mathrm{SO}(M)$ has a preferred basepoint $\mathbf{e}$.

6.1. **Triangular presentation complexes.** Recall that a *presentation* of a group G is a pair $(\mathcal{S},\mathcal{R})$, where $\mathcal{S}$ is a set of independent letters and $\mathcal{R}$ is a set of words in x and x^{-1} for $x \in \mathcal{S}$, such that the canonical quotient $\langle \mathcal{S} | \mathcal{R} \rangle$ is isomorphic to G. For a presentation of G, there is a naturally associated CW 2-complex K with a single vertex $*$ as the basepoint, called the *presentation complex*, such that the 1-cells are in correspondence with the generators, and the 2-cells are in correspondence with the relators. Hence the fundamental group $\pi_1(K,*)$ is naturally isomorphic to G. In fact, if $(X,*)$ is a pointed topological space, any homomorphism $\pi_1(K,*) \to \pi_1(X,*)$ can be realized by a map $(K,*) \to (X,*)$. Combinatorially, any 2-cell of K has polygonal boundary, and the number of edges is equal to the word length of the corresponding relator. A presentation is said to be *finite* if the sets of generators and relators are finite.

Lemma 6.3. *If $(K,*)$ is a presentation complex of $(M,*)$, then the presentation map*

$$f:\,(K,*)\to(M,*)$$

which induces the natural isomorphism on π_1 induces an epimorphism

$$f_*:\,H_2(K;\,\mathbf{Z})\to H_2(M;\,\mathbf{Z}).$$

Proof. This follows from the fact that M is aspherical. In fact, we may obtain an Eilenberg–MacLane space $K'\simeq K(\pi_1(M,*),1)$ by attaching to K cells of dimension greater than 2, and we may extend f to obtain a homotopy equivalence $f':(K',*)\to(M,*)$. This implies the surjectivity of the induced homomorphism f_* on the second homology. $\square$

Lemma 6.4. *If $(K,*)$ is a presentation complex of $(M,*)$, then the presentation map*

$$f:\,(K,*)\to(M,*)$$

lifts to a map

$$\hat{f}:\,(K,*)\to(\mathrm{SO}(M),\mathbf{e}).$$

Proof. This follows from the fact that $\pi_1(\mathrm{SO}(M))$ is the splitting extension of $\pi_1(M)$ by $\pi_1(\mathrm{SO}(3),I)\cong\mathbf{Z}_2$. Moreover, the homotopy classes of lifts of f are determined by the splittings, which are in bijection with $H^1(M;\mathbf{Z}_2)$ since $\pi_2(\mathrm{SO}(M),\mathbf{e})$ is trivial. $\square$

A presentation is said to be *triangular*, if the word length of the relators are at most 3. The complex $(K,*)$ associated to a triangular finite presentation is compact with only monogonal, bigonal, or triangular 2-cells, and we say that $(K,*)$ is *triangular* and *finite*. Note that any finite presentation gives rise to a triangular finite presentation, by adding a maximal collection of mutually non-intersecting diagonals to subdivide the 2-cells of K. Furthermore, if we assume that K minimizes the number of generators and the number of relators in the lexicographical order among triangular finite presentations of G, it is easy to see that K will not contain any monogonal 2-cells, and that any bigonal 2-cells of K will be attached to 1-cells representing elements of order 2.

For our application, it suffices to consider a specific triangular finite presentation of $\pi_1(M,*)$ associated to the triangular generating set guaranteed by Lemma 5.16 (2). In particular, the associated triangular finite presentation complex $(K,*)$ is a Δ-complex, and we denote the presentation map as

$$f:\,(K,*)\to(M,*),$$

which is unique up to homotopy relative to the basepoint.

6.2. **Panted complexes.** By a (topological) *panted complex* $\mathcal{K}$ we mean a compact CW space obtained from a finite disjoint union of circles by attaching finitely many disjoint pairs of pants via homeomorphisms from cuffs. Recording the pants decomposition of $\mathcal{K}$ as part of data, we will refer to the defining circles and pairs of pants as the *structure curves* and *structure pants* of $\mathcal{K}$. When a panted complex $\mathcal{K}$ is immersed into a closed hyperbolic 3-manifold M, we will say that the immersion is (R,ϵ)-*panted*, if the restriction to each pair of pants is (R,ϵ)-nearly regular up to homotopy, or ambiguously, we will say that $\mathcal{K}$ is an (R,ϵ)-*panted complex.* Note that any connected (R,ϵ)-panted surface is naturally an (R,ϵ)-panted complex. A *panted map* between two panted complexes is a map that sends each structural pair of pants homeomorphically onto a structural pair of pants of the target.

Lemma 6.5. *For any nontrivial element $\alpha \in H_2(\mathcal{K};\mathbf{Z})$, there is a closed oriented panted surface F and a panted map $F \to \mathcal{K}$, such that the fundamental class of F in $H_2(F;\mathbf{Z})$ is sent to α.*

Proof. Let $\mathcal{P}$ be the disjoint union of structure pants of $\mathcal{K}$, and $\mathcal{C}$ be the disjoint union of structure curves of $\mathcal{K}$. The long exact sequence of homology yields

$$\begin{tikzcd}
0 \arrow[r] & H_2(\mathcal{K};\mathbf{Z}) \arrow[r] & H_2(\mathcal{K},\mathcal{C};\mathbf{Z}) \arrow[r, "\partial_*"] & H_1(\mathcal{C}) \\
& & H_2(\mathcal{P},\partial\mathcal{P};\mathbf{Z}) \arrow[u, "\cong"] \arrow[r, "\partial_*"] & H_1(\partial\mathcal{P};\mathbf{Z}) \arrow[u, "\chi"']
\end{tikzcd}$$

where χ means the homomorphism induced by the *characteristic map* of the panted complex $\mathcal{K}$ that identifies the cuffs of structure pants with the structure curves. Now any element $\alpha \in H_2(\mathcal{K};\mathbf{Z})$ can be identified as an element α' of $H_2(\mathcal{P},\partial\mathcal{P};\mathbf{Z})$ in the kernel of $\chi\circ\partial_*$. Since $H_2(\mathcal{P},\partial\mathcal{P};\mathbf{Z})$ has a basis formed by the fundamental classes $[P]$ of the components $P \subset \mathcal{P}$, the element α' naturally yields a collection of copies of structure pants with suitable orientations, and $\chi(\partial_*[\alpha']) = 0$ implies that these copies of pants can be glued up along cuffs, resulting in a closed oriented panted surface F. The naturally induced panted map $F \to \mathcal{K}$ is as desired. □

6.3. **Homologous substitution.** Let $(K,*)$ be the CW complex associated to the triangular generating set guaranteed by Lemma 5.16 (2), (cf. Subsection 6.1), and $f:(K,*)\to(M,*)$ be the basepoint-preserving map associated to the presentation. Let τ_h be the conjugation provided by Lemma 5.16 (4). The presentation conjugated by τ_h induces an isomorphism $\tau_h\circ f_\sharp : \pi_1(K,*)\to\pi_1(M,*)$. We denote the corresponding basepoint-preserving map as

$$f^h : (K,*)\to(M,*).$$

It can be topologically obtained from f by pushing the image of $*\in K$ along the loop corresponding to $h^{-1}\in\pi_1(M,*)$.

Lemma 6.6. *There exists a CW complex K' obtained from K by attaching 1-cells, and a compact panted complex $\mathcal{K}$, and there exist maps*

$$(\mathcal{K},\emptyset)\longrightarrow(K',*)\longrightarrow(M,*)$$

satisfying the following.

- *The map $(K',*)\to(M,*)$ restricts to be the basepoint-preserving map*
$$f^h : (K,*)\to(M,*).$$
- *The map $\mathcal{K}\to K'$ induces an epimorphism*
$$H_2(\mathcal{K};\mathbf{Z}) \to H_2(K';\mathbf{Z}).$$
- *The composition $\mathcal{K}\to M$ yields an (R,ϵ)-panted complex.*

We prove Lemma 6.6 in the rest of this subsection.

6.3.1. *Construction of $\mathcal{K}$ and K'.* We construct $\mathcal{K}$, K' and maps $(\mathcal{K},\emptyset)\to(K',*)\to(M,*)$ following the construction of $\Psi:\pi_1(\mathrm{SO}(M),\mathbf{e})\to\mathbf{\Omega}_{R,\epsilon}$ in Subsection 5.2. Fix a lift of f^h into $\mathrm{SO}(M)$, denoted as

$$\hat{f}^h:(K,*)\to(\mathrm{SO}(M),\mathbf{e})$$

(Lemma 6.4). Note that by our assumption, K is a Δ-complex, so each 1-cell of K can be conveniently denoted by the element $g\in\pi_1(K,*)\cong\pi_1(M,*)$ that it represents, and each 2-cell of K can be conveniently denoted by a triangular relation $g_0g_1g_2=\mathrm{id}$ of three generators or their inverses. As K and f^h are provided from Lemma 5.16, for each 1-cell g of K, $\hat{f}^h(g)$ is a δ-sharp element. Thus there is an associated oriented ∂-framed segment $\mathfrak{s}_{\hat{f}^h(g)}$, or simply written as $\mathfrak{s}_{\hat{g}}$ in order to be consistent with the notation in Subsection 5.2.

Step 1. For each 1-cell g of K, fix a right-hand nearly regular tripod $\mathfrak{a}_0\vee\mathfrak{a}_1\vee\mathfrak{a}_2$ and an oriented ∂-framed segment $\mathfrak{b}$ as in the definition of Ψ. We construct a 2-dimensional Δ-complex $X(g)$ and a multcurve $L(g)$, together with maps

$$(L(g),\emptyset)\to(X(g),*)\to(M,*)$$

such that the composition yields an (R,ϵ)-multicurve $L(g)\looparrowright M$, which represents the (R,ϵ)-panted cobordism class $\Psi(\hat{g})\in\mathbf{\Omega}_{R,\epsilon}$, where $\hat{g}\in\pi_1(\mathrm{SO}(M),\mathbf{e})$ is the element $\hat{f}_\sharp(g)$.

The construction is as follows. Take the subcomplex $*\cup g$ of K; attach 1-cells $a_{i,i+1}$ for $i\in\mathbf{Z}_3$ and b, corresponding to the carrier segments of $\mathfrak{a}_{i,i+1}$ and $\mathfrak{b}$, respectively; attach a simplicial 2-cell with boundary the cycle $a_{01}a_{12}a_{20}$. The resulting 2-complex will be denoted as $X(g)$. There is a naturally induced map $(X(g),*)\to(M,*)$ extending $f^h|_{*\cup g}$. Let $L(g)=\overline{[\mathfrak{s}_{\hat{g}}\mathfrak{a}_{01}]}\sqcup\overline{[\mathfrak{s}_{\hat{g}}\mathfrak{a}_{12}]}\sqcup\overline{[\mathfrak{s}_{\hat{g}}\mathfrak{a}_{20}]}\sqcup\overline{[\mathfrak{s}_{\hat{g}}\mathfrak{b}]}\sqcup\overline{[\mathfrak{s}_{\hat{g}}\bar{\mathfrak{b}}]}$ be the multicurve of five components, which are the reduced cyclic concatenations of the defining oriented ∂-framed segments. The natural immersion $L(g)\looparrowright M$ can be homotoped to factor through $X(g)$ via a composition $(L(g),\emptyset)\to(X(g),*)\to(M,*)$ in the explicit way as indicated by the construction.

We define $L(g^{-1})$ to be $\overline{L(g)}$, and $X(g^{-1})$ for $X(g)$. Note that $L(g^{-1})$ is the representative of $\Phi(\hat{g}^{-1})\in\mathbf{\Omega}_{R,\epsilon}$ corresponding to the defining right-hand tripod $\mathfrak{a}_2^*\vee\mathfrak{a}_1^*\vee\mathfrak{a}_0^*$ and $\mathfrak{b}^*$ since the δ-sharp element $\hat{g}^{-1}$ is associated to $\bar{\mathfrak{s}}_{\hat{g}}^*$. Thus there are also maps

$$(L(g^{-1}),\emptyset)\to(X(g^{-1}),*)\to(M,*),$$

and the composition is homotopic to an (R,ϵ)-multicurve $L(g^{-1})\looparrowright M$, which represents $\Psi(\hat{g}^{-1})\in\mathbf{\Omega}_{R,\epsilon}$.

Step 2. For each simplicial 2-cell σ of K corresponding to a triangular relation $g_0g_1g_2=\mathrm{id}$, write

$$X(\partial\sigma)=X(g_0)\vee X(g_1)\vee X(g_2),$$

where the wedge is over the basepoint $*$, and

$$X(\sigma)=X(\partial\sigma)\cup\sigma$$

by identifying the copies of g_i in $X(g_i)$ and σ, and

$$L(\partial\sigma)=L(g_0)\sqcup L(g_1)\sqcup L(g_2).$$

We construct a 2-complex $Y(\sigma)$ obtained from $X(\sigma)$ by attaching 1-cells and a panted surface $F(\sigma)$ bounded by $L(\partial\sigma)$, together with maps

$$(F(\sigma), \emptyset) \to (Y(\sigma), *) \to (M, *),$$

such that the composition is homotopic to an (R, ϵ)-panted surface $F(\sigma) \looparrowright M$; moreover, the following diagram of maps commutes:

$$\begin{tikzcd}
F(\sigma) \arrow[r] & Y(\sigma) \\
L(\partial\sigma) \arrow[r] \arrow[u, "\cup"] & X(\partial\sigma). \arrow[u, "\cup"]
\end{tikzcd}$$

The construction is as follows. Since $[L(g_0)]_{R,\epsilon} + [L(g_1)]_{R,\epsilon} + [L(g_2)]_{R,\epsilon} = 0$ in $\mathbf{\Omega}_{R,\epsilon}$ (Lemma 5.19), there exists an (R, ϵ)-panted surface $F(\sigma)$ with $\partial F(\sigma)$ equal to $L(\partial\sigma)$. In fact, the construction of $F(\sigma)$ relies on Lemmas 5.18, 5.19. Checking the constructions there, we see that $F(\sigma)$ can be constructed based only on the Δ-complex $X(\sigma)$ and the map $X(\sigma) \to M$ induced from the constructed maps $X(g_i) \to M$ and the given map $f^h| : \sigma \to M$. Formally, we regard $X(\sigma)$ as a partially-Δ space over M where the partially-Δ structure is given by the entire Δ-complex $X(\sigma)$, and the map $X(\sigma) \to M$ is as described above (Definition 4.9) Then the construction of $F(\sigma)$ implies that there is a partially-Δ space X' over M which is an extension of $X(\sigma)$, such that there is a commutative diagram of maps:

$$\begin{tikzcd}
F(\sigma) \arrow[r] & X' \arrow[dr] & \\
L(\partial\sigma) \arrow[r] \arrow[u, "\cup"] & X(\partial\sigma) \arrow[r] \arrow[u, "\cup"] & M.
\end{tikzcd}$$

By the Spine Principle (Lemma 4.14), X' is 1-spined over $X(\partial\sigma)$, so we may replace X' with a CW complex $Y(\sigma)$, which is obtained from $X(\sigma)$ by attaching 1-cells. Then $F(\sigma)$, $Y(\sigma)$ and the involved maps from the diagram are as desired.

Step 3. Now we may naturally attach the disjoint union of all $F(\sigma)$ to the disjoint union of all $L(g)$ according to the attaching maps of K. The result is a panted complex $\mathcal{K}$. Similarly, we may attach the disjoint union of $Y(\sigma)$ to the disjoint union of all $X(g)$ by naturally identifying the copies of $X(g)$ (possibly marked by g^{-1}). The result is a CW 2-complex $Y(K)$ containing the subcomplex K. Moreover, there are naturally induced maps

$$(\mathcal{K}, \emptyset) \to (Y(K), *) \to (M, *).$$

The composition is homotopic to an (R, ϵ)-panted complex $\mathcal{K} \looparrowright M$; the restriction of $Y(K) \to M$ to K is f^h.

It is clear from the construction that $Y(K)$ deformation retracts relative to K to a CW subspace

$$K' \hookrightarrow Y(K),$$

which can be obtained from K by attaching 1-cells. Therefore, replacing $Y(K)$ with K', we obtain maps

$$(\mathcal{K}, \emptyset) \to (K', *) \to (M, *).$$

6.3.2. *Verifications.* To verify that $\mathcal{K}$, K' and the maps $(\mathcal{K},\emptyset) \to (K',*) \to (M,*)$ above are as desired, it suffices to prove that $\mathcal{K} \to K'$ is surjective on the second homology, as the other listed properties are obviously satisfied. By the construction, we may equivalently prove with $Y(K)$ instead of K'.

Write $\mathcal{C}$ for the disjoint union of all $L(g)$, and $X(K^{(1)})$ for the wedge of all $X(g)$ over $*$. There is a commutative diagram of homomorphisms

$$\begin{tikzcd}
0 \arrow[r] & H_2(\mathcal{K};\mathbf{Z}) \arrow[r] \arrow[d, "\phi"] & H_2(\mathcal{K},\mathcal{C};\mathbf{Z}) \arrow[r, "\partial_*"] \arrow[d, "\phi''"] & H_1(\mathcal{C};\mathbf{Z}) \arrow[d, "\phi'"] \\
0 \arrow[r] & H_2(Y(K);\mathbf{Z}) \arrow[r] & H_2(Y(K),X(K^{(1)});\mathbf{Z}) \arrow[r, "\partial_*"] & H_1(X(K^{(1)});\mathbf{Z}) \\
0 \arrow[r] & H_2(K;\mathbf{Z}) \arrow[r] \arrow[u, "\iota", "\cong"'] & H_2(K,K^{(1)};\mathbf{Z}) \arrow[r, "\partial_*"] \arrow[u, "\iota''", "\cong"'] & H_1(K^{(1)};\mathbf{Z}) \arrow[u, "\iota'", "\cup"']
\end{tikzcd}$$

where the rows are part of the long exact sequences of homology, and the homomorphisms ϕ, ϕ'', ϕ' are induced from the map $(\mathcal{K},\mathcal{L}) \to (Y(K), X(K^{(1)})$ of our construction, and the homomorphisms ι, ι'', ι' are induced from the natural inclusion $(K, K^{(1)}) \hookrightarrow (Y(K), X(K^{(1)}))$.

Write the quotient map defining $\mathcal{K}$ as

$$q : \bigsqcup_{\sigma \subset K} F(\sigma) \to \mathcal{K}.$$

Define a homomorphism

$$\psi'' : H_2(K,K^{(1)};\mathbf{Z}) \to H_2(\mathcal{K},\mathcal{C};\mathbf{Z})$$

by assigning

$$\psi''([\sigma]) = q_*[F(\sigma)]$$

where $[F(\sigma)] \in H_2(F(\sigma), L(\partial\sigma);\mathbf{Z})$ is the fundamental class. Define a homomorphism

$$\psi' : H_1(K^{(1)};\mathbf{Z}) \to H_1(\mathcal{C};\mathbf{Z})$$

by assigning

$$\psi'([g]) = q_*[L(g)]$$

where $[L(g)] \in H_2(L(g);\mathbf{Z})$ is the fundamental class. There is a commutative diagram

$$\begin{tikzcd}
H_2(\mathcal{K},\mathcal{C};\mathbf{Z}) \arrow[r, "\partial_*"] & H_1(\mathcal{C};\mathbf{Z}) \\
H_2(K,K^{(1)};\mathbf{Z}) \arrow[r, "\partial_*"] \arrow[u, "\psi''"] & H_1(K^{(1)};\mathbf{Z}). \arrow[u, "\psi'"']
\end{tikzcd}$$

Lemma 6.7. *With the notations above, $\phi' \circ \psi' = \iota'$ and $\phi'' \circ \psi'' = \iota''$.*

Proof. It is straightforward to check that $\phi'(\psi'([g])) = \iota'([g])$ for any 1-cell g of K, by the construction of $L(g)$. Since $H_1(K^{(1)};\mathbf{Z})$ is freely generated by $[g]$ where g runs over all 1-cells of K, we see that $\phi' \circ \psi' = \iota'$.

We claim that for any 2-cell σ of K, $\phi''(\psi''([\sigma])) = \iota''([\sigma])$. In fact, applying the discussion to the special case when K consists of a single 2-simplex σ together with the 1-skeleton $\partial\sigma$, we obtain the commutative diagrams

$$\begin{tikzcd}
0 \arrow[r] & H_2(F(\sigma), L(\partial\sigma); \mathbf{Z}) \arrow[r, "\partial_*"] \arrow[d, "\phi''_\sigma"] & H_1(L(\partial\sigma); \mathbf{Z}) \arrow[d, "\phi'_\sigma"] \\
0 \arrow[r] & H_2(Y(\sigma), X(\partial\sigma); \mathbf{Z}) \arrow[r, "\partial_*"] & H_1(X(\partial\sigma); \mathbf{Z}) \\
0 \arrow[r] & H_2(\sigma, \partial\sigma; \mathbf{Z}) \arrow[r, "\partial_*"] \arrow[u, "\iota''_\sigma", "\cong"'] & H_1(\partial\sigma; \mathbf{Z}) \arrow[u, "\iota'_\sigma", "\cup"']
\end{tikzcd}$$

where the rows are exact sequences, and

$$\begin{tikzcd}
H_2(F(\sigma), L(\partial\sigma); \mathbf{Z}) \arrow[r, "\partial_*"] & H_1(L(\partial\sigma); \mathbf{Z}) \\
H_2(\sigma, \partial\sigma; \mathbf{Z}) \arrow[r, "\partial_*"] \arrow[u, "\psi''_\sigma"] & H_1(\partial\sigma; \mathbf{Z}). \arrow[u, "\psi'_\sigma"]
\end{tikzcd}$$

Because

$$\partial_*(\phi''_\sigma \circ \psi''_\sigma([\sigma])) = \phi'_\sigma \circ \psi'_\sigma(\partial_*([\sigma])) = \iota'_\sigma(\partial_*([\sigma])) = \partial_*(\iota''_\sigma[\sigma]),$$

the injectivity of ∂_* in this case implies that

$$\phi''_\sigma \circ \psi''_\sigma([\sigma]) = \iota''_\sigma([\sigma]).$$

By naturality of Mayer–Vietoris sequences, it follows that

$$\phi'' \circ \psi''([\sigma]) = \iota''([\sigma]),$$

as claimed.

Since $H_2(K, K^{(1)}; \mathbf{Z})$ is freely generated by $[\sigma]$ where σ runs over all 2-cells of K, we conclude that $\phi'' \circ \psi'' = \iota''$. □

Now an easy diagram chase will show the surjectivity of

$$\phi : H_2(\mathcal{K}; \mathbf{Z}) \to H_2(Y(K); \mathbf{Z}).$$

In fact, we may identify $H_2(\mathcal{K}; \mathbf{Z})$ and $H_2(Y(K))$ as kernels of ∂_* in $H_2(\mathcal{K}, \mathcal{C}; \mathbf{Z})$ and $H_2(Y(K), X(K^{(1)}; \mathbf{Z})$, respectively. If $\alpha \in H_2(Y(K), X(K^{(1)}; \mathbf{Z})$ vanishes under ∂_*, $\beta = \psi'' \circ (\iota'')^{-1}(\alpha)$ in $H_2(\mathcal{K}, \mathcal{C}; \mathbf{Z})$ also vanishes under ∂_*, using the injectivity of ι'. Moreover, $\phi''(\beta) = \alpha$ by Lemma 6.7. This implies that ϕ'' is surjective between the kernels of ∂_*, or in other words, ϕ is surjective.

This completes the verification, and hence completes the proof of Lemma 6.6.

6.4. **Proof of Theorem 6.1.** We summarize the proof of Theorem 6.1 as follows. Let M be a closed oriented hyperbolic 3-manifold. By Lemma 6.6, there exists a CW complex K' obtained from a presentation complex K of $\pi_1(M, *)$ by attaching 1-cells, and a compact panted complex $\mathcal{K}$, together with maps

$$(\mathcal{K}, \emptyset) \longrightarrow (K', *) \longrightarrow (M, *),$$

satisfying the listed properties. In particular, the composed map $\mathcal{K} \to M$ is an (R, ϵ)-panted complex in M. It is surjective on the second homology by Lemmas 6.6 and 6.3. In other words, any element α of $H_2(M; \mathbf{Z})$ can be lifted to be an element $\tilde{\alpha}$ of $H_2(\mathcal{K};\mathbf{Z})$. Moreover, $\tilde{\alpha}$ can be represented by a panted surface via a panted map $F \to \mathcal{K}$ (Lemma 6.5), which yields an (R,ϵ)-panted surface via the map $\mathcal{K} \to M$. Therefore, the (R,ϵ)-panted surface $F \looparrowright M$ is an representative of α as desired. This completes the proof of Theorem 6.1.

7. Panted connectedness between curves

In this section, we show that in a closed hyperbolic 3-manifold M, the collection of (R,ϵ)-curves $\mathbf{\Gamma}_{R,\epsilon}$ are (R,ϵ)-*panted connected* in the sense of the following Proposition 7.1. This will be applied to show that an ubiquitous measure of (R,ϵ)-pants $\mu \in \mathcal{M}(\mathbf{\Pi}_{R,\epsilon})$ is irreducible in the proof of Theorem 2.10 (Subsection 8.2).

Proposition 7.1. *Let M be a closed hyperbolic 3-manifold. For all universally small positive ϵ and sufficiently large positive R depending on M and ϵ, the following holds. For any two curves $\gamma_0,\gamma_1 \in \mathbf{\Gamma}_{R,\epsilon}$, there exists a connected (R,ϵ)-panted subsurface $j: F \looparrowright M$, and ∂F contains two components homotopic to γ_0 and γ_1, respectively.*

The proof of Proposition 7.1 follows from an easy construction using the Connection Principle (Lemma 4.13). However, we leave it as a separate section as its statement has certain independent geometric interest.

Proof. We say that two curves $\gamma_0,\gamma_1 \in \mathbf{\Gamma}_{R,\epsilon}$ are (R,ϵ)-*panted connected* if there exists a connected (R,ϵ)-panted subsurface $j: F \looparrowright M$ as in the conclusion of the proposition. This defines an equivalence relation between curves in $\mathbf{\Gamma}_{R,\epsilon}$. Suppose that ϵ universally small and R is sufficiently large, for instance, as guaranteed by Lemma 5.16.

By splitting (Construction 4.15), every curve in $\mathbf{\Gamma}_{R,\epsilon}$ is (R,ϵ)-panted connected to a curve in $\mathbf{\Gamma}_{R,\frac{\epsilon}{10000}}$. It suffices to show that any two curves in $\mathbf{\Gamma}_{R,\frac{\epsilon}{10000}}$ are (R,ϵ)-panted connected.

Let γ_0,γ_1 are any two curves in $\mathbf{\Gamma}_{R,\frac{\epsilon}{10000}}$. By interpolating a pair of points of distance $\frac{R}{4}$ on γ with suitably assigned normal vectors, we may decompose γ_0 into a $(1,\frac{\epsilon}{10000})$-tame bigon $[\mathfrak{a}_-\mathfrak{a}_+]$ with $\mathfrak{a}_-$ of length $\frac{R}{4}$. Similarly, we decompose γ_1 into a $(1,\frac{\epsilon}{10000})$-tame bigon $[\mathfrak{b}_-\mathfrak{b}_+]$ with $\mathfrak{b}_+$ of length $\frac{R}{4}$. By the Connection Principle (Lemma 4.13), there exist oriented ∂-framed segments $\mathfrak{s}$ and $\mathfrak{t}$ of length $(\frac{\epsilon}{10000})$-close to $\frac{R}{4}$ and phase $(\frac{\epsilon}{10000})$-close to 0, so that $\mathfrak{a}_-,\mathfrak{s},\mathfrak{b}_+,\mathfrak{t}$ form a $(100,\frac{\epsilon}{100})$-tame cycle. Let γ' be the reduced cyclic concatenation $[\mathfrak{a}_-\mathfrak{s}\mathfrak{b}_+\mathfrak{t}]$, then $\gamma' \in \mathbf{\Gamma}_{R,\frac{\epsilon}{100}}$ by the Length and Phase Formula (Lemma 4.7). Furthermore, $[\mathfrak{a}_-\mathfrak{a}_+]$ and $[\mathfrak{a}_-(\mathfrak{s}\mathfrak{b}_+\mathfrak{t})]$ form an $(\frac{\epsilon}{100})$-swap pair, and $[\mathfrak{b}_-\mathfrak{b}_+]$ and $[(\mathfrak{s}\mathfrak{b}_-\mathfrak{t})\mathfrak{b}_+]$ form an $(\frac{\epsilon}{100})$-swap pair (Definition 4.16). By swapping (Construction 4.17), we see that γ_0 is (R,ϵ)-panted connected with γ', and that γ' is (R,ϵ)-panted connected with γ_1. Thus γ_0 and γ_1 are (R,ϵ)-panted connected. This completes the proof. □

An application of Proposition 7.1 is that we can replace any (R,ϵ)-panted surface with a connected one with the same boundary without changing its relative homology class.

Lemma 7.2. *If M is an oriented closed hyperbolic 3-manifold and (R,ϵ) are constants so that $\mathbf{\Gamma}_{R,\epsilon}$ is (R,ϵ)-panted connected in the sense of Proposition 7.1, then for any oriented compact (R,ϵ)-panted subsurface $j: F \looparrowright M$ bounded by an (R,ϵ)-multicurve L, there exists an oriented compact connected (R,ϵ)-panted subsurface $j': F' \looparrowright M$ bounded by L such that $j_*[F]$ equals $j'_*[F']$ in $H_2(M,L;\mathbf{Z})$.*

Proof. By induction, it suffices to prove the case when F has only two components F_1 and F_2.

Without loss of generality, we may assume that each F_i has a nonseparating glued cuff c_i of its pants structure. In fact, this is automatically true if F_i is closed. If some F_i is has a nonempty boundary component c, there is an (R,ϵ)-pants P with a boundary component c' homotopic to c. Take two oppositely oriented copies $P_\pm$ of P and glue up along the two cuffs other that $c'_\pm$. Denote the resulting (R,ϵ)-panted surface as Q, so ∂Q is $c'_+ \sqcup c'_-$. If P_+ is has the same orientation as that of P, we glue up Q and F_i identifying c'_- with $\bar{c}$. The resulting (R,ϵ)-panted surface F'_i has the same boundary as that of F_i up to homotopy, and F'_i is homologous to F_i in M relative to their boundary. After replacing F_i with F'_i, each component of F has a nonseparating glued cuff c as claimed.

Now suppose that $c_i \subset F_i$ is a nonseparating glued cuff for each F_i. Let E_i be the (R,ϵ)-panted surface obtained by cutting F_i along c_i, so ∂E_i has two components c_{i+} and c_{i-} homotopic to c and its orientation reversal, respectively. By Proposition 7.1, there exists an (R,ϵ)-panted surface W with boundary, so that there are two components d_1 and d_2 of ∂W homotopic to c_1 and c_2, respectively. Take a copy W_+ of W and a copy W_- of the orientation reversal of W. Denote the components of $\partial W_\pm$ corresponding to d_i as $d_{i\pm}$. We glue up $W_\pm$ along the opposite pairs of boundary components other than $d_{i\pm}$, and glue them with E_i by identifying $d_{i\pm}$ with $c_{i\mp}$, respectively. The resulting (R,ϵ)-panted surface F' has the same boundary as that of F, and F' is homologous to F in M relative to their boundary. Since F' is connected by the construction, we see that F' is a connected (R,ϵ)-panted surface as desired. $\square$

8. Bounded quasi-Fuchsian subsurfaces

In this section, we prove Theorem 2.10 by applying Theorem 5.2 and Propositions 6.1, 7.1 (Subsection 8.2); we prove Theorem 1.3 by applying Theorems 2.9 and 2.10, following the methodology of Section 2; we prove Theorem 1.4 by applying Theorems 5.2, 6.1 and Proposition 7.1. Subsection 8.1 contains some remarks about the formulation of Theorem 1.3

8.1. **Description of the problem.**

Generally speaking, the construction problem of geometrically finite surface subgroups in a Kleinian group is concerned about the existence of such surface subgroups subject to various conditions. For instance, one may ask about the existence specifying the boundary of the subsurface, or requiring it to represent some homology class. We will allow ourselves to pass to finite covers of the designated boundary loops or positive multiples of the homology class, and we will only look for immersed subsurfaces rather than embedded ones. In many motivating applications, these are the usual assumptions under similar circumstances. On the other hand, we will only consider closed hyperbolic 3-manifolds. This is due to the restriction of our current techniques, and it would certainly be interesting to study the construction problem for cusped hyperbolic 3-manifolds.

Let M be a closed hyperbolic 3-manifold. Suppose $\gamma_1, \cdots, \gamma_m$ are π_1-injectively immersed loops $\gamma_i : S^1 \looparrowright M$, and let $L : \sqcup^m S^1 \looparrowright M$ be their disjoint union:

$$L = \gamma_1 \sqcup \cdots \sqcup \gamma_m.$$

The relative homology of the mapping cone $H_*(M \cup_L \vee^m D^2, \vee^m D^2; \mathbf{Z})$ is well defined, depending only on the free homotopy classes of the loops. Without loss of generality, we may hence assume that they are embedded. Identifying L with its image, the relative homology of the cone becomes $H_*(M, L; \mathbf{Z})$ by excision. If

$$j : (F, \partial F) \looparrowright (M, L)$$

is an immersion of an oriented compact surface F, then F naturally represents a relative homology class

$$j_*[F, \partial F] \in H_2(M, L; \mathbf{Q}),$$

where the rational coefficient is taken since we are not interested in the torsion. Moreover, it is clear that the restriction of j on ∂F is a covering if F has no disk or sphere component and if j is π_1-injective on each component of F.

With the notations above, the construction problem that we are concerned about in this paper is the following:

Question 8.1. For any element $\alpha \in H_2(M, L; \mathbf{Q})$, is there a positive multiple of α represented by a connected oriented surface F and a π_1-injective quasi-Fuchsian immersion j?

The answer is affirmative as stated in Theorem 1.3.

8.2. **Proof of Theorem 2.10.** We summarize the proof of Theorem 2.10. Note that the 'furthermore' part follows immediately from the main statements because the boundary homomorphism ∂ is integral coefficiented over the natural basis of $\mathcal{M}(\mathbf{\Pi}_{R,\epsilon})$ and $\mathcal{M}(\mathbf{\Gamma}_{R,\epsilon})$.

To prove the first statement, it suffices to find an (R, ϵ)-panted surface representing any class $\alpha \in H_2(M, |\mathcal{L}|; \mathbf{Q})$ up to a scalar multiple. We may assume without loss of generality that α is integral. Under the boundary homomorphism

$$\partial : H_2(M, |\mathcal{L}|; \mathbf{Z}) \to H_1(|\mathcal{L}|; \mathbf{Z}),$$

$\partial\alpha$ can be represented by a multicurve L all of whose components are carried by components of $|\mathcal{L}|$. Note that there is a commutative diagram of homomorphisms

$$\begin{tikzcd} H_2(\mathrm{SO}(M); \mathbf{Z}) \arrow[r, "\partial"] \arrow[d] & H_1(\mathrm{SO}(M); \mathbf{Z}) \arrow[d] \\ H_2(M; \mathbf{Z}) \arrow[r, "\partial"] & H_1(M; \mathbf{Z}) \end{tikzcd}$$

Under the composition of the canonical isomorphism $\mathbf{\Omega}_{R,\epsilon} \in H_1(\mathrm{SO}(M); \mathbf{Z})$ (Theorem 5.2) and the projection $H_1(\mathrm{SO}(M); \mathbf{Z}) \to H_1(M; \mathbf{Z})$, $[L]_{R,\epsilon}$ is sent to 0 since L is a boundary. This means $2[L]_{R,\epsilon}$ is 0 in $\mathbf{\Omega}_{R,\epsilon}$, so there is an (R, ϵ)-panted surface F_L with boundary $2L$. Let

$$\beta = 2\alpha - [F],$$

in $H_2(M, |\mathcal{L}|; \mathbf{Z})$. Since $\partial\beta = 0$ in $H_1(|\mathcal{L}|; \mathbf{Z})$, β can be regarded as an element of $H_2(M; \mathbf{Z})$. By Theorem 6.1, β is represented by an (R, ϵ)-panted surface F_β. Therefore 2α is represented by the (R, ϵ)-panted surface $F_L \sqcup F_\beta$.

To prove the second statement, recall that by [KM1, Theorem 3.4], there is a measure of (R, ϵ)-nearly regular pants $\mu \in \mathcal{M}(\mathbf{\Pi}_{R,\epsilon})$ which is $(R, \frac{\epsilon}{2})$-nearly evenly footed (Definition 2.8), such that $\partial\mu$ is positive on any curve $\gamma \in \mathbf{\Gamma}_{R,\epsilon}$. Let μ_1 be the sum $\mu + \bar{\mu}$, where $\bar{\mu}(\{\Pi\}) = \mu(\{\bar{\Pi}\})$, so $\mu_1 \in \mathcal{BM}(\mathbf{\Pi}_{R,\epsilon})$. Let $\mu_2 \in \mathcal{BM}(\mathbf{\Pi}_{R,\epsilon})$ be the sum of all Π for all $\Pi \in \mathbf{\Pi}_{R,\epsilon}$. For some sufficiently large positive integer N, let

$$\mu_0 = N\mu_1 + \mu_2.$$

Then $\mu_0 \in \mathcal{BM}_{R,\epsilon}$ is ubiquitous, (R,ϵ)-nearly evenly footed. It is certainly rich as $\partial^\flat \mu_0$ vanishes in this case. By Proposition 7.1, μ_0 is irreducible. In fact, suppose otherwise that μ_0 were the sum $\mu'+\mu''$, such that $\partial\mu'$ and $\partial\mu''$ have disjoint supports on $\mathbf{\Gamma}_{R,\epsilon}$. Then for $\gamma',\gamma'' \in \mathbf{\Gamma}_{R,\epsilon}$ lying in the supports of $\partial\mu'$ and $\partial\mu''$ respectively, γ' and γ'' cannot appear simultaneously on an (R,ϵ)-panted surface F whose pants are all from the support of μ_0. However, μ_0 is ubiquitous so the support of μ_0 is $\mathbf{\Pi}_{R,\epsilon}$. Thus we reach a contradiction since γ' and γ'' should be (R,ϵ)-panted connected in the sense of the conclusion of Proposition 7.1. Therefore, $\mu_0 \in \mathcal{BM}(\mathbf{\Pi}_{R,\epsilon})$ is a measure as claimed in the second statement of Theorem 2.10.

It remains to prove that if $\xi \in \mathcal{ZM}(\mathbf{\Pi}_{R,\epsilon}, |\mathcal{L}|)$, then for some sufficiently large integer m,

$$\xi' = \xi + m\mu_0$$

in $\mathcal{ZM}(\mathbf{\Pi}_{R,\epsilon}, |\mathcal{L}|)$ is ubiquitous, irreducible, (R,ϵ)-nearly evenly footed, and rich. Technically, one may assume here that μ_0 is $(R,\frac{\epsilon}{2})$, and such a μ_0 can be achieved by using $\frac{\epsilon}{2}$ instead of ϵ in the construciton above. It is clear that ξ' is ubiquitous and irreducible as so is μ_0. On the other hand, μ_0 being ubiquitous also implies that $\partial^\sharp\mu_0$ is positive on $\mathscr{N}_\gamma$ for any curve γ, so when m is sufficiently large, the normalization of the measure $\partial^\sharp\xi'$ can be $(\frac{\epsilon}{2R})$-equivalent to the normalization of $\partial^\sharp\mu_0$ restricted to $\mathscr{N}_\gamma$, for all $\gamma \in \mathbf{\Gamma}_{R,\epsilon}$. Hence ξ' is (R,ϵ)-nearly evenly footed. It is also clear that ξ' is rich if m is so large that $\partial\xi'(\{\gamma\})$ is less than, for instance, $\frac{m}{3}\partial\mu_0(\{\gamma\})$ for all $\gamma \in \mathbf{\Gamma}_{R,\epsilon}$. This completes the proof of the second statement, and hence the proof of Theorem 2.10.

8.3. **Proof of Theorem 1.3.** We derive Theorem 1.3 from Theorems 2.10 and 2.9 as follows.

Let M be a closed hyperbolic 3-manifold, and $L \subset M$ be the union of finitely many mutually disjoint, π_1-injectively embedded loops.

Lemma 8.2. *If Theorem 1.3 is true when M is orientable, it holds in the general case as well.*

Proof. Assume that Theorem 1.3 is true for the orientable case. If M is not orientable, we consider an orientable 2-fold cover $\kappa : \tilde{M} \to M$, and let $\tilde{L}$ be the preimage of L. By excision, $H_2(M,L;\mathbf{Q})$ is isomorphic to $H_2(N;\mathbf{Q})$ where N is the compact 3-manifold with tori boundary obtained from M with L removed, and similarly, $H_2(\tilde{M},\tilde{L};\mathbf{Q})$ is isomorphic to $H_2(\tilde{N};\mathbf{Q})$ where $\tilde{N}$ is the 2-fold cover of N obtained from $\tilde{M}$ with $\tilde{L}$ removed. Because $H_2(\tilde{N};\mathbf{Q})$ surjects $H_2(N;\mathbf{Q})$ under the covering, the same holds for $H_2(\tilde{M},\tilde{L};\mathbf{Q})$ and $H_2(M,L;\mathbf{Q})$. Therefore, for any element $\alpha \in H_2(M,L;\mathbf{Q})$, we may take an element $\tilde{\alpha} \in H_2(\tilde{M},\tilde{L};\mathbf{Q})$ which is projected to be α. The orientable case implies that a positive integral multiple of $\tilde{\alpha}$ can be represented by an orientable compact π_1-injectively immersed quasi-Fuchsian surface $\tilde{j} : (F,\partial F) \looparrowright (\tilde{M},\tilde{L})$, so $\kappa\circ\tilde{j} : (F,\partial F) \looparrowright (M,L)$ represents a positive integral multiple of α as desired. □

By Lemma 8.2, we may assume without loss of generality that M is an oriented closed hyperbolic 3-manifold.

Lemma 8.3. *With the notations above, for any constant $\epsilon > 0$, and for any constant $\hat{R} > 0$, there exist some $R > \hat{R}$, such that every component of L has a finite cyclic cover which is homotopic to a curve of $\mathbf{\Gamma}_{R,\epsilon}$.*

Proof. Let $c_1, \cdots, c_n$ be the components of L. Let $\ell_k \in (0, +\infty)$ be the length of c_k, and $\varphi_k \in \mathbf{R} / 2\pi\mathbf{Z}$ be the phase of c_k. Then for any positive integer m_k, the length and phase of the m_k-fold cyclic cover of c_k are $m_k\ell_k$ and $m_k\varphi_i$, respectively. We must show that for any positive constant ϵ and $\hat{R}$ as above, there exists $R > \hat{R}$ and positive integers $m_1, \cdots, m_n$, such that the complex numbers $m_k(\ell_k + \varphi_k \mathrm{i})$ are all ϵ close to the real number R.

Consider the complex torus T obtained by $\mathbf{C}^n$ modulo the lattice spanned over $\mathbf{Z}$ by $2\pi \mathrm{i}\, \vec{e}_k$, and $(\ell_k + \varphi_k \mathrm{i})\, \vec{e}_k$, for $k = 1, \cdots, n$, where $\vec{e}_1, \cdots, \vec{e}_n$ denotes the standard basis. The origin of $\mathbf{C}^n$ is projected to be the basepoint $*$ of T. The diagonal ray

$$\vec{v} : [0, +\infty) \to \mathbf{C}^n$$

defined by $\vec{v}(r) = r(\vec{e}_1 + \cdots + \vec{e}_n)$ is projected to be a ray

$$v : [0, +\infty) \to T$$

start from $v(0) = *$. Then to find R and $m_1, \cdots, m_n$ as above, it suffices to show that for any positive constants ϵ and $\hat{R}$, there exists $R > \hat{R}$ so that $v(R)$ is ϵ-close to $*$. However, this is a well known fact, which can be derived easily from dynamics of geodesic flow on a Euclidean torus, so we omit the details. □

By Lemma 8.3, we may choose universally small positive ϵ and sufficiently large positive R, so that there is a finite cover $\tilde{L}$ of L with all components homotopic to curves of $\mathbf{\Gamma}_{R,\epsilon}$. We may also assume that Theorems 2.9, 2.10 can be applied with respect to ϵ and R. Let $|\mathcal{L}| \subset |\mathbf{\Gamma}_{R,\epsilon}|$ be all the unoriented curve classes which are realized by some component of $\tilde{L}$.

Consider the essential case when $\tilde{L}$ does not have two components that are homotopic to each other up to the orientation reversion. In this case, $|\mathcal{L}|$ is in correspondence with components of $\tilde{L}$. The relative homology $H_2(M, |\mathcal{L}|; \mathbf{Z})$ defined in Subsection 2.3 is naturally isomorphic to $H_2(M, \tilde{L}; \mathbf{Z})$ (cf. Subsection 8.1), and that there is a natural homomorphism $H_2(M, \tilde{L}; \mathbf{Z}) \to H_2(M, L; \mathbf{Z})$ induced by the covering $\tilde{L} \to L$. We may realize any homology class $\alpha \in H_2(M, L; \mathbf{Z})$ as follows. Passing to some large multiple of α, we may lift α to be an element

$$\tilde{\alpha} \in H_2(M, |\mathcal{L}|; \mathbf{Z}).$$

Then by Theorem 2.10 (1), there exists a measure $\mu \in \mathcal{ZM}(\mathbf{\Pi}_{R,\epsilon}, |\mathcal{L}|)$ which is mapped onto $\tilde{\alpha}$, and by Theorem 2.10 (2), we may assume that μ is ubiquitous, irreducible, (R, ϵ)-nearly evenly footed, and rich. Then there exists an oriented, connected, compact, π_1-injectively immersed quasi-Fuchsian subsurface:

$$j : F \looparrowright M,$$

which is (R, ϵ)-nearly regularly panted subordinate to a positive integral multiple of μ, by Theorem 2.9. It is clear from the construction that (up to homotopy) $[F]$ can be regarded as an element in $H_2(M, L; \mathbf{Z})$ representing a positive multiple of α.

From the essential case above one can derive the general case when $\tilde{L}$ may have components that are homotopic to each other up to the orientation reversion. We will only sketch the argument, as the tricks used below should be easy and less important.

Let $\tilde{L}_0$ be a maximal subunion of components of $\tilde{L}$ so that $\tilde{L}_0$ does not have two components that are homotopic to each other up to the orientation reversion. As in the essential case, $|\mathcal{L}|$ is in correspondence with components of $\tilde{L}$, so $H_2(M, |\mathcal{L}|; \mathbf{Z})$ can be realized as $H_2(M, \tilde{L}_0; \mathbf{Z})$. There is also a natural homomorphism $H_2(M, \tilde{L}_0; \mathbf{Z}) \to H_2(M, \tilde{L}; \mathbf{Z})$ induced by the inclusion $\tilde{L}_0 \subset \tilde{L}$. It is easy to verify that $H_2(M, \tilde{L}; \mathbf{Z})$ is generated by $H_2(M, \tilde{L}_0; \mathbf{Z})$ together with all $[A]$ where $A \looparrowright M$ is any annulus between two homotopic components of $\tilde{L}_0$. If the boundary of any such annulus A is a curve $\gamma \in \mathbf{\Gamma}_{R,\epsilon}$, we may take a null-homologous (R,ϵ)-panted surface E which has a glued cuff c homotopic to γ. Such an E can be obtained, for example, by gluing pairs of pants prescribed by an ubiquitous $\mu_0 \in \mathcal{BM}(\mathbf{\Pi}_{R,\epsilon})$. Cutting E along c and homotoping the two boundary components to the two components of ∂A gives rise to an (R,ϵ)-panted surface E_A such that $[E_A]$ equals $[A]$ as in $H_2(M, \tilde{L}; \mathbf{Z})$. Now for any homology class $\alpha \in H_2(M, L; \mathbf{Q})$, we may pass to a positive integral multiple of α and lift it as $\tilde{\alpha} \in H_2(M, \tilde{L}; \mathbf{Q})$. From the above $\tilde{\alpha}$ equals the sum of some $\tilde{\alpha}_0 \in H_1(M, \tilde{L}_0; \mathbf{Q})$ together with some positive rational multiple $[E_A]$. Possibly after passing to further positive multiple, the essential case implies that $\tilde{\alpha}_0$ can be represented by an (R,ϵ)-panted surface F_0 with ∂F_0 mapped to $\tilde{L}_0$. We may also represent the difference term $\tilde{\alpha} - \tilde{\alpha}_0$ by a union of (R,ϵ)-panted surfaces $E_{A_1}, \cdots, E_{A_s}$ as above. Thus the union $F = F_0 \cup E_{A_1} \cup \cdots \cup E_{A_s}$ is an (R,ϵ)-panted surface representing $\tilde{\alpha}$ in $H_2(M, \tilde{L}; \mathbf{Q})$. We may assume that F_0 and each $E_{A_1}, \cdots, E_{A_s}$ to be obtained by a (R,ϵ)-nearly unit-shearing gluing of a collection of (R,ϵ)-pants prescribed by a ubiquitous, irreducible, (R,ϵ)-nearly evenly footed, and rich measure, then modifying the gluing by a hybriding argument (cf. Lemma 3.9) will yield a connected π_1-injectively immersed quasi-Fuchsian surface F', which still represents $\tilde{\alpha}$. This completes the argument of the general case, and hence completes the proof of Theorem 1.3.

8.4. **Proof of Theorem 1.4.** We derive Theorem 1.4 from Theorems 5.2, 6.1 and Proposition 7.1 as follows. We point out that as those results rely only on the constructions of ∂-framed segments (Section 4), the input from dynamics necessary for the proof is only the mixing property of the frame flow on the closed hyperbolic 3-manifold M, but not the fact that the mixing rate is exponential.

The invariant σ can be defined for any null-homologous (R,ϵ)-multicurve as

$$\sigma(L) = \Phi([L]_{R,\epsilon}),$$

where $\Phi : \mathbf{\Omega}_{R,\epsilon} \to H_1(\mathrm{SO}(M); \mathbf{Z})$ is the canonical isomorphism by Theorem 5.2. Note that since L is null homologous, $\sigma(L)$ lies in the canonical submodule of $H_1(\mathrm{SO}(M); \mathbf{Z})$ coming from the center $\mathbf{Z}_2$ of $\pi_1(\mathrm{SO}(M))$. Hence $\sigma(L)$ has well defined value in $\mathbf{Z}_2$.

It follows that $\sigma(L_1 \sqcup L_2)$ equals $\sigma(L_1) + \sigma(L_2)$ because Φ is a homomorphism. It also follows that $\sigma(L)$ vanishes if and only if L is the boundary of an (R,ϵ)-panted subsurface F of M. Moreover, Proposition 7.1 implies that we may assume F to be connected (Lemma 7.2).

It remains to show the last statement in the conclusion of Theorem 1.4. Let L be an (R,ϵ)-multicurve with vanishing $\sigma(L)$. Fix an (R,ϵ)-panted surface F_0 bounded by L, and denote the relative homology class of F_0 as $\alpha_0 \in H_2(M, L; \mathbf{Z})$. For any homology class $\alpha \in H_2(M, L; \mathbf{Z})$ with $\partial\alpha$ equal to $[L] \in H_1(L; \mathbf{Z})$, there exists some $\beta \in H_2(M; \mathbf{Z})$ such that $\alpha = \alpha_0 + \beta$. By Theorem 6.1, β can also be represented by a closed (R,ϵ)-panted subsurface E. Thus we may take F to be $F_0 \sqcup E$ so that F is an oriented compact (R,ϵ)-panted subsurface of M representing α. By applying Lemma 7.2 again, we may substitute F with another oriented compact connected

(R, ϵ)-panted subsurface F' representing α, as desired. This completes the proof of Theorem 1.4.

9. Conclusions

In conclusion, we are able to construct homologically interesting connected immersed nearly geodesic nearly regularly panted subsurfaces in a closed hyperbolic 3-manifold M by knowning a finite presentation of its fundamental group. The existence of plenty of nearly regular pairs of pants in M is a consequence of the exponential mixing property of the frame flow, and is the essential reason for the connectedness and the π_1-injective quasi-Fuchsian property. Even if we did not know the mixing rate, the Connection Principle can still be deduced from the mixing property, so homologically interesting connected (R, ϵ)-panted subsurfaces can still be constructed.

We propose a few further questions regarding generalization of results from this paper.

Question 9.1. Is it possible to generalize Theorem 1.3 to other coefficients? For example, if $\mathbb{F}$ is any field, does every homology class $\alpha \in H_2(M, L; \mathbb{F})$ have represented an $\mathbb{F}$-oriented compact π_1-injectively immersed quasi-Fuchsian subsurface?

It seems that our argument can be modified without difficulty to confirm Question 9.1 when $\mathbb{F}$ is any field of characteristic other than 2. However, the $\mathbf{Z}_2$ coefficient case is not clear since the subsurface constructed might be non-orientable, so an unoriented (R, ϵ)-panted cobordism theory needs to be developed.

Question 9.2. Is it possible to generalize Theorems 5.2 and 6.1 to other dimensions? In particular, can we define and determine the (R, ϵ)-panted cobordism group $\mathbf{\Omega}_{R,\epsilon}(M)$ for any oriented closed hyperbolic manifold M?

We expect that Theorem 5.2 should hold in all dimensions at least 3. In dimension 2, it seems that $\mathbf{\Omega}_{R,\epsilon}(S)$ should be a splitting extension of $H_1(S; \mathbf{Z})$ by $\mathbf{Z}_2$. This is basically because the special orthonormal frame bundle $\mathrm{SO}(S)$ should be replaced with the special orthonormal frame bundle of a stabilization $T(S) \oplus \epsilon^1$ of the tangent bundle $T(S)$, in order that Lemma 5.12 holds.

The following two questions are much more difficult but significant. To answer Question 9.3, we expect a notion of good basic pieces playing the role of nearly regular pairs of pants. To answer Question 9.4, we need a modified version of the Connection Principle since the mixing property of frame flow no longer holds.

Question 9.3. How to construct (homologically interesting) connected π_1-injectively immersed geometrically finite submanifolds in a closed hyperbolic manifold M?

Question 9.4. How to construct (homologically interesting) connected π_1-injectively immersed quasi-Fuchsian subsurfaces in a cusped hyperbolic 3-manifold M of finite volume?

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