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HARMONIC SURFACES IN 3-MANIFOLDS AND THE SIMPLE LOOP THEOREM

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Abstract. Denote by $\mathfrak{M}(\Sigma)$ the space of hyperbolic metrics on a closed, orientable surface Σ and by $\mathfrak{M}(M)$ the space of negatively curved Riemannian metrics on a closed, orientable 3-manifold M. We show that the set of metrics for which the corresponding harmonic map is in Whitney's general position is an open, dense, and connected subset of $\mathfrak{M}(\Sigma)\times\mathfrak{M}(M)$. The main application of this result is the proof of the Simple Loop Theorem.

1. Introduction

Throughout the paper Σ denotes a closed, orientable, and smooth surface of genus ≥ 2, and M a closed, orientable 3-manifold M. Let $\mathfrak{M}(\Sigma)$ denote an open and connected subset of the space of $C^{\mathbf{n},\alpha}$-smooth *hyperbolic* Riemannian metrics on Σ. We let $\mathfrak{M}(M)$ denote an open and connected subset of the space of $C^{\mathbf{n},\alpha}$-smooth *negatively curved* Riemannian metrics on M, where $\mathbf{n}\geq 2$ and $\alpha\in(0,1)$ are fixed.

Remark. The set $\mathfrak{M}(\Sigma)\times\mathfrak{M}(M)$ has the structure of a Banach manifold. This enables one to apply the Sard-Smale Transversality Theorem and its relatives (see [7]). We let $C(\Sigma,M)$ denote the Banach manifold consisting of $C^{\mathbf{n}+1,\alpha}$-smooth mappings from Σ to M (harmonic maps corresponding to $C^{\mathbf{n},\alpha}$ metrics are naturally $C^{\mathbf{n}+1,\alpha}$-smooth).

Definition 1.1. *Let $\mathbf{f}$ be a homotopy class of mappings from Σ into M. We say that $\mathbf{f}$ is admissible if $\mathbf{f}_*(\pi_1(\Sigma_0))$ is not Abelian.*

Unless otherwise stated, $\mathbf{f}$ denotes an admissible homotopy class. Then, for each $\mu\in\mathfrak{M}(\Sigma)$, and $\nu\in\mathfrak{M}(M)$, there exists the unique harmonic map $f_{\mu,\nu}:(\Sigma,\mu)\to(M,\nu)$ in the homotopy class $\mathbf{f}$. Set

$$\mathfrak{M}=\mathfrak{M}(\Sigma)\times\mathfrak{M}(M),$$

and define

$$\mathfrak{M}^W=\{(\mu,\nu)\in\mathfrak{M}:\ f_{\mu,\nu}\ \text{ is in Whitney's general position}\}.$$

Our first main result is the following theorem.

Theorem 1.1. *Let $\mathbf{f}$ be an admissible homotopy class of mappings from a closed orientable surface Σ of genus at least seven into a closed, orientable 3-manifold M. Then $\mathfrak{M}^W$ is open, dense, and connected subset of $\mathfrak{M}$.*

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Remark. In this theorem it is implicitly assumed that M supports a metric of negative curvature. However, it suffices to assume that M supports a Riemannian metric ν of non-positive curvature such that every map in the homotopy class $\mathbf{f}$ maps some portion of Σ into the part of M where the sectional curvatures of ν are strictly negative. We do not prove this generalization in the present paper. Also, the assumption that M is closed can be relaxed. The proof of Theorem 1.1 goes through (word for word) assuming that M is a convex-cocompact hyperbolic 3-manifold.

The main application of Theorem 1.1 is the proof of the Simple Loop Theorem. A map $f : X \to Y$ from a closed surface X into a compact manifold Y is essential if the induced map $f_* : \pi_1(X) \to \pi_1(Y)$ is injective. We say that f is incompressible if there exists an essential simple closed loop on X which lies in the kernel of f_*.

Theorem 1.2 (The Simple Loop Theorem)**.** *Let $f : S \to M$ be an incompressible homotopy class of maps from a closed orientable surface S into a closed orientable negatively curved 3-manifold M. Then f is essential.*

Remark. The proof of Theorem 1.2 goes through when M is a convex cocompact hyperbolic 3-manifold.

In the first part of the paper (Sections 1-10) we prove Theorem 1.1. In the remaining sections we discuss the background, outline the idea, and prove Theorem 1.2 using Theorem 1.1. The first part of the paper is mathematically independent of the second part. However, the two parts complement one another and naturally belong in the same paper.

1.1. **Whitney's general position.** We begin by recalling the notion of Whitney's general position.

Definition 1.2. *A C^2 smooth map $f : \Sigma \to M$ is in Whitney's general position (or just general position) at a point $p \in \Sigma$ if*

1. *f is an immersion near p, or*
2. *for any choice of local coordinates (x_1, x_2) for which $\frac{\partial f}{\partial x_1}(p) = 0$, the vectors $\frac{\partial f}{\partial x_2}(p)$, $\frac{\partial^2 f}{\partial x_1 \partial x_1}(p)$, $\frac{\partial^2 f}{\partial x_1 \partial x_2}(p)$ are linearly independent.*

We say that $f : \Sigma \to M$ is in general position if it is in general position at every $p \in \Sigma$.

Remark. Whitney showed (the second page in [34]) that if the second condition holds for one pair of local coordinates, then it holds for any pair of local coordinates. Thus, assuming that f is not an immersion near p, to show that f is in general position at p, it suffices to find one pair of local coordinates such that the corresponding three vectors are linearly independent at $T_{f(p)}M$.

This technical (and seemingly uninformative) definition represents the analytical way of writing that the only singularities of f (the points where $\mathrm{Rank}(df) < 2$) are Whitney's umbrellas. These singularities are isolated and the rank of df is equal to one at such points. Consequently, surfaces in general position have at most finitely many points which are not regular.

Being in general position is an open condition (with respect to the C^2 metric on the space of mappings). Singularities of surfaces in general position are stable under small perturbations. This observation yields the following corollary.

Proposition 1.1. *Suppose $g_t : \Sigma \to M$, $0 \le t \le 1$, is a continuous (in the C^2 sense) path of mappings in general position. If g_0 is an immersion then so is every g_t.*

The significance of Definition 1.2 steams from the Whitney's result (Theorem 2 in [34]) that every C^2 smooth map can be perturbed (in the C^2 sense) to a map in general position. Let $\mathcal{U}$ denote a connected component of the space of C^2 mappings from Σ to M. By $\mathcal{U}^{\mathrm{W}}$ we denote its subset consisting of surfaces which are in general position. The Whitney's result states that $\mathcal{U}^{\mathrm{W}}$ is an open and everywhere dense subset of $\mathcal{U}$. However, $\mathcal{U}^{\mathrm{W}}$ is not connected!

Remark. This can be seen as follows. There exist immersions $f, g : \Sigma \to M$ which are smoothly homotopic to each other, but not through a regular homotopy. Given any immersion f, one can construct the second immersion g by endowing f with "kinks" (see [14]).

However, if we restrict to harmonic maps the outcome is different. This is the content of Theorem 1.1. So, the question is what prompts this difference between the harmonic and the general case. Heuristically speaking, the answer is the following. For a given harmonic map $f : (\Sigma, \mu) \to (M, \nu)$, let $z = x + \mathbf{i}\, y$ denote the local complex coordinate in which $\mathrm{Hopf}(f) \equiv 1$. Then, if f is not an immersion near p, we have

$$\frac{\partial f}{\partial y}(p) = 0, \quad \frac{\partial f}{\partial x}(p) \neq 0,$$
$$\frac{\partial f}{\partial x}(p) \perp \frac{\partial^2 f}{\partial y \partial y}(p), \quad \text{and} \quad \frac{\partial f}{\partial x}(p) \perp \frac{\partial^2 f}{\partial y \partial x}(p).$$

Therefore, to show that the three vectors $\frac{\partial f}{\partial x}(p), \frac{\partial^2 f}{\partial y \partial y}(p), \frac{\partial^2 f}{\partial y \partial x}(p)$ are linearly independent it suffices to show that the last two vectors are linearly independent. This suggests that the set of harmonic maps which are not in general position has larger codimension than the corresponding set in the general case (which means that the space of harmonic maps in general position has a bigger chance of being connected than its general counterpart).

This remarkable and easy to establish property of harmonic maps shows that harmonic maps are (relatively speaking) more likely to be in general position than general maps. But to turn this into a valid argument we need to find a coordinate free condition for harmonic maps to be in general position, and then compute the codimension of the space of harmonic surfaces which are not in general position.

1.2. **Harmonic surfaces.** Given a map $f : \Sigma \to M$, we let $\mathbf{F} = f^{-1}TM$ denote the pull back vector bundle over Σ. Sections of $\mathbf{F}$ are labeled as $\Gamma(\mathbf{F})$.

Suppose μ and ν are Riemannian metrics on Σ and M respectively. By $\nabla^{\mathbf{F}}$ we denote the pull back connection of the Levi-Civita connection ∇^{ν} on M, and by $\nabla^{\mathbf{T}}$ the induced connection on the tensor product $\mathbf{T} = T^*\Sigma \otimes \mathbf{F}$ (where $T^*\Sigma$ is endowed with the Levi-Civita connection of μ). The second fundamental form $\nabla^{\mathbf{T}} df$ of the map f is given by

$$\nabla^{\mathbf{T}} df(X, Y) = \nabla^{\mathbf{F}}_X df(Y) - df(\nabla^{\mu}_X Y) \tag{1}$$

where X, Y are vector fields on Σ. The bilinear form $\nabla^{\mathbf{T}} df$ is symmetric (see Corollary 2.13 in [8]), and we get

$$\nabla^{\mathbf{F}}_X df(Y) - \nabla^{\mathbf{F}}_Y df(X) = df([X, Y]) \tag{2}$$

where $[X,Y]$ is the Lie bracket.

Definition 1.3. *We say that $f:(\Sigma,\mu)\to(M,\nu)$ is harmonic if $\tau\equiv 0$, where the tension field $\tau=\tau(f,\mu,\nu)\in\Gamma(\mathbf{F})$ given by $\tau=Trace_{\mu}(\nabla^{\mathbf{T}}df)$.*

Recall that each Riemannian metric μ yields the unique marked Riemann surface Σ_μ on which μ becomes a conformal metric. With respect to a local complex coordinate $z=x+iy$ on Σ_μ, the harmonic map equation becomes

$$\nabla^{\mathbf{F}}_{\frac{\partial}{\partial x}} df\left(\frac{\partial}{\partial x}\right)+\nabla^{\mathbf{F}}_{\frac{\partial}{\partial y}} df\left(\frac{\partial}{\partial y}\right)=0. \tag{3}$$

Throughout the paper we abbreviate $f=f_{\mu,\nu}$ whenever possible. Thus, $f:(\Sigma,\mu)\to(M,\nu)$ always stands for a harmonic map, and we write $f_{\mu,\nu}$ only when it is necessary to point out the dependence on the metrics. The connection $\nabla^{\mathbf{F}}$ is often used so from now on we abbreviate

$$\nabla\stackrel{\text{def}}{=}\nabla^{\mathbf{F}}.$$

A word on singular points of f. A point $p\in\Sigma$ is called a *branch* point of f if $\operatorname{rank}(df)(p)=0$. A point $p\in\Sigma$ is called a *regular* point of f if $\operatorname{rank}(df)(p)=2$ (clearly f is an immersion near a regular point). The set of regular points of f is denoted by $\Sigma^{\text{reg}}(f)$. Assuming that $f_*(\pi_1(\Sigma))$ is not a cyclic subgroup in $\pi_1(M)$, it follows from Theorem 3 in [29] that $\Sigma^{\text{reg}}(f)$ is open and dense subset of Σ.

1.3. **The Hopf differential.** We let $TM^{\mathbb{C}}=TM\oplus\mathbf{i}\,TM$ denote the complexification of TM and $\mathbf{E}=f^{-1}TM^{\mathbb{C}}$ its pullback. Each section W of $\mathbf{E}$ is uniquely written as $W=\operatorname{Re}(W)+\mathbf{i}\operatorname{Im}(W)$, where $\operatorname{Re}(W)$ and $\operatorname{Im}(W)$ are the sections of $\mathbf{F}$.

Let $z=x+\mathbf{i}\,y$ denote a local complex parameter on Σ_μ. Set $df\left(\frac{\partial}{\partial x}\right)=f_x$, $df\left(\frac{\partial}{\partial y}\right)=f_y$, and

$$df\left(\frac{\partial}{\partial z}\right)=\frac{1}{2}df\left(\frac{\partial}{\partial x}-\mathbf{i}\,\frac{\partial}{\partial y}\right)=\frac{1}{2}(f_x-\mathbf{i}\,f_y)=f_z.$$

It is easily checked that

$$(f\circ h)_w=(f_z\circ h)\,h', \tag{4}$$

for a holomorphic map h such that $h(w)=z$. Therefore, f_z is a $\mathbf{E}$-valued $(1,0)$-form on Σ and

$$\operatorname{Hopf}(f)=\langle f_z,f_z\rangle,$$

is a holomorphic quadratic differential called the Hopf differential of f. Here $\langle\cdot,\cdot\rangle=\langle\cdot,\cdot\rangle_\nu$ is the inner product on $\mathbf{F}$ pulled back from TM (this inner product extends to the complex bilinear form on $\mathbf{E}$ which we also denote by $\langle\cdot,\cdot\rangle$). A map f is said to be minimal if it is harmonic and conformal at the same time. Thus, f is minimal if and only if $\operatorname{Hopf}(f)\equiv 0$.

We also note the formula for the derivative of the Hopf differential

$$\left(\operatorname{Hopf}(f)\right)_z=2\langle\nabla_z f_z,f_z\rangle. \tag{5}$$

This formula holds because ∇ is pull back of the Levi-Civita connection from (M,ν) (see (1.8) in [8]).

Remark. The complex vector bundle $\mathbf{E}$ supports the unique holomorphic structure such that the $(0,1)$ part of the connection $\nabla^{\mathbf{T}}$ agrees with the standard $\overline{\partial}$ (see Section 2 in [21]). The harmonic map equation becomes $\nabla_{\overline{z}}f_z\equiv 0$, that is, f_z is a holomorphic $\mathbf{E}$-valued 1 form.

1.4. **The E-valued quadratic differential $\mathbf{M}(f)$.** Given a (non-conformal) harmonic map $f:(\Sigma,\mu)\to(M,\nu)$, we let

$$\mathbf{M}(f)=-\frac{1}{2}\left(\nabla_z f_z-\frac{\langle\nabla_z f_z,f_z\rangle}{\langle f_z,f_z\rangle}f_z\right), \tag{6}$$

where z is a local complex parameter on Σ, and we abbreviate

$$\nabla_z \overset{\mathrm{def}}{=} \nabla_{\frac{\partial}{\partial z}}.$$

Remark. For a fixed local complex parameter z near p, the vector $\mathbf{M}(f)(p)\in\mathbf{E}_p$ is the projection of the vector $-\frac{1}{2}\nabla_z f_z$ onto the orthogonal complement of the vector $f_z(p)\in\mathbf{E}_p$.

Let

$$\nabla^{\mathrm{H}}df(X,Y)=\nabla^{\mathbf{F}}_X df(Y)-df(\nabla^{|\mathrm{Hopf}(f)|}_X Y)$$

denote the second fundamental form of the map f where instead of the metric μ we take the singular flat metric $|\mathrm{Hopf}(f)||dz^2|$ on Σ. It is routinely checked that

$$\mathbf{M}(f)=\nabla^{\mathrm{H}}df\left(\frac{\partial}{\partial z},\frac{\partial}{\partial z}\right). \tag{7}$$

Therefore

$$\mathbf{M}(f\circ h)=\left(\mathbf{M}(f)\circ h\right)(h')^2, \tag{8}$$

where h is a holomorphic map such that $h(w)=z$. Thus, $\mathbf{M}(f)$ is a well defined $\mathbf{E}$-valued quadratic differential on Σ minus the zeroes of the Hopf differential.

Remark. Since (M,ν) is negatively curved, it can be shown that providing f is not a constant function then $\mathbf{M}(f)\equiv 0$ if and only if f maps Σ onto a closed geodesic in M. This suggests that $\mathbf{M}(f)$ contains information about the geometry of the harmonic surface $f(\Sigma)$.

1.5. **Harmonic maps in general position.** For $p\in\Sigma$, we let $\mathrm{Rank}(f_z)(p)$ denote the dimension of the vector subspace of $\mathbf{F}_p$ spanned by the vectors $\mathrm{Re}(f_z)(p)$ and $\mathrm{Im}(f_z)(p)$. Similarly, $\mathrm{Rank}(\mathbf{M}(f))(p)$ is the dimension of the vector subspace of $\mathbf{F}_p$ spanned by $\mathrm{Re}(\mathbf{M}(f))(p)$ and $\mathrm{Im}(\mathbf{M}(f))(p)$.

From (4) and (8) we derive that $\mathrm{Rank}(f_z)(p)$ and $\mathrm{Rank}(\mathbf{M}(f))(p)$ do not depend on the choice of the complex parameter z. Clearly, $\mathrm{Rank}(f_z)(p)=\mathrm{Rank}(df)(p)$, and f is an immersion near p if and only if $\mathrm{Rank}(f_z)(p)=2$. The following is the key lemma.

Lemma 1.1. *A non-conformal harmonic map $f:(\Sigma,\mu)\to(M,\nu)$ is in general position at a point $p\in\Sigma$ if and only*

1. *$Rank(f_z)(p)=2$, or*
2. *$f_z(p)\neq 0$ and $Rank(\mathbf{M}(f))(p)=2$.*

If f is a conformal harmonic map, then f is in general position if and only if $f_z(p)\neq 0$ for every $p\in\Sigma$ (in which case f is an immersion).

Proof. A minimal map is in general position if and only if it is an immersion. On the other hand, the only singularities of minimal maps are branch points. Thus, if f has no branch points then it is an immersion. Therefore, the proposition holds for minimal maps. In the rest of the proof we assume that $\mathrm{Hopf}(f)$ is not identically zero.

Consider first the case when p is a zero of $\operatorname{Hopf}(f)$. Then either $f_z(p) = 0$, or f is conformal at p, in which case $\operatorname{Rank}(f_z)(p) = 2$. This confirms the lemma when p is a zero of $\operatorname{Hopf}(f)$.

It remains to analyze the case when p is not a zero of $\operatorname{Hopf}(f)$. The map f is an immersion near p if and only if $\operatorname{Rank}(f_z)(p) = 2$. Suppose $\operatorname{Rank}(f_z)(p) < 2$. Then $\operatorname{Rank}(f_z)(p) = 1$ because p is not a branch point (which can only occur at the zeroes of $\operatorname{Hopf}(f)$). In the remainder we show that f is in general position if and only if $\operatorname{Rank}(\mathbf{M}(f))(p) = 2$. Since $\operatorname{Rank}(\mathbf{M}(f))(p)$ is independent of the local complex parameter, we are free to choose the one that is most suitable to compute it.

Let z be the local parameter near p such that $\operatorname{Hopf}(f) \equiv 1$. Then $f_y(p) = 0$ and $f_x(p) \neq 0$. The identity (5) yields

$$\langle \nabla_z f_z, f_z \rangle \equiv 0. \tag{9}$$

This formula has two important corollaries. The first one

$$\mathbf{M}(f) = -\frac{1}{2} \nabla_z f_z, \tag{10}$$

is obtained by replacing (10) in (6). The second one

$$\operatorname{Re}(f_z)(p) \perp \operatorname{Re}\left(\mathbf{M}(f)\right)(p) \quad \text{and} \quad \operatorname{Re}(f_z)(p) \perp \operatorname{Im}\left(\mathbf{M}(f)\right)(p), \tag{11}$$

follows from the equality $f_z(p) = f_x(p) = \operatorname{Re}(f_z)(p)$, which in turn follows from $f_y(p) = 0$.

From (10) we compute

$$\operatorname{Re}(\mathbf{M}(f)) = -\frac{1}{2}(\nabla_x f_x - \nabla_y f_y), \qquad \operatorname{Im}(\mathbf{M}(f)) = -\frac{1}{2}(-\nabla_x f_x - \nabla_y f_x).$$

Together with (3) and (2), this yields

$$\operatorname{Re}(\mathbf{M}(f)) = \nabla_y f_y, \quad \operatorname{Im}(\mathbf{M}(f)) = \nabla_y f_x, \tag{12}$$

where we use the identity $[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}] = 0$.

Consider the flat metric induced by $\operatorname{Hopf}(f)$ near p and choose the normal coordinates on M near $f(p)$ for the metric ν on M. Then, with respect to these local coordinates, and using (12), we compute the second fundamental form of f and obtain that the following hold at the point p

$$\begin{aligned} \operatorname{Re}(f_z) &= \frac{\partial f}{\partial x}, & \operatorname{Im}(f_z) &= \frac{\partial f}{\partial y}, \\ \operatorname{Re}(\mathbf{M}(f)) &= \frac{\partial^2 f}{\partial y \partial y}, & \operatorname{Im}(\mathbf{M}(f)) &= \frac{\partial^2 f}{\partial y \partial x}. \end{aligned} \tag{13}$$

Remark. The left-hand side in each of the above equalities is a vector in $\mathbf{F}$, while the right-hand side is a vector in TM. The equality sign means that the image of the left-hand side under the bundle morphism $\mathbf{F} \to TM$ is equal to the right-hand side in TM.

Thus, the identities (13) imply that f is in general position at p if and only if the three vectors $\mathrm{Re}(f_z)(p)$, $\mathrm{Re}(\mathbf{M}(f))(p)$, and $\mathrm{Im}(\mathbf{M}(f))(p)$ are linearly independent in $\mathbf{F}_p$. From (11), and since $\mathrm{Re}(f_z)(p) \neq 0$, we conclude that these three vectors are linearly independent if and only the last two vectors $\mathrm{Re}(\mathbf{M}(f))(p)$ and $\mathrm{Im}(\mathbf{M}(f))(p)$ are linearly independent. But this is equivalent to $\mathrm{Rank}(\mathbf{M}(f))(p) = 2$, and we are done.

$\square$

1.6. **Vector bundles and the universal cover.** Since M is a closed, and orientable 3-manifold its tangent bundle is trivial. Therefore, there is an isomorphism $TM^{\mathbb{C}} \to \mathbb{C}^3 \times M$, and we let

$$\sigma : TM^{\mathbb{C}} \to \mathbb{C}^3$$

denote the projection onto the first factor which restricts to an isometry $\sigma : TM^{\mathbb{C}}_p \to \mathbb{C}^3$ between the inner products ν on $TM^{\mathbb{C}}_p$ and the standard inner product on $\mathbb{C}^3$.

Unless otherwise stated, the dimension (codimension) refers to the real dimension (codimension). Define the submanifold $\mathcal{E} \subset \mathbb{C}^3 \oplus \mathbb{C}^3$ by letting

$$\mathcal{E} = \{(A, B) \in \mathbb{C}^3 \oplus \mathbb{C}^3 : \langle A, B \rangle = 0, \text{ and } A \neq 0\}.$$

We also define the submanifold $\mathcal{L} \subset \mathcal{E}$ by

$$\mathcal{L} = \{(A, B) \in \mathcal{E} : \mathrm{Rank}(A) = 1, \text{ and } \mathrm{Rank}(B) < 2\}$$

(recall that $\mathrm{Rank}(C)$ denotes the dimension of the vector space spanned by $\mathrm{Re}(C)$ and $\mathrm{Im}(C)$). We observe that the codimension of $\mathcal{L}$ in $\mathcal{E}$ is equal to four.

To prove Theorem 1.1 we show that $\mathfrak{M} \setminus \mathfrak{M}^W$ is of codimension two as a subset of $\mathfrak{M}$. We construct the map

$$\Psi : \widehat{\Sigma} \times \mathfrak{M}' \to \mathcal{E},$$

where $\widehat{\Sigma}$ denotes the universal cover of Σ, and $\mathfrak{M}'$ a suitable open, dense, and connected subset of $\mathfrak{M}$. We then observe that

$$\widehat{\Sigma} \times (\mathfrak{M}' \setminus \mathfrak{M}^W) = \Psi^{-1}(\mathcal{L}).$$

Thus, the codimension of the submanifold $\widehat{\Sigma} \times (\mathfrak{M}' \setminus \mathfrak{M}^W)$ is four which implies that the codimension of the set $\mathfrak{M}' \setminus \mathfrak{M}^W$ is two. Of course, this argument assumes that Ψ is transverse to $\mathcal{L}$, and the remainder of the first part of the paper (after this section) is devoted to proving this.

Let $\widehat{\Sigma}$ denote the universal cover of Σ which is diffeomorphic to the unit disc $\mathbb{D}$. The lift of the metric μ to $\widehat{\Sigma}$ is also denoted by μ, and the lift of the harmonic map $f_{\mu,\nu}$ is also denoted by $f_{\mu,\nu} : (\widehat{\Sigma}, \mu) \to (M, \nu)$.

The Riemannian manifold $(\widehat{\Sigma}, \mu)$ is isometric to $(\mathbb{D}, \mathfrak{g})$, where $\mathfrak{g}$ the hyperbolic metric on $\mathbb{D}$. Moreover, we identify the Riemann surface Σ_μ with $\mathbb{D}/\Gamma_\mu$, where Γ_μ is a smoothly varying family of Fuchsian groups acting on $\mathbb{D}$. Let $z \in \mathbb{D}$ denote the complex parameter. This provides us with the canonical complex parameter $z_\mu = z$ on $\widehat{\Sigma}$ which depends only on μ (and not on ν).

1.7. **The map Ψ.** Recall that $\mathfrak{M} = \mathfrak{M}(\Sigma) \times \mathfrak{M}(M)$, where $\mathfrak{M}(\Sigma)$ and $\mathfrak{M}(M)$ are open and connected sets of hyperbolic metrics on Σ, and negatively curved Riemannian metrics on M respectively. By $\mathfrak{M}_0 \subset \mathfrak{M}$ we denote the set of pairs (μ, ν) such that $f_{\mu,\nu}$ has at least one branch point (a point where $\mathrm{Rank}(df)$ equals zero).

Let

$$\mathcal{X} = \widehat{\Sigma} \times (\mathfrak{M} \setminus \mathfrak{M}_0).$$

The harmonic map $f = f_{\mu,\nu}$ has no branch points when $(\mu, \nu) \in \mathfrak{M} \setminus \mathfrak{M}_0$. So, if f is a minimal map then it is an immersion and $\operatorname{Rank}(f_z) = 2$ everywhere on Σ. Similarly, if p is a zero of $\operatorname{Hopf}(f)$, or it is sufficiently close to a zero of f, then f is an immersion near p, and again $\operatorname{Rank}(f_z) = 2$. Thus, we can construct a smooth function $\xi : \mathcal{X} \to \mathbb{R}$ with the following properties:

- $\xi \equiv 1$ on an open subset of $\mathcal{X}$, containing every (p, μ, ν) such that $\operatorname{Rank}(f_z)(p) = 1$, where $f = f_{\mu,\nu}$,
- $\xi \equiv 0$ on an open set $\mathcal{X}$, containing every (p, μ, ν) such that either $f_{\mu,\nu}$ is a minimal map, or p is a zero of $\operatorname{Hopf}(f_{\mu,\nu})$ providing $f_{\mu,\nu}$ is not minimal,
- ξ is equivariant, meaning that $\xi(A(p), \mu, \nu) = \xi(p, \mu, \nu)$ for every $A \in \Gamma_\mu$.

Define the map $\Psi : \mathcal{X} \to \mathcal{E}$, by

$$\Psi(p, \mu, \nu) = (\sigma(f_z(p)),\, \sigma(\xi\, \mathbf{M}(f)(p)))$$

where $f = f_{\mu,\nu}$, $\xi = \xi(p, \mu, \nu)$, and $z = z_\mu$ (we evaluate f_z and $\mathbf{M}(f)$ with respect to z_μ). See the remark after the definition of $\mathbf{M}(f)$ as to why the image of Ψ lies in $\mathcal{E}$.

Remark. We use ξ to smoothly extend the map Ψ to the points in $\mathcal{X}$ at which $\mathbf{M}(f_{\mu,\nu})(p)$ is not well defined. However, in practice we are only concerned with the values of Ψ on the set where $\xi \equiv 1$.

From (4) and (8) we conclude that Ψ is equivariant in the sense that

$$\Psi(A(p), \mu, \nu) = \left(f_z(A(p))A'(p),\, \xi\, \mathbf{M}(f)(A(p))A'(p)^2\right). \tag{14}$$

We let

$$\mathcal{Q} = \Psi^{-1}(\mathcal{L}).$$

The set $\mathcal{Q}$ is invariant under the action of Γ_μ. That is, $(p, \mu, \nu) \in \mathcal{Q}$ if and only if $(A(p), \mu, \nu) \in \mathcal{Q}$ for every $A \in \Gamma_\mu$.

Recall that $\mathfrak{M}^W$ is the set of harmonic maps in general position. The significance of Ψ steams from the identity

$$(\mathfrak{M} \setminus \mathfrak{M}_0) \setminus \pi(\mathcal{Q}) = \mathfrak{M}^W, \tag{15}$$

where $\pi : \mathcal{X} \to \mathfrak{M}$ is the projection onto the second factor.

This is an immediate consequence of Lemma 1.1. Indeed, suppose $(\mu, \nu) \in (\mathfrak{M} \setminus \mathfrak{M}_0) \setminus \pi(\mathcal{Q})$. This means that for every $p \in \Sigma$, either $\operatorname{Rank}(f_z)(p) = 2$, or $\operatorname{Rank}(\mathbf{M}(f))(p) = 2$, or both. If $\operatorname{Rank}(f_z)(p) = 2$ then f is an immersion near p. Also, since f does not have branch points (for $(\mu, \nu) \in (\mathfrak{M} \setminus \mathfrak{M}_0)$), it follows that $(\mu, \nu) \in (\mathfrak{M} \setminus \mathfrak{M}_0) \setminus \pi(\mathcal{Q})$ if and only if $\operatorname{Rank}(f_z)(p) = 2$, or $\operatorname{Rank}(f_z)(p) = 1$ and $\operatorname{Rank}(\mathbf{M}(f))(p) = 2$. By Lemma 1.1 this is equivalent to $(\mu, \nu) \in \mathfrak{M}^W$, and we are done.

1.8. **The Transversality Theorem.** We show that Ψ is transverse to $\mathcal{L}$.

Theorem 1.3. *Suppose that genus of Σ is at least seven. Then, there exists an open, dense, and connected set $\mathfrak{M}' \subset \mathfrak{M}$ with the following properties*

(1) $\mathfrak{M}' \subset (\mathfrak{M} \setminus \mathfrak{M}_0)$ *(in particular, Ψ is well defined on $\mathfrak{M}'$),*

(2) *the restriction of Ψ to $\widehat{\Sigma} \times \mathfrak{M}'$ is transverse to $\mathcal{L}$.*

If we show that $\mathfrak{M}^W \cap \mathfrak{M}'$ is an open, dense, and connected subset of $\mathfrak{M}'$, then it follows that $\mathfrak{M}^W$ is an open, dense, and connected subset of $\mathfrak{M}$. In the remainder of the section we rename $\mathfrak{M}'$ to $\mathfrak{M}$, and assume that Ψ is transverse to on $\widehat{\Sigma} \times \mathfrak{M} = \mathcal{X}$.

We can now summarize the main idea behind the proof of Theorem 1.1, the idea we hinted at earlier. Suppose Ψ is transverse to $\mathcal{L}$. Then by the Transversality Theorem in Banach spaces, the set $\mathcal{Q}$ is a submanifold of $\mathcal{X}$ of codimension four because $\mathcal{L}$ has codimension four in $\mathcal{E}$. Therefore, the subset (if not quite a submanifold) $\pi(\mathcal{Q}) \subset \mathfrak{M}$ is of codimension at least two. It follows that $\pi(\mathcal{Q})$ is closed and nowhere dense in $\mathfrak{M}$, and in turn this means (see (15)) that $\mathfrak{M}^W$ is open and dense in $\mathfrak{M}$.

Remark. The above argument indicates that $\mathfrak{M}^W$ is a connected subset of $\mathfrak{M}$ because removing the closed, and nowhere dense set $\pi(\mathcal{Q})$ of codimension at least two does not disconnect $\mathfrak{M}$. But $\pi(\mathcal{Q})$ may not be a submanifold of $\mathfrak{M}$ so we need to work in $\mathcal{X}$ because $\mathcal{Q}$ is a submanifold of $\mathcal{X}$.

1.9. **Proof of Theorem 1.1.** We define $F : \mathfrak{M} \to C(\Sigma, M)$ by $F(\mu, \nu) = f_{\mu,\nu}$, where $f_{\mu,\nu} : (\Sigma, \mu) \to (M, \nu)$ is the unique harmonic map in the homotopy class $\mathbf{f}$. Then F is a $C^{\mathbf{n},\alpha}$-smooth map between the Banach manifolds $\mathfrak{M}$ and $C(\Sigma, M)$ (see [7]). This implies that Ψ is smooth and we can apply the Sard-Smale Transversality Theorem. In view of (15), to prove Theorem 1.1 it suffices to prove that $\mathfrak{M} \setminus \pi(\mathcal{Q})$ is open, dense, and connected subset of $\mathfrak{M}$.

From Theorem 1.3 and the Transversality Theorem for Banach manifolds (see Corollary 17.2 in [3]), we deduce that $\mathcal{Q}$ is a Banach submanifold of $\mathcal{X}$ of codimension four.

Remark. Let $g : A \to B$ be a smooth map between (real) Banach manifolds and $C \subset B$ a submanifold (not necessarily closed). If B is finite dimensional then $g^{-1}(C)$ is a submanifold of A whose codimension is equal to the codimension of C in B. If B is not finite dimensional additional assumptions are needed to ensure that $g^{-1}(C)$ is a submanifold of A.

Let $\pi_1 : \mathcal{Q} \to \mathfrak{M}$ denote the restriction of the projection $\pi : \mathcal{X} \to \mathfrak{M}$. Since $\mathcal{Q}$ is of codimension four, and $\widehat{\Sigma}$ is of dimension two, it follows that the codimension of the tangent subspace $d\pi_1(\mathcal{Q}) < T\mathfrak{M}$ is at least two. Here $d\pi_1$ is the derivative of π_1.

Therefore $\pi_1(\mathcal{Q})$ is a closed and nowhere dense subset of $\mathfrak{M}$. This also indicates that $\mathfrak{M}^W$ is connected. But $d\pi_1$ may not be an embedding so we can not claim that $\pi_1(\mathcal{Q})$ is a submanifold of $\mathfrak{M}$. Instead, we work in $\mathcal{X}$ and give a direct proof using the basic transversality theory.

Let $\gamma : [0, 1] \to \mathfrak{M}$ be a path whose endpoints lie in $\mathfrak{M}^W$. We show that γ can be perturbed (while keeping the endpoints fixed) to be entirely contained in $\mathfrak{M}^W$. First, we partition $[0, 1]$ into sufficiently small intervals whose image under γ is contained in a subset of $\mathfrak{M}$ which fits into a single chart in the model Banach space for $\mathfrak{M}$.

Suppose $(\mu_i, \nu_i) \in \mathfrak{M}$, $i = 0, 1$, are contained in this chart, and suppose $(\mu_0, \nu_0) \in \mathfrak{M}^W$. We show that one can perturb (μ_1, ν_1) ever so slightly, so that the straight line which connects (μ_0, ν_0) and the perturbed (μ_1, ν_1) is contained in $\mathfrak{M}^W$ (we are now in the model Banach space and the straight line refers to the linear combination $t(\mu_0, \nu_0) + (1-t)(\mu_1, \nu_1)$, $t \in [0, 1]$).

Let $U \subset \mathfrak{M}$ be a small neighborhood of (μ_1, ν_1) and consider the map $\beta : U \times (\widehat{\Sigma} \times [0, 1]) \to \mathcal{E}$ given by

$$\beta((\mu, \nu), (p, t)) = \Psi\big(t(p, \mu_0, \nu_0) + (1-t)(p, \mu, \nu)\big).$$

Here $t(p, \mu_0, \nu_0) + (1-t)(p, \mu, \nu)$ is the element of $\widehat{\Sigma} \times U$ for a fixed $t \in [0, 1]$. It follows from Theorem 1.3 that β is transverse to $\mathcal{L}$.

From the Parametric Transversality Theorem (see Theorem 19.1 in [3]), we conclude that for a generic point in $(\mu, \nu) \in U$, the evaluation map $\delta_{\mu,\nu} : \widehat{\Sigma} \times [0, 1] \to \mathcal{E}$ given by

$$\delta(p, t) = \beta((\mu, \nu), (p, t)),$$

is transverse to $\mathcal{L}$. Since the dimension of $\widehat{\Sigma} \times [0, 1]$ is three, and the codimension of $\mathcal{L}$ in $\mathcal{E}$ is four, it follows that $\delta(\widehat{\Sigma} \times [0, 1])$ is disjoint from $\mathcal{L}$. This implies that the path

$$\pi\left(\beta((\mu, \nu), (p, t))\right)) \subset \mathfrak{M}^W$$

connects (μ_0, ν_0) and (μ, ν).

As promised, we managed to perturb (μ_1, ν_1) to a nearby point (μ, ν) such that the straight line connecting (μ_0, ν_0) and (μ, ν) is entirely contained in $\mathfrak{M}^W$. Repeating this, we perturb γ to obtain the new path $\widehat{\gamma}$ which is entirely contained in $\mathfrak{M} \setminus \mathfrak{M}_0$.

Moreover, we can do this so that $\widehat{\gamma}(1)$ is as close to $\gamma(1)$ as we want to. Once the points $\gamma(1), \widehat{\gamma}(1) \in \mathfrak{M}^W$ are sufficiently close to each other to fit in a single chart, we can connect them by the straight segment $\gamma' \subset \mathfrak{M}^W$ (for $\mathfrak{M}^W$ is open). The concatenation $\gamma' \cdot \widehat{\gamma}$ is contained in $\mathfrak{M}^W$ and it connects $\gamma(0)$ and $\gamma(1)$. This shows that $\mathfrak{M}^W$ is connected. This completes the proof of Theorem 1.1.

1.10. **Outline.** It remains to prove Theorem 1.3. The central part of the proof is contained in sections two, three, and four. In the remaining sections we tie up loose ends using the methods introduced in the first four sections of the paper.

In Section 2 we compute the derivative $d\Psi$ which is needed in order to show that the map Ψ is transversal to $\mathcal{L}$. The formula for $d\Psi$ is obtained from the standard formulas computed by Eells-Lemaire in [7], [8].

Perhaps the most essential part in proving the transversality are the Reproducing formulas established in Section 3 (and again in Appendix A using the $\overline{\delta}$ method). These formulas recover the values of a section $V \in \Gamma(\mathbf{F})$, and its first and second derivatives, from the section $\mathbf{J}V$. The proof of these formulas rests on the fact that the Jacobi operator $\mathbf{J}$ is an isomorphism (which follows from the assumption that (M, ν) has negative curvature). This is one of the cornerstones of this paper.

In Section 4 we complete the proof of Theorem 1.3. Using the Reproducing formulas we show that if the theorem does not hold, then there exists a reproducing kernel $\mathcal{K}$ which is annihilated by all sections of $\mathbf{F}$ which arise from varying families of harmonic maps (when the metric on M is being varied). We then show this to be impossible using Proposition 4.1 (see Moore's paper [24] and book [23]) which states that there are sufficiently many variations ν_t.

The proofs in Section 4 are given modulo Lemma 4.3 which says that removing certain special metrics from $\mathfrak{M}$ does not disconnect it. This includes removing harmonic maps which have branch points and harmonic maps which are not somewhere injective. In Section 5 we deal with harmonic maps which have branch points. We show that $\mathfrak{M}_0$ is a meager set and that removing it does not disconnect $\mathfrak{M}$. The proof follows the exact same line as the proof that $\mathfrak{M}^W$ is connected, although the fact that $\mathfrak{M} \setminus \mathfrak{M}_0$ is open, dense, and connected is shorter and simpler to prove. In the particular, the auxiliary map Λ (which plays the role of Ψ) involves only the first derivative f_z.

One technical (but not very profound) complication that we have to address is that harmonic surfaces may not be somewhere injective. In Section 6 we show that the set of harmonic maps which are not somewhere injective, but do not factor through a holomorphic cover $\Sigma \to \Sigma_1$, is of dimension zero so it is pretty negligible (we settle for proving that it does not disconnect $\mathfrak{M}$, but this stronger fact can be established from our argument). The proof in Section 6 again follows the same line as above and it rests on the transversality theorem established in Section 7.

The proofs in sections 5 and 7 require more precise information about the reproducing kernels near their singularities. This is established in Section 8 (named the Appendix A) where we recompute the kernels using the standard $\overline{\partial}$ method.

In Appendix B (Section 9) we show that the set of exceptional Riemann surfaces (defined below) does not disconnect the Teichmüller space. This ought to be well known and it follows readily from the Riemann-Hurwitz formula. This is the only place where we use that Σ has genus ≥ 7 (it is likely this assumption can be relaxed).

In Appendix C (Section 10) we show that if the harmonic map $f_{\mu,\nu} : \Sigma \to M$ factors through a local conformal map between two neighborhoods on Σ, then Σ_μ is an exceptional Riemann surface. This is well known when f is a minimal map [12], and it is borderline known in general. We observe that if a harmonic map f has this property then its minimal suspension $F : \widehat{\Sigma} \to \widehat{M} \times T$ has the exact same property. Here T is the metric tree obtained from Hopf(f) (see Wolf's papers [36], [37]). The proof then reduces to the first case.

1.11. **Acknowledgment.** Beside the works of Eells-Lemaire [7] and Sampson [29], there are other instances where Transversality theory on Banach manifolds have been used to analyze spaces of harmonic maps. We mention just a few. In [4] Bohme-Tromba study harmonic discs which span a fixed curve in $\mathbb{R}^3$. In Theorem 1.19 in [4] they compute the dimension of the space of harmonic discs having a fixed number of branch points. Another example of this kind of investigation appear in works by Moore [24] and [23]. He shows that minimal surfaces with respect to generic Riemannian metrics on M (and without any curvature restrictions) do not have branch points (in [23] this was done for $\dim(M) \geq 4$, and in [24] extended to the case $\dim(M) = 3$).

2. Finding the derivative of Ψ

Before we compute the derivative of Ψ, we must find the derivative of $F : \mathfrak{M} \to C(\Sigma, M)$. First, we recall this well known computation [8]. We then compute $d\Phi$ in preparation for the proof of Theorem 1.3.

Remark. For our purposes it suffices to compute the restriction $dF : \{0\} \times T\mathfrak{M}(M)$. Throughout the paper, we neither compute, nor use the formula for the derivative dF in the $T\mathfrak{M}(\Sigma)$ direction.

2.1. **The Jacobi operator.** Let $f : (\Sigma, \mu) \to (M, \nu)$ be a smooth (not necessarily harmonic) map. The Jacobi Operator $\mathbf{J}_f = \mathbf{J} : \Gamma(\mathbf{F}) \to \Gamma(\mathbf{F})$ is given by (see Definition 4.4 in [8])

$$\mathbf{J}V = \Delta V - \mathrm{Trace}_\mu R(df, V)df, \qquad V \in \Gamma(\mathbf{F}). \tag{16}$$

Here Δ is the Laplacian induced by the connection $\nabla^{\mathbf{T}}$, and $R = R^M$ is the curvature tensor of the Levi-Civita connection of ν. If $z = x + \mathbf{i}\, y$ is a local complex parameter, and μ a conformal metric, then

$$\mathbf{J}V = \nabla_x \nabla_x V + \nabla_y \nabla_y V - |\mu|^{-1} \left(R(f_x, V)f_x + R(f_y, V)f_y \right). \tag{17}$$

The Jacobi operator is a second order strongly elliptic operator and it is self adjoint

$$\int_\Sigma \langle \mathbf{J}V, W \rangle \, dA = \int_\Sigma \langle V, \mathbf{J}W \rangle \, dA \tag{18}$$

for all $V, W \in \Gamma(\mathbf{F})$. Recall that $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_\nu$ is the inner product on $\mathbf{F}$ induced by the metric ν on M. We integrate over Σ with respect to the volume form $dA = dA_\mu$ corresponding to μ.

A section $V \in \Gamma(\mathbf{F})$ is called a Jacobi field if $\mathbf{J}V \equiv 0$. One of the cornerstones of this paper is the following well known fact.

Proposition 2.1. *Suppose $(\mu, \nu) \in \mathfrak{M}$ and let $f = f_{\mu,\nu}$. Then, the Jacobi operator $\mathbf{J}_f : \Gamma(\mathbf{F}) \to \Gamma(\mathbf{F})$ is an isomorphism.*

Remark. The proof of this proposition rests on the assumption that metrics in $\mathfrak{M}(M)$ have negative sectional curvatures. See Section 7 in [29] and [7]. A version of this result for non-positively curved metrics (or metrics sufficiently close to being non-positively curved) can be proved.

2.2. **The derivative dF.** Next, we compute the derivative of $F : \mathfrak{M} \to C(\Sigma, M)$. Fix $(\mu, \nu) \in \mathfrak{M}$ and set $f = f_{\mu,\nu}$. The tangent space $T_{\mu,\nu}\mathfrak{M}$ splits as $T_{\mu,\nu}\mathfrak{M} = T_\mu \mathfrak{M}(\Sigma) \times T_\nu \mathfrak{M}(M)$, while the tangent space $T_f C(\Sigma, M)$ is the space of vector fields along the image surface $f(\Sigma)$, and we have the natural identification $T_f C(\Sigma, M) = \Gamma(\mathbf{F})$. We are only interested in computing the restriction of dF to $\{0\} \times T\mathfrak{M}(M)$.

Pick a tangent vector $\dot{\nu} \in T_\nu \mathfrak{M}(M)$, and let $\nu_t \in \mathfrak{M}(M)$ be a variation such that

$$\frac{\partial \nu_t}{\partial t} = \dot{\nu}.$$

Let $f_t : (\Sigma, \mu) \to (M, \nu_t)$ denote the corresponding path of harmonic mappings $f_t = f_{\mu,\nu_t}$. Then, there exists a unique section $V \in \Gamma(\mathbf{F})$ such that

$$f_t(p) = \mathbf{exp}_{f(p)} \left(tV(p) \right) + O(t^2), \tag{19}$$

where the exponential map $\mathbf{exp}$ is defined with respect to ν. The section V is uniquely determined by the equation (see formula (2.7) in [7], or (4) in [29], or [18])

$$\mathbf{J}V = \mathcal{G}_{\mu,\nu}(\dot{\nu}), \tag{20}$$

where

$$\mathcal{G}_{\mu,\nu}(\dot{\nu}) = \frac{\partial \tau}{\partial t}. \tag{21}$$

Here $\tau = \tau(f_{\mu,\nu}, \mu, \nu_t)$ is the tension of field of the fixed map $f_{\mu,\nu}$ with respect to the varying metrics ν_t.

Remark. The section V is uniquely determined by (20) because $\mathbf{J}$ is an isomorphism.

We write $dF(0, \dot{\nu}) = dF(\dot{\nu}) = V \in \Gamma(\mathbf{F})$. The condition (22) is more succinctly written as

$$dF(\dot{\nu}) = V = \frac{\partial f_t}{\partial t}\Big|_{t=0}. \tag{22}$$

2.3. **The derivative of a varying family of maps.** Let $g : \Sigma \to M$ be a smooth map and choose any section $U \in g^{-1}TM$. Set

$$g^t(p) = \mathbf{exp}_{g(p)}\left(tU(p)\right), \tag{23}$$

where $\mathbf{exp}$ is computed with respect to a Riemannian metric ν on M (note that $g = g^0$). We let $G : \Sigma \times \mathbb{R} \to M$ be given by $G(p,t) = g^t(p)$, and let ∇^G denote the connection on $G^{-1}TM$ induced by the Levi-Civita connection for the metric ν on M.

In this subsection we compute the first and second derivatives of the family of maps g^t without assuming that g^t are harmonic maps. In the next subsection we apply these results to our case.

Fix local complex parameter $w = u + \mathbf{i}\, v$ near $p \in \Sigma$. Then (see Proposition 2.4 in [8])

$$\nabla^G_{\frac{\partial}{\partial t}} g^t_w \,\big|_{t=0} = \nabla_w U, \tag{24}$$

where ∇ is the connection on $g^{-1}TM$ induced by the Levi-Civita connection for the metric ν on M.

Suppose ν_t is a smooth family of metrics on M. By ∇^t where we denote connection on $(g^t)^{-1}TM$ induced by the Levi-Civita connection for ν_t. We have the following formula.

Proposition 2.2. *For simplicity of the notation we let $x_1 = u$ and $x_2 = v$. Then for fixed $i, j \in \{1,2\}$, we have*

$$\nabla^G_{\frac{\partial}{\partial t}}\left(\nabla^t_{x_i} g^t_{x_j}(p)\right) = \nabla_{x_i}\nabla_{x_j} U(p) + A_{ij}(p) + B_{ij}(p), \tag{25}$$

where the coordinates of $A_{ij}(p) \in \mathbf{F}_p$ are

$$A^{\gamma}_{ij} = \left(\dot{\Gamma}^{\gamma}_{\alpha\beta}\frac{\partial g^{\alpha}}{x_i}\frac{\partial g^{\beta}}{x_j}\right),$$

with $\gamma = 1, 2, 3$, and

$$\dot{\Gamma}^{\gamma}_{\alpha\beta} = \frac{\partial\left({}^{\nu_t}\Gamma^{\gamma}_{\alpha\beta}\right)}{\partial t}.$$

The vector $B_{ij}(p) \in \mathbf{F}_p$ is given by

$$B_{ij}(p) = R\left(g_{x_i}(p), U(p)\right) g_{x_j}(p).$$

Remark. Since the second fundamental form of g^t is symmetric, we have

$$\nabla^G_{\frac{\partial}{\partial t}}\left(\nabla^t_{x_i} g^t_{x_j}(p)\right) = \nabla^G_{\frac{\partial}{\partial t}}\left(\nabla^t_{x_j} g^t_{x_i}(p)\right).$$

Therefore the right-hand sides in the corresponding formulas (25) have to agree. This can be verified by observing that $A_{ij}(p) = A_{ji}(p)$ since $\dot{\Gamma}^{\gamma}_{\alpha\beta} = \dot{\Gamma}^{\gamma}_{\beta\alpha}$ (for the Levi-Civita connection is torsion free), and that

$$\nabla_{x_i}\nabla_{x_j}U(p) - \nabla_{x_j}\nabla_{x_i}U(p) = B_{ji}(p) - B_{ij}(p),$$

which follows from the Bianchi identities.

Proof. For fixed $\gamma \in \{1,2,3\}$, and $i, j \in \{1,2\}$, the corresponding component of the second fundamental form $\nabla^t dg^t$ is given by

$$\left(\nabla^t_{x_i} dg^t\left(\frac{\partial}{\partial x_j}\right)\right)^{\gamma} = \frac{\partial^2 (g^t)^{\gamma}}{\partial x_i x_j} - {}^{\mu}\Gamma^k_{ij}\frac{\partial (g^t)^{\gamma}}{x_k} + {}^{\nu_t}\Gamma^{\gamma}_{\alpha\beta}\frac{\partial (g^t)^{\alpha}}{x_i}\frac{\partial (g^t)^{\beta}}{x_j},$$

(see the third displayed formula on page 15 in Section (2.5) in [8]). As usual, we abbreviate $\nabla_{\frac{\partial}{\partial x_i}} = \nabla_{x_i}$. Then,

$$\left(\nabla^t_{x_i} dg^t\left(\frac{\partial}{\partial x_j}\right)\right)^{\gamma} = \left(\nabla_{x_i} dg^t\left(\frac{\partial}{\partial x_j}\right)\right)^{\gamma} + t\left(\dot{\Gamma}^{\gamma}_{\alpha\beta}\frac{\partial g^{\alpha}}{x_i}\frac{\partial g^{\beta}}{x_j}\right) + O(t^2).$$

We obtain

$$\nabla^G_{\frac{\partial}{\partial t}}\left(\nabla^t_{x_i} g^t_{x_j}\right) = \nabla^G_{\frac{\partial}{\partial t}}\left(\nabla_{x_i} g^t_{x_j}\right)(p) + A_{ij}(p). \tag{26}$$

It remains to compute

$$\nabla^G_{\frac{\partial}{\partial t}}\left(\nabla_{x_i} g^t_{x_j}\right)(p).$$

As shown by Eells-Lemaire (see the bottom of the page 27 in Section 4 in [8])

$$\left(\nabla_{\frac{\partial}{\partial t}}\nabla_{x_i} g^t_{x_j}\right)(p) = \nabla_{x_i}\nabla_{x_j}U(p) + R\left(g_{x_i}(p), U(p)\right) g_{x_j}(p).$$

Thus, we get

$$\nabla^G_{\frac{\partial}{\partial t}}\left(\nabla_{x_i} g^t_{x_j}\right)(p) = \nabla_{x_i}\nabla_{x_j}U(p) + B_{ij}(p).$$

Together with (26) this implies (25).

□

2.4. **A varying family of harmonic maps.** We now compute the second derivative of g^t assuming it is harmonic with respect to the metric ν_t on M.

Proposition 2.3. *Let $g^t : (\Sigma, \mu) \to (M, \nu_t)$ be a family of harmonic maps where (Σ, μ) is a fixed Riemann surface. Suppose that Rank$(g_w)(p) = 1$, and that w is the local complex parameter near p such that Hopf$(g) \equiv 1$ (recall that $g = g^0$). Then*

$$\nabla^G_{\frac{\partial}{\partial t}}\left(\nabla^t_w g^t_w(p)\right) = -\nabla_v\nabla_v U(p) - \mathbf{i}\,\nabla_u\nabla_v U(p). \tag{27}$$

Proof. We have

$$\nabla^t_w g^t_w(p) = \frac{1}{2}\left(\left(\nabla^t_u g^t_u(p) - \nabla^t_v g^t_v(p)\right) - \mathbf{i}\left(\nabla^t_u g^t_v(p) + \nabla^t_v g^t_u(p)\right)\right).$$

Since g^t is harmonic we apply (3) and find that $\nabla^t_u g^t_u(p) = -\nabla^t_v g^t_v(p)$. Moreover, the second fundamental form $\nabla^t dg^t$ is symmetric and we get $\nabla^t_u g^t_v(p) = \nabla^t_v g^t_u(p)$. Replacing this in the above formula yields

$$\nabla^t_w g^t_w(p) = -\nabla^t_v g^t_v(p) - \mathbf{i}\,\nabla^t_u g^t_v(p). \tag{28}$$

On the other hand, we are assuming that $\mathrm{Rank}(g_w)(p) = 1$ and $\mathrm{Hopf}(g) \equiv 1$ near p. Thus, $g_v(p) = 0$. Replacing this into (25) implies

$$\nabla^G_{\frac{\partial}{\partial t}} \left(\nabla^t_v g^t_v(p)\right) = \nabla_v \nabla_v U(p),$$

$$\nabla^G_{\frac{\partial}{\partial t}} \left(\nabla^t_u g^t_v(p)\right) = \nabla_u \nabla_v U(p),$$

because the corresponding A's and B's vanish since $g_v(p) = 0$. Replacing this into (28) proves the proposition.

□

2.5. **The derivative of Φ.** Fix a point $(p_0, \mu_0, \nu_0) \in \mathcal{X}$ and let $N \subset \mathcal{X}$ be a small neighborhood of this point. Define the auxiliary map

$$\Phi : N \to \mathcal{E}$$

by

$$\Phi(p, \mu, \nu) = \left(\sigma(f_w(p)),\, -\frac{1}{2}\sigma\left(\nabla_w f_w(p) - \frac{\langle \nabla_w f(p)_w, f_w(p)\rangle}{\langle f_w(p), f_w(p)\rangle} f_w(p)\right)\right),$$

where $f = f_{\mu,\nu}$. Recall that $\sigma : TM^{\mathbb{C}} \to \mathbb{C}^3 \oplus \mathbb{C}^3$ is the projection. Here w is the fixed local complex parameter near p_0 such that $\mathrm{Hopf}(f^0) \equiv 1$, where $f^0 = f_{\mu_0,\nu_0}$. The map Φ is closely related to the map Ψ (we explain this connection below). Right now, we compute the derivative of Φ.

Proposition 2.4. *Suppose that $\mathrm{Rank}(f^0_w)(p_0) = 1$. Let V be the section of $\mathbf{F}$ determined by the equation $\mathbf{J}V = \mathcal{G}_{\mu,\nu}(\dot\nu)$. Assuming $V(p_0) = 0$, we have*

$$d\Phi(0,0,\dot\nu) = \left(\sigma(\nabla_w V(p_0)),\, -\frac{1}{2}\sigma\left(\varphi(0,0,\dot\nu)(p_0) - \langle \nabla_w f^0_w(p_0), \nabla_w V(p_0)\rangle f^0_w(p_0)\right)\right), \tag{29}$$

where

$$\varphi(0,0,\dot\nu)(p_0) = \nabla^G_{\frac{\partial}{\partial t}}\left(\nabla^t_w f^t_w\right)(p_0) - \langle \nabla^G_{\frac{\partial}{\partial t}}\left(\nabla^t_w f^t_w\right)(p_0), f^0_w(p_0)\rangle f^0_w(p_0).$$

Proof. Since Φ maps N into $\mathcal{E} \subset \mathbb{C}^3 \oplus \mathbb{C}^3$, the derivative $d\Phi(0,0,\dot\nu)$ is a vector in the tangent space $T\mathcal{E} \subset T(\mathbb{C}^3 \oplus \mathbb{C}^3)$. Since we assume $V(p_0) = 0$, the vector $d\Phi(0,0,\dot\nu) \in T(\mathbb{C}^3 \oplus \mathbb{C}^3)$ can be identified with the vertical lift of the corresponding vector from $\mathbb{C}^3 \oplus \mathbb{C}^3$.

Moreover, we can use any connection to differentiate $\Phi(p_0, \mu_0, \nu_t)$ because the points $f^t(p_0)$ and $f^0(p_0)$ are $O(t^2)$ close to each other so for the purposes of differentiation we may assume they are the same. For the same reason the derivative of σ does not enter the computation either. We use the connection ∇^G defined above.

Combining Proposition 2.3 and formula (24) we obtain (29). We also use the facts that $\langle \nabla_w f^0_w(p_0), f^0_w(p_0)\rangle = 0$ and $\langle f^0_w(p_0), f^0_w(p_0)\rangle = 1$ which hold because w is the natural parameter for the map f^0 (this takes care of extra terms that appear after differentiating $\Phi(p_0, \mu_0, \nu_t)$).

□

2.6. **The derivative of Ψ.** Recall that $z = z_\mu$ is the complex parameter on $\widehat{\Sigma}$ depending only on μ. We let α be the local conformal map near p_0 such that $\alpha(z) = w$. Then $\lambda(p) f_w(p) = f_z(p)$ and $\mathbf{M}(f)(p) = \lambda^2(p)\mathbf{M}(f \circ \alpha)(p)$, where $\lambda(p) = \alpha'(p)$ (thus $\lambda(p) \neq 0$). This yields the formula

$$\Psi(p_0,\mu_0,\nu_t) = \left(\lambda\sigma(f^t_w(p_0)),\, -\frac{\lambda^2}{2}\sigma\left(\nabla_w f^t_w(p_0) - \frac{\langle \nabla_w f^t_w(p_0), f^t_w(p_0)\rangle}{\langle f^t_w(p_0), f^t_w(p_0)\rangle} f^t_w(p_0)\right)\right),$$

where $\lambda = \lambda(p_0)$. Together with Proposition 2.4 this yields the following proposition.

Proposition 2.5. *Suppose that Rank$(f^0_z)(p_0)$ = Rank(f^0_w) = 1. Let V be the section of $\mathbf{F}$ determined by the equation $\mathbf{J}V = \mathcal{G}_{\mu,\nu}(\dot{\nu})$. Assuming $V(p_0) = 0$, we have*

(30)

$$d\Psi(0,0,\dot{\nu}) = \left(\lambda\sigma(\nabla_w V(p_0)),\, -\frac{\lambda^2}{2}\sigma\big(\varphi(0,0,\dot{\nu})(p_0) - \langle \nabla_w f^0_w(p_0), \nabla_w V(p_0)\rangle f^0_w(p_0)\big)\right),$$

where

$$\varphi(0,0,\dot{\nu})(p_0) = \nabla^G_{\frac{\partial}{\partial t}}\left(\nabla^t_w f^t_w\right)(p_0) - \langle \nabla^G_{\frac{\partial}{\partial t}}\left(\nabla^t_w f^t_w\right)(p_0), f^0_w(p_0)\rangle f^0_w(p_0).$$

Remark. Recall the function ξ from the definition of Ψ. Note that $\xi \equiv 1$ on the set where $\mathrm{Rank}(f_z)(p) < 2$, so ξ does not figure in the formula for $d\Psi$ as we are only computing it at the points where $\mathrm{Rank}(f_z)(p) < 2$ (they are the only points in $\mathcal{X}$ that are mapped to $\mathcal{L}$ under the map Ψ).

Proposition 2.6. *Suppose that $(p,\mu,\nu) \in \mathcal{X}$ is such that Rank$(f_z)(p) = 1$. Let $Z, W \in \mathbf{E}_p$ be any two vectors and suppose that there exists $\dot{\nu}$ such that $V(p) = 0$, $\nabla_w V(p) = Z$, and $\nabla_v\nabla_v V(p) + \mathbf{i}\,\nabla_u\nabla_v V(p) = W$, where $\mathbf{J}V = \mathcal{G}_{\mu,\nu}(\dot{\nu})$ and w the local parameter near p such that Hopf$(f) \equiv 1$ near p. Then $d\Psi : T_{(p,\mu,\nu)}\mathcal{X} \to T\mathcal{E}$ is surjective, and in particular Ψ is transverse to $\mathcal{L}$ at those points.*

Proof. Follows readily from Proposition 2.3, Proposition 2.5, and the description of the tangent space of $\mathcal{E}$. Indeed, let $(A,B) \in \mathcal{E}$. Then $T_{(A,B)}\mathcal{E}$ is the vector space $T_{(A,B)}\mathcal{E} = \{(X,Y) \in \mathbb{C}^3 \oplus \mathbb{C}^3 : \langle A, Y\rangle + \langle X, B\rangle = 0\}$.

Letting

$$A = \lambda\sigma(f_w(p_0)), \quad B = -\frac{\lambda^2}{2}\sigma\left(\nabla_w f_w(p_0)\right),$$

we note that the image of $d\Psi : T_{(p,\mu,\nu)}\mathcal{X} \to T\mathcal{E}$ contains every pair (X,Y) of the form

$$X = \lambda Z, \qquad Y = \left(\left(\frac{\lambda^2}{2}W\right) - \frac{\langle \left(\frac{\lambda^2}{2}W\right), A\rangle}{\langle A, A\rangle}A - \frac{\langle B, X\rangle}{\lambda^2}A\right).$$

Here we use $\langle A, A\rangle = \lambda^2$. It is now clear that each pair $(X,Y) \in T_{(A,B)}\mathcal{E}$ is picked up for some values of Z, W. The proposition is proved.

$\square$

3. The Reproducing formulas

As stated in (21), if $V = dF(\dot{\nu})$ then $\mathbf{J}V = \mathcal{G}_{\mu,\nu}(\dot{\nu})$. To prove Lemma 4.1 we need to be able to compute the values of V, $\nabla_z V$, and $\nabla_z \nabla_z V$, at a point $p \in \Sigma$, in terms of $\mathbf{J}V$. This is the content of the Reproducing Formulas we establish in this section. These formulas hold for any harmonic surface $f : (\Sigma, \mu) \to (M, \nu)$ whose Jacobi operator is an isomorphism.

3.1. **Interior elliptic regularity theory.** We recall standard estimates and facts from the Elliptic Regularity Theory that we use. See Sections 10.1-10.4 in [25] for the references.

For $1 \le p < \infty$ we let $L^p(\mathbf{F})$ be the space of sections that are bounded in the L^p norm $||\cdot||_p$ and by $W^{2,p}(\mathbf{F})$ the Sobolev space of sections (with the Sobolev norm $||\cdot||_{2,p}$) whose second derivatives are in $L^p(\mathbf{F})$. For $0 < \alpha < 1$ and an integer $k \ge 0$, we let $C^{k,\alpha}(\mathbf{F})$ denote the Holder space of sections, with the Holder norm $||\cdot||_{k,\alpha}$, whose k-th derivatives are α-Holder.

Below we recall some basic facts and estimates. The Sobolev inequality compares different Sobolev norms.

Proposition 3.1. *Let $1 < p < 2$ and $2 < q$ be such that $1 - \frac{2}{q} = 2 - \frac{2}{p}$. Then there exists a constant $C > 0$ such that*

$$||V||_{1,q} \le C||V||_{2,p}, \quad V \in \Gamma(\mathbf{F}).$$

The following is the Morrey inequality which relates the Sobolev and the Holder norms (see Theorem 10.2.25 and it's proof in [25]).

Proposition 3.2. *For every $p > 2$ there exists a constant C such that*

$$||V||_{0,\alpha} \le C||V||_{1,p}, \quad V \in \Gamma(\mathbf{F})$$

where $\alpha = 1 - \frac{2}{p}$.

We recall the basic Elliptic Interior estimates.

Lemma 3.1. *Let $p > 1$ and $0 < \alpha < 1$. Then there exists a constant $C > 0$ such that for every $V \in \Gamma(\mathbf{F})$ the Interior Elliptic estimates*

$$||V||_{2,p} \le C(||\mathbf{J}V||_p + ||V||_p) \tag{31}$$

$$||V||_{2,\alpha} \le C(||\mathbf{J}V||_{0,\alpha} + ||V||_{0,\alpha}) \tag{32}$$

and the Poincaré inequality

$$||\mathbf{J}V||_p \le C||V||_p \tag{33}$$

hold. Moreover, for every $p > 1$ and $0 < \alpha < 1$, the operator $\mathbf{J}$ extends to continuous operator $\mathbf{J} : W^{2,p}(\mathbf{F}) \to L^p(\mathbf{F})$ and $\mathbf{J} : W^{2,\alpha}(\mathbf{F}) \to C^{0,\alpha}(\mathbf{F})$. If $\mathbf{J} : \Gamma(\mathbf{F}) \to \Gamma(\mathbf{F})$ is an isomorphism then so are the extensions.

Given an open set $\Omega \subset \Sigma$ and $V \in \Gamma(\mathbf{F})$, we write $||V||_{p,\Omega}$, $||V||_{2,p,\Omega}$, and $||V||_{k,\alpha,\Omega}$, for the above defined norms of the restriction of V to Ω. Let Ω' be another open subset of Σ which is compactly contained in Ω. The following versions of (31) and (32) hold

$$||V||_{2,p,\Omega'} \le C(||\mathbf{J}V||_{p,\Omega} + ||V||_{p,\Omega}) \tag{34}$$

$$||V||_{2,\alpha,\Omega'} \le C(||\mathbf{J}V||_{0,\alpha,\Omega} + ||V||_{0,\alpha,\Omega}) \tag{35}$$

Finally we state the regularity of L^p weak solutions of the equation $\mathbf{J}V = 0$.

Proposition 3.3. *Let $\Omega \subset \Sigma$ be an open set and suppose that for some $1 < p < \infty$ we have $||V_{p,\Omega}|| < \infty$. If $\mathbf{J}V = 0$ weakly on Ω, then V is smooth on Ω and $\mathbf{J}V \equiv 0$. Moreover, if (Σ, μ) and (M, ν) are real analytic, then so is V.*

3.2. **The zeroth derivatives.** Throughout the rest of this section we fix a harmonic map $f : (\Sigma, \mu) \to (M, \nu)$ such that $\mathbf{J}_f = \mathbf{J}$ is an isomorphism. Fix a point $x \in \Sigma$ and let $\mathbf{F}_x$ denote the fiber of $\mathbf{F}$ over x and $T_x\Sigma$ the tangent space to Σ at x.

Lemma 3.2. *We let $U \in \mathbf{F}_x$ be a unit vector. Then there exists a smooth section $\mathcal{K} = \mathcal{K}(U, x) : \Sigma \setminus \{x\} \to \mathbf{F}$ such that*

$$\langle V(x), U \rangle = \int\limits_{\Sigma} \langle \mathbf{J}V, \mathcal{K} \rangle \, dA$$

for every section $V \in \Gamma(\mathbf{F})$. Moreover, $\mathbf{J}\mathcal{K}(p) = 0$ for every $p \in \Sigma \setminus \{x\}$ and $\mathcal{K} \in L^r(\mathbf{F})$ for every $r \geq 1$.

Proof. Let $V \in \Gamma(\mathbf{F})$ and $q > 2$. Set $\alpha = 1 - \frac{2}{q}$. Applying first Proposition 3.2, and then Proposition 3.1, to estimate the norm $||V||_{0,\alpha}$ from above, we obtain

$$|V(x)| \leq ||V||_{0,\alpha} \leq C|||V||_{1,q} \leq C_1|||V||_{2,p},$$

where p is determined by $p = \frac{2q}{q+2}$. Using (31) and (33) to estimate the norm $||V||_{2,p}$ from above, yields the inequality

$$|V(x)| \leq C_2||\mathbf{J}V||_p. \tag{36}$$

Define $\lambda : \Gamma(\mathbf{F}) \to \mathbb{R}$ by

$$\lambda(W) = \langle V(x), U \rangle$$

where $V = \mathbf{J}^{-1}W$. From (36) we get that for each $W \in \Gamma(\mathbf{F})$ the inequality

$$\begin{aligned} |\lambda(W)| &= |\langle V(x), U \rangle| \leq |U| \, |V(x)| \\ &\leq C_2||\mathbf{J}V||_p = C_2||W||_p \end{aligned}$$

holds. Since $\Gamma(\mathbf{F})$ is dense in $L^r(\mathbf{F})$ it follows that λ extends to a bounded linear functional on $L^p(\mathbf{F})$. By the Riesz Representation Theorem there exists $\mathcal{K} \in L^r(\mathbf{F})$ such that

$$\lambda(W) = \int\limits_{\Sigma} \langle W, \mathcal{K} \rangle \, dA.$$

Here $\frac{1}{p} + \frac{1}{r} = 1$, so by choosing the appropriate $q > 2$ we can reach every $r > 2$.

It remains to show $\mathcal{K}$ is smooth, and that $\mathbf{J}\mathcal{K} = 0$ away from x . Let $V \in \Gamma(\mathbf{F})$ be any section which is equal to zero near x. From the definition of $\mathcal{K}$, and using that $\mathbf{J}$ is self-adjoint, we obtain

$$0 = \langle V(x), U \rangle = \int\limits_{\Sigma} \langle \mathbf{J}V, \mathcal{K} \rangle \, dA = \int\limits_{\Sigma} \langle V, \mathbf{J}\mathcal{K} \rangle \, dA.$$

Therefore, $\mathcal{K}$ is a weak solution of $\mathbf{J}\mathcal{K} = 0$ on $\Sigma \setminus \{x\}$. The lemma now follows from Proposition 3.3.

□

3.3. **The first derivatives.**

Lemma 3.3. *We let $X_1 \in T_x\Sigma$ and $U \in \mathbf{F}_x$ be unit vectors. Then there exists a smooth section $\mathcal{K} = \mathcal{K}(X_1, U, x) : \Sigma \setminus \{x\} \to \mathbf{F}$ such that*

$$\langle (\nabla_{X_1} V)(x), U \rangle = \int\limits_{\Sigma} \langle \mathbf{J}V, \mathcal{K} \rangle \, dA$$

for every section $V \in \Gamma(\mathbf{F})$. Moreover, $\mathbf{J}\mathcal{K}(p) = 0$ for every $p \in \Sigma \setminus \{x\}$ and $\mathcal{K} \in L^r(\mathbf{F})$ for every $1 \leq r < 2$.

Remark. In Appendix A we manually compute $\mathcal{K}$ using the ∂-bar technique applied to the complexified bundle $\mathbf{F} \otimes \mathbb{C}$ and the Koszul-Malgrange holomorphic structure on it.

Proof. Let $V \in \Gamma(\mathbf{F})$ and let $q > 2$ and set $\alpha = 1 - \frac{2}{q}$. Applying Proposition 3.2 to estimate the norm $||\nabla_{X_1} V||_{0,\alpha}$ from above, we get

$$|(\nabla_{X_1} V)(x)| \leq ||\nabla_{X_1} V||_{0,\alpha} \leq C|||\nabla_{X_1} V||_{1,q} \leq C||V||_{2,q}.$$

In the last inequality we used $|X_1| = 1$. Using (31) and (33) to estimate the norm $||V||_{2,q}$ from above yields the inequality

$$|(\nabla_{X_1} V)(x)| \leq C_1 ||\mathbf{J}V||_q. \tag{37}$$

Define $\lambda : \Gamma(\mathbf{F}) \to \mathbb{R}$ by

$$\lambda(W) = \langle (\nabla_{X_1} V)(x), U \rangle$$

where $V = \mathbf{J}^{-1} W$. From (38) we get that for each $W \in \Gamma(\mathbf{F})$ the inequality

$$\begin{aligned} |\lambda(W)| &= |\langle (\nabla_{X_1} V)(x), U \rangle| \leq |U| \, |(\nabla_{X_1} V)(x)| \\ &\leq C_1 ||\mathbf{J}V||_q = C_1 ||W||_q \end{aligned}$$

holds. Since $\Gamma(\mathbf{F})$ is dense in $L^q(\mathbf{F})$ it follows that λ extends to a bounded linear functional on $L^q(\mathbf{F})$. Let $1 < r < 2$ be given by the formula $\frac{1}{r} + \frac{1}{q} = 1$. By the Riesz Representation Theorem there exists $\mathcal{K} \in L^r(\mathbf{F})$ such that

$$\lambda(W) = \int\limits_{\Sigma} \langle W, \mathcal{K} \rangle \, dA.$$

That $\mathcal{K}$ is smooth and $\mathbf{J}\mathcal{K} = 0$ on $\Sigma \setminus \Omega$ is proved in exactly the same way as in the previous lemma.

$\square$

3.4. **The second derivatives.**

Proposition 3.4. *We let $X_1, X_2 \in T_x\Sigma$ and $U \in \mathbf{F}_x$ be unit vectors and denote by Ω a neighborhood (with C^1 boundary) of the point $x \in \Sigma$. Then there exists a smooth section $\mathcal{K}_\Omega = \mathcal{K}(X_1, X_2, U, x, \Omega) : \Sigma \setminus \Omega \to \mathbf{F}$ such that*

$$\langle (\nabla_{X_1} \nabla_{X_2} V)(x), U \rangle = \int\limits_{\Sigma} \langle \mathbf{J}V, \mathcal{K}_\Omega \rangle \, dA$$

for every section $V \in \Gamma(\mathbf{F})$ for which $\mathbf{J}V = 0$ on Ω. Moreover, $\mathbf{J}\mathcal{K}(p) = 0$ for every $p \in \Sigma \setminus \Omega$.

Proof. Let $\Gamma_0(\mathbf{F}) \subset \Gamma(\mathbf{F})$ be the subset containing sections which are equal to 0 on Ω, and let $V \in \Gamma(\mathbf{F})$ be such that $\mathbf{J}V \in \Gamma_0(\mathbf{F})$. Choose an open set $\Omega' \subset \Omega$ which contains x. From the definition of the Holder norm and using the assumption $|X_1|, |X_2| = 1$ we obtain

$$|(\nabla_{X_1}\nabla_{X_2}V)(x)| \leq ||V||_{2,\alpha,\Omega'}$$

for every $0 < \alpha < 1$. We use (35) to estimate $||V||_{2,\alpha,\Omega'}$ from above. Since $\mathbf{J}V = 0$ on Ω, we derive the estimate

$$|(\nabla_{X_1}\nabla_{X_2}V)(x)| \leq ||V||_{2,\alpha,\Omega'} \leq C||V||_{0,\alpha,\Omega} \leq C||V||_{0,\alpha}.$$

Let $q > 2$ and set $\alpha = 1 - \frac{2}{q}$. The previous inequality together with Proposition 3.2 yields

$$|(\nabla_{X_1}\nabla_{X_2}V)(x)| \leq C_1||V||_{1,q} \leq C_1||V||_{2,q}.$$

Together with (31) and (33) this gives

$$|(\nabla_{X_1}\nabla_{X_2}V)(x)| \leq C_2||\mathbf{J}V||_q. \tag{38}$$

Define $\lambda : \Gamma_0(\mathbf{F}) \to \mathbb{R}$ by

$$\lambda(W) = \langle(\nabla_{X_1}\nabla_{X_2}V)(x), U\rangle$$

where $V = \mathbf{J}^{-1}W$. Then

$$\begin{aligned}|\lambda(W)| &= |\langle(\nabla_{X_1}\nabla_{X_2}V)(x), U\rangle| \leq |U|\,|(\nabla_{X_1}\nabla_{X_2}V)(x)| \\ &\leq C_2||\mathbf{J}V||_q = C_2||W||_q.\end{aligned}$$

Denote by $L^q_0(\mathbf{F})$ the L^q sections of $\mathbf{F}$ supported in $\Sigma \setminus \Omega$. Since Ω has C^1 boundary $\Gamma_0(\mathbf{F})$ is dense in $L^q_0(\mathbf{F})$ (this is a standard result in the theory of Sobolev spaces, see [25]). Hence, λ extends to a bounded linear functional on $L^q_0(\mathbf{F})$. Let $1 < p < 2$ be given by the formula $\frac{1}{p} + \frac{1}{q} = 1$. Again, by the Riesz Representation Theorem there exists $\mathcal{K}_\Omega \in L^p_0(\mathbf{F})$ such that

$$\lambda(W) = \int\limits_{\Sigma\setminus\Omega} \langle W, \mathcal{K}_\Omega\rangle \, dA.$$

That $\mathcal{K}$ is smooth and $\mathbf{J}\mathcal{K} = 0$ on $\Sigma \setminus \Omega$ is proved in exactly the same way as above. $\square$

Let $\Omega(n)$ be a decreasing sequence of domains such that

$$\bigcap \Omega(n) = \{x\}.$$

If $n \geq m$ then $\mathcal{K}_{\Omega(n)} = \mathcal{K}_{\Omega(m)}$ on $\Omega(m)$. This follows from the uniqueness part of the Riesz Representation Theorem. Therefore, the sequence $\mathcal{K}_{\Omega(n)}$ converges to a smooth section $\mathcal{K} : \Sigma \setminus \{x\} \to \mathbf{F}$ such that $\mathbf{J}\mathcal{K} = 0$ away from x.

Let $V \in \Gamma(\mathbf{F})$ such that $\mathbf{J}V(x) = 0$. Denote by $\Gamma_n(\mathbf{F}) \subset \Gamma(\mathbf{F})$ the subset containing sections which are equal to 0 on $\Omega(n)$. Then one can find a sequence $V_n \in \Gamma_n(\mathbf{F})$ such that $\mathbf{J}V_n \to \mathbf{J}V$ in the $C^{0,\alpha}$-norm for every $0 < \alpha < 1$ (but not in the C^1 sense).

By Proposition 3.1 we have that $\mathbf{J} : W^{2,\alpha}(\mathbf{F}) \to C^{0,\alpha}(\mathbf{F})$ is a continuous isomorphism. Thus, $V_n \to V$ in the $W^{2,\alpha}$-norm which implies that $\nabla_{X_1}\nabla_{X_2}V_n \to$

$\nabla_{X_1}\nabla_{X_2}V$. Combining this with Proposition 3.4 yields the equality

$$\langle(\nabla_{X_1}\nabla_{X_2}V)(x), U\rangle = p.v. \int_{\Sigma} \langle \mathbf{J}V, \mathcal{K}\rangle \, dA.$$

The integral on the right hand side in the previous formula is defined as the Cauchy Principal Value. By construction, the limit

$$\lim_{n\to\infty} \int_{\Sigma\setminus\Omega(n)} \langle \mathbf{J}V, \mathcal{K}\rangle \, dA$$

exists whenever $\mathbf{J}V(p) = 0$, and V is smooth on Σ. This enables us to define

$$p.v. \int_{\Sigma} \langle \mathbf{J}V, \mathcal{K}\rangle \, dA = \lim_{n\to\infty} \int_{\Sigma\setminus\Omega(n)} \langle \mathbf{J}V, \mathcal{K}\rangle \, dA.$$

We have just established the following lemma.

Lemma 3.4. *We let $X_1, X_2 \in T_x\Sigma$ and $U \in \mathbf{F}_x$ be unit vectors. There exists a smooth section $\mathcal{K} = \mathcal{K}(X_1, X_2, U, x) : \Sigma \setminus \{x\} \to \mathbf{F}$ such that*

$$\langle(\nabla_{X_1}\nabla_{X_2}V)(x), U\rangle = p.v. \int_{\Sigma} \langle \mathbf{J}V, \mathcal{K}\rangle \, dA$$

for every $V \in \Gamma(\mathbf{F})$ for which $\mathbf{J}V(x) = 0$. Moreover, $\mathbf{J}\mathcal{K} = 0$ away from x.

Remark. The fact the the limit on the right-hand side exists (in the displayed formula in Lemma 3.4) tells us something about the growth of the kernel $\mathcal{K}$ near x. For example, if $\mathcal{K}$ is the real part of a meromorphic section of $\mathbf{E}$, then such section can have at most the second order pole at x. We use this observation in the next section.

Remark. In fact, one can compute $\mathcal{K}$ explicitly (see Appendix A) and show that it has a second order pole at x.

4. Proof of Theorem 1.3

Using Reproduction Formulas one can prescribe the 2-jet of a section $V \in \mathbf{F}$ at a point $p \in \Sigma$ by choosing appropriate $\mathbf{J}V$. This is exactly what is required to prove Theorem 1.3, except that in this case we have the constraint $\mathbf{J}V = \mathcal{G}(\dot\nu)$. So, we need to prove that there are sufficiently many tangent vectors $\dot\nu$ such that when we replace $\mathbf{J}V = \mathcal{G}(\dot\nu)$ in the Reproduction Formulas we can recover any given 2-jet at any given point on Σ.

This problem reduces to the following question: Does there exist a (non-trivial) reproducing kernel $\mathcal{K}$ which is annihilated by all possible sections of the form $\mathcal{G}(\dot\nu)$. To answer this question we employ the following result of Moore. Suppose that $\mathcal{K}$ is annihilated by all sections of the form $\mathcal{G}(\dot\nu)$, where $\dot\nu$ has a support in some open disc $\Omega \subset \Sigma^{\mathrm{reg}}(f)$. Then $\mathcal{K}$ is local section of the holomorphic line bundle $\mathbf{L} < \mathbf{E}$ on Ω.

Assuming $\mathcal{K}$ is real analytic (which is the case providing μ and ν are real analytic) we show that $\mathcal{K}$ is a holomorphic section of $\mathbf{L}$ on $\Sigma \setminus \{p\}$. Therefore, we need to ensure that the kernel $\mathcal{K}$ does not agree with a holomorphic local section of the line bundle $\mathbf{L}$ on some open subset of Σ.

4.1. **Special metrics.** We recall the notion of somewhere injective maps (originating in symplectic topology). A map $f : \Sigma \to M$ is somewhere injective if there exists an open subset $\Omega \subset \Sigma$ on which the restriction of f is injective.

Definition 4.1. *We say that a Riemann surface Σ is exceptional if one of the following holds:*

1. *Σ is a branched holomorphic cover of another Riemann surface of genus at least two,*
2. *Σ has an anti-holomorphic involution,*
3. *Σ is a branched holomorphic cover of the Riemann sphere of degree at most four.*

Definition 4.2. *We say that a pair of metrics $(\mu, \nu) \in \mathfrak{M}$ is special if it satisfies one of the following conditions:*

1. *$f_{\mu,\nu}$ is not somewhere injective,*
2. *$(\mu, \nu) \in \mathfrak{M}_0$ (that is, $f_{\mu,\nu}$ has a branch point),*
3. *the Riemann surface Σ_μ is exceptional.*

The above conditions are not mutually exclusive and there are pairs (μ, ν) which simultaneously satisfy several of them. By $\mathfrak{M}^{\mathrm{spc}}$ we denote the set of pairs $(\mu, \nu) \subset \mathfrak{M}$ which are not special. We prove below that $\mathfrak{M}^{\mathrm{spc}}$ is open, dense, and connected subset of $\mathfrak{M}$.

4.2. **Prescribing the 2-jet of a harmonic variation.** The next couple of lemmas show the image $dF(T\mathfrak{M}^{\mathrm{spc}})$ is sufficiently rich.

Lemma 4.1. *Suppose that genus of Σ is at least seven, and let $(\mu, \nu) \in \mathfrak{M}^{spc}$ be a pair of real analytic metrics such that $f_{\mu,\nu}$ is not minimal. Fix a local complex coordinate z near a point $p \in \Sigma_\mu$. Then, for any two vectors $Z, W \in \mathbf{E}_p$, we can find $\dot{\nu} \in T_\nu \mathfrak{M}(M)$ such that*

$$\nabla_z V(p) = Z, \qquad -\nabla_y \nabla_y V(p) - \mathbf{i}\, \nabla_x \nabla_y V(p) = W. \tag{39}$$

where $\mathcal{G}(\dot{\nu}) = V$ (we abbreviate $\mathcal{G} = \mathcal{G}_{\mu,\nu}$).

Remark. We explain the reasons we need to remove the special metrics to prove Lemma 4.1. The reason to exclude harmonic surfaces which are not somewhere injective is Proposition 4.1 below. This includes pairs (μ, ν) such that Σ_μ satisfies the first and the second condition in the definition of exceptional Riemann surfaces. The reason we exclude pairs (μ, ν) such that Σ_μ satisfies the third condition in the definition of exceptional Riemann surfaces is the proof of Proposition 4.4. We exclude the pairs from $\mathfrak{M}_0$ because the map Ψ is not defined on $\mathfrak{M}_0$. However, Lemma 4.1 holds for every real analytic pair $(\mu, \nu) \in \mathfrak{M}_0$ providing $f_{\mu,\nu}$ is not minimal.

4.3. **Proof of Lemma 4.1.** Lemma 4.1 follows from the following stronger result.

Lemma 4.2. *Suppose that genus of Σ is at least seven, and let $(\mu, \nu) \in \mathfrak{M}^{spc}$ be a pair of real analytic metrics such that $f_{\mu,\nu}$ is not minimal. Fix a local complex coordinate near a point $p \in \Sigma_\mu$. Then, for any five vectors $Z_j \in \mathbf{F}_p$, $j = 1, ...5$, we* *can find* $\dot{\nu} \in T_\nu \mathfrak{M}(M)$ *such that*

$$V(p) = Z_1, \qquad \nabla_x V(p) = Z_2, \qquad \nabla_y V(p) = Z_3, \tag{40}$$
$$\nabla_x \nabla_y V(p) = Z_4, \quad \nabla_y \nabla_y V(p) = Z_5,$$

where $\mathcal{G}(\dot{\nu}) = V$.

We prove Lemma 4.2 by contraposition. Suppose it is false, and that there are five vectors $Z_1, ..., Z_5 \in \mathbf{F}_p$, such that (40) fails for every V of the form $V = \mathcal{G}(\dot{\nu})$, where $\dot{\nu} \in T_\nu \mathfrak{M}(M)$. Considering the induced inner product on $\oplus_1^5 \mathbf{F}$, we can find a five-tuple of vectors $U_1, ..., U_5 \in \mathbf{F}_p$ (not all of them equal to the zero vector), which is orthogonal to every vector $V = \mathcal{G}(\dot{\nu})$. This yields the identity

$$\langle V(p), U_1 \rangle + \langle \nabla_x V(p), U_2 \rangle + \langle \nabla_y V(p), U_3 \rangle +$$
$$\langle \nabla_x \nabla_y V(p), U_4 \rangle + \langle \nabla_y \nabla_y V(p), U_5 \rangle = 0,$$

for every such V.

We invoke the Reproducing Formulas from Lemma 3.2, Lemma 3.3, and Lemma 3.4. Applying these three formulas, we find a smooth section $X : \Sigma \setminus \{p\} \to \mathbf{F}$, such that at the point p we have

$$\langle W, U_1 \rangle + \langle \nabla_x W, U_2 \rangle + \langle \nabla_y W, U_3 \rangle + \langle \nabla_x \nabla_y W, U_4 \rangle$$
$$+ \langle \nabla_y \nabla_y W, U_5 \rangle = \int_\Sigma \langle \mathbf{J} W, X \rangle \, dA, \tag{41}$$

for every $W \in \Gamma(\mathbf{F})$ such that $\mathbf{J} W(p) = 0$. Moreover, $\mathbf{J} X(p) = 0$ for every $x \in \Sigma \setminus \{p\}$, and X has at most the second order pole at p.

Claim 4.1. *We can find a section* $W \in \Gamma(\mathbf{F})$ *so that the six vectors* $W(p)$, $\nabla_x W(p)$, $\nabla_y W(p)$, $\nabla_x \nabla_y W(p)$, $\nabla_y \nabla_y W(p)$, *and* $\nabla_x \nabla_x W(p)$, *are equal to any six vectors in* $\mathbf{F}_p$. *Moreover, the section* $X : \Sigma \setminus \{p\} \to \mathbf{F}$ *from* (41) *is not identically equal to zero on* Σ.

Proof. To simplify the notation, in this proof we let $x = x_1$ and $y = x_2$. Fix $a, b \in \{1, 2\}$, we let $\alpha = \alpha_{(a,b)}$ be a real valued function defined near $p \in \Sigma$ such that

$$\alpha(p) = \alpha_{x_i}(p) = \alpha_{x_i x_j}(p) = 0, \qquad (i,j) \neq (a,b),\, (b,a)$$

and

$$\alpha_{x_a x_b}(p) = 1.$$

Let U be a any local section of $\mathbf{F}$ near p, and set $W = \alpha U$. Using the Leibniz rule, we get

$$(\alpha U)(p) = \nabla_{x_i}(\alpha U)(p) = \nabla_{x_i} \nabla_{x_j}(\alpha U)(p) = 0, \qquad (i,j) \neq (a,b),\, (b,a)$$

and

$$\nabla_{x_a} \nabla_{x_b}(\alpha U)(p) = \nabla_{x_b} \nabla_{x_a}(\alpha U)(p) = U(p).$$

Summing up the corresponding vectors $\alpha_{(a,b)} U$ over different pairs (a,b), we get

$$W_2 = \sum_{a,b=1}^{2} \alpha_{(a,b)} U.$$

This shows that there exists a section W_2 such that

$$\nabla_{x_1}\nabla_{x_1}W_2(p), \nabla_{x_1}\nabla_{x_2}W_2(p), \nabla_{x_2}\nabla_{x_2}W_2(p),$$

are equal to any three vectors in $\mathbf{F}_p$, while

$$W_2(p) = \nabla_{x_1}W_2(p) = \nabla_{x_2}W_2(p) = 0.$$

Remark. Since $\alpha_{x_i x_j} = \alpha_{x_j x_i}$, it follows that $\nabla_{x_i}\nabla_{x_j}W_2(p) = \nabla_{x_j}\nabla_{x_i}W_2(p)$. Thus, we can only prescribe the value of one of them and not the values of the two vectors individually.

Similarly, we can find a section W_1 such that $\nabla_1 W_1(p)$ and $\nabla_2 W_1(p)$ are equal to any two vectors in $\mathbf{F}_p$ that we want, while $W_1(p) = 0$. Moreover, we can find W_0 such that $W_0(p)$ equals any vector in $\mathbf{F}_p$ that we need. Adding up the appropriate sections W_0, W_1, and W_2, we obtain the section W which satisfies the condition of this claim. The claim is proved.

To show that X is not identically zero, we find W such that the right-hand side of (41) is positive. We still have the freedom to prescribe the value of the vector $\nabla_x\nabla_x W(p) + \nabla_y\nabla_y W(p)$. We do this so that $\mathbf{J}W(p) = 0$ (see (17)). Therefore, there is a section $W \in \Gamma(\mathbf{F})$ such that the left-hand side in (41) is not zero, thus X is not identically zero.

□

We conclude

$$\int\limits_{\Sigma} \langle \mathbf{J}V, X\rangle \, dA = 0, \tag{42}$$

for every $V = \mathcal{G}(\dot{\nu})$, where $\dot{\nu} \in T_\nu\mathfrak{M}(M)$. We show this is not possible when $(\mu,\nu) \in \mathfrak{M}^{\mathrm{spc}}$, providing (μ,ν) are real analytic.

4.4. **The wealth of harmonic variations.** Recall that f_z can be viewed as a local holomorphic section of the vector bundle $\mathbf{E}$ (with respect to the corresponding holomorphic structure on $\mathbf{E}$). By $\mathbf{L}$ we denote the holomorphic line bundle induced by f_z.

For $\Omega \subset \Sigma$, we say that a local section $V : \Omega \to \mathbf{F}$ is tangential if it is tangent to $f(\Omega)$ at the regular points of f. While there are no global Jacobi fields in $\Gamma(\mathbf{F})$ (since $\mathbf{J}$ is an isomorphism), there are local sections $V : \Omega \to \mathbf{F}$ which are (local) Jacobi fields, that is $\mathbf{J}V = 0$ on Ω. For example, the real (or imaginary) parts of local holomorphic sections of the line bundle $\mathbf{L}$ are examples of local Jacobi fields in $\mathbf{F}$. These local Jacobi fields are tangential.

Remark. However, not every tangential local Jacobi field $V : \Omega \to \mathbf{F}$ is the real (or imaginary) part of local holomorphic section of $\mathbf{L}$!

The following proposition is due to Moore (see Lemma 3.1 in [23]). The computation was repeated in Lemma 5.4.2 in [24] ($\mathcal{G}$ agrees with the Metric Deformation Operator defined on page 311 [24]).

Proposition 4.1. *Let $\Omega \subset \Sigma^{reg}(f)$ be an open subset of the regular set of f, and assume $f = f_{\mu,\nu}$ is injective on Ω (and thus an embedding). Suppose X is a smooth section $\Omega \to \mathbf{F}$. If*

$$\int\limits_{S} \langle \mathcal{G}(\dot{\nu}), X\rangle \, dA = 0$$

for every $\dot\nu \in T_\nu \mathfrak{M}(M)$ whose support is contained in Ω, then each point $p \in \Omega$ has a neighborhood on which X equals the real part of a local holomorphic section of $\mathbf{L}$.

Remark. Actually, it suffices to assume that X is locally L^2 (the proof then follows from the Moore's computation and the Weyl's lemma).

Proof. The reader can find the proof in the above cited sources. Here we only offer a brief sketch. Also, later in the paper we prove a version of this proposition for maps which are not somewhere injective, where we give more details.

Let $z = x_1 + \mathbf{i}\, x_2$ be a local complex parameter on Ω. Then, we can find local coordinates (u_1, u_2, u_3) near $f(\Omega) \subset M$ with the following properties

(1) $u_1 \circ f = x_1$, $u_2 \circ f = x_2$, and $u_3 \circ f = 0$, (thus, $f(\Omega)$ is contained in the coordinate plane $u_3 = 0$),

(2) $\nu_{13} = \nu_{23} = 0$ and $\nu_{33} \equiv 1$, when restricted to the plane $u_3 = 0$,

(3) the metric μ is conformal on Ω.

This normalization provides us with the identification of Ω with $f(\Omega)$. In these coordinates $\frac{\partial f^i}{\partial x_j} = 1$ iff $i = j$, and is zero otherwise, at every point in Ω. Therefore, the second derivatives of f are all equal to zero.

Remark. This choice of coordinates was used by Moore (see page 17 in [23] or page 311 in [24]). It is clear that one can arrange for the first two conditions to hold. To arrange that (3) holds simultaneously we solve the Beltrami equation (in the plane $u_3 = 0$) to obtain a diffeomorphism h which provides the transition to the new coordinate system in which μ is in conformal form. This diffeomorphism h is then extended from the plane $u_3 = 0$ to a 3-dimensional neighborhood of $f(\Omega)$ by letting $h(u_1, u_2, u_3) = (h(u_1, u_2), u_3)$. We conjugate everything by h and verify that the three conditions are satisfied.

On Ω (which is identified with $f(\Omega)$) we write X as

$$X = \sum_{i=1}^{3} X^i \frac{\partial}{\partial u_i},$$

where X^i are the real valued functions on Ω. Choosing suitable $\dot\nu$ one first shows that $X^3 \equiv 0$ (this is the part where the injectivity of f on Ω is crucial). Thus, X is a tangential section of $\mathbf{F}$ over Ω.

Furthermore, one shows that for any two real valued test functions α and β on Ω, we have

$$\int\limits_{\Omega} \alpha \left(\frac{\partial X^1}{\partial x_1} - \frac{\partial X^2}{\partial x_2} \right) + \beta \left(\frac{\partial X^1}{\partial x_2} + \frac{\partial X^2}{\partial x_1} \right) du_1 du_2 = 0.$$

It follows that

$$\left(\frac{\partial X^1}{\partial x_1} - \frac{\partial X^2}{\partial x_2} \right) \equiv \left(\frac{\partial X^1}{\partial x_2} + \frac{\partial X^2}{\partial x_1} \right) \equiv 0 \tag{43}$$

on Ω. Let

$$Z = (X^1 + iX^2) \left(\frac{\partial}{\partial x_1} - \mathbf{i} \frac{\partial}{\partial x_2} \right).$$

Then $\mathrm{Re}(Z) = X$. It remains to show that Z is a holomorphic section of $\mathbf{L}$. First of all, in these coordinates we have

$$f_z = \frac{1}{2}\left(f_{x_1} - \mathbf{i}\, f_{x_2}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - \mathbf{i}\,\frac{\partial}{\partial x_2}\right).$$

On the other hand, the equations (43) are the Cauchy-Riemann equations and therefore $h = X^1 + \mathbf{i}\, X^2$ is a holomorphic function on Ω. Thus $Z = hf_z$ is a holomorphic section of $\mathbf{L}$.

□

4.5. **Proof of Lemma 4.2.** Since μ and ν are real analytic so is X (see Proposition 3.3). On the other hand, we know from Proposition 4.1 that X agrees with the real part of a local holomorphic section of $\mathbf{L}$ on an open set $\Omega \subset \Sigma$.

Let $\gamma \subset \Sigma$ be an arc whose initial point belongs to Ω. It follows from the unique continuation theorem that X is the real part of a holomorphic section of $\mathbf{L}$ in some neighborhood of γ. Thus, we can show that X is the real part of a holomorphic section of $\mathbf{L}$ in some neighborhood of any point on Σ.

But in general this does not mean that there is a single holomorphic section of $\mathbf{L}$ whose real part agrees with X. However, this holds true in our case as the following proposition shows.

Proposition 4.2. *Let W denote a local (not necessarily holomorphic) section of $\mathbf{L}$. Then Re(W) is not identically zero.*

Proof. With respect to some local complex parameter $z = x + \mathbf{i}\, y$, the section W is written as $W = hf_z$ for some complex valued function $h = h_1 + \mathbf{i}\, h_2$. Then $\mathrm{Re}(W) = h_1 f_x + h_2 f_y = 0$. If $\mathrm{Re}(W) \equiv 0$, then f_x and f_y are linearly dependent vectors and $\mathrm{Rank}(df) < 2$ at these points. This is impossible since the set of regular points is open and dense in Σ.

□

Thus, we have shown the following proposition.

Proposition 4.3. *X is the real part of a holomorphic section*

$$Z : \Sigma \setminus \{p\} \to \mathbf{L}.$$

The proof of Lemma 4.2 follows from the next proposition.

Proposition 4.4. *Suppose that $f = f_{\mu,\nu}$ is not a minimal map and that there are no holomorphic maps from Σ_μ to the Riemann sphere of degree at most four. Let $p \in \Sigma$. Then, there is no holomorphic section $Z : \Sigma \setminus \{p\} \to \mathbf{L}$ such that Z has at most the second order pole at p.*

Proof. Suppose Z is a such a section. Set $\varphi = \langle Z, Z\rangle$. We show that φ is a (non-constant) holomorphic function on $\Sigma \setminus \{p\}$. In local coordinates

$$Z\, dz = hf_z$$

for some local holomorphic function h. Therefore,

$$\varphi\, dz^2 = h^2 \langle f_z, f_z\rangle.$$

Since f is not minimal it follows that $\langle f_z, f_z\rangle$ is not identically equal to zero. Thus, φ is a non-constant holomorphic function on $\Sigma \setminus \{p\}$.

If Z has at most the second order pole at p it follows that φ has at most the fourth order pole at p (and no other poles on Σ). But then $\varphi : \Sigma \to \mathbf{S}^2$ is a branched cover of degree at most four, where $\mathbf{S}^2$ is the Riemann sphere. Such pairs (μ, ν) are excluded from $\mathfrak{M}^{\rm spc}$. This concludes the proof.

$\square$

4.6. **Proof of Theorem 1.3.** In the remainder of the paper we establish the following lemma.

Lemma 4.3. *The set $\mathfrak{M}^{spc}$ is an open, dense, and connected subset of $\mathfrak{M}$.*

Assuming this lemma, we first prove Theorem 1.3. .

Definition 4.3. *We let $\mathfrak{M}' \subset \mathfrak{M}^{spc}$ be the maximal subset of $\mathfrak{M}^{spc}$ such that the restriction $\Psi : \mathfrak{M}' \to \mathcal{E}$ is transverse to $\mathcal{L}$.*

Clearly, $\mathfrak{M}'$ is an open subset of $\mathfrak{M}^{\rm spc}$. Moreover, the following holds.

Proposition 4.5. *Suppose $(\mu, \nu) \in \mathfrak{M}^{spc}$ is a pair of real analytic metrics. Then $(\mu, \nu) \in \mathfrak{M}'$.*

Proof. If $f_{\mu,\nu}$ is a minimal immersion and $(\mu, \nu) \in \mathfrak{M}^{\rm spc}$, then $(\mu, \nu) \in \mathfrak{M}'$ because $\Psi(p, \mu, \nu) \notin \mathcal{L}$ for such (μ, ν). We note that Lemma 4.1 applies to every other $(\mu, \nu) \in \mathfrak{M}^{\rm spc}$. Assume now that $f_{\mu,\nu}$ is not a minimal immersion. The proof of the proposition now follows directly from Lemma 4.1 and Proposition 2.6.

$\square$

Thus, $\mathfrak{M}'$ is an open subset of $\mathfrak{M}^{\rm spc}$. Let us show that $\mathfrak{M}'$ is also a dense and connected subset of $\mathfrak{M}^{\rm spc}$. Together with Lemma 4.3 this will imply that $\mathfrak{M}'$ is an open, dense, and connected subset of $\mathfrak{M}$, which yields Theorem 1.3.

Real analytic pairs of metrics are dense among all pairs of metrics. Thus $\mathfrak{M}'$ is dense in $\mathfrak{M}^{\rm spc}$. It remains to show it is connected. Suppose $(\mu_i, \nu_i) \in \mathfrak{M}'$, $i = 0, 1$.

Since $\mathfrak{M}^{\rm spc}$ is connected (as per Lemma 4.3 above), there exists a path $\gamma : [0, 1] \to \mathfrak{M}^{\rm spc}$ connecting (μ_0, ν_0) and (μ_1, ν_1). We can then find a nearby path $\widehat{\gamma} : [0, 1] \to \mathfrak{M}^{\rm spc}$ such that the pairs of metrics in the image $\widehat{\gamma}([0, 1])$ are real analytic (we can arrange that $\widehat{\gamma}$ is real analytic in t but this is not needed). From Proposition 4.5 we find that $\widehat{\gamma}$ maps into $\mathfrak{M}'$, that is, $\widehat{\gamma} : [0, 1] \to \mathfrak{M}'$.

We construct such $\widehat{\gamma} : [0, 1] \to \mathfrak{M}'$ so that the points $\gamma(0) = (\mu_0, \nu_0)$, and $\widehat{\gamma}(0)$ live in the same chart in $\mathfrak{M}'$. Thus, there exists a path $\alpha : [0, 1] \to \mathfrak{M}'$ connecting $\gamma(0)$ and $\widehat{\gamma}(0)$ (in fact, we can choose α to be the straight line in the corresponding local chart of $\mathfrak{M}'$). Similarly, we can arrange that there exists a path $\beta : [0, 1] \to \mathfrak{M}'$ connecting $\gamma(1) = (\mu_1, \nu_1)$ and $\widehat{\gamma}(1)$. The concatenation $\alpha \cdot \widehat{\gamma} \cdot \beta$ maps $[0, 1]$ into $\mathfrak{M}'$ and connects (μ_0, ν_0) and (μ_1, ν_1).

Thus, $\mathfrak{M}'$ is an open, dense, and connected subset of $\mathfrak{M}^{\rm spc}$ (and therefore of $\mathfrak{M}$), and we are done with the proof of Theorem 1.3.

5. Harmonic maps with branch points

In Appendix B we prove Proposition 5.1 stating that removing metrics (μ, ν) such that Σ_μ is exceptional does not disconnect $\mathfrak{M}$.

Proposition 5.1. *We let $\mu \in \mathfrak{M}'(\Sigma)$ if Σ_μ is not exceptional. If the genus of Σ is at least seven, then $\mathfrak{M}'(\Sigma)$ is an open, dense, and connected subset of $\mathfrak{M}(\Sigma)$.*

The proposition follows readily from the Riemann-Hurwitz formula, and it is the only place in the argument where we need to assume the genus of Σ is at least seven. In view of this proposition, we may assume $\mathfrak{M} = \mathfrak{M}(\Sigma) \times \mathfrak{M}(M)$, where $\mathfrak{M}(\Sigma)$ does not contain any metrics μ such that Σ_μ is an exceptional surface.

We let
$$\mathcal{J} = \{(\mu,\nu) \in \mathfrak{M} : \ f_{\mu,\nu} \text{ is not somewhere injective}\}.$$
To prove Lemma 4.3 it remains to show that $\mathfrak{M} \setminus (\mathfrak{M}_0 \cup \mathcal{J})$ is an open, dense, and connected subset of $\mathfrak{M}$.

5.1. **Localizing the problem.**

Proposition 5.2. *Suppose every pair of real analytic metrics $(\mu,\nu) \in \mathfrak{M}$ has a neighborhood $\mathcal{D} \subset \mathfrak{M}$ such that $\mathcal{D} \setminus (\mathfrak{M}_0 \cup \mathcal{J})$ is an open, dense, and connected subset of $\mathcal{D}$. Then $\mathfrak{M} \setminus (\mathfrak{M}_0 \cup \mathcal{J})$ is an open, dense, and connected subset of $\mathfrak{M}$.*

Proof. The proof is similar to the proof in Subsection 4.6. To simplify the notation, in this proof we let $\mathfrak{M}' = \mathfrak{M} \setminus (\mathfrak{M}_0 \cup \mathcal{J})$. Since the real analytic metrics are dense it follows that $\mathfrak{M}'$ is open and dense in $\mathfrak{M}$. It remains to show $\mathfrak{M}'$ is connected.

Suppose $(\mu,\nu),(\mu',\nu') \in \mathfrak{M}'$. Since $\mathfrak{M}$ is path connected, there exists a path $\gamma : [0,1] \to \mathfrak{M}$ connecting (μ,ν) and (μ',ν'). We then find a real analytic path $\widehat{\gamma} : [0,1] \to \mathfrak{M}$ such that $\widehat{\gamma}(0)$ is close to $\gamma(0) = (\mu,\nu)$, and $\widehat{\gamma}(1)$ close to $\gamma(1) = (\mu',\nu')$, so that there exist paths $\alpha,\beta : [0,1] \to \mathfrak{M}'$ connecting $\gamma(0)$ and $\widehat{\gamma}(0)$, and $\gamma(1)$ and $\widehat{\gamma}(1)$ respectively.

Thus, to show that $\mathfrak{M}'$ is connected it suffices to show that the endpoints $(\mu,\nu),(\mu',\nu') \in \mathfrak{M}'$ of a real analytic path $\gamma : [0,1] \to \mathfrak{M}$ can be connected by a path contained in $\mathfrak{M}'$.

For each $t \in [0,1]$ by $\mathcal{D}_t \subset \mathfrak{M}$ we denote the neighborhood of the point $(\mu_t,\nu_t) \in \mathfrak{M}$ such that $\mathcal{D}_t \cap \mathfrak{M}'$ is an open, dense and connected subset of $\mathcal{D}_t$ (such $\mathcal{D}_t$ exists by the assumption in the statement of the proposition we are proving). After passing to a finite sub-cover, we have found finitely many such open and connected sets $\mathcal{D}_j$, $j = 0,...,k$, which cover the image $\gamma(([0,1])$. We label $\mathcal{D}_j$ so that $\mathcal{D}_0$ contains (μ,ν), $\mathcal{D}_k$ contains (μ',ν'), and $\mathcal{D}_j$ has non-empty intersection with $\mathcal{D}_{j-1}$ and $\mathcal{D}_{j+1}$, when $0 < j < k$.

Then, we can find points
$$(\mu_j,\nu_j) \in (\mathcal{D}_j \cap \mathcal{D}_{j+1} \cap \mathfrak{M}')$$
for each $0 \leq j \leq k-1$. By the assumption tha-t $\mathcal{D}_j \cap \mathfrak{M}'$ is connected (and thus path connected) we can find paths $\alpha_j : [0,1] \to \mathcal{D}_j \cap \mathfrak{M}'$ connecting the points (μ_{j-1},ν_{j-1}) and (μ_j,ν_j). Moreover, we find paths $\alpha : [0,1] \to \mathcal{D}_0 \cap \mathfrak{M}'$ connecting the points (μ,ν) and (μ_0,ν_0), and $\alpha' : [0,1] \to \mathcal{D}_k \cap \mathfrak{M}'$ connecting the points (μ',ν') and (μ_k,ν_k). Concatenating the paths α, α_0,....,α_k, and α', we obtain the new path in $\mathfrak{M}'$ which connects (μ,ν) and (μ',ν').

□

The proof of the following lemma is postponed until the next section.

Lemma 5.1. *The set $\mathfrak{M} \setminus \mathcal{J}$ is an open, dense, and connected subset of $\mathfrak{M}$.*

Assuming Lemma 5.1, we are allowed to let $\mathfrak{M} = \mathfrak{M} \setminus \mathcal{J}$. In the remainder of this section we prove:

Lemma 5.2. *The set $\mathfrak{M} \setminus \mathfrak{M}_0$ is open, dense, and connected subset of $\mathfrak{M}$.*

5.2. **Proof of Lemma 5.2.** The obvious way to try to prove Lemma 5.2 is by considering the map $\widehat{\Sigma} \times \mathfrak{M} \to TM^{\mathbb{C}}$ given by $(p,\mu,\nu) \to f_z(p)$. Let $L \subset TM^{\mathbb{C}}$ be the zero sub-bundle $M \times \{0\} < TM^{\mathbb{C}}$. Then $\mathfrak{M}_0 \subset \Psi^{-1}(L)$. If we can show that Ψ is transverse to L that would imply that $\Psi^{-1}(L)$ is submanifold of $\mathfrak{M}$ of codimension six.

Using Proposition 4.4, one can show that Ψ is transverse to L at points (p,μ,ν) where $f_{\mu,\nu}$ is not a minimal map. But we need to modify the strategy to deal with the minimal maps.

Let N_1 and N_2 denote two vector fields on M with the values in $TM^{\mathbb{C}}$. Let $P \subset \Sigma$ be an open disc and z a local complex parameter on P. Consider the map $\Lambda : P \times \mathfrak{M} \to \mathbb{C}^2$ given by

$$\Lambda(p,\mu,\nu) = (\langle f_z, (N_1 \circ f)\rangle(p),\, \langle f_z, (N_2 \circ f)\rangle(p))\,. \tag{44}$$

Proposition 5.3. *Fix $p \in \Sigma$ and let $(\mu,\nu) \in \mathfrak{M}$ be real analytic. Suppose $f_z(p) = 0$ where $f = f_{\mu,\nu}$. Then there are vector fields N_1 and N_2 on M such that the linear map*

$$d\Lambda : T_{(p,\mu,\nu)}(P \times \mathfrak{M}) \to T_{0_{\mathbb{C}^2}}\mathbb{C}^2$$

is surjective.

We first prove Lemma 5.2 given Proposition 5.3.

Claim 5.1. *For each (p,μ_0,ν_0) (with (μ_0,ν_0) real analytic), there exist neighborhoods $P \subset \Sigma$ and $\mathcal{D} \subset \mathfrak{M}$, of p and (μ_0,ν_0) respectively, and the map Λ given by (44), such that $0_{\mathbb{C}^2}$ is a regular value of Λ when restricted to $P \times \mathcal{D}$.*

Proof. Suppose first that $f_z(p) \neq 0$. Then we can choose vector fields N_1 and N_2 such that $\Lambda(p,\mu_0,\nu_0) \neq (0,0)$. Therefore, $0_{\mathbb{C}^2}$ is a regular value of Λ because it is not in the image of Λ when we restrict it to a sufficiently small neighborhood $P \times \mathcal{D}$.

If $f_z(p) = 0$ we apply Proposition 5.3 to get the correct map Λ. Since being surjective is an open condition, there exists a neighborhood $P \times \mathcal{D}$ such that $0_{\mathbb{C}^2}$ is a regular value for Λ on $P \times \mathcal{D}$.

□

Keeping (μ_0,ν_0) fixed, we obtain the corresponding neighborhoods P_p and $\mathcal{D}_p$ for each $p \in \Sigma$. Extracting a finite subcover $P_1,...,P_k$ of Σ, we get finitely many neighborhoods $\mathcal{D}_j$ of the point (μ_0,ν_0). We intersect them and obtain the neighborhood $\mathcal{D}$ of (μ_0,ν_0) such that Λ_j is a submersion when restricted to $P_j \times \mathcal{D}$.

We show that $\mathcal{D} \setminus \mathfrak{M}_0$ is an open, dense, and connected subset of $\mathcal{D}$. In view of Proposition 5.2 that is enough to prove Lemma 5.2. Set

$$\mathcal{P}_j = \Lambda_j^{-1}(0_{\mathbb{C}^2}),$$

and $\mathcal{X} = \Sigma\times\mathcal{D}$. It follows from Claim 5.1 that $\mathcal{P}_j = \Lambda_j^{-1}(0)$ is a Banach submanifold of $P_j \times \mathcal{D}$, and thus a Banach submanifold of $\mathcal{X}$.

Let $\pi : \mathcal{X} \to \mathcal{D}$ denote the projection onto the second coordinate Then

$$\mathcal{D} \cap \mathfrak{M}_0 \subset \pi\left(\bigcup_{j=1}^{k} \mathcal{P}_j\right). \tag{45}$$

This is clear because if p is a branched point of f then $f_z(p) = 0$ and thus $\Lambda_j(p,\mu,\nu) = (0,0)$. Therefore $\pi^{-1}(\mathcal{D} \cap \mathfrak{M}_0)$ is contained in the union $\bigcup_{j=1}^{k} \mathcal{P}_j$

of finitely many submanifolds of $\mathcal{X}$ of codimension four (or zero if some $\mathcal{P}_j$ is empty). Repeating the exact same argument as in Subsection 1.9 above, one shows that $\mathcal{D} \setminus \pi\left(\bigcup_{j=1}^{k} \mathcal{P}_j\right)$ is open, dense, and connected in $\mathcal{D}$. Together with (45) this yields the proof of Lemma 5.2.

5.3. **Proof of Proposition 5.3.** Suppose $(\mu,\nu) \in \mathfrak{M}$ is real analytic and let $p \in \Sigma$. Fix a complex parameter z near p such that $z(p)=0$. Then, near 0 we have

$$f_z(z) = z^k G(z)$$

where $G(z)$ is a local section of $\mathbf{E}$ such that $G(0) \neq 0$, and $k \geq 0$.

Choose vectors fields N_1, N_2 with values in $TM^{\mathbb{C}}$, such that

$$(N_1 \circ f)(p),\ (N_2 \circ f)(p),\ G(p) \in \mathbf{E}_p, \quad \text{are linearly independent.} \tag{46}$$

Suppose $\dot{\nu} \in T\mathfrak{M}(M)$ and let $V = \mathcal{G}(\dot{\nu})$. Since $f_z(p)=0$, we have

$$d\Lambda(\dot{\nu}) = \left(\langle \nabla_z V, (N_1 \circ f)\rangle(p),\ \langle \nabla_z V, (N_2 \circ f)\rangle(p)\right). \tag{47}$$

The derivative $d\Lambda$ has more terms which come from differentiating $\langle \cdot, \cdot \rangle$ and $N_j \circ f$, but since $f_z(p)=0$ all these extra terms vanish.

Thus, to prove Proposition 5.3 it suffices to show that given any pair of complex numbers λ_1, λ_2 there exists $\dot{\nu}$ such that for $\mathcal{G}(\dot{\mu}) = V$ we have $\langle \nabla_z V, (N_j \circ f)\rangle(p) = \lambda_j$, $j=1,2$. The proof then follows from (47).

The proof is by contraposition (and very similar to the proof of Lemma 4.2 in the previous section). Suppose that for some λ_1 and λ_2 there is no such $\dot{\nu}$. In view of (47) (and by duality), there exist $\alpha, \beta \in \mathbb{C}$, not both of them zero, such that

$$\langle \nabla_z V,\ (\alpha(N_1 \circ f) + \beta(N_2 \circ f))\,\rangle(p) = 0 \tag{48}$$

for every $V = \mathcal{G}(\dot{\nu})$. We invoke the Reproducing Formula from Lemma 3.3 (also see Appendix A). We find the corresponding reproducing kernel $\mathcal{K} : \Sigma \setminus \{p\} \to \mathbf{E}$, such that

$$\langle \nabla_z W,\ (\alpha(N_1 \circ f) + \beta(N_2 \circ f))\,\rangle(p) = \int\limits_{\Sigma} \langle \mathbf{J}W, \mathcal{K}\rangle\, dA, \tag{49}$$

for every section W of $\mathbf{F}$ (or $\mathbf{E}$). Moreover, $\mathcal{K} \in L^r(\mathbf{E})$ for every $1 \leq r < 2$. We note that $\mathcal{K}$ is not identically zero since we can find a section W so that the left-hand side of (49) is not zero (see Claim 4.1).

From (48) we conclude that

$$\int\limits_{\Sigma} \langle \mathbf{J}V, \mathcal{K}\rangle\, dA = 0,$$

for every $V = \mathcal{G}(\dot{\nu})$. Since V is “real” (meaning that V is a section of $\mathbf{F}$) we conclude

$$\int\limits_{\Sigma} \langle \mathbf{J}V, \mathrm{Re}(\mathcal{K})\rangle\, dA = \int\limits_{\Sigma} \langle \mathbf{J}V, \mathrm{Im}(\mathcal{K})\rangle\, dA = 0. \tag{50}$$

Then, by the same argument as in the proof of Lemma 4.2, we show using (50) that $\mathrm{Re}(\mathcal{K}) = \mathrm{Re}(Z_1)$ and $\mathrm{Im}(\mathcal{K}) = \mathrm{Re}(Z_2)$, where Z_1 and Z_2 are meromorphic sections of $\mathbf{L}$ which have at most the first order poles at p (for $Z_j \in L^r(\mathbf{E})$ for every $r < 2$).

In Appendix A we apply the $\overline{\partial}$ method and compute $\mathcal{K}$ manually. Let

$$(\alpha(N_1 \circ f) + \beta(N_2 \circ f)) = N.$$

In a local chart where $p = 0$, we obtain (see formula (69) in Appendix A)

$$\mathcal{K}(z) = \frac{N(0)}{z} + B(z), \tag{51}$$

where B is a local C^1 section of $\mathbf{E}$.

On the other hand, Z_1 and Z_2 are meromorphic sections of $\mathbf{L}$ which have at most the first order poles at p. Recall that near $z = 0$ we have $f_z(z) = z^k G(z)$, for some local section $G(z)$ of $\mathbf{L}$. Thus, for z near $p = 0$, we have

$$Z_j(z) = \frac{c_j G(0)}{z} + D_j(z) \tag{52}$$

for some complex numbers c_1 and c_2, and bounded local sections D_j of $\mathbf{E}$.

Since $\mathrm{Re}(\mathcal{K}) = \mathrm{Re}(Z_1)$ and $\mathrm{Im}(\mathcal{K}) = \mathrm{Re}(Z_2)$, from (51) and (52) we find

$$\mathrm{Re}\left(\frac{N(0)}{z}\right) = \mathrm{Re}\left(\frac{c_1 G(0)}{z}\right), \quad \mathrm{Im}\left(\frac{N(0)}{z}\right) = \mathrm{Re}\left(\frac{c_2 G(0)}{z}\right),$$

because B and D_j are bounded sections near $z = 0$. Replacing $z = 1$ in the first equality, and $z = \mathbf{i}$ in the second yields

$$\mathrm{Re}\,(N(0)) = \mathrm{Re}\,(c_1 G(0)), \quad -\mathrm{Re}\,(N(0)) = \mathrm{Im}\,(c_2 G(0)),$$

and we conclude $\mathrm{Re}\,(c_1 G(0)) = -\mathrm{Im}\,(c_2 G(0))$. Similarly we get $\mathrm{Im}\,(c_1 G(0)) = \mathrm{Re}\,(c_2 G(0))$. This shows $c_1 = \mathbf{i}\, c_2$ and

$$N(0) = c_1 G(0).$$

This means that $N(p)$ and $G(p)$ are linearly dependent because N is a linear combination of $(N_1 \circ f)(p)$ and $(N_2 \circ f)(p)$. But this contradicts (46) and we are finished.

6. Somewhere injective maps

In this section we prove Lemma 5.1 which says that $\mathfrak{M} \setminus \mathcal{J}$ is an open, dense, and connected subset of $\mathfrak{M}$. First we show that for a typical regular point p of f, the set $f^{-1}(f(p))$ consists of finitely many pre-images which are all regular points. We call them super-regular points.

6.1. **Super Regular points.** Recall that $\Sigma^{\mathrm{reg}}(f)$ is the regular set of a harmonic map f. Suppose $f(p) = f(q)$ for two regular points $p, q \in \Sigma^{\mathrm{reg}}(f)$. We say that the inner products $\mu(p)$ and $\mu(q)$ are conformal to each other via f, if the tangent planes $df(T_p\Sigma)$ and $df(T_q\Sigma)$ agree in $T_{f(p)}M$, and if the push forward $f_*\mu(p)$ is a scalar multiple of $f_*\mu(q)$.

Definition 6.1. *Define the set* $\Sigma^{\mathcal{SR}}(f) \subset \Sigma$ *by letting* $x_0 \in \Sigma^{\mathcal{SR}}(f)$ *if*

$$f^{-1}(f(x)) = \{x_0, x_1, ..., x_k\} \subset \Sigma^{reg}(f),$$

and the restrictions $\mu(x_i)$ *and* $\mu(x_j)$ *are not conformal via* f *when* $i \neq j$. *We let* $(p, \mu, \nu) \in \mathcal{SR} \subset \Sigma \times \mathfrak{M}$ *if* $p \in \Sigma^{\mathcal{SR}}(f_{\mu,\nu})$.

The following are the key properties of $\mathcal{SR}$ and $\Sigma^{\mathcal{SR}}$.

Proposition 6.1. *The set $\mathcal{SR}$ is open in $\Sigma \times \mathfrak{M}$. Moreover, $\Sigma^{\mathcal{SR}}(f)$ is an open and dense subset of Σ.*

Proof. It is elementary to see that $\mathcal{SR}$ is open in $\Sigma \times \mathfrak{M}$ (by the same token $\Sigma^{\mathcal{SR}}(f)$ is open in Σ). It remains to prove $\Sigma^{\mathcal{SR}}(f)$ is dense in Σ.

For a fixed $f = f_{\mu,\nu}$, we define $A = A(\mu, \nu) \subset \Sigma$ by letting $x \in A$ if $f^{-1}(f(x)) \subset \Sigma^{\mathrm{reg}}(f)$. Clearly, A is an open subset of Σ.

Claim 6.1. *A is dense in Σ.*

Proof. If not, there exists an open set $\Omega \subset \Sigma \setminus A$ which consists of regular points of f. Then, for every $x \in \Omega$ the set $f^{-1}(f(x))$ contains a singular point of f. We show this is impossible.

By reducing Ω we may assume that the restriction of f to Ω is an embedding into M. We then find a small neighborhood $N \subset M$ of the patch $f(\Omega)$, such that the nearest point projection $\pi : N \to f(\Omega)$ is well defined. Set $S = f^{-1}(N)$. Then $S \subset \Sigma$ is an open set.

Let $g = \pi \circ f$. Then $g : S \to f(\Omega)$ is a map between 2-dimensional manifolds. Observe that if $y \in S$ is a singular point of f, then y is a singular point for g. By our assumption, for each $u \in f(\Omega)$ the set $f^{-1}(u)$ contains a singular point of f. Therefore each $f^{-1}(u)$ contains a singular point of g. Thus, each point in $f(\Omega)$ is the image of a singular point $y \in S$ of the map g.

On the other hand, the ordinary Sard's theorem implies that the image of the set of singular points of g has measure zero in $f(\Omega)$ (this refers to the 2-dimensional measure). Therefore, the set of singular values can not be the whole of $f(\Omega)$. This is a contradiction and we conclude that A is dense.

$\square$

We now show that $\Sigma^{\mathcal{SR}}(f)$ is dense in A. Together with the previous claim this proves the proposition. Note that the set $f^{-1}(f(x))$ is finite providing $x \in A$. Indeed, if $|f^{-1}(f(x))| = \infty$, then the (closed) set $f^{-1}(f(x))$ contains its own accumulation point, at which the rank of *df* is necessarily strictly less than two (because f can not be injective near such a point).

We show that on a dense subset of A the conformality condition holds. Suppose on the contrary that on an open subset $\Omega \subset A$ we have that for every $p \in \Omega$ there exists $q \in f^{-1}(f(p))$ such that $\mu(p)$ and $\mu(q)$ are conformal to each other via f. By reducing the size of Ω if necessary, we may also assume that $|f^{-1}(f(x))|$ is a constant function on Ω, and that the restriction of f to Ω is an embedding (here we use that $|f^{-1}(f(x))|$ is finite when $x \in A$).

With these assumptions, we can find a continuous function $f : \Omega \to \Sigma^{\mathrm{reg}}(f)$ such that $f(p) = f(q)$, where $q = h(p)$. Moreover $h : \Omega \to \Sigma^{\mathrm{reg}}(f)$ is a diffeomorphism onto its image and $f = f \circ h$ on Ω. In particular, $\mu(p)$ and $\mu(h(p))$ are conformal via f. This means that the metrics μ and $h^*\mu$ are pointwise conformally equivalent on Ω, and thus h is a conformal map.

But then, the Riemann surface Σ_μ covers another closed Riemann surface of genus ≥ 2, or it admits an anti-holomorphic involution! This is well known providing that f is a minimal map (by Gulliver-Osserman-Royden [12]), and is borderline known in general. We state and prove this fact in Lemma 10.1 in Appendix C.

So, if the metrics μ and $h^*\mu$ are pointwise conformally equivalent on an open set then Σ_μ is exceptional. But such pairs (μ,ν) have already been excluded from $\mathfrak{M}$. This proves that there is no such set Ω and we are done.

□

6.2. **The map Θ.** Let $P, Q \subset \Sigma$ be two disjoint open embedded discs in Σ. For $\delta > 0$, we let

$$\mathcal{D}(P,Q,\delta) = \{(\mu,\nu) \in \mathfrak{M} :\ d_\mu(f(P), f(Q)) > \delta, \text{ and } P, Q \subset \Sigma^{\mathcal{SR}}(f_{\mu,\nu})\}.$$

It follows from Proposition 6.1 that $\mathcal{D}(P,Q,\delta)$ is an open subset of $\mathfrak{M}$. By Proposition 6.1 we know that for each (μ,ν), the set $\Sigma^{\mathcal{SR}}(f_{\mu,\nu})$ is dense in Σ, and therefore non-empty.

Thus, each $(\mu,\nu) \in \mathfrak{M}$ is contained in some $\mathcal{D}(P,Q,\delta)$, for some discs P, Q, and some $\delta > 0$. In view of Proposition 5.2, to show that $\mathfrak{M} \setminus \mathcal{J}$ is connected it suffices to prove that every $\mathcal{D} \setminus \mathcal{J}$ is connected. Here $\mathcal{D}$ is a connected component of some $\mathcal{D}(P,Q,\delta)$.

In the rest of the proof of Lemma 5.1 we assume that $\mathfrak{M} = \mathcal{D}$. Set $\mathcal{Y} = \Sigma^2 \times (P \times Q \times \times \mathfrak{M})$. Here $\Sigma^2 = \Sigma \times \Sigma$ and we let $M^2 = M \times M$. Define the map $\Theta : \mathcal{Y} \to M^2 \times M^2$ by

$$\Theta(r,s,p,q,\mu,\nu) = (f(r), f(s), f(p), f(q)),$$

where we abbreviate $f = f_{\mu,\nu}$.

Let

$$L = \{((u_1,u_2),(v_1,v_2)) \in (M^2 \times M^2) :\ u_1 = v_1 \ \text{ and } \ u_2 = v_2\}.$$

The significance of Θ and L is contained in the fact

$$\pi^{-1}(\mathcal{J}) \subset \bigcup_{i,j=1}^{k} \Theta^{-1}(L), \tag{53}$$

where $\pi : \mathcal{Y} \to \mathfrak{M}$ is the projection onto the last factor. Indeed, suppose $(\mu,\nu) \in \mathcal{J}$. Since $\mathfrak{M} = \mathcal{D}$ and $P, Q \subset \Sigma^{\mathcal{SR}}(f)$, it follows that for each pair of points $(p,q) \in P \times Q$ there exists a pair of points $(r,s) \in \Sigma \times \Sigma$ such that $f(p) = f(r)$ and $f(q) = f(s)$. Thus $\Theta(r,s,p,q,\mu,\nu) \in L$.

6.3. **Proof of Lemma 5.1.** To complete the argument we need another transversality lemma which is proved in the next section.

Lemma 6.1. *The restriction of* Θ *to the set* $\Theta^{-1}(L)$ *is a submersion. In particular,* Θ *is transverse to* L.

We already know $\mathfrak{M} \setminus \mathcal{J}$ is open (for $\mathcal{J}$ is clearly closed). It remains to show it is dense and connected. Fix $(p,q,\mu,\nu) \in P \times Q \times \mathfrak{M}$, and define $\theta : \Sigma^2 \to M^2 \times M^2$ by $\theta(r,s) = \Theta(r,s,p,q,\mu,\nu)$.

Again using the Parametric Transversality Theorem (see Theorem 19.1 in [3]), we conclude that for a generic $(p,q,\mu,\nu) \in P \times Q \times \mathfrak{M}$ the corresponding map θ is transverse to L. But L is of codimension six, while Σ^2 has dimension four, thus $\theta(\Sigma^2)$ is disjoint from L. This implies that (μ,ν) does not belong to $\mathcal{J}$. This is true for generic (p,q,μ,ν), and therefore a generic pair (μ,ν) does not live in $\mathcal{J}$. We have just established that $\mathcal{J}$ is nowhere dense.

The previous argument can be pushed further to show that $\mathfrak{M} \setminus \mathcal{J}$ is connected. The argument goes the same way as in the proof of Theorem 1.1, and the subsequent proof that $\mathfrak{M} \setminus \mathfrak{M}_0$ is connected. We briefly rehearse it again.

Let $\gamma : [0,1] \to \mathfrak{M}$ be a path whose endpoints lie in $\mathfrak{M} \setminus \mathcal{J}$. We show that γ can be perturbed (while keeping its endpoints fixed) to be entirely contained in $\mathfrak{M} \setminus \mathcal{J}$. First, we partition $[0,1]$ into sufficiently small intervals whose image under γ is contained in a sufficiently small subset of $\mathfrak{M}$ which fits into a single chart in the model Banach space for $\mathfrak{M}$.

Suppose $(\mu_i, \nu_i) \in \mathfrak{M}$, $i = 0, 1$, are contained in this chart, and suppose $(\mu_0, \nu_0) \in \mathfrak{M} \setminus \mathcal{J}$. We show that one can perturb (μ_1, ν_1) ever so slightly so that the straight line connecting (μ_0, ν_0) and the perturbed (μ_1, ν_1) is contained in $\mathfrak{M} \setminus \mathcal{J}$.

Let $U \subset (P \times Q \times \mathfrak{M})$ be a small neighborhood of (μ_1, ν_1) and consider the map $\beta : \big([0,1] \times \Sigma^2\big) \times U \to M^2 \times M^2$ given by

$$\beta(t, r, s, p, q, \mu, \nu) = \Theta\left(t(r, s, p, q, \mu_0, \nu_0) + (1-t)(r, s, p, q, \mu, \nu)\right).$$

Here $t(r, s, p, q, \mu_0, \nu_0) + (1-t)(r, s, p, q, \mu, \nu)$ is the corresponding element of $\Sigma^2 \times U$, for fixed $t \in [0,1]$.

The map β is a submersion since Θ is. Again using the Parametric Transversality Theorem, we conclude that for a generic point in $(p, q, \mu, \nu) \in U$ the map $\delta : [0,1] \times \Sigma^2 \to M^2 \times M^2$, given by

$$\delta(t, r, s) = \beta(t, r, s, p, q, \mu, \nu),$$

is transverse to L. Since the dimension of $[0,1] \times \Sigma^2$ is five and the codimension of L is six, it follows that $\delta([0,1] \times \Sigma)$ is disjoint from L. This implies that the path

$$\pi\left(\beta(t, r, s, p, q, \mu, \nu)\right)) \subset \mathfrak{M} \setminus \mathcal{J}$$

connects (μ_0, ν_0) and (μ, ν). Recall that $\pi : \mathcal{Y} \to \mathfrak{M}$ is the projection onto the last factor.

As promised, we managed to perturb (μ_1, ν_1) to a nearby point (μ, ν) such that the straight line connecting (μ_0, ν_0) and (μ, ν) is entirely contained in $\mathfrak{M} \setminus \mathcal{J}$. The rest of the argument is identical as before. This completes the proof of Lemma 5.1.

7. Proof of Lemma 6.1

7.1. **The derivative of Θ.** Consider the derivative $d\Theta : T\mathcal{Y} \to \mathbf{F}^2 \times \mathbf{F}^2$. The tangent space $T\mathcal{Y}$ splits as $T(\Sigma^2 \times P \times Q) \times T\mathfrak{M}$. The restriction $d\Theta : T(\Sigma^2 \times P \times Q) \times \{0\} \to \mathbf{F}^2 \times \mathbf{F}^2$ is given by $df_r \times df_s \times df_p \times df_q$, where df's are the derivatives of f at the corresponding points. Since all four points are regular, the image of $d\Theta$ contains every quadruple of vectors $(Z_1, Z_2, Z_3, Z_4) \in \mathbf{F}^2 \times \mathbf{F}^2$ which are tangent to the surface $f(\Sigma)$ at the corresponding points.

Let $\dot{\nu} \in T\mathfrak{M}(M)$, and $\mathcal{G}(\dot{\nu}) = V$. The restriction of $d\Theta$ to $T\mathfrak{M}(M)$ is given by (see (22))

$$d\Theta\left(0, \dot{\nu}\right) = \left(V(r),\, V(s),\, V(p),\, V(q)\right). \tag{54}$$

Suppose $\Theta(r, s, p, q, \mu, \nu) \in L$. Then $p, q \in \Sigma^{\mathcal{SR}}(f_{\mu,\nu})$, and the pairs $\mu(p)$ and $\mu(r)$, and $\mu(q)$ and $\mu(s)$, are not conformal via f. In particular, all four points p, q, r, s are regular for f and $f(p) \neq f(q)$.

To simplify the notation, we rename the points as $z_1 = p$, $z_2 = r$, $w_1 = q$, $w_2 = s$. The proof of Lemma 6.1 is by contraposition. We already observed that all tangential vectors in $\mathbf{F}^2 \times \mathbf{F}^2$ are being picked up by $d\Theta$. Suppose the lemma is wrong. Then, there are four vectors $Z_i \in \mathbf{F}_{z_i}$ and $W_i \in \mathbf{F}_{w_i}$, not all of them zero, such that

(1) Z_i and W_i are normal to the surface $f(\Sigma)$ at the points $f(z_i)$ and $f(w_i)$ respectively,

(2) for every V such that $\mathbf{J}V = \mathcal{G}(\dot{\nu})$, the following holds

$$\sum_{i=1}^{2} \left(\langle V(z_i), Z_i\rangle + \langle V(w_i), W_i\rangle\right) = 0. \tag{55}$$

We now invoke the Reproducing Formula from Lemma 3.2. Applying this formula four times in a row, and adding up the results, we find a smooth section $X : \Sigma \setminus \{z_1, z_2, w_1, w_2\} \to \mathbf{F}$, such that

$$\sum_{i=1}^{2} \left(\langle W(z_i), Z_i\rangle + \langle W(w_i), W_i\rangle\right) = \int_{\Sigma} \langle \mathbf{J}W, X\rangle \, dA, \tag{56}$$

for every $W \in \Gamma(\mathbf{F})$. Moreover, $\mathbf{J}X(p) = 0$ for every $p \in \Sigma \setminus \{z_1, z_2, w_1, w_2\}$ and $X \in L^p(\Gamma(\mathbf{F}))$ for every $p \geq 1$. Furthermore, X is not identically equal to zero because there are sections $W \in \Gamma(\mathbf{F})$ such that the left-hand side in (56) is not zero.

On the other hand, from (55) and (56) we conclude that

$$\int_{\Sigma} \langle \mathbf{J}V, X\rangle \, dA = 0, \tag{57}$$

for every $\mathbf{J}V = \mathcal{G}(\dot{\nu})$.

Until now we had the assumption that $f_{\mu,\nu}$ is somewhere injective, but we do not anymore. The main issue here is that when we choose a variation $\dot{\nu}$ supported in some neighborhood of the point $f(z_1)$, then $\mathcal{G}(\dot{\nu})$ is not supported only in a neighborhood of z_1, but also in a neighborhood of z_2 and any other point which is mapped to $f(z_1)$.

The kernel X may have singularities at z_1 and z_2, while the other preimages of $f(z_1)$ do not figure very much in the computation because X is smooth at those points. However, since $\mu(z_1)$ and $\mu(z_2)$ are not conformal via f, we can choose such $\dot{\nu}$ so that $\mathcal{G}(\dot{\nu})$ is zero at z_2 but non-zero at z_1. This way we eliminate the singularity of X at z_1. Repeating this shows that X is a global Jacobi field which is not possible unless $X \equiv 0$.

7.2. **The time derivative of the tension field.** Let Ω be a small neighborhood of z_1 such that $f : \Omega \to M$ is an embedding. We let (u_1, u_2, u_3) denote local coordinates near $f(z_1) \in M$ such that

(1) $f(z_1) = (0, 0, 0)$,

(2) the surface $f(\Omega)$ is tangent to the plane $u_3 = 0$ at $f(z_1)$,

(3) $\nu_{13} = \nu_{23} = 0$ and $\nu_{33} = 1$, when restricted to the plane $u_3 = 0$,

Set $f^{-1}(f(z_1)) = \{z_1, z_2, z_3, ..., z_m\}$. For $\epsilon > 0$, we let $D(\epsilon)$ denote the disc of radius ϵ in the plane $u_3 \equiv 0$, and let $D_\epsilon = D(\epsilon) \times (-\sqrt{\epsilon}, \sqrt{\epsilon})$. Since $z_k \in \Sigma^{\mathrm{reg}}(f)$, for ϵ small enough we have

$$f^{-1}(D_\epsilon) = \bigcup_{k=1}^{m} \Omega_k,$$

where $\Omega_k = \Omega_k(\epsilon)$ is the corresponding neighborhood of z_k.

Near each z_k the reproducing kernel X can be expressed as a linear combination of the sections $f^{-1}\left(\frac{\partial}{\partial u_j}\right)$, $j = 1,2,3$. We let X_k denote the restriction of X to the neighborhood Ω_k, and by X_k^j we denote the real valued function on Ω_k such that

$$X_k = X_k^1 f^{-1}\left(\frac{\partial}{\partial u_1}\right) + X_k^2 f^{-1}\left(\frac{\partial}{\partial u_2}\right) + X_k^3 f^{-1}\left(\frac{\partial}{\partial u_3}\right).$$

In local coordinates on Σ (not necessarily holomorphic), the tension field τ is given by

$$\tau^\gamma(f,\mu,\nu) = \mu^{ij}\left(\frac{\partial^2 f^\gamma}{\partial x_i \partial x_j} - {}^{\mu}\Gamma_{ij}^k \frac{\partial f^\gamma}{\partial x_k} + {}^{\nu}\Gamma_{\alpha\beta}^\gamma(f)\frac{\partial f^\alpha}{\partial x_i}\frac{\partial f^\beta}{\partial x_j}\right) \tag{58}$$

where $\gamma = 1,2,3$, and we use the standard summation convention. Here μ^{ij} are the components of the inverse of the metric tensor μ. From here on easily computes $\mathcal{G}^\gamma(\dot\nu)$ to be (see [7] and Section 7 in [29])

$$\mathcal{G}^\gamma(\dot\nu) = \mu^{ij}\dot\Gamma_{\alpha\beta}^\gamma \frac{\partial f^\alpha}{\partial x_i}\frac{\partial f^\beta}{\partial x_j}. \tag{59}$$

Here

$$\dot\Gamma_{\alpha\beta}^\gamma = \lim_{t\to 0} \frac{\partial\ {}^{\nu_t}\Gamma_{\alpha\beta}^\gamma}{\partial t}$$

where ν_t is the metric variation which corresponds to $\dot\nu$.

Assume $\dot\nu$ is supported on D_ϵ. Using that $\nu_{13} = \nu_{23} = 0$ and $\nu_{33} = 1$ when $u_3 = 0$, the equalities (57) and (59) yield

$$\int_\Sigma \langle \mathbf{J}V, X\rangle \, dA = \sum_{k=1}^{m} \int_{\Omega_i} \left(\mu^{ij}\dot\Gamma_{\alpha\beta}^3(f)\frac{\partial f^\alpha}{\partial x_i}\frac{\partial f^\beta}{\partial x_j}\right) X_k^3 \, dA, \tag{60}$$

since the pre-image of D_ϵ is the union of Ω_k's.

7.3. **Osculating harmonic patches.** Possible singularities of X are the points z_1, z_2, w_1, w_2. On the other hand we know that $f(z_1) = f(z_2)$, $f(w_1) = f(w_2)$, and $f(z_1) \neq f(w_1)$. Thus, the reproducing kernel X is smooth near the point z_k when $k > 2$. The size of the area of Ω_i is either comparable to ϵ^2 if $f(\Omega_k)$ is tangent to the plane $u_3 = 0$, or to $\epsilon^{\frac{5}{2}}$ if not. Thus

$$\int_{\Omega_k} \langle \mathbf{J}V, X\rangle \, dA = O(\epsilon^2). \tag{61}$$

However, X may have a singularity near z_1, and/or z_2. In Appendix A we compute the reproducing kernel for the zeroth derivative near its singularity. From the formula (70) below, we find that for z near z_k, $k = 1,2$, the following holds

$$X(z) = \frac{1}{2\pi}\left(\log\frac{1}{|z|}\right) Z_k + B(z), \tag{62}$$

where $B(z)$ is a C^1 section of $\mathbf{F}$ near 0, and $Z_k \in \mathbf{F}_{z_k}$ is the vector normal to the surface $f(\Sigma)$ at $f(z_k)$.

By construction, the patch $f(\Omega_1)$ is tangent to the plane $u_3 = 0$ at $f(z_1)$ (in particular, Z_1 is normal to the plane $u_3 = 0$ because $\dot{\nu}_{12} = \dot{\nu}_{13} = 0$). If $f(\Omega_2)$ is not tangent to the plane $u_3 = 0$, then the size of the area of Ω_2 is $O(\epsilon^{\frac{5}{2}})$. Together with (62) we get

$$\int\limits_{\Omega_2} \langle \mathbf{J}V, X\rangle \, dA = o(\epsilon^2).$$

Using this and (61) we rewrite (60) as

$$\int\limits_{\Sigma} \langle \mathbf{J}V, X\rangle \, dA = \sum_{k=1}^{2} \sigma(k) \int\limits_{\Omega_k} \left(\mu^{ij} \dot{\Gamma}^{3}_{\alpha\beta}(f) \frac{\partial f^\alpha}{\partial x_i} \frac{\partial f^\beta}{\partial x_j}\right) X^3_k \, dA + O(\epsilon^2), \tag{63}$$

where $\sigma(k) = 1$ if $f(\Omega_k)$ is tangent to $u_3 = 0$, and $\sigma(k) = 0$ if not (by definition, $\sigma(1) = 1$).

At the point z_k where $f(\Omega_k)$ is tangent to $u_3 = 0$, we choose local coordinates (x_1, x_2) so that at z_k we have $\frac{\partial f^\alpha}{\partial x_i}(z_k) = 1$ if $\alpha = i$, and 0 otherwise (the metric μ may not be conformal in these new coordinates). The formula (63) becomes

$$\int\limits_{\Sigma} \langle \mathbf{J}V, X\rangle \, dA = \sum_{k=1}^{2} \sigma(k) \int\limits_{\Omega_k} \left(\mu^{ij} \dot{\Gamma}^{3}_{ij}(f)\right) X^3_k \, dA + O(\epsilon^2), \tag{64}$$

We are ready to finish the proof of Lemma 6.1.

Remark. We observe that the constant in the $O(\epsilon^2)$ notation depends only on the supremum norm of $\dot{\Gamma}^{3}_{ij}(f)$.

7.4. **Proof of Lemma 6.1.** For $\alpha, \beta = 1, 2$, we let $\varphi_{\alpha\beta} = \varphi_{\beta\alpha}$ denote three smooth real valued functions supported in D_ϵ, such that on $D(\epsilon) \times (-\epsilon, \epsilon)$ the functions $\varphi_{\alpha\beta}$ do not depend on u_3. If $f(\Omega_k)$ is tangent to the plane $u_3 = 0$, then $f(\Omega_k)$ is contained in $D(\epsilon) \times (-\epsilon, \epsilon)$ when ϵ is small enough (recall that $D_\epsilon = D(\epsilon) \times (-\sqrt{\epsilon}, \sqrt{\epsilon})$).

We let

$$\dot{\nu}^{\alpha\beta}(u_1, u_2, u_3) = -2u_3 \varphi_{\alpha\beta}(u_1, u_2, u_3),$$

and let every other $\dot{\nu}^{\alpha\beta}$ to be zero. Applying $\nu_{13} = \nu_{23} = 0$ and $\nu_{33} = 1$ when $u_3 = 0$, we get (see [23], [24])

$$\dot{\Gamma}^{3}_{\alpha\beta} = \varphi_{\alpha\beta}, \quad \text{on} \quad D(\epsilon) \times (-\epsilon, \epsilon).$$

Replacing this back into (64) yields

$$\int\limits_{\Sigma} \langle \mathbf{J}V, X\rangle \, dA = \sum_{k=1}^{2} \sigma(k) \int\limits_{\Omega_k} \left(\mu^{ij} \varphi_{ij}(f)\right) X^3_k \, dA + O(\epsilon^2), \tag{65}$$

We now use the fact that the restrictions of the metric μ at the points z_1 and z_2 are not conformal to each other via f. By the choice of local coordinates this means that the matrix $\mu^{ij}(z_1)$ is not a multiple of the matrix $\mu^{ij}(z_2)$. Thus we can choose φ_{ij} uniformly (with respect to ϵ) bounded above, such that

$$\sum_{i,j=1}^{2} \mu^{ij}(z_1)\varphi_{ij}(f(z_1)) = 1,$$

and

$$\sum_{i,j=1}^{2} \mu^{ij}(z_2)\varphi_{ij}(f(z_2)) = 0.$$

Together with (62), we conclude the following holds near z_1

$$\mu^{ij}(x_1,x_2)\varphi_{ij}(f(x_1,x_2))X_1^3(x_1,x_2) = \frac{1}{2\pi}\left(\log\frac{1}{|x|}\right) Z_1^3 + B(0) + O(|x|), \tag{66}$$

where $|x| = \sqrt{|x_1|^2 + |x_2|^2}$.

On the other hand, near z_2 we have

$$\mu^{ij}(x_1,x_2)\varphi_{ij}(f(x_1,x_2)) = O(|x|),$$

which yields

$$\mu^{ij}(x_1,x_2)\varphi_{ij}(f(x_1,x_2))X_2^3(x_1,x_2) = \left(\log\frac{1}{|x|}\right) O(|x|) = o(1), \tag{67}$$

where $o(1) \to 0$ when $\epsilon \to 0$. Replacing this back into (65) yields

$$\begin{aligned}\int\limits_{\Sigma} \langle \mathbf{J}V, X\rangle\, dA &= \frac{1}{2\pi}\int\limits_{\Omega_1}\left(\log\frac{1}{|x|}\right) Z_1^3\, dA(x_1,x_2) + o(1)\mathrm{Area}(\Omega_2) + O(\epsilon^2)\\ &= \frac{1}{2\pi}\int\limits_{\Omega_1}\left(\log\frac{1}{|x|}\right) Z_1^3\, dA(x_1,x_2) + O(\epsilon^2).\end{aligned}$$

Since the left-hand side is equal to 0 by the assumption we made at the beginning, we conclude $Z_1^3 = 0$. But then $Z_1 = 0$ (for Z_1 is a normal vector to the surface $f(\Omega_1)$ at $f(z_1)$). Similarly we show Z_2, W_1, and W_2 are zero. This is a contradiction and we are done.

8. Appendix A

8.1. **The first derivative.** Fix a vector $U \in \mathbf{F}_p$. Our goal is to compute the reproducing kernel $X : \Sigma \setminus \{p\} \to \mathbf{E}$ such that

$$\langle (\nabla_z W(p), U\rangle = \int\limits_{\Sigma} \langle \mathbf{J}W, X\rangle\, dA$$

for every section $W \in \Gamma(\mathbf{F})$.

Denote by Ω a disc neighborhood of p and by Ω' an even smaller neighborhood of p which is compactly contained in Ω. For $n \in \mathbb{N}$ large, we let μ_n denote a $\mathbf{E}$-valued 1-form, supported on $\{|z| < \frac{1}{n}\} \subset \Omega'$, and such that

$$\int_{\Sigma} d\zeta \wedge \mu_n = U.$$

Let $T_n \in \Gamma(\mathbf{E})$ be a section whose support is contained in Ω and such that $z \in \Omega'$ we have

$$T_n(z) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{d\zeta \wedge \mu_n}{z-\zeta}.$$

Furthermore, we can choose T_n so it converges to a section $T : \Sigma \setminus \{p\} \to \mathbf{E}$ which is compactly supported in Ω and such that for $z \in \Omega'$ we have

$$T(z) = -\frac{1}{\pi z}\, U(p).$$

Set

$$\Phi_n^1 = \frac{1}{\sigma^2}\left(\nabla_z \nabla_{\overline{z}} T_n - \nabla_z \mu_n\right), \quad \Phi_n^2 = \frac{1}{\sigma^2} R\left(T_n, f_z\right) f_{\overline{z}},$$

and $\Phi_n = \Phi_n^1 + \Phi_n^2$. Here R is the curvature tensor on M (the complexified version), and σ^2 the density of the conformal metric μ on Σ_μ.

Claim 8.1. *For every $1 < p < 2$, the sequence of norms $||\Phi_n||_p$ is uniformly bounded. Moreover, let $\Psi_n = J^{-1}(\Phi_n)$. Then after passing onto a subsequence if necessary, Ψ_n converges to a section $\Psi \in L^{2,p}(\mathbf{E})$.*

Proof. It follows

$$\nabla_{\overline{z}} T_n(z) = \mu_n(z), \quad z \in \Omega'.$$

Thus, the section Φ_n^1 is equal to zero on Ω'. Therefore, the sequence Φ_n^1 converges to a smooth section in $\Gamma(\mathbf{E})$ (because T_n converges to T, and T is smooth away from p). It follows that $||\Phi_n^1||_p$ is uniformly bounded.

The sections Φ_n^2 converge to a section that is smooth away from 0. Near 0, we have the estimate

$$||\Phi_n^2(z)|| \leq C\frac{1}{|z|}$$

for some constant $C > 0$ and for every n. Thus $||\Phi_n^2||_p$ is uniformly bounded as well, when $1 < p < 2$ (but not necessarily for $p \geq 2$).

The second part of the claim now follows from Proposition 3.1.

□

Let $W \in \Gamma(\mathbf{E})$. In local coordinates, the complexified Jacobi operator is given by (see [21] or [24])

$$\mathbf{J}W = \nabla_z \nabla_{\overline{z}} - R(W, f_z) f_{\overline{z}}.$$

Since J is self-adjoint we get

$$\begin{aligned}
\int_\Sigma \langle \mathbf{J}W, T_n\rangle\, dA &= \int_\Sigma \langle W,\, \nabla_z \nabla_{\overline{z}} T_n\rangle + \int_\Sigma \langle W,\, R\left(T_n, f_z\right) f_{\overline{z}}\rangle \\
&= \int_\Sigma \langle W,\, \Phi_n\rangle\, dA + \int_\Sigma \langle W,\, \nabla_z \mu_n\rangle \\
&= \int_\Sigma \langle \mathbf{J}W,\, \Psi_n\rangle\, dA + \int_\Sigma \langle W,\, \nabla_z \mu_n\rangle.
\end{aligned}$$

We now compute the second term on the right-hand side. Integration by parts yields

$$\int_\Sigma \langle W,\, \nabla_z \mu_n\rangle = -\int_\Sigma \langle \nabla_z W,\, \mu_n\rangle.$$

From the defining property of μ_n (and the fact that the supports of μ_n shrink to p) we get

$$\int_{\Sigma} \langle \nabla_z W,\, \mu_n \rangle \longrightarrow \langle \nabla_z W(p), U\rangle.$$

Replacing this into the above equality yields

$$\int_{\Sigma} \langle \mathbf{J}W, T_n\rangle \, dA - \int_{\Sigma} \langle \mathbf{J}W,\, \Psi_n\rangle \, dA \longrightarrow \; -\langle \nabla_z W(p), U\rangle.$$

Passing onto the subsequence from Claim 8.1, we get

$$\int_{\Sigma} \langle \mathbf{J}W, (\Psi - T)\rangle \, dA = \langle \nabla_z W(p), U\rangle. \tag{68}$$

Therefore, $X = \Psi - T$. The section $\Psi \in L^{2,p}(\mathbf{E})$, thus it is C^1 on Σ. It follows that

$$X(z) = \frac{U}{\pi z} + B(z), \tag{69}$$

where $B(z)$ is a C^1 section of $\mathbf{E}$ near 0.

8.2. **The zeroth derivative.** Fix a vector $U \in \mathbf{F}_p$. Our second goal is to compute the reproducing kernel $X : \Sigma \setminus \{p\} \to \mathbf{F}$ such that

$$\langle (W(p), U\rangle = \int\limits_{\Sigma} \langle \mathbf{J}W, X\rangle \, dA$$

for every section $W \in \Gamma(\mathbf{F})$. Following the same argument as above we derive

$$X(z) = \frac{1}{2\pi}\left(\log \frac{1}{|z|}\right) U + B(z), \tag{70}$$

where $B(z)$ is a C^1 section of $\mathbf{F}$ near 0.

9. Appendix B

Let $\mathcal{T}(\Sigma)$ be the Teichmüller space of Σ. By $\mathcal{T}'(\Sigma)$ we denote the set of marked Riemann surfaces which are not exceptional. We show $\mathcal{T}'(\Sigma)$ is an open, dense, and connected subset of $\mathcal{T}(\Sigma)$.

By $\mathcal{M}(\Sigma)$ we denote the Moduli space of Riemann surfaces homeomorphic to Σ (then $\mathcal{T}(\Sigma)$ is its universal cover). Let Σ_1 be closed surface. Each topological type of a branched covering $\pi : \Sigma \to \Sigma_1$ with the branch set $B \subset \Sigma$, yields the embedding $\pi^* : \mathcal{M}(\Sigma_1, \pi(B)) \to \mathcal{M}(\Sigma)$. The Riemann-Hurwitz formula reads

$$2g(\Sigma) - 2 = d(2g(\Sigma_1) - 2) + \sum_{p \in \Sigma} (n_p - 1)$$

where $g(\cdot)$ is the the genus, $d > 1$ the degree of π, and n_p the order of branching at the point p. The previous formula implies the inequality

$$3g(\Sigma) - 3 \geq d(3g(\Sigma_1) - 3) + \frac{3}{2}|B|. \tag{71}$$

On the other hand,

$$\dim_{\mathbb{C}} \mathcal{M}(\Sigma) = 3g(\Sigma) - 3, \quad \dim_{\mathbb{C}} \mathcal{M}(\Sigma_1, \pi(B)) = 3g(\Sigma_1) - 3 + |B|. \tag{72}$$

We first consider the case when the genus of Σ_1 is at least two. From (71) and (72) we get

$$\operatorname{codim}_{\mathbb{C}} \pi^* \left(\mathcal{M}(\Sigma_1, \pi(B))\right) \geq 3(d-1) + \frac{1}{2}|B| \geq 3.$$

Therefore, the complex codimension of $\mathcal{M}(\Sigma_1, B)$ is at least three. There are finitely many topological types of branched coverings $\Sigma \to \Sigma_1$, so we conclude that if $\Sigma_\mu \in \mathcal{M}(\Sigma)$ is a branched covering of another Riemann surface of genus at least two, then Σ_μ lives in the union of finitely many sub-orbifolds of $\mathcal{M}(\Sigma)$, each of them of real codimension greater or equal to six.

Suppose now that Σ_1 is the 2-sphere and $d \leq 4$. From (71), and since $d \leq 4$, we derive the estimate $2g(\Sigma) + 6 \geq |B|$. But $\dim_{\mathbb{C}} \mathcal{M}(\Sigma_1, \pi(B)) = |B| - 3$, so in this case we get

$$\operatorname{codim}_{\mathbb{C}} E\left(\mathcal{M}(\Sigma_1, \pi(B))\right) = 3g(\Sigma) - |B| \geq g(\Sigma) - 6.$$

Therefore, the complex codimension of $\mathcal{M}(\Sigma_1, B)$ is at least 1 when $g(\Sigma) \geq 7$. There are finitely many topological types of branched coverings $\Sigma \to \Sigma_1$ of degree at most four. We conclude that if $\Sigma_\mu \in \mathcal{M}(\Sigma)$ is a branched covering of the 2-sphere of degree at most four, then Σ_μ lives in the union of finitely many sub-orbifolds of $\mathcal{M}(\Sigma)$, each of them of real codimension greater or equal to two.

The set of Riemann surfaces in $\mathcal{M}(\Sigma)$ that admit an anti-holomorphic involution agrees with the "Real" Moduli space which is the space of real algebraic curves. There are finitely many topological types of anti-holomorphic involutions and each of them contributes a component to the "Real" Moduli space. The dimension of each component is half the dimension of $\mathcal{M}(\Sigma)$, and we find that if the genus Σ is ≥ 2, the "Real" Moduli space has real codimension greater or equal to three.

Putting this together we see that exceptional surfaces live the union of finitely many sub-orbifolds of $\mathcal{M}(\Sigma)$, each of them of real codimension greater or equal to two. The complement of this set is open dense and path connected. Lifting this set to $\mathcal{T}$ yields the proposition.

10. Appendix C

We prove the following lemma.

Lemma 10.1. *Suppose there are points $x_0, x_1 \in \Sigma^{reg}(f)$, with disjoint neighborhoods Ω_0 and Ω_1, and a conformal diffeomorphism $h : \Omega_0 \to \Omega_1$ such that $f = f \circ h$ on Ω_0. Then μ is an exceptional metric.*

As we said, if the harmonic map $f : (\Sigma, \mu) \to (M, \nu)$ is minimal this result was proved by Gulliver-Osserman-Royden [12]. Moore gave a very similar proof (see page 282 in [23]) incorporating the formula of Micallef-White (see Appendix in [22]). The proof is essentially based on the classical estimates by Hartman-Wintner. We explain the main idea in the next subsection.

On the other hand, each harmonic map has its minimal suspension which was introduced and studied in detail by Wolf [36], [37]. One can readily prove the general case of Lemma 10.1 by reducing it to the minimal case by replacing the harmonic map with its harmonic suspension. This is done in the last subsection.

10.1. **The minimal case.** Suppose $f : \Sigma \to N$ be a minimal map from a surface Σ into a Riemannian manifold N of dimension ≥ 3 (but not < 3). Let $x_0, x_1 \in \Sigma^{\mathrm{reg}}(f)$ and consider the local minimal surfaces $f(\Omega_0)$ and $f(\Omega_1)$. Assuming $f(x_0) = f(x_1)$, Micallef-White (based on a key observation by Lawson and Osserman) compute the non-parametric form of $f(\Omega_0)$ and $f(\Omega_1)$ near $f(x_0)$.

Using the theorem of Hartman-Wintner, they show that up to a certain order the two surfaces differ by a harmonic polynomial. In particular, if the two surfaces $f(\Omega_0), f(\Omega_1) \subset N$ agree on an open set which contains $f(x_0) = f(x_1)$ as one of its accumulation points, then $f(\Omega_0) = f(\Omega_1)$ in some neighborhood of $f(x_0) = f(x_1)$.

Since f is minimal, it follows that $\Sigma^{\mathrm{reg}}(f)$ is equal to Σ minus finitely many branch points. Thus, the above argument shows that the minimal map f has The Unique Continuation Property (in the sense of Gulliver-Osserman-Royden) as defined in Definition 1.7 in [12]. In the case when f is minimal, the proof of Lemma 10.1 now follows from the following result of Gulliver-Osserman-Royden, assuming that the diffeomorphism $h : \Omega_0 \to \Omega_1$ from Lemma 10.1 is orientation preserving.

Lemma 10.2. *Let S be a smooth closed surface and N a smooth manifold, and suppose we are given a branched immersion $f : S \to N$ which has the unique continuation property. If there are disjoint open sets $\Omega, \Omega' \subset S$, and an orientation preserving diffeomorphism $h : \Omega \to \Omega'$, such that $f \circ h = f$ holds on Ω, then there exists a closed surface S_1, a map $f_1 : S_1 \to N$, and a branched covering $\pi : S \to S_1$, such that $f = f_1 \circ \pi$.*

Remark. The covering map π has the same structure as h, while f_1 is similar to f. In particular, suppose S is a Riemann surface and f a harmonic map. If h is holomorphic then so is π. If f is harmonic, then so is f_1.

If every diffeomorphism $h : \Omega_0 \to \Omega_1$ from Lemma 10.1 is orientation reversing (for all choices of Ω_0 and Ω_1 such that $f(\Omega_0) = f(\Omega_1)$), then using the unique continuation as above we can show that h extends to an anti-conformal involution of Σ_μ. This completes the proof of the lemma when f is minimal.

10.2. **The minimal suspension of a harmonic map.** Let Φ be the lift of the Hopf differential Hopf(f) to the universal cover $\widehat{\Sigma}$. We say that two points $z, w \in \widehat{\Sigma}$ are equivalent, and write $z \sim w$, if z and w can be connected by an arc which is completely contained within a complete vertical leaf of Φ. We let $T = \left(\widehat{\Sigma}/\sim\right)$ be the leaf space, and denote by $\pi : \widehat{\Sigma} \to T$ the corresponding projection. The space T is an $\mathbb{R}$-tree endowed with the distance given by $d(z, w) = m(\gamma)$, where γ is any arc connecting z and w which is transverse to the vertical foliation. By m we denote the transverse measure coming from Φ.

The projection $\pi : \widehat{\Sigma} \to T$ is a harmonic map (as the map of $\widehat{\Sigma}$ into the metric space (T, d)), and the Hopf differential of p is equal to $-\Phi$. Let $\widehat{f} : \widehat{\Sigma} \to \widehat{M}$ be a lift of f, and let $F : \widehat{\Sigma} \to \widehat{M} \times T$ be given by $F(p) = (\widehat{f}, \pi(p))$. The map F is minimal and it is called the Minimal Suspension of f. The key reason we are interested in F is because if $h : \Omega_0 \to \Omega_1$ is as in the statement of Lemma 10.1, then the equality

$$F \circ \widehat{h} = \widehat{h}, \quad \text{holds on} \quad \widehat{\Omega_0}, \tag{73}$$

where $\widehat{h} : \widehat{\Omega_0} \to \widehat{\Omega_1}$ is a lift to the universal cover (the equality (73) holds because $\widehat{h} : \widehat{\Omega_0} \to \widehat{\Omega_1}$ is conformal to begin with, and it would fail without this assumption). So, we have nearly reduced the problem to the minimal case. The only issue we need to address is the fact that T is not a manifold.

Recall the following observation by Wolf (see page 450 in [37]). Let $q \in \widehat{\Sigma}$ be a point which is not a zero of Φ. Then there are open neighborhoods $A \subset \widehat{\Sigma}$ and

$B \subset \widehat{M}$, of q and $\widehat{f}(q)$ respectively, such that the map F factors as $F = \iota \circ F_1$. Here $F_1 : A \to B \times (-\epsilon, \epsilon)$ is a harmonic map into the 4-manifold $B \times (-\epsilon, \epsilon)$. The Riemannian metric on $B \times (-\epsilon, \epsilon)$ is obtained as the product of the metric ν from B and the Euclidean metric on $(-\epsilon, \epsilon)$. By $\iota : (-\epsilon, \epsilon) \to T$ we denote the isometric inclusion.

So, although T is not a manifold (it is not even locally compact in general), the minimal map F_1 maps a neighborhood A into the 4-manifold $B \times (-\epsilon, \epsilon)$ (this 4-manifold is then isometrically included in $\widehat{M} \times T$). So, for all intense and purposes, all that matters for us is the map F_1. In particular, the identity (73) becomes

$$F_1 \circ \widehat{h} = \widehat{h}, \quad \text{holds on} \quad \widehat{\Omega_0}. \tag{74}$$

We can now apply the argument from the previous subsection to complete the proof of Lemma 10.1.

11. The Simple Loop Theorem

The Simple Loop Theorem for surfaces was established by Gabai [10]. Hass [13] proved it for compact Seifert fibered 3-manifolds. This was generalized by Rubinstein-Wang [27] to compact graph manifolds. Very recently Zemke [38] proved the theorem for 3-manifolds that admit a geometric structure modeled on *Sol* (there is overlap between results in [27] and [38] but the methods used in these two papers are entirely different). The Simple Loop Theorem does not hold for higher dimensional manifolds (in particular, it fails in every closed hyperbolic manifold of dimension at least four).

A closed 3-manifold is geometric if it is modeled on one of the eight standard geometries. Combining Theorem 1.2 and the results of Hass, Rubinstein-Wang, and Zemke, we establish the Simple Loop Theorem for geometric 3-manifolds.

Theorem 11.1. *Suppose M is a a closed orientable geometric 3-manifold and S a closed orientable surface. Assume $f : S \to M$ is an incompressible map. Then f is essential.*

The only remaining case of the Simple Loop Conjecture is that of closed mixed manifold, i.e. closed 3-manifolds which contain essential tori such that at least one piece of the corresponding JSJ-decomposition is atoridal (hyperbolic). Every mixed 3-manifold admits a metric of non-positive curvature that is strictly negatively curved on the atoridal part. This was shown by Leeb [17] (see also Theorem 4.3 in Bridson's paper [5]). It is expected that the methods of this paper will yield the result in this remaining case as well (again through the application of Theorem 1.1 which can be extended to cover this case).

Remark. Since every compact 3-manifold essentially embeds into its double (which is closed), it suffices to prove the Simple Loop Theorem when the target 3-manifold is closed.

11.1. **Outline of the proof.** Suppose $f_0 : S \to M$ is an incompressible map. The least area map in this homotopy class is a minimal immersion (which for simplicity we denote by $f_0 : S \to M$). For any finite cover $\pi : \Sigma \to S$ the corresponding lift $f : \Sigma \to M$ given by $f = f_0 \circ \pi$ is a minimal immersion (although it may be the least area map in its homotopy class anymore). Then, by Theorem 1.1 harmonic immersions are dense among all harmonic maps homotopic to f. This is how we utilize the condition that f_0 is incompressible.

Now, suppose that $f_0 : S \to M$ is not essential. We can find a finite cover $\pi : \Sigma \to S$ such that the lift $f : \Sigma \to M$ collapses an embedded pair of pants $\Pi \subset \Sigma$. This means that the map $f : \Pi \to M$ is homotopic to the familiar map $g : \Pi \to R$, which is a degree two holomorphic branched covering of the annulus R (the map g has a single branch point of order two which is contained inside Π). In particular, $f_*(\pi_1(\Sigma))$ is a cyclic subgroup of $\pi_1(M)$ where $f_* : \pi_1(\Sigma) \to \pi_1(M)$ is the induced map between fundamental groups.

By construction, f is an immersion. We vary the metrics on Σ and M, and consider all harmonic maps $f_{\mu,\nu}$ in this homotopy class. The idea is that for suitably chosen metrics on Σ and M, the harmonic map $f_{\mu,\nu}$ exhibits the behavior similar to that of the map g (one can think that the restriction of $f_{\mu,\nu}$ to Π is a slight perturbation of g). But g is a branched immersion with the branch point of order two, and as such it can not be approximated by immersions (see [20]). This would contradict that $f_{\mu,\nu}$ can be approximated by immersions (which is a consequence of the assumption that f_0 is incompressible as described above).

Remark. Branched immersions can be approximated by immersions if the order of the branch point is odd. In fact, Anderson gave examples [2] where the branched immersion and the approximating immersions are all minimal maps. The blow-down of the Enneper surface provides an example of such behavior.

We actually find finite covers $\Sigma_1 \to S$ and $M_1 \to M$ such that the lift $f^1 : \Sigma_1 \to M_1$ semi-collapses a four-holed sphere $\Sigma^0 \subset \Sigma_1$ onto a pair of pants $N^0 \subset N$, where N is an embedded quasi-Fuchsian surface $N \subset M_1$ (we use the Surface Subgroup Theorem to construct such N and the LERF-ness of 3-manifold groups to promote it to an embedded surface in the finite cover M_1).

We choose the appropriate metrics on Σ_1 so that the cuffs of Σ^0 become pinched. The metric on M_1 are chosen so that N is a totally geodesic surface in M_1 and the cuffs of N^0 become pinched. We then show that the limiting harmonic map $f_\infty : \Sigma^0 \to N^0$ is a proper harmonic map between the corresponding punctured surfaces. Harmonic maps between surfaces are well understood which in turn yields that once $f_{\mu,\nu}$ is close enough to the limiting map f_∞, it can not be approximated by immersions.

11.2. **Collapsed pairs of pants and four holed spheres.** Let Π be an immersed pair of pants in a closed surface S (all immersed surfaces in S are assumed to be essentially immersed). Denote by $A_i \subset S$, $i = -1, 0, 1$, the cuffs of Π (which are immersed closed curves in S). By A we denote the *figure eight curve* (as in Figure 1) which is also immersed in S.

Definition 11.1. *Let $f : S \to M$ denote a map and Π an immersed pair of pants in S. We say that a map $f : S \to M$ collapses an immersed pair of pants Π if A lies in the kernel of $f_* : \pi_1(S) \to \pi_1(M)$ but the cuffs A_i do not.*

Remark. Notice that $\pi_1(\Pi)$ is a free group of rank two which injects into $\pi_1(S)$. Moreover, we can choose $a_i \in \pi_1(S)$ representing the homotopy classes of the curves A_i, such that $a_{-1}a_1a_0 = 1_{\pi_1(S)}$. Then, f collapses the immersed pair of pants Π if and only if there exists a non-trivial element $c \in \pi_1(M)$ such that $f_*(a_{-1}) = f_*(a_1) = c$ and $f_*(a_0) = c^{-2}$. Let $C \subset M$ be an oriented closed curve whose homotopy class is represented by c. Then (up to homotopy) $f : A_1 \to C$ is orientation preserving homeomorphism, $f : A_{-1} \to C$ is orientation reversing homeomorphism, and $f : A_0 \to C$ is an orientation preserving double covering.

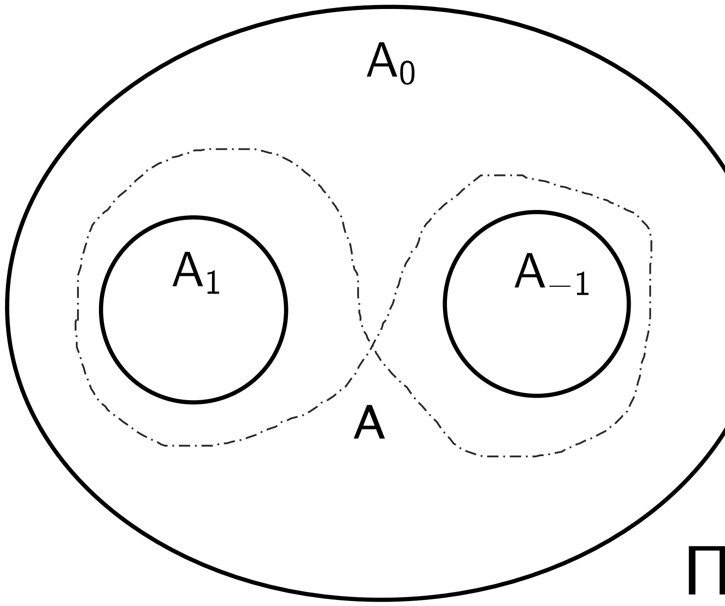

FIGURE 1. Topological pair of pants with the dotted figure eight curve

Suppose Π' is another immersed pair of pants in S such that Π and Π' share the common cuff A_1 (but have no other cuffs in common). Moreover, suppose that Π and Π' are on the opposite sides of A_1. Then $H = \Pi \cup \Pi'$ is an immersed four-holed sphere.

Definition 11.2. *Let $f : S \to M$ denote a map and $H = \Pi \cup \Pi'$ an immersed four-holed sphere in S (as defined above). We say that f semi-collapses H if it collapses Π, but the restriction $f : \Pi' \to M$ is essential.*

11.3. **The existence of partially collapsed four holed spheres.** The connection between semi-collapsed four-holed spheres and the Simple Loop Problem is provided by the following lemma.

Lemma 11.1. *Suppose $f : S \to M$ is an incompressible and non-essential map from a closed, orientable surface S of genus at least two into M. Then f semi-collapses an immersed four-holed sphere.*

Proof. We first establish the existence of a collapsed pair of pants. Endow S with a hyperbolic metric and suppose α is a closed geodesic with the smallest self-intersection number k among all closed geodesics from the kernel of f_*. Since f is incompressible, one concludes that $k > 0$. We show that α is the figure eight curve in an immersed pair of pants which is collapsed by f.

Let $E \subset \mathbb{R}^2$ be the figure eight curve in the plane (like A in Figure 1). Consider E as an oriented (singular) planar manifold (it is a regular 1-manifold away from the intersection point). It was observed by Koberda-Santharoubanea (see Lemma 4.1 in [16]) that every non-simple closed geodesic in S is the image of a smooth immersion $\iota : E \to S$. This immersion extends to some neighborhood of E which is homeomorphic to a topological pair of pants Π_0. Therefore, ι extends to an immersion $\iota : \Pi_0 \to S$.

Denote by E_i the cuffs of Π_0. The curve $A_i = \iota(E_i)$, $i = -1, 1$, is homotopic to the piecewise geodesic arc which has one non-smooth point and at most $k-1$ self-intersection points. Therefore, A_i is homotopic to the closed geodesic $\alpha_i \subset S$ which has at most $k-1$ self-intersection points. On the other hand, the curve $A_0 = \iota(E_0)$ is homotopic to the piece-wise geodesic arc which has two non-smooth points (a bigon) and exactly $k-1$ self-intersection points. Again, we conclude A_0 is homotopic to to the closed geodesic α_0 which has no more than $k-1$ self-intersection points.

It remains to observe that α_i's (for $i = -1, 0, 1$) are not in the kernel of f_* since their self-intersection numbers are strictly smaller than k. On the other hand, by definition α is in the kernel of f_*. Thus, we have constructed an immersed pair of pants Π (given by $\iota : \Pi_0 \to S$) which is collapsed by f.

It remains to construct the second pair of pants Π' such that the restriction $f : \Pi' \to M$ is essential. Fix $a_1 \in \pi_1(S)$ which represents the homotopy class determined by the cuff A_1. Suppose that the corresponding component of the lift of Π to the universal cover $\widehat{S}$ lies to the left of the oriented geodesic axis(a_1). We construct Π' so that the corresponding lift $\widehat{\Pi'}$ lies to the right of axis(a_1).

Let $B \subset S$ be a simple closed curve, and choose $b \in \pi_1(S)$ which represents the homotopy class of B, and such that axis(b) lies to right of axis(a_1) (in particular, the axes of a_1 and b are disjoint). Since B is a simple closed curve, we have $f_*(b) \neq 0$ in $\pi_1(M)$. Moreover, using the standard ping-pong argument we can find an integer $n \geq 1$ such that the group generated by $f_*(a_1)$ and $f_*(b^n)$ is a free group on two generators.

Let $b_1 = b^n$. Then the group generated by a_1 and b_1 is the fundamental group of an immersed pair of pants $\Pi' \subset S$ (because the axes of a_1 and b_1 are disjoint). By construction, the restriction of f to Π' is essential. Also, the pairs of pants Π and Π' share the same cuff A_1, but do not share any other cuffs. So, $H = \Pi \cup \Pi'$ is an essentially immersed four-holed sphere which is semi-collapsed by f. This completes the proof of Lemma 11.1.

□

11.4. **Harmonic maps between covers.**

Lemma 11.2. *Let $f : \Sigma \to M$ be a map whose homotopy class $[f]$ is admissible, and suppose that f semi-collapses an embedded four-holed sphere in Σ. Then there exist:*

- *finite covers Σ_1 and M_1 of Σ and M respectively, and the lift $f^1 : \Sigma_1 \to M_1$ of f,*
- *a hyperbolic metric $\mathfrak{g}_1$ on Σ_1,*
- *a negatively curved Riemannian metric $\mathfrak{h}_1$ on M_1 which can be connected to the hyperbolic metric $\mathfrak{h}$ through a path of negatively curved metrics on M_1,*

such that the harmonic map $f_{\mathfrak{g}_1,\mathfrak{h}_1} : \Sigma_1 \to M_1$, homotopic to f^1, can not be approximated (in the C^2 sense) by a sequence of smooth immersions.

11.5. **Proof of Theorem 1.2.** Let $f_0 : S \to M$ be an incompressible map of a closed, orientable surface S into a closed, orientable 3-manifold M endowed with the hyperbolic metric $\mathfrak{h}$. Schoen-Yau [31], and independently Sacks-Uhlenbeck [28], showed that there exists a hyperbolic metric $\mathfrak{g}_0$ on S such that $f_{\mathfrak{g}_0,\mathfrak{h}} : (S, \mathfrak{g}_0) \to (M, \mathfrak{h})$ is the least area (minimal) map in the homotopy class of f_0. Building on the work of Osserman [26], Gulliver [11] proved that $f_{\mathfrak{g}_0,\mathfrak{h}}(S)$ is an immersed surface. Together with Gabai's Simple Loop Theorem for surfaces, this implies that $f_{\mathfrak{g}_0,\mathfrak{h}}$ is a minimal immersion (see [13]).

To prove Theorem 1.2 we need to prove that f_0 is essential. The proof is by contraposition. Suppose that f_0 is not essential. Then by Lemma 11.1, f_0 semi-collapses a four-holed sphere $H = \Pi \cup \Pi'$ which is immersed in S. Using the LERF property for surface groups (proved by Scott [33]), we can find a finite cover $\pi : \Sigma \to S$ where H lifts to an embedded four-holed sphere.

Let f be the lift of f_0, that is $f = f_0 \circ \pi$ (the lift f may not be incompressible any longer). By $\mathfrak{g}$ we denote the lift of $\mathfrak{g}_0$, and by $f_{\mathfrak{g},\mathfrak{h}} : (\Sigma, \mathfrak{g}) \to (M, \mathfrak{h})$ the corresponding harmonic map in the homotopy class of f. Then $f_{\mathfrak{g},\mathfrak{h}} = f_{\mathfrak{g}_0,\mathfrak{h}} \circ \pi$, and so $f_{\mathfrak{g},\mathfrak{h}}$ is a minimal immersion (but $f_{\mathfrak{g},\mathfrak{h}}$ may not be the least area map in the homotopy class of the lift f).

Since f semi-collapses an embedded four-holed sphere, we can employ Lemma 11.2. There exist finite covers Σ_1 and M_1, of Σ and M respectively, the lift f^1 of f, a hyperbolic metric $\mathfrak{g}_1$ on Σ_1, and a negatively curved Riemannian metric $\mathfrak{h}_1$ on M_1, such that the harmonic map $f_{\mathfrak{g}_1,\mathfrak{h}_1} : (\Sigma_1, \mathfrak{g}_1) \to (M_1, \mathfrak{h}_1)$ can not be approximated (in the C^2 sense) by a sequence of smooth immersions.

On the other hand, the homotopy class $[f^1]$ is admissible. Let $\mathfrak{g}'$ be the lift of $\mathfrak{g}$ to Σ_1. The minimal immersion $f_{\mathfrak{g},\mathfrak{h}} : (\Sigma, \mathfrak{g}) \to (M, \mathfrak{h})$ lifts to the corresponding minimal immersion $f_{\mathfrak{g}',\mathfrak{h}} : (\Sigma_1, \mathfrak{g}') \to (M_1, \mathfrak{h})$. Since $(\mathfrak{g}', \mathfrak{h}) \in \mathfrak{M}$, and $f_{\mathfrak{g}',\mathfrak{h}}$ is an immersion, it follows from Theorem 1.1 and Proposition 1.1 that for every $(\mu, \nu) \in \mathfrak{M}^W$ the harmonic map $f_{\mu,\nu}$ is an immersion.

Therefore, harmonic immersions are dense in $\mathfrak{M}$, and since $(\mathfrak{g}_1, \mathfrak{h}_1) \in \mathfrak{M}$ (for $\mathfrak{h}$ and $\mathfrak{h}_1$ are in the same connected component), we conclude that $f_{\mathfrak{g}_1,\mathfrak{h}_1}$ can be approximated (in the C^2 sense) by immersions. This is a contradiction and the proof of the Simple Loop Theorem follows.

12. Nearly hyperbolic metrics on M and harmonic limits

12.1. **Nearly hyperbolic metrics from embedded surfaces.** Recall that the hyperbolic metric on the underlying closed 3-manifold M is denoted by $\mathfrak{h}$. The following lemma will be used in the next section to prove Lemma 12.2 which is stated later in the section.

Lemma 12.1. *Denote by $\iota : N \to M$ a quasi-Fuchsian immersion of an oriented closed surface N into the 3-manifold M. Let ν denote a hyperbolic metric on N and let $K > 0$. Then there exists a finite cover $M_1 \to M$, and a negatively curved Riemannian metric $\mathfrak{h}_1$ on M_1, with the following properties*

- *the immersion ι lifts to an isometric embedding $\iota_1 : (N, \nu) \to (M_1, \mathfrak{h}_1)$,*
- *the metrics $\mathfrak{h}$ and $\mathfrak{h}_1$ live in the same connected component of negatively curved Riemannian metics on M.*

- *the collar $\mathcal{N}_K(\iota_1(N)) = \{p \in M_1 : d_{\mathfrak{h}_1}(p, \iota_1(N)) < K\}$ is embedded in M_1, and the metric $\mathfrak{h}_1$ is hyperbolic on $\mathcal{N}_K(\iota_1(N))$.*

Moreover, there exists a finite cover $\Sigma_1 \to \Sigma$ such that the map $f : \Sigma \to M$ lifts to the map $f^1 : \Sigma_1 \to M_1$.

Remark. The metric $\mathfrak{h}_1$ is hyperbolic in the collar $\mathcal{N}_K(\iota_1(N))$ where it agrees with the pull-back of the hyperbolic metric from the Fuchsian manifold N^3. The metrics $\mathfrak{h}_1$ and $\mathfrak{h}$ are equal to each other outside some even larger collar about $\iota_1(N) \subset M_1$. In particular, the sectional curvatures of $\mathfrak{h}_1$ are uniformly very close to being equal to -1 on the whole of M_1, but $\mathfrak{h}_1$ is not a slight perturbation of the underlying hyperbolic metric $\mathfrak{h}$ on M_1.

Proof. The hyperbolic metric ν on N yields the Fuchsian 3-manifold N^3 (the corresponding hyperbolic metric on N^3 is also denoted by ν). The immersion $\iota : N \to M$ can be extended to an immersion $\iota : N^3 \to M$. Moreover, given any $\epsilon > 0$, we can choose the immersion $\iota : (N^3, \nu) \to (M, \mathfrak{h})$ to be ϵ-nearly locally isometric map outside some collar neighborhood of N in N^3.

This last claim is proved as follows. Denote by $\widehat{\iota} : \mathbb{H}^2 \to \mathbb{H}^3$ the lift of the immersion $\iota : N \to M$ to the map between the universal covers. Denote by $\partial\widehat{\iota} : \partial\mathbb{H}^2 \to \partial\mathbb{H}^3$ the corresponding boundary map. Here we identify $\partial\widehat{N}$ with $\partial\mathbb{H}^2$ (the boundary of the hyperbolic plane $\mathbb{H}^2$) and $\widehat{M}$ with $\partial\mathbb{H}^3$ (the boundary of the hyperbolic 3-space $\mathbb{H}^3$). The boundary map $\partial\widehat{\iota}$ is quasi-symmetric and is determined by the homotopy class of the immersion $\iota : N \to M$.

We can choose the extension $\iota : N^3 \to M$ such that $\widehat{\iota} : \partial\mathbb{H}^3 \to \partial\mathbb{H}^3$ is differentiable on $\partial\mathbb{H}^3 \setminus \partial\mathbb{H}^2$. Then ι is nearly locally isometric away from some collar neighborhood of N.

Let $\mathcal{U} \subset N^3$ be a collar neighborhood of N which contains the collar neighborhood $\mathcal{N}_K(N) = \{p \in N^3 : d_\nu(p, N) < K\}$, and such that ι is ϵ-nearly locally isometric map outside $\mathcal{U}$. Applying the LERF property of 3-manifold groups (proved by Agol [1]), we can choose a finite cover $M_1 \to M$ such that the immersion $\iota : \mathcal{U} \to M$ lifts to the embedding $\iota_1 : \mathcal{U} \to M_1$.

When ϵ is small enough (ϵ depending only on the quasi-symmetric constant of $\widehat{\iota}$) we can interpolate between the Riemannian metrics $(\iota_1)_*\nu$ and $\mathfrak{h}$ near the boundary of the embedded collar $\iota_1(\mathcal{U})$ to obtain a new negatively curved Riemannian metric $\mathfrak{h}_1$ on M_1. The metric $\mathfrak{h}_1$ agrees with $(\iota_1)_*\nu$ on the collar $\mathcal{N}_K(\iota_1(N)) = \iota(\mathcal{N}_K(N))$, and with the hyperbolic metric $\mathfrak{h}$ outside the embedded collar $\iota_1(\mathcal{U})$. In particular, the metrics $\mathfrak{h}_1$ and $\mathfrak{h}$ can be connected by a path of negatively curved metrics on the 3-manifold M_1 which is obtained from the pointwise interpolation.

Now that we have found the cover M_1, we select the corresponding finite cover $\Sigma_1 \to \Sigma$ such that $f : \Sigma \to M$ lifts to a map between the covers Σ_1 and M_1.

□

Whenever feasible we identify N and $\iota_1(N)$ (since $\iota_1(N)$ is embedded in M_1).

12.2. **Extracting the 2-dimensional harmonic limit.** The key feature Lemma 11.2 is that for a suitable choice of metrics (after passing to suitable covers) the sequence of harmonic maps converges to a harmonic map between surfaces. The following definition lays out the framework for extracting such a sequence.

Definition 12.1. *Let $\iota : N \to M$ be a quasi-Fuchsian immersion of a closed surface N into M. Suppose $\Sigma^0 \subset \Sigma$ and $N^0 \subset N$ are essentially embedded subsurfaces and*

$h^0 : (\Sigma^0, \partial\Sigma^0) \to (N^0, \partial N^0)$ a proper map. We say that f partially factors through the pair of maps (ι, h^0) if there exists a map h (homotopic to f) whose restriction to Σ^0 factors as $h = \iota \circ h^0$.

Claim 12.1. *Suppose Σ_1 and M_1 are finite covers of Σ and M respectively, and $f^1 : \Sigma_1 \to M_1$ the lift of $f : \Sigma \to M$. Furthermore, suppose that a quasi-Fuchsian immersion $\iota : N \to M$ lifts to the embedding $\iota_1 : N \to M_1$. If f partially factors through (ι, h^0), then f^1 partially factors through (ι_1, h^0).*

Proof. Since the immersion $\iota : N \to M$ lifts to the embedding $\iota_1 : N \to M_1$, and f partially factors through (ι, h^0), we conclude that Σ^0 has a degree one lift to Σ_1. We fix one such degree one lift, and (also) denote it by $\Sigma^0 \subset \Sigma_1$. Furthermore, the map h lifts to the map $h_1 : \Sigma_1 \to M_1$, which factors as $h_1 = \iota_1 \circ h^0$ when restricted to $\Sigma^0 \subset \Sigma_1$. This confirms the claim.

□

Lemma 12.2. *Suppose that $f : \Sigma \to M$ partially factors through (ι, h^0), where $\iota : N \to M$ is a quasi-Fuchsian immersion. Let ν_n be any sequence of hyperbolic metrics on N such that the restrictions of ν_n to N^0 converge to the complete hyperbolic metric ν_∞ on N^0. Then there exist finite covers Σ_n and M_n, of Σ and M respectively, a hyperbolic metric $\mathfrak{g}_n$ on Σ_n, and a negatively curved metric $\mathfrak{h}_n$ on M (in the same connected component as $\mathfrak{h}$), with the following properties:*

1. *the restrictions of $\mathfrak{g}_n$ to Σ^0 converge to the complete hyperbolic metric $\mathfrak{g}_\infty$ on Σ^0,*
2. *the restrictions $f_n : \Sigma^0 \to M_n$ converge to a proper harmonic map $f_\infty : (\Sigma^0, \mathfrak{g}_\infty) \to (N^0, \nu_\infty)$, which is an immersion when restricted to some neighborhood of each cusp in the boundary of Σ^0.*

Here $f_n : (\Sigma_n, \mathfrak{g}_n) \to (M_n, \mathfrak{h}_n)$ are the corresponding harmonic maps homotopic to the lift of f.

Remark. The boundary components of $\partial\Sigma^0$ and ∂N^0 are cusps with respect to the limiting metrics $\mathfrak{g}_\infty$ and ν_∞ respectively. Furthermore, $f_\infty : (\Sigma^0, \mathfrak{g}_\infty) \to (N^0, \nu_\infty)$ is the unique harmonic map of finite energy homotopic to the fixed map $h^0 : \Sigma^0 \to N^0$. The map f_∞ is an immersion near each cusp, but it may not be injective there. Moreover, away from the cusps f_∞ may not be an immersion at all.

In the remainder of this section we prove Lemma 11.2 assuming Lemma 12.2 (which is proved in the next section).

12.3. **Semi-collapsing the four-holed sphere into a quasi-Fuchsian surface.** The finite covers Σ_1 and M_1, and the metrics $\mathfrak{g}_1$ and $\mathfrak{h}_1$ from Lemma 11.2 will be chosen from the corresponding sequences in Lemma 12.2. But first, we need to show that f partially factors through a pair (ι, h^0).

Suppose that $f : \Sigma \to M$ semi-collapses an embedded four-holed sphere $H = \Pi \cup \Pi' \subset \Sigma$. We let $\Sigma^0 = H$. The cuffs of Π are labelled as A_{-1}, A_0, and A_1, while the cuffs of Π' are $A_1 = B_1$, B_2, and B_3. Thus, the boundary curves of Σ^0 are A_0, A_{-1}, B_2, and B_3.

Proposition 12.1. *There exist a quasi-Fuchsian immersion $\iota : N \to M$, an embedded pair of pants $N^0 \subset N$, and a proper map $h^0 : (\Sigma^0, \partial\Sigma^0) \to (N^0, \partial N^0)$, such that f partially factors through (ι, h^0).*

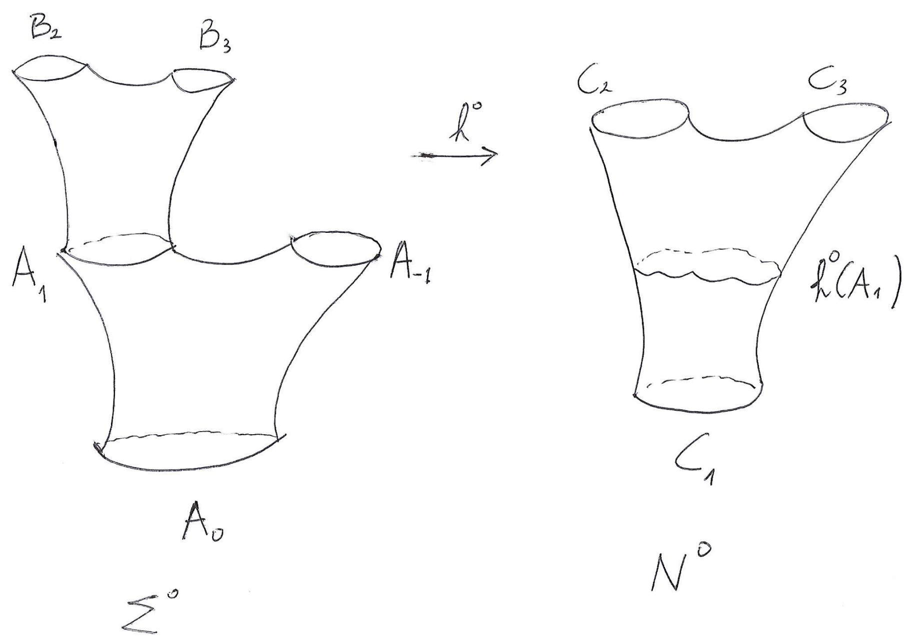

FIGURE 2. The map $h^0 : (\Sigma^0, \partial\Sigma^0) \to (N^0, \partial N^0)$ is a proper map between surfaces with boundary.

Proof. By construction, the restriction of f to Π' is essential. That is, $f(\Pi')$ is a skewed pair of pants in M (skewed pants are determined by their half-lengths). Surfaces constructed by Kahn-Markovic in the proof of the Surface Subgroup Theorem [15] are nearly geodesic. However, by a minor (and well understood) modification of the argument one can construct a quasi-Fuchsian immersion $\iota : N \to M$ of a closed surface N such that $f(\Pi')$ is included in $\iota(N)$. More precisely, there exists an embedded pair of pants $N^0 \subset N$, and a homeomorphism $h^0 : \Pi' \to N^0$, such that the restriction $f : \Pi' \to M$ is homotopic to $\iota \circ h^0 : \Pi' \to M$.

Label the cuffs of N^0 as C_1, C_2, and C_3. The map h^0 is extended to the map $h^0 : \Sigma^0 \to N^0$ as follows. Let $h^0 : A_1 \to C_1$ be any orientation preserving homeomorphism between the two cuffs. We let $h^0 : A_{-1} \to C_1$ be any orientation reversing homeomorphism, and $h^0 : A_0 \to C_1$ an orientation preserving degree two covering map. We then extend h^0 to the interior of Π to obtain the proper map $h^0 : (\Sigma^0, \partial\Sigma^0) \to (N^0, \partial N^0)$. This extension is possible because f collapses Π.

□

12.4. **Proof of Lemma 11.2.** Consider the finite covers $\Sigma_n \to \Sigma$ and $M_n \to M$, and the harmonic maps $f_n : (\Sigma_n, \mathfrak{g}_n) \to (M_n, \mathfrak{h}_n)$ with respect to the corresponding metrics from Lemma 12.2. The restrictions $f_n : \Sigma^0 \to M_n$ converge to the harmonic map $f_\infty : (\Sigma^0, \mathfrak{g}_\infty) \to (N^0, \nu_\infty)$ which is homotopic to h^0. We show that for n

sufficiently large, the map f_n can not be approximated by immersions. But first, we replace the four times punctured sphere $\Sigma^0 = (\Sigma^0, \mathfrak{g}_\infty)$ by the four-holed sphere Σ^1 compactly contained in Σ^0.

Proposition 12.2. *There exists a four-holed sphere Σ^1, which is compactly and essentially embedded in Σ^0, and a pair of pants N^1, which is compactly and essentially embedded in N^0, such that the restriction $f_\infty : (\Sigma^1, \partial\Sigma^1) \to (N^1, \partial N^1)$ is a proper map between surfaces with boundary.*

Proof. The limiting harmonic map f_∞ is an immersion near each cusp on Σ^0. Moreover, f_∞ is an orientation preserving homeomorphism near the cusps B_2, B_3, an orientation reversing homeomorphism near A_{-1}, while it is a degree two orientation preserving immersion near A_0. Let $\beta_2, \beta_3 \subset \Sigma^0$ be two (disjoint) simple closed curves surrounding the cusps B_2 and B_3 respectively. We let $\gamma_2 = f_\infty(\beta_2)$ and $\gamma_3 = f_\infty(\beta_3)$. If β_2 and β_3 are sufficiently close to the corresponding cusps, then $\gamma_2, \gamma_3 \subset N^0$ are disjoint simple closed curves surrounding the cusps C_2 and C_3 in the boundary of N^0.

The map f_∞ is an orientation preserving degree two immersion near A_0 that continuously extends to the point A_0 which becomes the branch point of order two. As such, f_∞ is topologically conjugated to the map $z \to z^2$ in a small neighborhood of the cusp A_0. Thus, there exists a simple closed curve α_0 surrounding the cusp A_0 such that $f_\infty(\alpha_0)$ is a simple closed curve surrounding the cusp C_1 in the boundary of N^0. We let $\gamma_1 = f_\infty(\alpha_0)$. Then $f_\infty : \alpha_0 \to \gamma_1$ is a degree two covering map.

We let $\alpha_{-1} = f_\infty^{-1}(\gamma_1)$. Then α_{-1} is a simple closed curve surrounding the cusp A_{-1} and $f_\infty : \alpha_{-1} \to \gamma_1$ is an orientation reversing homeomorphism. We let $\Sigma^1 \subset \Sigma^0$ be the essentially embedded four-holed sphere bounded by the curves α_0, α_{-1}, β_2, β_3. Likewise, we let N^1 be the pair of pants bounded by the curves γ_1, γ_2, and γ_3. The restriction $f_\infty : (\Sigma^1, \partial\Sigma^1) \to (N^1, \partial N^1)$ is a proper map between surfaces with boundary.

□

We show that when n is large the map f_n can not be approximated by immersions. The proof is by contraposition. Suppose on the contrary that for every fixed n, there exists a sequence of immersions $g_m : \Sigma_n \to M_n$, $m \in \mathbb{N}$, approximating f_n. Then the restrictions $g_m : \Sigma^1 \to M$ approximate the restriction $f_\infty : \Sigma^1 \to N^1$ uniformly.

The surface N^1 is embedded in N, which in turn is embedded in M_1. Let $\mathcal{N} \subset M$ be a tubular neighborhood of the quasi-Fuchsian surface N such that the nearest point projection $\pi : \mathcal{N} \to N$ is well defined. The restriction $g_m : \Sigma^1 \to M$ approximates f_∞ on Σ^1. On the other hand, f_∞ is an immersion near each boundary component of Σ^1, thus we can deform g_m ever so slightly (in the C^2 sense) so that the new immersion (also denoted by) $g_m : \Sigma^1 \to M$ is a vertical lift of f_n with respect to π is some neighborhood of each boundary curve of Σ^1 (see [20]). Moreover, we can arrange that the surface $g_m(\Sigma)$ intersects itself transversally (in particular, $g_m(\Sigma^1)$ intersects itself transversally).

Thus, we can view the new approximating maps g_m as proper maps $g_m : (\Sigma^1, \partial\Sigma^1) \to (\mathcal{N}^1, \partial\mathcal{N}^1)$. Here $\mathcal{N}^1 = \pi^{-1}(N^1)$, and $\mathcal{N}^1$ homeomorphic to $N^1 \times (-1, 1)$. Consider the double arcs of the immersed (and self-transverse) surface $g_m(\Sigma^1) \subset \mathcal{N}^1$. A point in $\partial\mathcal{N}^1$ is an endpoint of such a double arc if and only if it lies in the intersection between the curves in $g_m(\partial\Sigma^1)$ (including the self-intersection points of such curves).

For m large enough, the curves $g_m(\beta_2)$, $g_m(\beta_3)$, and $g_m(\alpha_{-1})$ are embedded and mutually disjoint curves in $\partial\mathcal{N}^1$. Thus they do not yield any intersection points. On the other hand, the curves $g_m(\alpha_0)$ and $g_m(\alpha_{-1})$ converge to the same simple closed curve γ_1 when $m \to \infty$. We can perturb g_m so that $g_m(\alpha_0) \cap g_m(\alpha_{-1}) = \emptyset$. We do this by pushing γ_1 off of itself to obtain a nearby (but disjoint) curve γ_1'. We then perturb g_m near the boundary curve α_{-1} so that the curves $g_m(\alpha_{-1})$ converge to γ_1'.

Therefore, the set of endpoints of double arcs agrees with the set of self-intersection points of the curve $g_m(\alpha_0)$. Note that the sequence of curves $g_m(\alpha_0)$ is contained in the annulus $\gamma_1 \times (-1,1) \subset \partial\mathcal{N}^1$. Moreover, $g_m(\alpha_0)$ intersects itself transversally and the restrictions $g_m : \alpha_0 \to \gamma_1 \times (-1,1)$ converge to the double covering of the core curve γ_1. This implies that the number of self-intersection points is odd (see again [20]). Such a thing is only possible if at least one of the double arcs ends in a branch point. This is a contradiction to the assumption that g_m is an immersion, and the proof is complete.

13. Proof of Lemma 12.2

In the first part of this section we consider a fixed map $f : \Sigma \to M$, where (as usual) Σ is a closed surface and M a closed 3-manifold with a fixed negatively curved metric ν. We let $\alpha_0, \alpha_1, ..., \alpha_k \subset \Sigma$ denote a collection of disjoint simple closed curves and define the embedded subsurface $\Omega = \Sigma \setminus \cup_{i=0}^{k} \alpha_i$, which may be disconnected.

We construct the family of surfaces Σ_r, and Riemannian metrics μ_r on Σ_r, such that (Σ_r, μ_r) converge to the limiting Riemannian surface with nodes $(\Sigma_\infty, \mu_\infty)$ (the "limit" of each curve α_i yields a node on Σ_∞). We then show that the family of harmonic maps $f_r = f_{\mu_r,\nu} : (\Sigma_r, \mu_r) \to (M, \nu)$ converges (after passing onto a subsequence) to the limiting harmonic map $f_\infty : (\Sigma_\infty, \mu_\infty) \to (M, \nu)$, which opens up the nodes and "maps" them to the closed geodesics in M homotopic to $f(\alpha_i)$.

This is the key step in proving Lemma 12.2. We apply these findings to the sequence of covers M_n and the associated metrics $\mathfrak{h}_n$. The metrics $\mathfrak{g}_n$ are chosen as the hyperbolic metrics conformally equivalent to suitable μ_r. This implies that the restrictions of harmonic maps $f_{\mathfrak{g}_n,\mathfrak{h}_n} : (\Sigma^0, \mathfrak{g}_n) \to (M_n, \mathfrak{h}_n)$ converge to the map $f_\infty : (\Sigma^0, \mu_\infty) \to (N^0, \nu_\infty)$, which is the finite energy harmonic map homotopic to h^0. Using the standard theory of harmonic maps between surfaces we then show that f_∞ is proper and an immersion near each cusp in the boundary of the four-holed sphere Σ^0.

13.1. The family of metrics μ_r.

We let $\gamma_i \subset (M, \nu)$ denote the geodesic homotopic to $f(\alpha_i)$. Note that γ_i need not be primitive. Let μ be a hyperbolic metric on Σ such that $\alpha_i \subset \Sigma$ are geodesics with respect to μ, and such that

$$\mathbf{l}_\mu(\alpha_i) = \mathbf{l}_\nu(\gamma_i). \tag{75}$$

Given $r > 0$, we construct the surface Σ_r, and the metric μ_r on Σ_r as follows. Cut out each α_i from Σ and insert the Euclidean cylinder $S_i \times [0, r]$ instead, where

S_i is the circle of circumference $\mathbf{l}_\mu(\alpha_i)$ (thus, S_i is identified with α_i). We let

$$\Sigma_r = \Omega \cup \bigcup_{i=0}^{k} (S_i \times [0,r]) \,.$$

We construct the Riemannian metric μ_r on Σ_r by letting it agree with μ on Ω, and with the Euclidean metric on each cylinder $S_i \times [0,r]$ (we smooth out μ_r near the edges of the cylinder). The equality (75) implies

$$\operatorname{Area}\left((S_i \times [0,r]), \mu_r\right) = r\,\mathbf{l}_\mu(\gamma_i). \tag{76}$$

Let $h : \Sigma \to M$ be a map, homotopic to f, which maps α_i locally isometrically onto γ_i. If γ_i is primitive then the restriction $h : \alpha_i \to \gamma_i$ will be an isometry, but in general $h : \alpha_i \to \gamma_i$ is a locally isometric covering map.

Define $h_r : \Sigma_r \to M$ to be the map which agrees with h on Ω, and which we extend to the map $h_r : S_i \times [0,r] \to M$ by letting $h_r(p,s) = h(p)$. Here $p \in S_i \equiv \alpha_i$, and $s \in [0,r]$. Observe that each restriction $h_r : S_i \times [0,r] \to M$ is a harmonic map, while the restriction of h_r to Ω does not depend on r.

Denote by $f_r : (\Sigma_r, \mu_r) \to (M, \nu)$ the homotopic harmonic map. Since harmonic maps between negatively curved closed manifolds are energy minimizers, we obtain

$$\int\limits_{\Sigma_r} |df_r|^2 \, dA_{\mu_r} \leq \int\limits_{\Sigma_r} |dh_r|^2 \, dA_{\mu_r}. \tag{77}$$

The auxiliary map h_r is very energy efficient, especially on the cylinders $S_i \times [0,r]$. As we shall see, this implies that f_r and h_r are close to each other. But first, we estimate the energy of f_r on the fixed compact set Ω.

Proposition 13.1. *For every $r > 0$, the inequality*

$$\int\limits_{\Omega} |df_r|^2 \, dA_{\mu_r} \leq \int\limits_{\Omega} |dh_r|^2 \, dA_{\mu_r}$$

holds.

Proof. From the Cauchy-Schwartz inequality we obtain

$$\begin{aligned}
\int\limits_{\Sigma_r \setminus \Omega} |df_r| \, dA_{\mu_r} &\leq \sqrt{\int\limits_{\Sigma_r \setminus \Omega} |df_r|^2 \, dA_{\mu_r}} \sqrt{\int\limits_{\Sigma_r \setminus \Omega} dA_{\mu_r}} \\
&= \sqrt{\int\limits_{\Sigma_r \setminus \Omega} |df_r|^2 \, dA_{\mu_r}} \sqrt{r \sum_{i=0}^{k} \mathbf{l}_\mu(\gamma_i)}.
\end{aligned}$$

The last equality follows from (76). We summarize this as

$$\int\limits_{\Sigma_r \setminus \Omega} |df_r| \, dA_{\mu_r} \leq \sqrt{\int\limits_{\Sigma_r \setminus \Omega} |df_r|^2 \, dA_{\mu_r}} \sqrt{r \sum_{i=0}^{k} \mathbf{l}_\mu(\gamma_i)}. \tag{78}$$

For $s \in [0,r]$, set $\beta_i(s) = f_r\left(S_i \times \{s\}\right)$. Then

$$r \sum_{i=0}^{k} \mathbf{l}_\nu(\gamma_i) \le \sum_{i=0}^{k} \int_0^r \mathbf{l}_\nu(\beta_i(s))\, ds \le \int_{\Sigma_r \setminus \Omega} |df_r|\, dA_{\mu_r},$$

where in the first inequality we use the estimate $\mathbf{l}_\nu(\beta_i(s)) \ge \mathbf{l}_\nu(\gamma_i)$ (for γ_i is a geodesic). We use this inequality to estimate from below the left-hand side in (78). We get

$$r \sum_{i=0}^{k} \mathbf{l}_\nu(\gamma_i) \le \int_{\Sigma_r \setminus \Omega} |df_r|^2. \tag{79}$$

But

$$\int_{\Sigma_r \setminus \Omega} |dh_r|^2\, dA_{\mu_r} = \int_{\Sigma_r \setminus \Omega} dA_{\mu_r} = r \sum_{i=0}^{k} \mathbf{l}_\nu(\gamma_i)$$

because $|dh_r| = 1$, since h_r projects $S_i \times [0,r]$ onto γ_i (locally isometrically when restricted to each $S_i \times \{s\}$). Combining this with (79) gives

$$\int_{\Sigma_r \setminus \Omega} |dh_r|^2\, dA_{\mu_r} \le \int_{\Sigma_r \setminus \Omega} |df_r|^2\, dA_{\mu_r},$$

which together with (77) yields the proposition.

□

13.2. **The harmonic limit map.** Suppose $g, h : A \to B$ are homotopic maps between Riemannian manifolds A and B. We define the distance function $\mathbf{d}(g,h)(p)$, $p \in A$, in the usual way. Lift g and h to the corresponding maps $\widehat{g}$ and $\widehat{h}$ between the universal covers of A and B respectively, and define $\mathbf{d}(g,h)(p)$ as the distance between $\widehat{g}(p)$ and $\widehat{h}(p)$. Since g and h are homotopic, this distance function is equivariant and therefore well defined on A. Moreover, the estimate (see page 368 in [30])

$$\Delta \mathbf{d}(g,h)(p) \ge -(|\tau(g)|(p) + |\tau(h)|(p)) \tag{80}$$

holds, where τ denotes the tension field.

Proposition 13.2. *There exists a constant $C > 0$, independent of r, such that*

$$\mathbf{d}(f_r, h_r)(p) \le C \tag{81}$$

for every $p \in \Sigma_r$. After passing onto a subsequence, we have

$$f_r \to f_\infty, \qquad r \to \infty,$$

where $f_\infty : \Sigma_\infty \to M$ is the limiting harmonic map. The Hopf differential Hopf(f_∞) has the second order pole at the node α_i. The associated leading coefficient of Hopf(f_∞) is real and equal to $\frac{1}{4}\mathbf{l}_\nu^2(\gamma_i)$.

Proof. From Proposition 13.1 we have the uniform (independent of r) upper bound on the energy of f_r on Ω. Since (M, ν) has strict negative curvature, we conclude that f_r is a normal family of harmonic maps when restricted to Ω. After passing to a subsequence, we obtain a harmonic limit $f_\infty : \Omega \to M$.

Moreover, from the Courant-Lebesgue lemma we find that f_∞ is absolutely continuous on $\overline{\Omega}$. This implies that for each $p \in \overline{\Omega}$, the estimate

$$\mathbf{d}(f_\infty, h)(p) \leq C \tag{82}$$

holds, for some constant $C > 0$.

On the other hand, the maps h_r and f_r are homotopic. Since Σ_r is a closed surface the distance function $\mathbf{d}(h_r, f_r)$ achieves its maximum at some point $q \in \Sigma_r$. Then

$$\Delta \mathbf{d}(h_r, f_r)(q) \geq 0.$$

It follows from (80) that $q \in \Omega$. But then, from (82) we conclude $\mathbf{d}(f_r, h_r)(p) \leq C$, that is, the estimate (81) holds.

Having bounded the distance between f_r and h_r by a constant independent of r, we conclude that after passing to a subsequence the harmonic limit $f_\infty : \Sigma_\infty \to M$ exists. Moreover, the distance function $\mathbf{d}(h_\infty, f_\infty)$ is uniformly bounded and subharmonic in the infinite cylinder $S_i \times [0, \infty)$. Let $D_{\mu_\infty}(p_m, b)$ be the disc of a fixed radius $b > 0$, where $p_m \in \Sigma_\infty$ is a sequence of points converging to a node. Then the sequence of functions $\mathbf{d}(h_\infty, f_\infty)(p)$, $p \in D_{\mu_\infty}(p_m, b)$, converges to a constant $C_1 \leq C$.

This implies that on $D_{\mu_\infty}(p_m, b)$, the harmonic map f_∞ is very close to $R \circ h_\infty$, where $R : \gamma_i \to \gamma_i$ is a rotation. Therefore, $\mathrm{Hopf}(f_\infty)$ converges to $\mathrm{Hopf}(h_\infty)$ on $D_{\mu_\infty}(p_m, b)$ (recall that h_∞ is harmonic outside Ω). But $\mathrm{Hopf}(h_\infty)$ is easily computed and the second part of the proposition follows (see also Section 3 in [35]).

$\square$

13.3. **Proof of Lemma 12.2.** According to the assumptions in Lemma 12.2, we are given a map $f : \Sigma \to M$ which partially factors through (ι, h^0). Here $\iota : N \to M$ is a quasi-Fuchsian immersion and $h^0 : \Sigma^0 \to N^0$ a proper map between two surfaces with boundary (embedded in Σ and N respectively). Moreover, ν_n denotes a sequence of hyperbolic metrics on N such that the restrictions of ν_n to N^0 converge to the complete hyperbolic metric ν_∞ on N^0 so that the components of ∂N^0 are cusps. We have to find finite covers Σ_n and M_n, of Σ and M respectively, a hyperbolic metric $\mathfrak{g}_n$ on Σ_n, and a negatively curved metric $\mathfrak{h}_n$ on M which satisfy the corresponding properties.

We apply Lemma 12.1 and obtain the finite covers Σ_n and M_n, and the corresponding negatively curved metric $\mathfrak{h}_n$ on M_n. The constant K can be anything for now, but soon we shall specify it (and apply Lemma 12.1 again to construct the embedded collar of the required size). Quasi-Fuchsian surface N is now embedded in M_n. It remains to find the metric $\mathfrak{g}_n$ on Σ_n and verify the stated properties.

To simplify the notation, we temporarily let $\Sigma_n = \Sigma'$, $M_n = M'$, $\nu_n = \nu'$, and $\mathfrak{h}_n = \mathfrak{h}'$. Note that the metric ν' agrees with $\mathfrak{h}'$ when restricted to N. Let $\alpha_0, \alpha_1, ..., \alpha_k \subset \Sigma$ denote the boundary curves of the embedded subsurface $\Sigma^0 \subset \Sigma'$. Then Σ^0 is a connected component of $\Omega = \Sigma' \setminus \cup_{i=0}^k \alpha_i$.

As above, we consider the metric μ_r on Σ'_r, and the corresponding harmonic maps $f'_r : (\Sigma'_r, \mu_r) \to (M', \mathfrak{h}')$. We let Σ^0_r denote the union of Σ^0 and the cylinders $S_i \times [0, r]$ attached to its cuffs α_i. As per Proposition 13.2, $f'_r \to f'_\infty$ when $r \to \infty$, where $f'_\infty : (\Sigma^0_\infty, \mu_\infty) \to (M, \mathfrak{h}')$ is the harmonic map from the punctured surface $(\Sigma^0_\infty, \mu_\infty)$.

Choose the parameter K from Lemma 12.1 to be equal to the constant C from (81) in Proposition 13.2. We apply Lemma 12.1 again (and pass onto further covers) to ensure that for each r, the image $f'_r(\Sigma^0_r)$ is contained in a fixed embedded collar $\mathcal{U} \subset M'$ around $N \subset M'$.

Claim 13.1. *The map f'_∞ maps the punctured surface $(\Sigma^0_\infty, \mu_\infty)$ onto the geodesic subsurface (N^0, ν') of N (the cuffs of N^0 are geodesics on N).*

Proof. By the choice of the cover M', we know that the surface (N, ν') isometrically embeds in $(M', \mathfrak{h}')$. Moreover, the auxiliary maps $h_r : \Sigma'_r \to M'$ can be chosen so the image $h_r(\Sigma^0_r)$ is contained in $N^0 \subset N$ for every r.

The metric $\mathfrak{h}'$ is hyperbolic on $\mathcal{U}$. Let $I : \mathcal{U} \to \mathcal{U}$ be the reflection through the totally geodesic subsurface $N \subset \mathcal{U}$ (we can choose $\mathcal{U}$ so it is invariant under this reflection). Since $f'_r(\Sigma^0_r)$ is contained in $\mathcal{U}$, the map $\overline{f'}_r = I \circ f'_r$ is well defined on Σ^0_r. Furthermore, $\overline{f'}_r : (\Sigma^0_r, \mu_r) \to (M', \mathfrak{h}')$ is harmonic (for $\mathfrak{h}'$ is invariant under I on $\mathcal{U}$).

Consider the distance function $\mathbf{d}(f'_\infty, \overline{f'}_\infty)$. From (80) we find that $\mathbf{d}(f'_\infty, \overline{f'}_\infty)$ is subharmonic on Σ^0_∞. Let $p_k \in \Sigma^0_\infty$ be a sequence converging to a node. The sequence $f_\infty(p_k)$ converges to the corresponding geodesic $\gamma_i \subset N$. Hence $\mathbf{d}(f'_\infty, \overline{f'}_\infty)$ tends to zero when we approach the nodes (because I restricts to the identity on the geodesics γ_i). By the maximum principle we conclude that $\mathbf{d}(f'_\infty, \overline{f'}_\infty) \equiv 0$, that is $f'_\infty \equiv \overline{f'}_\infty$.

This implies that the image $f'_\infty(\Sigma^0_\infty)$ is contained in the totally geodesic surface N. Then $f'_\infty : (\Sigma^0_\infty, \mu_\infty) \to (N^0, \nu')$ is the harmonic map from the punctured surface Σ^0 onto the pair of pants N^0 (since N^0 is convex). ☐

Remark. In [35] Wolf studied this map in details. As shown by Wolf [35], the map $f'_\infty : (\Sigma^0_\infty, \mu_\infty) \to (N^0, \nu')$ is uniquely determined by the fact that its Hopf differential has the second order pole, and that the leading coefficient at the puncture α_i is real and equal to $\frac{1}{4}\mathbf{l}^2_\nu(\gamma_i)$ (see Proposition 13.2). He also shows it is closely approximated near the cusps by certain model maps which are explicitly computed. Many other information about this map were provided in [35].

Thus, for fixed $n \in \mathbb{N}$, we have shown that the sequence of harmonic maps $f^n_r : (\Sigma^0_r, \mu_r) \to (N^0, \nu_n)$ converges to the limiting harmonic map $f^n_\infty : (\Sigma^0_\infty, \mu_\infty) \to (N^0, \nu_n)$. We now let $n \to \infty$. Then the harmonic maps f^n_∞ converge (see the remark at the end of [19]) to a harmonic map $f_\infty : (\Sigma^0_\infty, \mu_\infty) \to (N^0, \nu_\infty)$ between the corresponding cusped surfaces.

Therefore, $\operatorname{Hopf}(f^n_\infty)$ is a sequence of holomorphic quadratic differentials on the four-holed sphere $(\Sigma^0_\infty, \mu_\infty)$ with the second order poles at the cusps. Since $\mathbf{l}_{\nu_n}(\gamma_i) \to 0$, it follows from Proposition 13.2 that the limiting holomorphic quadratic differential $\operatorname{Hopf}(f_\infty)$ can have at most first order poles. That f_∞ is a proper map and an immersion near each cusp is readily deduced from standard methods. Since we could not locate it in the literature, we give a short proof in the following subsection.

It remains to say that for each n we choose r_n large enough so that the sequence $f^n_{r_n}$ converges to f_∞. We let $\mathfrak{g}_n$ be the hyperbolic metric on Σ_n which is conformally equivalent to μ_{r_n}. This completes the proof of Lemma 12.2.

13.4. **Finite energy maps between cusped surfaces.** Since Hopf(f_∞) has at most first order poles, it follows that f_∞ has finite total energy

$$\int_{\Sigma^0} \mathbf{e}(f_\infty)\, dA_{\mathfrak{g}_\infty} < \infty.$$

This is because $||\text{Hopf}(f_\infty)||(p) << \mathbf{e}(f_\infty)(p)$ only at the points p where f_∞ is very close to being conformal. At those point we use standard estimates from quasiconformal maps to show that the energy is finite over those areas. We leave details to the reader.

Schoen-Yau proved in Theorem 1 in [30] that harmonic maps of finite total energy between cusped surfaces are unique in their homotopy class. This enables us to construct a new sequence of harmonic maps g_n, which converges to the same limit f_∞, and which is slightly more convenient because of the symmetries we enforce.

Let μ_n denote a hyperbolic on Σ^0 such that

$$\mathbf{l}_{\mu_n}(\alpha_i) = \mathbf{l}_{\nu_n}(\gamma_i). \tag{83}$$

We double Σ^0 to obtain the closed surface Σ^1 with the hyperbolic metric μ_n. Likewise, the double of N^0 is denoted by N^1.

The map $h^0 : \Sigma^0 \to N^0$ yields the corresponding map $h : \Sigma^1 \to N^1$ between the doubles. We choose $h_n : \Sigma^1 \to N^1$ to be a local isometry in a fixed equidistant tubular neighborhood of each closed geodesic α_i. The restriction of h_n near α_i is a locally isometric covering. Moreover, we can arrange that on Σ^0 we have $h_n \to h_\infty$, where $h_\infty : (\Sigma^0, \mu_\infty) \to (N^0, \nu_\infty)$ is a local isometry in a fixed horocyclic neighborhood of each cusp in the boundary of Σ^0.

Let $f_n : \Sigma^1 \to N^1$ be the harmonic map homotopic to h. Then the restrictions of f_n to Σ^0 converge to the harmonic map $f_\infty : (\Sigma^0, \mu_\infty) \to (N^0, \nu_\infty)$. Moreover, $\mathbf{d}(f_n, h_n)$ does not achieve its maximum in the equidistant neighborhoods of the cusps because both f_n and h_n are harmonic there which implies that $\mathbf{d}(f_n, h_n)$ is subharmonic there. Thus, for each n the maximum is achieved on some compact set of uniformly bounded diameter in Σ^1.

This shows that $\mathbf{d}(f_n, h_n)$ remains uniformly bounded in n (because both f_n and h_n converge to their limits there). Consequently $\mathbf{d}(f_\infty, h_\infty)$ is a bounded function. An immediate corollary of this is that f_∞ is proper. A standard argument then shows that f_∞ is an immersion near each cusp. Indeed, similarly as in the proof of Proposition 13.2 we conclude that the function $\mathbf{d}(f_\infty, h_\infty)$ behaves like a constant function when approaching a cusp in the boundary of Σ^0. Passing to the universal cover $\mathbb{H}^2$, we find that on large discs which are close to a cusp, the lift $\widehat{f_\infty}$ is close to $\widehat{h}_\infty \circ R$. Here R is an isometry of $\mathbb{H}^2$. Since h_∞ is an immersion there, so is f_∞. This completes the proof.

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