BIHOLOMORPHIC MAPS BETWEEN TEICHMÜLLER SPACES

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Abstract. In this paper we study biholomorphic maps between Teichmüller spaces and the induced linear isometries between the corresponding tangent spaces. The first main result in this paper is the following classification theorem. If M and N are two Riemann surfaces which are not of exceptional type, and if there exists a biholomorphic map between the corresponding Teichmüller spaces $Teich(M)$ and $Teich(N)$, then M and N are quasiconformally related. Also, every such biholomorphic map is geometric. In particular, we have that every automorphism of the Teichmüller space $Teich(M)$ must be geometric. This result, generalize the previously known results (see [3], [5] , [7]), and enable us to positively answer the well known conjecture which states that the group of automorphisms of $Teich(M)$ is isomorphic to the Mapping class group of M, whenever the surface M is not of exceptional type. In order to prove the above results, we develop a method for studying linear isometries between L^1 type spaces. Our focus is on studying linear isometries between Banach spaces of integrable holomorphic quadratic differentials, which are supported on Riemann surfaces. Our main result in this direction (Theorem 1.1) states that if M and N are Riemann surfaces of non-exceptional type, then every linear isometry between $A^1(M)$ and $A^1(N)$ is geometric. That is, every such isometry is induced by a conformal map between M and N.

1. Introduction

In this paper, unless it is specified otherwise, every Riemann surface is assumed to carry the unique complete metric of curvature -1, which we call the hyperbolic metric. If M and N is a pair of Riemann surfaces, by $z = x + iy$ and $z_1 = x_1 + iy_1$ we denote local parameters on M and N respectively.

Let M be a Riemann surface, and let $A^1(M)$ denote the Banach space of all integrable holomorphic quadratic differentials $\phi(dz)^2 = \phi$, with the Banach norm

$$||\phi||_1 = \int\limits_M |\phi| dxdy.$$

This space is also sometimes called the Bergman space $A^1(M)$. Let N be another Riemann surface, and denote by $A^1(N)$ the corresponding Bergman space. One of the main goals of this paper is to characterize surjective linear isometries between the Banach spaces $A^1(M)$ and $A^1(N)$.

Definition. A surjective linear isometry $T : A^1(M) \to A^1(N)$ is said to be geometric, if there exists a conformal map $\alpha : M \to N$, and a complex number θ, $|\theta| = 1$,

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such that

$$T^{-1}(\phi) = \theta(\phi \circ \alpha)(\alpha')^2$$

for every $\phi \in A^1(N)$.

The following problem has been extensively studied (see [7], [2], [3], [5]).

Problem 1.1. *Let M and N be two Riemann surfaces which are not of exceptional type. Then, every surjective linear isometry $T : A^1(M) \to A^1(N)$ is geometric.*

We also mention the following special case of Problem 1.1, which was introduced in [3].

Problem 1.2. *Let M_1 and M_2 be two Riemann surfaces, quasiconformally related to a Riemann surface M, and assume that M is not of exceptional type. Then, every surjective linear isometry $T : A^1(M_1) \to A^1(M_2)$ is geometric.*

Remark. A Riemann surface M is said to be of exceptional type, if it is of finite hyperbolic area and $2g + n \leq 4$, where g is the genus of M, and n the number of points in the boundary of M. Also, we say that two Riemann surfaces are conformally or quasiconformally related, if there exists a conformal or quasiconformal map between them. If Problem 1.2 has a positive answer for M, we say that M has the isometry property.

It has been known since Royden [7], that if M and N are closed Riemann surfaces (which are not of exceptional type) then all linear isometries between $A^1(M)$ and $A^1(N)$ are geometric. This result was extended in [2] to the case of all Riemann surfaces of finite hyperbolic area, that is all Riemann surfaces where the dimension of the space $A^1(M)$ is finite. In recent years, Problem 1.1 has been positively solved in many important cases. In [3], it has been proved that Problem 1.1 has a positive solution, if M and N are of finite topological type (finitely generated fundamental group). In [5] it was proved that Problem 1.1 has a positive solution for all surfaces of finite genus. In this paper we settle the general case.

Theorem 1.1. *Suppose that Riemann surfaces M and N are of non- exceptional type. Let $T : A^1(M) \to A^1(N)$ be a surjective linear isometry. Then the isometry T is geometric. The surfaces M and N are conformally related and therefore homeomorphic.*

Remark. Note that in the above theorem we do not assume that M and N are homeomorphic. A Riemann surface M is said to have finite analytical type if it can be obtained from a closed surface by deleting at most finitely many points. Therefore we derive a positive solution to Problem 1.1, that is, every Riemann surfaces which is not of exceptional type has the isometry property. Clearly, the positive solution to Problem 1.1 implies that Problem 1.2 has a positive solution as well.

Problem 1.1 is interesting in its own right. However, the importance of this problem also comes from the Teichmüller theory. Denote by $Teich(M)$ the Teichmüller space of M. Let N be another Riemann surface. One of the central question in Teichmüller theory, is to describe all biholomorphic maps between $Teich(M)$ and $Teich(N)$. In particular, it is important to describe biholomorphic maps of

$Teich(M)$ onto itself. These maps are called automorphisms of $Teich(M)$. They form the group called $Aut(Teich(M))$.

Every quasicnoformal map $g : M \to N$, induces a biholomorphic map $\rho_g : Teich(M) \to Teich(N)$. This map is called geometric (see [3], [4] for the definition of the notion of geometric maps between Teichmüller spaces). We define the Mapping class group $MC(M)$ to be the group of all classes of quasiconformal maps $g : M \to M$. Every element g of the group $MC(M)$ induces the automorphism ρ_g of $Teich(M)$. The following conjecture is well known.

Conjecture 1.1. *If the Riemann surface M is not of exceptional type then*

$$Aut(Teich(M)) = MC(M).$$

In his celebrated paper [7] Royden showed that in the case when M and N are closed surfaces of finite genus (and not of exceptional type), then every biholomorphic map $F : Teich(M) \to Teich(N)$ is geometric. In particular, every automorphism of $Teich(M)$ is geometric, and represented by an element of $MC(M)$. This is a very important result in the theory of finitely dimensional Teichmüller spaces. Since then, the question of extending this theorem to other Riemann surfaces, and their Teichmüller spaces, was studied.

Remark. The Royden's result about automorphisms of Teichmüller spaces was extended to Riemann surfaces of finite genus in [3],[5] by answering Problem 1.1 positively in these cases. A Riemann surface is of finite genus, if there exists a closed Riemann surface M^*, such that M can be conformally embedded in M^*.

Our next theorem settles this conjecture.

Theorem 1.2. *Suppose that M is a Riemann surface which is not of exceptional type. Then*

$$Aut(Teich(M)) = MC(M).$$

If Riemann surfaces M and N are not quasiconformally related, one still can ask if there exists a biholomorphic map $F : Teich(M) \to Teich(N)$. The following classification theorem is a corollary of Theorem 1.1 and some important results from [3] (see section 9. for details).

Theorem 1.3. *Suppose that Riemann surfaces M and N are not of exceptional type, and let $F : Teich(M) \to Teich(N)$ be a biholomorphic map. Then, we have that M and N are quasiconformally related, and the map F is geometric.*

The method we develop to prove Theorem 1.1 is different than methods used in [3], [5], [7]. Although in this paper we are focused on linear isometries between L^1 Bergman spaces of quadratic differentials on Riemann surfaces, parts of the proof of Theorem 1.1 could be used in studying linear isometries between subspaces of general L^p type spaces (here $p > 0$, and p is not an even integer). We now list main steps in the proof of Theorem 1.1.

Theorem 2.1 is the first crucial step toward the proof of Theorem 1.1. It is particularly interesting because it holds in the context of arbitrary L^1 spaces. In Theorem 3.3 we prove an important separation property of the Bergman spaces $A^1(M)$, where M is of non-exceptional type. In a way, Theorem 3.1 states that the space $A^1(M)$ contains enough elements to completely determine the underlying Riemann surface M.

In Section 4. as a direct corollary of Theorem 2.1 and Theorem 3.3, we construct the map β with the following properties. The map β is defined almost everywhere on M, and it is a bijection onto some set of the full measure in N. In particular, β is the measure preserving, that is the following holds. Let F be a measurable subset of M. Then for every $\phi \in A^1(M)$ we have

$$\int\limits_F |\phi| dxdy = \int\limits_{\beta(F)} |T(\phi)| dx_1 dy_1. \tag{1.1}$$

The identity (1.1) shows that the map β is the map we could use to prove Theorem 1.1. The main obstacle is that we do not know whether β is defined everywhere on M, and in particular, whether β is conformal. In sections 5,6,7, and 8, we show that there exists a conformal map $\alpha : M \to N$, such that $\alpha = \beta$ almost everywhere on M. With this, we conclude the proof of Theorem 1.1. In the last section of this paper we give the proofs of Theorem 1.2 and Theorem 1.3.

I would like to thank the referee for a number of valuable comments and suggestions. In particular, I am grateful to the referee for pointing out to me that Lemma 3.2 holds for all Riemann surfaces of non-exceptional type. Also, the previous version of this paper contained the proof of Theorem 2.1. I am thankful to the referee for bringing to my attention the paper of Rudin [8] where this theorem was originally proved.

2. Preliminary results

In the following theorem the measures μ, and ν are finite and positive measures on sets X and Y respectively. The nature of these sets plays no role in what follows, so we will not specify them. The following result was proved in [8].

Theorem 2.1. *Let $f_1, ..., f_n \in L^1(\mu)$, and $g_1, ..., g_n \in L^1(\nu)$, be such that for every n-tuple of complex numbers α_k, $k = 1, 2...n$, we have*

$$\int |1 + \sum_{k=1}^{n} \alpha_k f_k| d\mu = \int |1 + \sum_{k=1}^{n} \alpha_k g_k| d\nu. \tag{2.1}$$

Then, for every bounded Borel measurable function (and for every real valued non-negative Borel function) $u : \mathbf{C}^n \to \mathbf{C}$, we have that

$$\int u(f_1, ..., f_n) d\mu = \int u(g_1, ..., g_n) d\nu. \tag{2.2}$$

Furthermore, let $I : X \to \mathbf{C}^n$ and $J : Y \to \mathbf{C}^n$ be the maps $I = (f_1, ..., f_n)$ and $J = (g_1, ..., g_n)$ respectively. Then we have

$$\mu(I^{-1}(E)) = \nu(J^{-1}(E)), \tag{2.3}$$

for every Borel set E in $\mathbf{C}^n$.

Remark. In Rudin's paper [8], the above theorem was proved in a more general context. In fact, it holds if instead $L^1(\mu)$ we consider $L^p(\mu)$ where $p > 0$, and p is not an even integer.

The following two lemma are easy corollaries of the above theorem.

Lemma 2.1. *Suppose that ϕ_k, $k=0,1,2...n$, $n \in \mathbf{N}$, are elements of $A^1(M)$ and ψ_k, $k=0,1,2...n$, are elements of $A^1(N)$, such that for every n-tuple of complex numbers α_k, $k=1,2...n$ we have*

$$||\phi_0 + \sum_{k=1}^{n} \alpha_k \phi_k||_1 = ||\psi_0 + \sum_{k=1}^{n} \alpha_k \psi_k||_1 \tag{2.4}$$

If neither ϕ_0 nor ψ_0 is constantly zero, then for every real valued non-negative Borel function $u : \mathbf{C}^n \to \mathbf{C}$, we have

$$\int_M u(\frac{\phi_1}{\phi_0}, ... \frac{\phi_n}{\phi_0})|\phi_0| dx dy = \int_N u(\frac{\psi_1}{\psi_0}, ... \frac{\psi_n}{\psi_0})|\psi_0| dx_1 dy_1.$$

Proof. Set $d\mu = |\phi_0| dx dy$ and $d\nu = |\psi_0| dx_1 dy_1$, where $z = x + iy$ and $z_1 = x_1 + iy_1$ are local parameters on M, and N respectively. μ and ν are well defined measures on M and N respectively. The equality (2.3) can be written as

$$\int_M |1 + \sum_{k=1}^{n} \alpha_k \frac{\phi_k}{\phi_0}| d\mu = \int_N |1 + \sum_{k=1}^{n} \alpha_k \frac{\psi_k}{\psi_0}| d\nu.$$

The proof now follows from the previous theorem. □

By the Lebesgue measure $\mathbf{m}_M$ on a Riemann surface M we consider the measure which has the following properties. The universal covering surface of M is the unit disc. Let F be a subset M which is contained in an injectively embedded hyperbolic disc on M. Then we say that F is Lebesgue measurable on M, if the set in the unit disc, which represents a lift of F, is Lebesgue measurable as a subset of the complex plane. The measure of the set F is equal to its hyperbolic area. Note that, the measure μ introduced above, has the same families of measurable sets and also the same families of sets of measure zero, as the measure $\mathbf{m}_M$. Since these are the only two types of measures on Riemann surfaces we deal with in the remainder of this paper, we will use terms like "measurable set ", "set of measure zero", "set of positive measure", etc., without specifying the measure.

Set

$$I(p) = (\frac{\phi_1}{\phi_0}(p), ... \frac{\phi_n}{\phi_0}(p))$$

for $p \in M$, and

$$J(q) = (\frac{\psi_1}{\psi_0}(q), ... \frac{\psi_n}{\psi_0}(q))$$

for $q \in N$. We have

Lemma 2.2. *Under the same set of assumptions as in the previous lemma, we have that the following holds. Let F be a compact set in M. Let $u : \mathbf{C}^n \to \mathbf{C}$ be the function defined to be 1 on the set $I(F)$, and to be equal to 0 elsewhere in $\mathbf{C}^n$. Then, we have*

$$\int_M (u \circ I) d\mu = \int_{I^{-1}(I(F))} d\mu = \int_{J^{-1}(I(F))} d\nu = \int_N (u \circ J) d\nu.$$

Remark. The set $I^{-1}(I(F))$ is the set of all points on M, which are mapped in the set $I(F)$ by the map I. Clearly, F is contained in $I^{-1}(I(F))$. Also the set $J^{-1}(I(F))$ is the set of all points on N, which are mapped in the set $I(F)$ by the map J. However, in general this set may as well be empty, and it is a corollary of this lemma that the set $J^{-1}(I(F))$ has positive measure in N whenever F has positive measure in M.

Proof. Since the map I is continuous, the set $I(F)$ is a compact set in $\mathbf{C}^n$. This implies that the function u is a Borel function. The proof of the lemma now follows from Lemma 2.1.

□

3. Separation properties of the space $A^1(M)$

Let S_M be the subset of a Riemann surface M defined as follows.

Definition. We say that $p_0 \in S_M$, if for every point $p \in M$, $p \neq p_0$, there exists a differential $\phi \in A^1(M)$, such that $\phi(p)\phi(p_0) = 0$, and not both $\phi(p)$, and $\phi(p_0)$ are equal to 0.

Although $\phi \in A^1(M)$ is not a function on M, the statements $\phi(p) = 0$, and $\phi(p) \neq 0$ are well defined, and so the above definition is independent of the choice of a local parameter.

Denote by $\Gamma(M)$ the Hilbert space of all integrable holomorphic $(1,0)$ forms on M. $\Gamma(M)$ is also known as the space of all Abelian square integrable forms on M. The Hilbert norm of $\varphi dz = \varphi$, $\varphi \in \Gamma(M)$, is given by

$$||\varphi||_2 = \left(\int\limits_M |\varphi|^2 \right)^{\frac{1}{2}} = ||\varphi^2||_1^{\frac{1}{2}}.$$

Theorem 3.1. *Suppose that the dimension of the space $\Gamma(M)$ is at least 3. Then the set $M \setminus S_M$ is a discrete set of points in M. Furthermore, if p' is a puncture in the boundary of M then p' is not a cluster point of the set $M \setminus S_M$.*

Remark. By saying that p' is a puncture in the boundary of M we mean that there exists another Riemann surface $\tilde{M}$ such that the surface M is obtained from $\tilde{M}$ by deleting the point p'. The assumption that the dimension of the space $\Gamma(M)$ is at least 3 is satisfied if the genus of the surface M is infinite. The assumption is also satisfied by many surfaces of finite genus, but an even sharper theorem holds for such surfaces (see Theorem 3.2).

Proof. We say that a point $p_0 \in D_M$ if $p_0 \in M$ and if there exists $\phi_0 \in \Gamma(M)$ with a zero of order two at p_0, that is

$$\phi_0(z - z_0) = A_0(z - z_0)^2 + o((z - z_0)^2), A_0 \neq 0, \tag{3.1}$$

where z is close to z_0. Here $z(p) = z$ is a local parameter near p_0, and $z(p_0) = z_0$.

We first show that D_M is contained in S_M. Let $p_0 \in M$. Denote by φ_0 the *adjoint kernel* differential with the reference point p_0. It is well known (see [1], and page 43 in [10]) that this differential exists for every Riemann surfaces (not necessarily hyperbolic). It has the following properties. φ_0 is a $(1,0)$ meromorphic form on M, with the second order pole at p_0. Furthermore, if U denotes an arbitrary open set containing p_0 then

$$\int_{M\setminus U} |\varphi_0|^2 < \infty. \tag{3.2}$$

In particular, (3.2) implies that p_0 is the only pole of the differential φ_0.

Remark. The existence of the above differential was proved in [1] (also see [10]). For an example if M is the complex plane , and $p_0 = 0$, then the differential $\varphi_0 = \frac{1}{z^2}$. φ_0 clearly has the second order pole at 0, and $(\varphi_0)^2$ is integrable on the sphere minus some open set containing 0. In particular, if the space $\Gamma(M)$ is not trivial then the the kernel differential φ_0 is characterized as the unique (up to the multiplication by a non zero constant) differential with the above properties and which annihilates the space $\Gamma(M)$ in the sense of the standard inner product (computed as a Cauchy principial value).

Now suppose that $p_0 \in D_M$. Denote by ϕ_0 the differential satisfying (3.1), and by φ_0 the kernel differential with the reference point p_0. Let $p \neq p_0$ be an arbitrary point in M. If $\phi_0(p) \neq 0$, and since $\phi_0^2 \in A^1(M)$ we conclude that $p_0 \in S_M$. If $\phi_0(p) = 0$, then we have that $\phi_0(p)\varphi_0(p) = 0$. From (3.1) it follows that $\phi_0(p_0)\varphi_0(p_0) \neq 0$. Also, from (3.1), and (3.2) it follows that $\phi_0\varphi_0 \in A^1(M)$. We conclude that $p_0 \in S_M$ in this case too.

To finish the proof it is enough to show that $M \setminus D_M$ is a discrete set of points in N. Also, if p' is a puncture in the boundary of M, we have to show that p' is not a cluster point of the set $M \setminus D_M$. Since $\Gamma(M)$ is at least of the dimension 3, we can choose 3 linearly independent differentials from $\Gamma(M)$. Denote them by φ, ϕ, ψ. We have that φ, ϕ, and ψ are holomorphic at every puncture in the boundary of M.

Let p' be a point of M or a puncture on the boundary of M, and let z be a local coordinate in a neighborhood U of p'. For $z \in U$, set

$$\begin{aligned} a(z)\varphi(z) + b(z)\phi(z) + c(z)\psi(z) &= 0, \\ a(z)\varphi'(z) + b(z)\phi'(z) + c(z)\psi'(z) &= 0, \\ a(z)\varphi''(z) + b(z)\phi''(z) + c(z)\psi''(z) &= 1. \end{aligned} \tag{3.3}$$

This system of equations can be solved for $a(z), b(z), c(z)$ at every point z where the Wronskian determinant of φ, ϕ, ψ is not zero. Since φ, ϕ, ψ are linearly independent, the set of zeroes of the Wronskian determinant is a discrete set in M, and also this set can not accumulate at any puncture in the boundary of M.

Let $p \in M$ be a point where the system (3.3) has a solution. Then we have that $p \in D_M$. This implies that the set $M \setminus D_M$ can not accumulate at any point of M (or a puncture in the boundary of M). $\square$

In the remainder of this section we deal with the case of finite genus surfaces. The following lemma is well known. I thank the referee for pointing out this lemma and its proof to me.

Lemma 3.1. *Let M^* be compact Riemann surface of genus g, and let D be a divisor of degree $n \geq 0$ on M^*.Suppose $2g + n \geq 5$. Let V be the vector space of meromorphic quadratic differentials φ on M^* such that $(\varphi) + D \geq 0$ (here (φ) denotes the divisor of φ). If p and q are distinct points of M^* that do not belong to the support of D, then there is a φ in V that is zero at p but not at q.*

Proof. Let ω be a nontrivial one-form on M^*. We have that $\varphi \in V$ if and only if $\varphi = f\omega^2$, where f is a meromorphic function on M^* with $(f) + 2(\omega) + D \geq 0$.

Let E be the divisor $2(\omega) + D$. Then V has dimension $dim(E)$. Also, the space of φ in V such φ has a zero at p has dimension $dim(E - p)$, and the space of φ in V with zeroes at both p and q has dimension $dim(E - p - q)$.

By hypothesis, $2g + n \geq 5$, so the divisors $(\omega) - E$, $(\omega) - E + p$, and $(\omega) - E + p + q$ all have negative degree. Therefore, the Riemann-Roch theorem implies that $dim(V) = dim(E) = 3q + n - 3$, $dim(E - p) = dim(V) - 1$, and $dim(E - p - q) = dim(V) - 2$. In particular, the space of φ in V with zeroes at both p and q has lower dimension that the subspace with zeroes at p. $\square$

Theorem 3.2. *Suppose M has finite genus and is not of exceptional type. Then $M = S_M$*

Remark. It is easy to verify that the above theorem does not hold for exceptional type surfaces.

Proof. Let M^* be the closed surface of the same genus as M, such that M is obtained from M^* by deleting points $p_1, ..., p_k \in M^*$ (the set of points $p_1, ..., p_k$ may be empty). Define the divisor D on M^* by $D = p_1 + ... + p_k$. The theorem now follows from the previous lemma. $\square$

Combining the previous two theorems we obtain the following theorem.

Theorem 3.3. *Suppose that M is a Riemann surface of non-exceptional type. Then the set $M \setminus S_M$ is a discrete set of points in M. Furthermore, if p' is a puncture in the boundary of M then p' is not a cluster point of the set $M \setminus S_M$.*

4. Construction of the map β, and its properties.

From now on, we assume that M, and N are Riemann surfaces of non-exceptional type. We do not assume that M and N are homeomorphic.

Let $T : A^1(M) \to A^1(N)$ be a surjective linear isometry. Since $A^1(M)$ is a separable Banach space, we can choose a sequence of distinct differentials (ϕ_k), $k = 0, 1, 2...$, such that for every $\phi \in A^1(M)$ there exists a subsequence of (ϕ_k), which is norm converging to ϕ, in the norm on $A^1(M)$. In addition, we choose ϕ_0 and ϕ_1 to be linearly independent.

Denote by M_1 the set S_M minus the two discrete sets of points. The first one is the set of points where ϕ_0 has zeroes and the second is where the meromorphic function $\frac{\phi_1}{\phi_0}$ has the first derivative equal to 0. It follows from Theorem 3.3 that $M \setminus M_1$ is a discrete set of points in M. Note that if p' is a puncture in the boundary of M, then the function $\frac{\phi_1}{\phi_0}$ is well defined meromorphic function in a given neighborhood of p'. Therefore, the set of points where ϕ_0 has zeroes, and also the set of points where $\frac{\phi_1}{\phi_0}$ has the first derivative equal to 0, does not have the point p' as its cluster point. Since we proved in Section 3. that the set S_M has the same property, we conclude that the set $M \setminus M_1$ does not have the point p' as its cluster point. We define the set N_1 in N in the same way, using the sequence $T(\phi_k)$ in $A^1(N)$. The set $N \setminus N_1$ has the same directness properties as $M \setminus M_1$.

The notation and assumptions introduced above remain valid until the end of this paper. In this section we will work with M_1, and N_1, and we will return to M, and N later.

For every $n \in \mathbf{N}$, we define the maps $I_n : M_1 \to \mathbf{C}^n$ and $J_n : N_1 \to \mathbf{C}^n$, by

$$I_n(p) = (\frac{\phi_1}{\phi_0}(p), ..., \frac{\phi_n}{\phi_0}(p)),$$

for $p \in M_1$, and

$$J_n(q) = (\frac{T(\phi_1)}{T(\phi_0)}(q), ..., \frac{T(\phi_n)}{T(\phi_0)}(q)),$$

for $q \in N_1$. I_n and J_n are holomorphic maps. In particular, from the definition of M_1 it follows that for every $p \in M_1$ there is a neighborhood U of p, and a biholomorphic map map $\tilde{I_n} : U \times \mathbf{C}^{n-1} \to \mathbf{C}^n$, such that $I_n(p) = \tilde{I_n}(p, 0)$. This implies that if F is a Borel set on M_1 then $I_n(F)$ is a Borel set in $\mathbf{C}^n$. A similar observation holds for N_1.

Also, define the maps $I_\infty : M_1 \to \mathbf{C}^{\mathbf{N}}$ and $J_\infty : N_1 \to \mathbf{C}^{\mathbf{N}}$ as

$$I_\infty(p) = (\frac{\phi_1}{\phi_0}(p), ...),$$

for $p \in M_1$, and

$$J_\infty(q) = (\frac{T(\phi_1)}{T(\phi_0)}(q), ...),$$

for $q \in N_1$. The product topology of pointwise convergence makes the space $\mathbf{C}^{\mathbf{N}}$ a Fréchet space. We have that maps I_∞ and J_∞ are holomorphic, and they map Borel sets to Borel sets for the same reason that the maps I_n and J_n do.

Suppose that $z = x + iy$, and $z_1 = x_1 + iy_1$ are local parameters on M, and N respectively. Set $d\mu = |\phi_0| dxdy$, and $d\nu = |T(\phi_0)| dx_1 dy_1$. μ, ν are well defined measures on M, and N respectively.

Lemma 4.1. *Let $p \in M_1$, and $q \in N_1$. Then*

$$\cap_{n=1}^{\infty} I_n^{-1}(I_n(p)) = \{p\},$$

$$\cap_{n=1}^{\infty} J_n^{-1}(J_n(q)) = \{q\}.$$

In other words, the maps I_∞ and J_∞ are injective.

Proof. Suppose that there exists $p' \in M$, $p \neq p'$, such that $p' \in \cap_{n=1}^{\infty} I_n^{-1}(I_n(p))$. Since ϕ_n is dense in $A^1(M)$, we conclude that

$$\frac{\phi}{\phi_0}(p) = \frac{\phi}{\phi_0}(p'), \tag{4.1}$$

for every $\phi \in A^1(M)$. From Theorem 3.3 we know that there exists $\psi \in A^1(M)$ such that either $\psi(p) = 0$, and $\psi(p') \neq 0$, or $\psi(p) \neq 0$, and $\psi(p') = 0$. But this contradicts (4.1). The proof for N_1 is similar. $\square$

Let G_M be the set of all points $p \in M_1$ such that the set

$$\cap_{n=1}^{\infty} J_n^{-1}(I_n(p))$$

is a nonempty subset of N_1. Also, let G_N be the set of all points $q \in N_1$ such that the set

$$\cap_{n=1}^{\infty} I_n^{-1}(J_n(q))$$

is a nonempty subset of M_1.

Observe that

$$G_M = I_\infty^{-1}(J_\infty(N_1)), G_N = J_\infty^{-1}(I_\infty(M_1)),$$

and

$$I_\infty(G_M) = J_\infty(G_N) = I_\infty(M_1) \cap J_\infty(N_1).$$

In particular we have that G_M and G_N are Borel sets.

Lemma 4.2. *We have*

$$\mu(M_1 \setminus G_M) = \nu(N_1 \setminus G_N) = 0.$$

Proof. Let K_1 be an arbitrary compact set in N_1. Suppose that there exists a compact set $F \subset M_1$ such that $\mu(F) > 0$, and $F \cap G_M$ is an empty set. Note that $J_n^{-1}(I_n(F))$ contains $J_{n+1}^{-1}(I_{n+1}(F))$. Set

$$A_n = \nu(J_n^{-1}(I_n(F)) \cap K_1).$$

From the assumptions on the set F, we find that the set

$$\cap_{n=1}^{\infty} J_n^{-1}(I_n(F))$$

is an empty set. Therefore we have $A_n \to 0$ when $n \to \infty$.

From Lemma 2.3 we have

$$\mu(F) \leq \mu\left(I_n^{-1}(I_n(F))\right) = \int\limits_{I_n^{-1}(I_n(F))} d\mu = \int\limits_{J_n^{-1}(I_n(F))} d\nu = \nu\left(J_n^{-1}(I_n(F))\right) \qquad (4.2)$$

The *liminf* of the sequence of values of the integrals on the right hand side in (4.2) when $n \to \infty$, is at most equal to $\nu(N_1 \setminus K_1)$. By varying the set K_1 we obtain a contradiction.

Therefore, we have proved that $\mu(M_1 \setminus G_M) = 0$. Similarly we prove $\nu(N_1 \setminus G_N) = 0$. $\square$

If $p \in G_M$ we have that $\cap_{n=1}^{\infty} J_n^{-1}(I_n(p))$ is a non empty subset of N_1. From Lemma 4.1 we see that there exists a unique point $q \in N_1$, such that

$$\cap_{n=1}^{\infty} J_n^{-1}(I_n(p)) = \{q\}.$$

Set $\beta(p) = q$. In particular, we conclude that β is a well defined bijection between G_M, and G_N. We also have $\beta = J_\infty^{-1} \circ I_\infty$.

Lemma 4.3. *The map $\beta : G_M \to G_N$ satisfies the following conditions:*

1. *β is a bijection, and both β and β^{-1} are measurable maps. Also, β and β^{-1} map sets of measure zero onto sets of measure zero.*

(2) *The equality*

$$\frac{T(\psi)}{T(\phi)}(\beta) = \frac{\psi}{\phi}$$

holds on G_M, for every $\phi, \psi \in A^1(M)$.

(3) *Let $p \in G_M$, and $q \in N_1$. If*

$$\frac{T(\psi)}{T(\phi)}(q) = \frac{\psi}{\phi}(p)$$

holds for every $\phi, \psi \in A^1(M)$, then $q = \beta(p)$.

(4) *Let $q \in G_N$, and $p \in M_1$. If*

$$\frac{T(\psi)}{T(\phi)}(q) = \frac{\psi}{\phi}(p)$$

holds for every $\phi, \psi \in A^1(M)$, then $p = \beta^{-1}(q)$.

(5) *If F is a measurable subset of G_M, then*

$$\int\limits_{F} |\phi| = \int\limits_{\beta(F)} |T(\phi)|$$

holds for every $\phi \in A^1(M)$. In particular, we have $\mu(F) = \nu(\beta(F))$.

Proof. It follows from Lemma 4.1 that β is bijection between G_M, and G_N. From the definition of G_M, G_N, I_n, J_n, and since ϕ_n is dense in $A^1(M)$, we see that

$$\frac{T(\phi)}{T(\phi_0)}(\beta) = \frac{\phi}{\phi_0}$$

on G_M. Since for $\phi, \psi \in A^1(M)$ we have

$$\frac{\phi}{\phi_0} / \frac{\psi}{\phi_0} = \frac{\phi}{\psi},$$

it follows that β satisfies the condition (2). The fact that β satisfies the conditions (3) and (4) follows directly from Lemma 4.1.

It is left to show that β satisfies the condition (5). Let F be a compact set in G_M. Let $u_n : \mathbf{C}^n \to \mathbf{R}$ be the function given by

$$u_n(w_1, ..., w_n) = |w_1|,$$

where $(w_1, ..., w_n) \in \mathbf{C}^n$. Let v_n be the function defined as the product of u_n and the characteristic function of the set $I_n(F)$. From the proof of Lemma 2.2 applied to the function v_n we find

$$\int\limits_{I_n^{-1}(I_n(F))} |\phi_1| dx dy = \int\limits_{J_n^{-1}(I_n(F))} |T(\phi_1)| dx_1 dy_1. \tag{4.3}$$

In particular, since $\beta(F)$ is a Borel set we have that the above integrals are well defined. Since $I_{n+1}^{-1}(I_{n+1}(F))$ is contained in $I_n^{-1}(I_n(F))$, from Lemma 4.1 we have that the set

$$\cap_{n=1}^{\infty} I_n^{-1}(I_n(F)),$$

is equal to F. Now, from Lemma 4.2 we conclude that

$$\left| \int\limits_{I_n^{-1}(I_n(F))} |\phi_1| dxdy - \int\limits_{F} |\phi_1| dxdy \right| \to 0, n \to \infty.$$

Similarly we obtain

$$\left| \int\limits_{J_n^{-1}(I_n(F))} |T(\phi_1)| dxdy - \int\limits_{\beta(F)} |T(\phi_1)| dx_1 dy_1 \right| \to 0, n \to \infty.$$

So far we have verified (5) whenever F is a compact subset of G_M. Applying the standard argument we obtain that (5) holds for every measurable set F contained in G_M. From (5) we also conclude that β maps the sets of measure zero onto sets of measure zero. This proves the claim (1).

In the same way, we can show that (5) holds for every ϕ_n, and from the density of the sequence (ϕ_n) we have that (5) holds for every $\phi \in A^1(M)$.

If we take $\phi = \phi_0$, we see that β, and β^{-1} are measure preserving maps, that is the equality $\mu(F) = \nu(\beta(F))$ holds for every measurable set F in G_M. $\square$

5. Construction of the map α, and its properties

Define the sets E_M, and E_N as follows.

Definition. We say that a point $p \in M$ belongs to E_M, if there exists an open set U containing p and the the holomorphic injection $\alpha : U \to N$, such that the equality

$$\frac{T(\psi)}{T(\phi)} \circ \alpha = \frac{\psi}{\phi} \tag{5.1}$$

holds on U for every $\phi, \psi \in A^1$. In the similar way we define the set E_N.

It follows that the set E_M is an open subset of M. E_N is an open subset of N. We show that the map α is well defined holomorphic bijection between E_M and E_N.

To show that α is well defined we need to consider two different open sets U_1, and U_2 and two holomorphic injections $\alpha : U_1 \to N$, and $\alpha_2 : U_1 \to N$, which both satisfy (5.1), and to show that α_1, and α_2 agree on $U_1 \cap U_2$. Combining the fact that $M \setminus M_1$ is a discrete set and items (3), and (4) from Lemma 4.3 together with (5.1), we conclude that $\alpha_k = \beta$ almost everywhere on U_k, for $k = 1, 2$. This implies that α_1, and α_2 agree almost everywhere on $U_1 \cap U_2$. Similarly we show that α is a bijection between E_M and E_N.

Lemma 5.1. *We have*

1. *Let $p_0 \in M_1$, and let $(p_n)_{n \in \mathbf{N}}$ be a sequence of points such that $p_n \in G_M$ and $p_n \to p_0$. If there is a point $q_0 \in N_1$ such that $\beta(p_n) \to q_0$, then the points p_0 and q_0 belong to E_M and E_N respectively.*
2. *The sets E_M and E_N have the full measure in M and N respectively. The map $\alpha : E_M \to E_N$ is a holomorphic bijection between E_M and E_N.*

(3) *We have that if* $p \in E_M \cap M_1$, *and* $\alpha(p) \in N_1$, *then* $p \in G_M$. *Similarly, if* $q \in E_N \cap N_1$, *and* $\alpha^{-1}(q) \in M_1$, *then* $q \in G_N$. *The maps* α, *and* β *agree almost everywhere on* M. *Also,* α^{-1}, *and* β^{-1} *agree almost everywhere on* N.

(4) *The equality*
$$|(T(\phi)\circ\alpha)(\alpha')^2| = |\phi|$$
holds on E_M *for every* $\phi \in A(M)$.

Proof. Since $p_n \in G_M$, from the property (2) of Lemma 4.3, we conclude that
$$\frac{T(\psi)}{T(\phi)}(\beta(p_n)) = \frac{\psi}{\phi}(p_n), \tag{5.2}$$
for every $\psi, \phi \in A^1(M)$. Since $p_n \to p_0 \in M_1$, and $\beta(p_n) \to q_0 \in N_1$, we have
$$\frac{T(\psi)}{T(\phi)}(q_0) = \frac{\psi}{\phi}(p_0),$$
for every $\phi, \psi \in A^1(M)$. On the other hand, from the definition of M_1, and N_1 we know that there are open neighborhoods U_0, and V_0 of points p_0, and q_0 respectively, such that the functions $\frac{\phi_1}{\phi_0}(p)$, and $\frac{T(\phi_1)}{T(\phi_0)}(q)$ are univalent, as maps from U_0 and V_0 into $\mathbf{C}$ respectively. By modifying the neighborhoods U_0, and V_0 if necessary, we see that there exists a conformal map $\tilde{\alpha} : U_0 \to V_0$, such that the relation
$$\frac{T(\phi_1)}{T(\phi_0)} \circ \tilde{\alpha} = \frac{\phi_1}{\phi_0} \tag{5.3}$$
holds on U_0. Here we choose U_0, and V_0 small enough so they belong to M_1, and N_1 respectively.

Since $\tilde{\alpha}$ is conformal on U_0, it follows from (5.2), and (5.3), that $\tilde{\alpha}(p_n) = \beta(p_n)$, for all but finitely many $n \in \mathbf{N}$. Also, the equality
$$\frac{T(\psi)}{T(\phi)}(\tilde{\alpha}(p_n)) = \frac{\psi}{\phi}(p_n)$$
holds for all but finitely many $n \in \mathbf{N}$, and for every $\phi, \psi \in A^1(M)$. Since $p_n \to p_0$ we conclude that
$$\frac{T(\psi)}{T(\phi)}(\tilde{\alpha}(p)) = \frac{\psi}{\phi}(p)$$
for every $p \in U_0$, and for every $\phi, \psi \in A^1(M)$. From the property (3) of Lemma 4.3 and from the above equality, together with the fact that U_0 is a subset of M_1, and V_0 is a subset of N_1, we conclude that β is defined in U_0, and that U_0 is an open subset of G_M. This proves the property (1) of this lemma.

The map β is a measurable map. In fact, if we ignore some sets of measure zero we can think of the map β as a map from the interior of a fundamental polygon for M to the the interior of such polygon for N. This makes β an L^1 function on a plane region, and we can talk about the set of Lebesgue points of the map β on the interior of the fundamental polygon for M. The corresponding points on the surface M will be called the Lebesgue points for β. Note that this definition of the Lebesgue points on M for β depends on the choice of the fundamental polygon for M. Nevertheless, this set has the full measure in M.

Since β maps sets of positive measure onto sets of positive measure, we conclude that for almost every Lebesgue point $p_0 \in G_M$ the following holds. There exists a point $q_0 \in G_N$, and a sequence of distinct points $(p_n)_{n\in\mathbf{N}}$, such that $p_n \in G_M$, $p_n \to p_0$, and $\beta(p_n) \to q_0$. Therefore every such Lebesgue point is contained in E_M, and we conclude that E_M has the full measure in M.

We have already shown that α is the well defined holomorphic bijection between E_M, and E_N. Also, if $p \in E_M \cap M_1$, and $\alpha(p) \in N_1$, it follows form the definition of G_M, that $p \in G_M$. Similarly, we show the corresponding result for $E_N \cap N_1$. We have seen that α agrees with β almost everywhere on any open set subset of E_M. Since E_M has the full measure in M we conclude that α, and β agree almost everywhere on M. In the same way we show that α^{-1} agrees with β^{-1} almost everywhere on N.

For every $\phi \in A^1(M)$ we define

$$L(\phi) = (T(\phi) \circ \alpha)(\alpha')^2.$$

The differential $L(\phi)$ is defined on E_M. Also, it follows from the property (5) of Lemma 4.3, that for every measurable set F which is a subset of E_M, we have

$$\int\limits_F |\phi| dxdy = \int\limits_F |L(\phi)| dxdy. \tag{5.4}$$

Since ϕ, and $L(\phi)$ are holomorphic forms on E_M, and E_M is an open set in M, it follows from (5.4) that

$$|(T(\phi) \circ \alpha)(\alpha')^2| = |\phi|$$

holds everywhere on E_M, and for every $\phi \in A^1(M)$.

□

Lemma 5.2. *Let $p' \in M$, and assume that there exists an open set U which contains p', and such that $U \setminus \{p'\}$ is contained in E_M. Then $p' \in E_M$.*

Proof. Denote by $\tilde{\alpha}$ the extension of the map α on the set U. Since $U \setminus \{p'\}$ is a subset of E_M, we have that $\tilde{\alpha}$ satisfies the equality (5.1) on $U \setminus \{p'\}$. If $\tilde{\alpha}(p')$ belongs to N, then $\tilde{\alpha}$ satisfies (5.1) on U as well and by the definition of E_M we have $p' \in E_M$. So, we may assume that $\tilde{\alpha}(p')$ is not in N. Then $\tilde{\alpha}(p')$ has to be a puncture in the boundary of N.

It is well known that there exists $\phi \in A(N)$ such that ϕ has a first order pole at a given puncture in the boundary of M, in this case the puncture $\tilde{\alpha}(p')$. From the item (4) of the previous lemma we see that the equality

$$|(\phi) \circ \tilde{\alpha})(\tilde{\alpha}')^2| = |T^{-1}(\phi)|$$

holds on $U \setminus \{p'\}$, and therefore on U as well. Since $T^{-1}(\phi)$ is holomorphic at p' we obtain a contradiction. □

Let us repeat that the sets $M \setminus M_1$, and $N \setminus N_1$ are discrete sets of points in M and N respectively. We have also shown that if p' is a puncture in the boundary of M then p' is not a cluster point of the set $M \setminus M_1$. The same holds for punctures in the boundary of N.

Lemma 5.3. *The sets $M \setminus E_M$, and $N \setminus E_N$ have no isolated points. Furthermore, assume that $(p_n)_{n\in\mathbf{N}}$ is sequence of distinct points, such that every $n \in \mathbf{N}$, $p_n \in E_M$, and such that $p_n \to p_0$, where $p_0 \in M_1 \setminus E_M$. Then, the sequence $\alpha(p_n)$ has no cluster points in N_1.*

The same conclusion holds for the set E_N.

Proof. The fact that $M \setminus E_M$, and $N \setminus E_N$ have no isolated points follows directly from Lemma 5.2.

Assume that the sequence $\alpha(p_n)$ has a cluster point $q \in N_1$. By passing to a subsequence if necessary, we may assume $\alpha(p_n) \to q_0$, where $q_0 \in N_1$. Since $p_0 \in M_1$, from the item (1) of Lemma 5.1, we conclude that $p_0 \in E_M$, which is a contradiction. This proves the second part of this lemma. □

6. The boundary behavior of the map α

Let Γ_N be the Fuchsian group acting on the unit disc $\mathbf{\Delta}$, such that $\mathbf{\Delta}/\Gamma_N$ is conformally equivalent to the surface N. Denote by $P : \mathbf{\Delta} \to N$ the universal covering map.

Let $(z_n)_{n\in\mathbf{N}}$ be a sequence of distinct points in $\mathbf{\Delta}$ such that

(1) $|z_n| \to 1$, $n \to \infty$.
(2) There exists a sequence $r_n > 1$ such that $r_n \to \infty$ for $n \to \infty$, and the covering map $P : \mathbf{\Delta} \to N$ is univalent in the hyperbolic disc of radius r_n, centered at z_n.
(3) If by q_n we denote the points on N such that $q_n = P(z_n)$, then there exists a sequence of distinct points (p_n), $p_n \in E_M \cap M_1$, for every $n \in \mathbf{N}$, such that $\alpha(p_n) = q_n$.

Our aim in this section is to show that under the stated assumptions the sequence p_n can not have a cluster point in M_1.

If M has a positive genus, we have that the space $\Gamma(M)$ of Abelian differentials of finite norm, is of positive dimension. From the Virtanen's theorem (see page 43. in [10]), it follows that for every point $p \in M$, there exists $\varphi \in \Gamma(M)$ such that $\varphi(p) \neq 0$. Then $(\varphi)^2(p) \neq 0$. We conclude that for every point $p \in M$ there exists $\phi \in A^1(M)$ such that $\phi(p) \neq 0$.

If M is a planar region, then using the fact that M is not of exceptional type, for every point $p \in M$, we can construct a rational differential of M which does not have zero at p.

For each p_n denote by $Z_n(M)$ the space of all $\phi \in A^1(M)$ such that $\phi(p_n) = 0$. Also, for every q_n we denote by $Z_n(N)$ the space corresponding to q_n. We see that $Z_n(M)$ is a vector subspace of $A^1(M)$ of codimension 1. Repeating the same argument, we conclude that $Z_n(N)$ is a vector subspace of $A^1(N)$ of codimension 1.

We have $\alpha(p_n) = q_n$. Since ϕ_0 is $\neq 0$ on M_1 (ϕ_0 was determined at the beginning of Section 4.), from (5.1) we conclude that $T(Z_n(M)) = Z_n(N)$. Let $\lambda_n : A^1(M) \to \mathbf{C}$ be a linear functional of the Banach norm equal to 1 which annihilates the subspace $Z_n(M)$, that is $\lambda_n(Z_n(M)) = 0$. Let $\lambda_n' : A^1(N) \to \mathbf{C}$ be the linear functional such that $\lambda_n(\phi) = \lambda'_n(T(\phi))$ for every $\phi \in A^1(M)$. Since T is a linear isometry, we have that λ'_n is of norm 1, and by definition λ' annihilates the subspace $Z_n(N)$ of $A^1(N)$.

We now show that from the assumptions on the sequence z_n, and the group Γ_N, it follows that for every $\phi \in A^1(N)$, we have that $\lambda'_n(\phi) \to 0$.

From the assumption (2) it follows that the covering map P is univalent on the hyperbolic disc $\mathbf{D}_n$ centered at z_n, and of the hyperbolic radius 1. Let B_n be a Möbius transformation of the unit disc onto itself such that $B_n(z_n) = 0$. Define the function η_n to be equal to $\frac{\overline{(B'_n)}}{B'_n}$ on $\mathbf{D}_n$. For $A \in \Gamma_N$ set η_n to be equal to

$$\left(\frac{\overline{(B'_n)}}{B'_n} \circ A \right) \frac{\overline{(A')}}{A'}$$

on the disc $A^{-1}(\mathbf{D}_n)$. Elsewhere in the unit disc we define η_n to be equal to 0.

Denote by $A^1(\mathbf{\Delta})$ the Banach space of all integrable analytic functions on the unit disc. Also, denote by Θ the standard Poincaré operator $\Theta : A^1(\mathbf{\Delta}) \to A^1(N)$. The operator Θ is surjective and its Banach norm is less or equal to 1. We have that the covering map P is univalent on $\mathbf{D}_n$, and since η_n is equivariant, we see that η_n represents the lift of a certain $(-1, 1)$ measurable form on N.

Let $\omega \in A^1(\mathbf{\Delta})$. From the Mean value theorem for analytic functions, we obtain that

$$\int_{\mathbf{\Delta}} \eta_n \omega = d \sum_{A \in \Gamma_N} (\omega \circ A)(A')^2(z_n),$$

where $d > 0$ is some constant. Therefore, we have that

$$\int_{\mathbf{\Delta}} \eta_n \omega = 0,$$

if and only if $\Theta(\omega) \in Z_n(N)$, $\omega \in A^1(\mathbf{\Delta})$.

Therefore, there exists $t_n \in \mathbf{C}$, such that for each $\omega \in A^1(\mathbf{\Delta})$ we have

$$\int_{\mathbf{\Delta}} t_n \eta_n \omega = \lambda'_n(\Theta(\omega)). \tag{6.1}$$

Set $\omega_n = (B'_n)^2$. We have that the Banach norm $||\omega_n||_1$ is equal to π. Also, from the properties of the sequence z_n, it follows that

$$\int_{\mathbf{\Delta}} \eta_n \omega_n = \int_{D_n} \eta_n \omega_n + \sum_{A \in (\Gamma_N \setminus Id)} \int_{A(D_n)} \eta_n \omega_n.$$

From the properties of the sequence z_n it follows that as $n \to \infty$ the hyperbolic distance between z_n and any point $A(z_n)$, $A \in (\Gamma_N \setminus Id$, uniformly tends to ∞. From the construction of the function ω_n, it follows that

$$\sum_{A \in (\Gamma_N \setminus Id)} \int_{A(D_n)} |\omega_n| \to 0,$$

and we have

$$\left| \int_{\mathbf{\Delta}} \eta_n \omega_n \right| \to c_1$$

when $n \to \infty$, and $z_n \to x$. Here, c_1 stands for the Euclidean area of the hyperbolic disc of the hyperbolic radius 1, centered at the origin in the unit disc.

We have constructed the sequence $\varphi_n = \Theta(\omega_n) \in A^1(N)$, such that $||\varphi_n||_1 \leq \pi$, and

$$\lim_{n\to\infty} \frac{|\lambda'(\varphi_n)|}{|t_n|} = c_1.$$

Since $|\lambda'(\varphi_n)| \leq \pi$, we conclude that $limsup(|t_n|) \leq \frac{\pi}{c_1}$.

We have that $t_n\eta_n$ is uniformly bounded on the unit disc (the bound does not depend on n), and we have that $t_n\eta_n$ converges to 0 pointwise on the unit disc. This conclusion, together with (6.1), yields that for any fixed $\omega \in A^1(\mathbf{D})$, we have

$$\int_{\mathbf{\Delta}} t_n\eta_n\omega = \lambda'_n(\Theta(\omega)) \to 0, n \to \infty.$$

Since Θ is surjective, we have shown that $\lambda'_n(\phi) \to 0$, $n \to \infty$, for every $\phi \in A^1(N)$.

We have

Lemma 6.1. *With the notation and assumptions stated above we have that the sequence p_n has no cluster points in M_1.*

Proof. Assume now that the sequence p_n has a cluster point in M_1. By passing onto a subsequence if necessary, we may assume that there exists a point $p_0 \in M_1$ such that $p_n \to p_0$. Let $\psi_0 \in A^1(M)$ such that $\psi_0(p_0) \neq 0$. We show that the sequence $\lambda_n(\psi_0)$ does not converge to 0.

To see that $\lambda_n(\psi_0)$ does not converge to 0, note that for n large enough the functional λ_n is a multiple of the linear functional

$$\psi \to \frac{\psi}{\psi_0}(p_n), \psi \in A^1(M).$$

Since the above functional converges to the linear functional

$$\psi \to \frac{\psi}{\psi_0}(p_0), \psi \in A^1(M)$$

which is a non-zero linear functional, we see that the sequence $\lambda_n(\psi_0)$ does not converge to 0. $\square$

7. Connected components of the set E_M

Let $p_0 \in E_M$, and let $E_{M,0}$ be the maximal connected component of the set E_M which contains p_0. That is, the set $E_{M,0}$ is the union of all connected components of E_M containing the point p_0.

Lemma 7.1. *Let γ be a Jordan curve in $E_{M,0}$ such that the Jordan domain Ω_0 bounded by γ is a well defined planar subregion of M_1, and such that $p_0 \in \Omega_0$. Then Ω_0 is contained in $E_{M,0}$.*

Proof. Set $D_0 = \Omega_0 \cap E_{M,0}$. Our aim is to show $D_0 = \Omega_0$. Since γ is contained in $E_{M,0}$ we see that γ is an isolated boundary component of D_0. Also, set $\tilde{D_0} = \alpha(D_0)$. As a Riemann surface, the region $\tilde{D_0}$ is embedded in the surface N. We see that $\alpha(\gamma)$ is an isolated component of the boundary of the region $\tilde{D_0}$ in N. We now show that this is the only component of the boundary $\partial\tilde{D_0}$ in N. This means that the region $\tilde{D_0}$ is a well defined *end* of N in the sense of [10] , page 3.

Let B be a component of the boundary $\partial\tilde{D_0}$ in N. In particular, B is a closed set in N. If the component B is different from $\alpha(\gamma)$, we conclude that $B \cap E_N$ is an empty set. Choose an arbitrary point $q' \in B$. Let $(q_n)_{n \in \mathbf{N}}$ be a sequence of distinct points in $\tilde{D_0}$ such that $q_n \to q'$. Let p' be a cluster point of the sequence $\alpha^{-1}(q_n)$. Then p' belongs to the closure of the set D_0. Since α is conformal and B is a boundary component different from $\alpha(\gamma)$, p' can not belong to the curve γ.

Assume now that $q' \in N_1$. Since $q_n \to q'$, $q_n \in E_N$, and $q' \in N$, from Lemma 5.3 we conclude that the sequence $\alpha^{-1}(q_n)$ has no cluster points in M_1. But this is not possible since p' does not belong to γ, and the closure of D_0 is contained in M_1.

So, the only possibility left is that $q' \in N \setminus N_1$. Therefore we have that the whole component B has to be a subset of $N \setminus N_1$. But B is connected, so it has to reduce to a single point q'. Then q' is an isolated point in the set $N \setminus E_N$ which is not possible by Lemma 5.3.

Therefore we have proved that $\tilde{D_0}$ is a regularly embedded region in N, that is $\tilde{D_0}$ is an end in N. The Jordan curve $\alpha(\gamma)$ is the boundary of $\tilde{D_0}$. Also, since D_0 is a planar region, we conclude that $\alpha(D_0) = \tilde{D_0}$ (as a Riemann surface) is a planar region too. In other words, $\tilde{D_0}$ is a planar end in N. Then, it is well known (see [10] page 258., and Theorem 5A on page 269.) that there exists a Riemann surface N^*, and a compact set S in N^*, such that

(1) $N^* \setminus S = N$
(2) $\tilde{D_0} \cup S = \Omega_0^*$ is a Jordan region bounded by $\alpha(\gamma)$

We also define the set E to be the boundary of the set S in N^*.

Remark. It is elementary to construct the surface N^* (see [10] page 258.). The surface N^* is called an extension of the surface N.

By construction there exists a Jordan region Ω_0^*, which is a subset of N^* and which is bounded by the Jordan curve $\alpha(\gamma)$. But in general this by no means suggests that the map α can be conformally extended to map Ω_0 onto Ω_0^*. But in our case this is true, and it is a corollary of the existence of the linear isometry $T : A^1(M) \to A^1(N)$. However, we do not have to go that far in order to prove this lemma.

If the set E defined above is not empty , then it can not consist of finitely many points. If E consisted of finitely many points, then the map α^{-1} could be extended over E, and map E into $M \setminus E_M$. But every point in $\alpha^{-1}(E)$ is then an isolated point of the set $M \setminus E_M$ which is not possible by Lemma 5.3.

Since E contains infinitely many points, we can choose a finite set E' which is a subset of E, and such that E' contains at least 5 points. Then, we can choose two linearly independent $\tilde{\phi}, \tilde{\psi} \in A^1(N^* \setminus E')$. The function $\frac{\tilde{\psi}}{\tilde{\phi}}$ is a well defined meromorphic function on $N^* \setminus E'$, and in particular there are only finitely many points in Ω_0^*, in which the function $\frac{\tilde{\psi}}{\tilde{\phi}}$ has poles or its first derivative is equal to 0. Since $\tilde{\phi}, \tilde{\psi} \in A^1(N^* \setminus E')$, and E' is a subset of E, we have $\tilde{\phi}, \tilde{\psi} \in A^1(N)$ (it is understood that the restrictions of $\tilde{\phi}, \tilde{\psi}$ on N are in $A^1(N)$). The function $\frac{T^{-1}(\tilde{\psi})}{T^{-1}(\tilde{\phi})}$ is a well defined meromorphic function on M, and in particular on there are only finitely many points Ω_0 in which the function $\frac{T^{-1}(\tilde{\psi})}{T^{-1}(\tilde{\phi})}$ has poles or its first derivative is equal to 0. From Lemma 5.1 we have that the equality

$$\frac{T^{-1}(\tilde{\psi})}{T^{-1}(\tilde{\phi})} = \frac{\tilde{\psi}}{\tilde{\phi}} \circ \alpha \tag{7.1}$$

holds on D_0.

Since α is a conformal map between D_0, and $\tilde{D_0}$, we see that one can choose points, p in the closure of D_0 (and p is not in γ), and $q \in (E \setminus E')$, such that the following holds. There exist a sequence of distinct points $(q_n)_{n\in\mathbf{N}}$, $q_n \in \tilde{D_0}$, such that $q_n \to q$, and $\alpha^{-1}(q_n) \to p$. Also, we can arrange that the functions $\frac{T^{-1}(\tilde{\psi})}{T^{-1}(\tilde{\phi})}$, and $\frac{\tilde{\psi}}{\tilde{\phi}}$ are univalent and bounded in some (sufficiently small) neighborhoods of the points p, and q respectively. Denote these neighborhoods by U, and V respectively. Since $q_n \in E_N$, and from (7.1), it follows that

$$\frac{T^{-1}(\tilde{\psi})}{T^{-1}(\tilde{\phi})}(p) = \frac{\tilde{\psi}}{\tilde{\phi}}(q).$$

Therefore, by modifying U and V if necessary, we see that there exists a conformal map $\tilde{\alpha} : U \to V$, such that the equality

$$\frac{T^{-1}(\tilde{\psi})}{T^{-1}(\tilde{\phi})} = \frac{\tilde{\psi}}{\tilde{\phi}} \circ \tilde{\alpha} \tag{7.2}$$

holds on U. Set $U' = \alpha^{-1}(V \cap \tilde{D_0}) \cap U$. Since $q_n \to q$, and $\alpha^{-1}(q_n) \to p$, we see that U' is a non empty open set. Then, from (7.1), and (7.2), and since both α and $\tilde{\alpha}$ are injections, we conclude that α, and $\tilde{\alpha}$ agree on U'. From Lemma 5.1 (item (4)), we see that the equality

$$|(\phi \circ \tilde{\alpha})(\tilde{\alpha}')^2| = |T^{-1}(\phi)| \tag{7.3}$$

holds on U' for every $\phi \in A^1(N)$.

We now choose $\varphi \in A^1(N^* \setminus (E' \cup \{q\}))$ such that φ has the first order pole at q. From (7.3) it follows that

$$|(\varphi \circ \tilde{\alpha})(\tilde{\alpha}')^2| = |T^{-1}(\varphi)|$$

in $U \setminus \{g\}$. Therefore $T^{-1}(\varphi)$ has the first order pole at p. Since $T^{-1}(\varphi) \in A^1(M)$ we have that $T^{-1}(\varphi)$ is holomorphic at p, which is a contradiction.

So, we have proved that the set E is an empty set and therefore it follows that $\Omega_0 = D_0$. This proves the lemma. □

8. Proof of Theorem 1.1

In this section we show that $E_M = M$, $E_N = N$, and that α is a conformal map between M, and N.

Let $p_0 \in E_M$, and same as before, let $E_{M,0}$ be the maximal connected component of the set E_M which contains p_0. Also, let $\mathbf{D}_0$ be a hyperbolic disc centered at p_0 which is contained in a larger concentric hyperbolic disc which is injectively embedded in M_1. Set $\Omega = \mathbf{D}_0 \cap E_{M,0}$, and denote by Ω_0 the connected component of the set Ω which contains p_0. We have that Ω_0 is an open connected subset of $\mathbf{D}_0$.

Lemma 8.1. $\mathbf{D}_0 = \Omega_0$.

Proof. We assume that $\mathbf{D}_0 \neq \Omega_0$.

It follows from Lemma 7.1 that Ω_0 is a simply connected region. We define the following partition of the boundary $\partial\Omega_0$. Set $C = E_M \cap \partial\Omega_0$, and set $B = \partial\Omega_0 \setminus C$. From the construction, we have that C is an open subset of the circle $\partial\mathbf{D}_0$. Therefore, B is a compact set in M.

We recall the following statement proved in Section 5. Suppose that a point $q \in N$ belongs to the boundary of $\alpha(\Omega_0)$. If in addition, we assume that q does not belong to E_N, then from Lemma 5.3 we conclude that $q \in N \setminus N_1$. Therefore, we have shown that, if $(p_n)_{n\in\mathbf{N}}$ is a sequence of points in Ω_0, such that $p_n \to B$, then the sequence $\alpha(p_n)$ has no cluster points in N_1. In particular, the only possible accumulation points (in N) of the sequence $\alpha(p_n)$ belong to the set $N \setminus N_1$.

Let $f : \mathbf{\Delta} \to \Omega_0$ be a Riemann map with $f(0) = 0$. Let C_1 be the open subset of the unit circle $\partial\mathbf{\Delta}$, such that the map f extends analytically onto C_1, and $f : C_1 \to C$. Denote by B_1 the complement of of the set C_1 with respect to the unit circle.

First, we study the case when the 1-dimensional Lebesgue measure of the set B_1 is zero. We may assume that $\mathbf{D}_0$ is in the complex plane (take a lift of $\mathbf{D}_0$ in the universal cover of M). Since the measure of B_1 is zero, we have that C_1 is dense in $\partial\mathbf{\Delta}$, and B_1 is a totally disconnected subset of the unit circle. But then, the Riemann map $f : \mathbf{\Delta} \to \Omega_0$, $f(0) = 0$, can be conformally extended (by the reflection principle) onto $\mathbf{C} \setminus B_1$. Now, the extended map $f : \mathbf{C} \setminus B_1 \to \mathbf{C}$, maps the region $\mathbf{C} \setminus B_1$ into the region $\mathbf{C} \setminus B$. Since B_1 is of measure zero, it follows (see [1]) that the extended map f is a Möbius transformation and that $f(\mathbf{\Delta}) = \mathbf{D}_0$, which is a contradiction.

Next, we obtain a contradiction to the assumption that B_1 has a positive measure. Set $\tilde{g} = \alpha \circ f$. Since α maps the domain Ω_0 into N_1, we have that the map $\tilde{g} : \mathbf{\Delta} \to N_1$ is a holomorphic injection. Denote by $g_1 : \mathbf{\Delta} \to \mathbf{\Delta}$, the lift of the map $\tilde{g}$ into the universal cover (the unit disc) of the surface N_1 (note here that the points in $N \setminus N_1$ are isolated points in the boundary of N_1). Let $(z_n)_{n\in\mathbf{N}}$ be any sequence in the unit disc $\mathbf{\Delta}$ such that z_n converges to the set B_1. Then, as we already pointed out, from Lemma 5.3 it follows that $\alpha(f(z_n))$ does not have cluster points in N_1. This implies that $|g_1(z_n)| \to 1$, $n \to \infty$. This last fact, together with the fact that the measure of B_1 is positive, implies (see Corollary 6.20 in [6]) that the map g_1 is conformal at almost every point of the set B_1 (see [6] for the notion of being conformal at boundary points). Let X_1 be the subset of B_1 whose elements are the points where the map g_1 is conformal. We just saw that X_1 has the full measure in B_1.

Let $x \in X_1$. From Proposition 4.10 from [6], we have that smooth curves in a Stolz angle at x are mapped onto smooth curves, and the angles between curves are preserved. In particular, for every sequence $z_n \in \mathbf{\Delta}$ such that $z_n \to x$ and z_n is contained in a given Stolz angle at x, we have that $r_n \to \infty$. Here r_n is the radius of the maximal hyperbolic disc centered at $g_1(z_n)$, which is contained in $g_1(\mathbf{\Delta})$. Let $P_1 : \mathbf{\Delta} \to N_1$ be the corresponding universal covering map. Then, we conclude that the sequence $P_1(g_1(z_n))$ can not accumulate at any isolated puncture in the boundary of N_1 (this includes the points of the set $N \setminus N_1$). This follows from the fact that if q_n is a sequence on a given Riemann surface which converges to a puncture in the boundary of that surface, then the injectivity radius at q_n goes to zero.

Denote by $g : \mathbf{\Delta} \to \mathbf{\Delta}$, the lift of the map $\tilde{g}$ into the universal cover (the unit disc) of the surface N and let $P : \mathbf{\Delta} \to N$ be the corresponding universal covering map (note here that the points in $N \setminus N_1$ are not in the boundary of N). Let $x \in X_1$, and let z_n be a sequence in $\mathbf{\Delta}$, $z_n \to x$, and z_n is contained in a given Stolz angle at x. Since we have that the corresponding sequence $P(g(z_n)) = P_1(g_1(z_n))$ does not accumulate at any point in the set $N \setminus N_1$, and by using Lemma 5.3 again, we find that $|g(z_n)| \to 1$. This means that if the function g has a radial limit at some point $x \in X_1$, then this limit lies on the unit circle.

On the other hand, the map g has radial limits almost everywhere on the unit circle. Since X_1 is of positive measure, and all the radial limits at points from X_1 lie on the unit circle, from Corollary 6.20 from [6], we conclude that the map g is conformal almost everywhere on X_1. Denote by X the subset of X_1 which consists of points where g is conformal. We just saw that X has the full measure in X_1.

Now, we again apply Proposition 4.10 from [6]. In a similar fashion as above, we have that for every $x \in X$, and every sequence $z_n \in \mathbf{\Delta}$, $z_n \to x$, and z_n is contained in a given Stolz angle at x, we have that $r_n \to \infty$. Here r_n is the radius of the maximal hyperbolic disc which is centered at $g(z_n)$ and contained in $g(\mathbf{\Delta})$. Then the sequence $\alpha^{-1}(P(g(z_n)))$ remains in the disc $\mathbf{D}_0$ which is compactly contained in M_1. From Lemma 6.1 we obtain a contradiction.

Therefore, we have proved that $\Omega_0 = \mathbf{D}_0$. □

The previous lemma implies that every injectivelly embedded hyperbolic disc in M_1 belongs to E_M. Therefore we have that M_1 is contained in E_M. By Lemma 5.3 there are no isolated points in E_M, so we conclude that $E_M = M$, and similarly $E_N = N$. Therefore, $\alpha : M \to N$ is a conformal map, which by Lemma 5.1 satisfies the equality

$$|(T(\phi) \circ \alpha)(\alpha')^2| = |\phi|$$

for every $\phi \in A^1(M)$. Therefore, for every $\phi \in A^1(M)$ there exists a complex number θ, $|\theta| = 1$ such that the equality

$$(T(\phi) \circ \alpha)(\alpha')^2 = \theta\phi$$

holds on M. Since T is a linear map we find that the corresponding θ is the same for all $\phi \in A^1(M)$. This proves Theorem 1.1.

9. The proofs of Theorem 1.2 and Theorem 1.3

The both theorems are direct corollaries of Theorem 1.1 and the following propositions proved in [3].

Proposition 9.1. *Let M and N be two quasiconformally related Riemann surfaces which are not of exceptional type. If M has the isometry property then every biholomorphic map between $Teich(M)$ and $Teich(N)$ is geometric.*

The above proposition shows, that if Problem 1.1 has a positive answer for a Riemann surface M, then Conjecture 1.1 holds for M. Since Theorem 1.1 gives a positive answer to Problem 1.1, the proof of Theorem 1.2 follows from Proposition 9.1.

$Teich(M)$ is a complex Banach manifold. The tangent space at the origin of $Teich(M)$ is isometric to the Banach dual of the space $A^1(M)$. The following was proved in [3].

Proposition 9.2. *Suppose that M and N are not of exceptional type. If $F : Teich(M) \to Teich(N)$ is a biholomorphic map, then there exist Riemann surfaces M' and N', which are quasiconformally related with M and N respectively, and there exists a linear isometry $T : A^1(M') \to A^1(N')$.*

We now prove Theorem 1.3. Let $F : Teich(M) \to Teich(N)$ be a biholomorphic map. If M and N are not of exceptional type we need to show that F is geometric.

First we show that the existence of the map F implies that M and N are quasiconformally related. We apply Proposition 9.2. Then from Theorem 1.1 we have that the linear isometry $T : A^1(M') \to A^1(N')$ is geometric and therefore by the definition M and N are conformally related. But this implies that M and N are quasiconformally related.

Now from Proposition 9.1 we conclude that F is geometric. This proves Theorem 1.3.

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