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UNIQUENESS OF MINIMAL DIFFEOMORPHISMS BETWEEN SURFACES

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Abstract. We prove that there exists at most one minimal diffeomorphism in a given homotopy class between any two closed Riemannian surfaces. This results was previously known only under the assumption that the Riemannian metrics have constant Gaussian curvature. Along the way, we prove the New Main Inequality which substantially strengthens the classical Reich-Strebel inequality for quasiconformal maps.

1. Introduction

1.1. **Uniqueness of minimal diffeomorphisms.** A map $f : (M_1, \sigma_1) \to (M_2, \sigma_2)$ between two Riemannian manifolds is called *minimal* if graph(f) is a minimal submanifold of the product Riemannian manifold $(M_1 \times M_2, \sigma_1 \times \sigma_2)$. When f is a diffeomorphism it is called a minimal diffeomorphism (in this case the inverse map f^{-1} is also a minimal diffeomorphism). Minimal maps are closely related to harmonic maps but are more subtle. In general, much less is known regarding the existence and (especially) uniqueness of minimal maps.

The most studied case is that of minimal diffeomorphisms between Riemannian surfaces. Let $F : (M_1, \sigma_1) \to (M_2, \sigma_2)$ be a homeomorphism between Riemannian surfaces M_1 and M_2. The basic questions are whether there exists a minimal diffeomorphism homotopic to F, and whether such a diffeomorphism is unique. Much like in the case of harmonic maps, assuming that the Riemannian metrics σ_1 and σ_2 have negative Gaussian curvature yields the existence of a minimal diffeomorphism in the prescribed homotopy class. The proof is an adaptation of the standard Schoen-Yau method from [13].

Furthermore, if both σ_1 and σ_2 have constant (negative) Gaussian curvatures then it is a theorem of Schoen (see Proposition 2.12 in [12]) that f is unique in its homotopy class (this argument relies on the work of Micallef-Wolfson [9], see also the paper by Wan [16]). This

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result has been extended and generalized in various directions. Brendle [1] showed the uniqueness of minimal diffeomorphisms between certain domains in the hyperbolic plane, while Labourie [4] proved that given a Hitchin representation in a split real Lie group of rank two, there exists a unique equivariant minimal surface in the corresponding symmetric space. See also the work of Lee [5] extending the Schoen's result to certain other maps beside diffeomorphisms, and the work by Lee-Wang [6], [7] discussing the uniqueness of minimal sub-manifolds in higher dimensions.

However, all proofs of the uniqueness of the minimal diffeomorphism f depend heavily on the assumption that the curvatures of σ_i's are constant. The purpose of this paper is to show that the uniqueness result holds for arbitrary Riemannian metrics σ_1 and σ_2. In particular, the assumption that the curvature is constant is redundant (nor do we need to assume that the curvatures are negative).

Theorem 1.1. *Let (M_1, σ_1) and (M_2, σ_2) denote two closed Riemannian surfaces of genus at least two. There exists at most one minimal diffeomorphism $f : (M_1, \sigma_1) \to (M_2, \sigma_2)$ in any given homotopy class.*

Remark. It is well known that a harmonic map between negatively curved closed manifolds is unique in its homotopy class. In general, this uniqueness result does not hold without the curvature assumption. It is therefore somewhat surprising that a harmonic diffeomorphism between closed Riemannian surfaces is unique in its homotopy class. Theorem 1.1 can been seen as an analogue of Theorem 4 from [8], but the present argument is significantly more involved which is not surprising given the subtle nature of minimal maps.

Adding the assumption that the curvatures are negative we obtain both the existence and the uniqueness of minimal diffeomorphisms. The following theorem follows immediately from the existence result discussed above and Theorem 1.1.

Theorem 1.2. *Let $F : (M_1, \sigma_1) \to (M_2, \sigma_2)$ be a homeomorphism between two negatively curved closed Riemannian surfaces. Then there exists a unique minimal diffeomorphism $f : (M_1, \sigma_1) \to (M_2, \sigma_2)$ homotopic to F.*

1.2. **The New Main Inequality.** Given a diffeomorphism $f : S \to S'$, between Riemann surfaces S and S', we let μ_f denote the Beltrami dilatation of f. Recall that μ_f is a $(-1,1)$ form on S which is expressed as $\mu_f = \frac{f_{\bar z}}{f_z}\frac{d\bar z}{dz}$, in local coordinates.

Let ϕ denote a holomorphic quadratic differential on S. Then ϕ is a $(2,0)$ form ($\phi = \phi\, dz^2$ in local coordinates). Suppose that $f : S \to S$ is a diffeomorphism homotopic to the identity. The classical Reich-Strebel Inequality [11] states:

$$\operatorname{Re}\int_S \frac{\mu_f}{1-|\mu_f|^2}\phi \leq \int_S \frac{|\mu_f|^2}{1-|\mu_f|^2}|\phi|. \tag{1}$$

In this paper we prove the substantially stronger inequality which we call the New Main Inequality.

Lemma 1.1. *Suppose $f_1, f_2 : S \to S'$ are mutually homotopic diffeomorphisms between Riemann surfaces S and S'. Then for every holomorphic quadratic differential ϕ on S, we have:*

$$\left|\int_S \phi\left(\frac{\mu_{f_1}}{1-|\mu_{f_1}|^2} - \frac{\mu_{f_2}}{1-|\mu_{f_2}|^2}\right)\right| \leq \int_S |\phi|\left(\frac{|\mu_{f_1}|^2}{1-|\mu_{f_1}|^2} + \frac{|\mu_{f_2}|^2}{1-|\mu_{f_2}|^2}\right). \tag{2}$$

The New Main Inequality is stronger than the Reich-Strebel inequality. In particular, we derive (1) from (2) by letting $f = f_1$, and f_2 be the identity mapping on S.

Remark. The New Main Inequality holds for arbitrary homotopic (rel boundary) quasiconformal maps $f_1, f_2 : S \to S'$, where S and S' are any two Riemann surfaces (possibly non-compact, possibly with boundary). This extension can be proved from Lemma 1.1 by standard approximation techniques. This extended New Main Inequality and its ramifications to the theory of quasiconformal maps are not being discussed further in this paper.

The Reich-Strebel inequality is a generalization of the classical Groetzch argument and its proof is based on the extremal length method. On the other hand, the proof of the New Main Inequality follows from the Schoen uniqueness theorem for minimal diffeomorphisms between hyperbolic Riemann surfaces and the solvability of certain PDE's involving the Hopf differentials of harmonic diffeomorphisms between hyperbolic Riemann surfaces (this theory was developed by Wan [15], Tam-Wan [14], and Wolf [17]). Thus, we produce a completely different proof of the Reich-Strebel inequality.

1.3. **Organization of the paper.** From now on, all surfaces are assumed to be closed, and all maps are assumed to be orientation preserving. Furthermore, given a Riemannian surface (M,σ), after passing to new local coordinates, we always assume that M is a Riemann surface and σ a conformal metric.

In Section 2 we compute the difference between the total energies of different mappings, and use this to compute useful upper and lower bounds. In Section 3 we prove prove Theorem 1.1 combining these estimates with the New Main Inequality.

In Section 4 we first recall two facts specific to harmonic diffeomorphism between Riemann surfaces endowed with the hyperbolic metrics. The first one is that the total energy of the pair of harmonic diffeomorphisms, which define the minimal diffeomorphism, minimizes the energy over the Teichmuüller space. The second one is the existence of a harmonic diffeomorphism with prescribed Hopf's differential. We then prove Lemma 1.1 using these facts and some estimates from Section 2.

2. The energy of a map

Let S and M denote two Riemann surfaces and suppose M is endowed with a conformal metric σ. Throughout this section we assume that $h : S \to M$ is a smooth map. Let

$$|\partial h|^2 = (\sigma \circ h)|h_z|\,|dz|^2, \qquad |\bar{\partial} h|^2 = (\sigma \circ h)|h_{\bar{z}}|^2\,|dz|^2.$$

Define the energy density

$$\mathbf{e}(h) = |\partial h|^2 + |\bar{\partial} h|^2,$$

and the (total) energy of h

$$\mathcal{E}(h) = \int\limits_S \mathbf{e}(h).$$

We also set

$$\mathbf{H}(h) = (\sigma \circ h) h_z \overline{(h_{\bar{z}})}\, dz^2.$$

Observe that $\mathbf{H}(h)$ is a (not necessarily holomorphic) quadratic differential on S.

We note the following elementary proposition.

Proposition 2.1. *If h is a diffeomorphism then the pointwise inequalities*

$$|\bar{\partial} h|^2 \leq |\mathbf{H}(h)| < |\partial h|^2, \tag{3}$$

hold everywhere on S.

Remark. We reiterate that the second inequality in (3) is strict.

The following proposition is used later in the proof of the New Main Inequality.

Proposition 2.2. *Suppose h is a diffeomorphism and $\eta : S \to \mathbb{R}$ a bounded measurable function on S. Then*

$$\int_S \mathbf{e}(h)\,\eta \le 2\left(\int_S |\mathbf{H}(h)|\,\eta\right) + ||\eta||_\infty \mathcal{A}(M,\sigma), \tag{4}$$

where $||\eta||_\infty$ denotes the essential supremum of η and $\mathcal{A}(M,\sigma)$ the σ-area of M.

Proof. Since $|\partial h|^2 - |\bar{\partial} h|^2$ is the Jacobian of h, we have

$$\int_S (|\partial h|^2 - |\bar{\partial} h|^2) = \mathcal{A}(M,\sigma).$$

This yields the estimate

$$\int_S (|\partial h|^2 - |\bar{\partial} h|^2)\,\eta \le ||\eta||_\infty \mathcal{A}(M,\sigma),$$

which can be written as

$$\int_S \mathbf{e}(h)\,\eta \le 2\left(\int_S |\bar{\partial} h|^2\,\eta\right) + ||\eta||_\infty \mathcal{A}(M,\sigma).$$

But, from (3) we have the pointwise estimate $|\bar{\partial} h|^2 \le |\mathbf{H}(h)(z)|$. Replacing this into the previous inequality proves the proposition.

□

2.1. **Variation of the total energy.** Let S' denote another Riemann surface and $f : S \to S'$ a diffeomorphism. In this subsection we estimate (above and below) the difference $\mathcal{E}(h \circ f^{-1}) - \mathcal{E}(h)$.

Proposition 2.3. *The difference between the total energies of h and $h \circ f^{-1}$ is computed as*

$$\mathcal{E}(h \circ f^{-1}) - \mathcal{E}(h) = -4\,\mathrm{Re}\left(\int_S \mathbf{H}(h)\frac{\mu_f}{1-|\mu_f|^2}\right) + 2\int_S \mathbf{e}(h)\frac{|\mu_f|^2}{1-|\mu_f|^2}. \tag{5}$$

Proof. This proposition is elementary and well known (for example see (1.1) in [11]). The interested reader can first verify the following pointwise identity (see [2] for useful formulas)

$$\mathbf{e}(h\circ f^{-1}) = \big(\mathbf{e}(h)\circ f^{-1}\big)J_{f^{-1}} + 2\big(\mathbf{e}(h)\circ f^{-1}\big)J_{f^{-1}}\frac{(|\mu_f|^2\circ f^{-1})}{1-(|\mu_f|^2\circ f^{-1})}$$
$$-4\,\mathrm{Re}\left(\big(\mathbf{H}(h)\circ f^{-1}\big)J_{f^{-1}}\frac{(\mu_f\circ f^{-1})}{1-(|\mu_f|^2\circ f^{-1})}\right),$$

where $J_{f^{-1}}$ denotes the Jacobian of f^{-1}. The proposition then follows by integration.

□

We are interested in the following corollary which provides the lower bound on $\mathcal{E}(h\circ f^{-1})-\mathcal{E}(h)$.

Corollary 2.1. *Suppose that $f: S\to S'$ is not a biholomorphism. Then the strict inequality*

$$\mathcal{E}(h\circ f^{-1})-\mathcal{E}(h) > -4\,\mathrm{Re}\left(\int\limits_S \mathbf{H}(h)\frac{\mu_f}{1-|\mu_f|^2}\right) + 4\int\limits_S |\mathbf{H}(h)|\frac{|\mu_f|^2}{1-|\mu_f|^2} \tag{6}$$

holds. On the other hand, if f is biholomorphic then $\mathcal{E}(h\circ f^{-1})=\mathcal{E}(h)$.

Proof. From (3) we deduce the strict pointwise inequality $\mathbf{e}(h) > 2|\mathbf{H}(h)|$ everywhere on S. Replacing this in the second integral on the right hand side of the equality (5), and using the fact that $|\mu_f|$ is positive on a set of positive measure proves the first part of the corollary. On the other hand, the equality $\mathcal{E}(h\circ f^{-1})=\mathcal{E}(h)$ is obvious when f is a biholomorphism.

□

Next, we produce the upper bound on $\mathcal{E}(h\circ f^{-1})-\mathcal{E}(h)$.

Corollary 2.2. *We have*

$$\mathcal{E}(h\circ f^{-1})-\mathcal{E}(h) \leq -4\,\mathrm{Re}\left(\int\limits_S \mathbf{H}(h)\frac{\mu_f}{1-|\mu_f|^2}\right) + 4\int\limits_S |\mathbf{H}(h)|\frac{|\mu_f|^2}{1-|\mu_f|^2}$$
$$+\frac{||\mu_f||^2_\infty}{1-||\mu_f||^2_\infty}{}_\infty\,\mathcal{A}(M,\sigma). \tag{7}$$

Proof. From (4) we find

$$\int_S \mathbf{e}(h)\frac{|\mu_f|^2}{1-|\mu_f|^2} \le 2\int_S |\mathbf{H}(h)|\frac{|\mu_f|^2}{1-|\mu_f|^2} + \left|\left|\frac{|\mu_f|^2}{1-|\mu_f|^2}\right|\right|_\infty \mathcal{A}(M,\sigma)$$
$$= 2\int_S |\mathbf{H}(h)|\frac{|\mu_f|^2}{1-|\mu_f|^2} + \frac{||\mu_f||^2_\infty}{1-||\mu_f||^2_\infty}\mathcal{A}(M,\sigma).$$

Applying this inequality to the second integral on the right hand side of the equality (5) proves the corollary.

□

3. Proof of Theorem 1.1

3.1. **Harmonic and minimal diffeomorphisms.** We begin by recalling the definitions of harmonic and minimal diffeomorphisms. A diffeomorphism $h : S \to (M,\sigma)$ is a harmonic map if and only if $\mathbf{H}(h)$ is a holomorphic quadratic differential on S. In this case we refer to $\mathbf{H}(h)$ as the Hopf differential.

On the other hand, let $h_i : S \to (M_i,\sigma_i)$, $i = 1,2$, be two diffeomorphisms. The induced diffeomorphism $g = h_2 \circ h_1^{-1}$ is minimal if and only if both h_i's are harmonic, and if $\mathbf{H}(h_1) = -\mathbf{H}(h_2)$. Moreover, every minimal diffeomorphism $g : (M_1,\sigma_1) \to (M_2,\sigma_2)$ arises in this way (see [12]).

Given a pair of diffeomorphisms $h_i : S \to (M_i,\sigma_i)$, we define the total energy of the pair (h_1,h_2) by

$$\mathcal{E}(h_1,h_2) = \mathcal{E}(h_1) + \mathcal{E}(h_2).$$

Proposition 3.1. *Let $h_i : S \to (M_i,\sigma_i)$ be two harmonic diffeomorphisms such that $\mathbf{H}(h_1) = -\mathbf{H}(h_2)$. Suppose we are given another two diffeomorphisms $\widehat{h}_i : S' \to M_i$, $i = 1,2$, such that at least one of the diffeomorphisms $\widehat{h}_i^{-1} \circ h_i$ is not biholomorphic. Then*

$$\mathcal{E}(h_1,h_2) < \mathcal{E}(\widehat{h}_1,\widehat{h}_2).$$

Proof. Define $f_i : S \to S'$ by $f_i = \widehat{h}_i^{-1} \circ h_i$. Then at least one diffeomorphism f_i, $i = 1,2$, is not biholomorphic. Thus, the inequality (6) from Corollary 2.1 yields the strict inequality

$$\sum_{i=1}^{2}\left(\mathcal{E}(\widehat{h}_i)-\mathcal{E}(h_i)\right) > \sum_{i=1}^{2}\left(-4\,\mathrm{Re}\left(\int_S \mathbf{H}(h_i)\frac{\mu_{f_i}}{1-|\mu_{f_i}|^2}\right) + 4\int_S |\mathbf{H}(h_i)|\frac{|\mu_{f_i}|^2}{1-|\mu_{f_i}|^2}\right).$$

Set $\phi = \mathbf{H}(h_1) = -\mathbf{H}(h_2)$. The previous inequality then becomes

$$\mathcal{E}(\widehat{h}_1,\widehat{h}_2) - \mathcal{E}(h_1,h_2) > -4\,\mathrm{Re}\left(\int\limits_S \phi \frac{\mu_{f_1}}{1-|\mu_{f_1}|^2}\right) + 4\int\limits_S |\phi| \frac{|\mu_{f_1}|^2}{1-|\mu_{f_1}|^2}$$

(8)

$$+ \left(-4\,\mathrm{Re}\left(\int\limits_S (-\phi) \frac{\mu_{f_2}}{1-|\mu_{f_2}|^2}\right) + 4\int\limits_S |\phi| \frac{|\mu_{f_2}|^2}{1-|\mu_{f_2}|^2}\right).$$

Since ϕ is a holomorphic quadratic differential (because h_i's are harmonic), we may apply the New Main Inequality (2) from Lemma 1.1, and conclude that the right hand side in (8) is non-negative. This implies the strict inequality

$$\mathcal{E}(\widehat{h}_1,\widehat{h}_2) - \mathcal{E}(h_1,h_2) > 0,$$

which proves the proposition.

□

3.2. **Proof of Theorem 1.1.** Suppose $g, \widehat{g} : (M_1,\sigma_1) \to (M_2,\sigma_2)$ are two homotopic minimal diffeomorphisms. We need to show $\widehat{g} = g$. Let $h_i : S \to M_i$ be the harmonic diffeomorphisms such that $g = h_2 \circ h_1^{-1}$, and $\widehat{h}_i : S' \to M_i$ the harmonic diffeomorphisms such that $\widehat{g} = \widehat{h}_2 \circ \widehat{h}_1^{-1}$. Set $A_i = \widehat{h}_i^{-1} \circ h_i$, $i = 1, 2$.

Claim 1. *The diffeomorphisms A_1 and A_2 are conformal.*

Proof. Suppose that at least one of A_i's is not a conformal diffeomorphism. We argue by contradiction. Since g is a minimal diffeomorphisms, we have $\mathbf{H}(h_1) = -\mathbf{H}(h_2)$. From Proposition 3.1 we find that

$$\mathcal{E}(h_1,h_2) < \mathcal{E}\big(\widehat{h}_1,\widehat{h}_2\big).$$

Likewise, $\widehat{g}$ is a minimal diffeomorphisms so $\mathbf{H}(\widehat{h}_1) = -\mathbf{H}(\widehat{h}_2)$. Since at least one of A_i^{-1}'s is not a conformal diffeomorphism, again using Proposition 3.1 we find that

$$\mathcal{E}(h_1,h_2) > \mathcal{E}\big(\widehat{h}_1,\widehat{h}_2\big).$$

The last two inequalities contradict each other which proves the claim.

□

From the definition of A_i's, we have

$$g = h_2 \circ h_1^{-1} = \widehat{h}_2 \circ (A_2 \circ A_1^{-1}) \circ \widehat{h}_1^{-1}.$$

Since $\widehat{g} = \widehat{h}_2 \circ \widehat{h}_1^{-1}$, and since g is homotopic to $\widehat{g}$, we conclude that $A_2 \circ A_1^{-1}$ is homotopic to the identity map on $S = S'$. But both A_1 and A_2 are conformal, so it follows that $A_2 \circ A_1^{-1}$ is the identity map. This shows that $g = \widehat{g}$ and the theorem is proved.

4. The proof of the new main inequality

4.1. **The energy functional on $\mathbf{T_g}$.** In this subsection we recall Schoen's theorem that a diffeomorphism which is minimal with respect to the hyperbolic metrics (constant curvature -1), is unique in its homotopy class. In fact we need a quantitative version which states that the total energy of the pair of harmonic diffeomorphisms (which define the minimal diffeomorphism) minimizes the energy.

Denote by $\Sigma_{\mathbf{g}}$ a smooth surface of genus $\mathbf{g}$. We let $\mathbf{T_g}$ denote the Teichmüller space of marked complex structures on $\Sigma_{\mathbf{g}}$, where S_τ is the marked Riemann surface corresponding to $\tau \in \mathbf{T_g}$.

Let M_i, $i = 1, 2$, denote a pair of Riemann surfaces and denote by σ_i the hyperbolic metric on M_i. Also, let $G_i : \Sigma_{\mathbf{g}} \to M_i$ be a homeomorphism. For $\tau \in \mathbf{T_g}$, we let $h_i^\tau : S_\tau \to (M_i, \sigma_i)$ denote the harmonic diffeomorphism homotopic to G_i.

The next theorem follows from Proposition 2.12 in [12].

Theorem 4.1. *There exists a unique $\tau \in \mathbf{T_g}$ such that $\mathbf{H}(h_1^\tau) = -\mathbf{H}(h_2^\tau)$. Moreover, for every $\tau' \in \mathbf{T_g}$ the inequality*

$$\mathcal{E}(h_1^{\tau'}, h_2^{\tau'}) \geq \mathcal{E}(h_1^\tau, h_2^\tau) \tag{9}$$

holds .

We record the following simple corollary.

Corollary 4.1. *Let $\tau' \in \mathbf{T_g}$, and let $f_i : S_\tau \to S_{\tau'}$ be any diffeomorphism, $i = 1, 2$, such that $h^\tau \circ f_i^{-1}$ is homotopic to $h_i^{\tau'}$. Then*

$$\mathcal{E}(h^\tau \circ f_1^{-1}, h^\tau \circ f_2^{-1}) \geq \mathcal{E}(h_1^\tau, h_2^\tau). \tag{10}$$

Proof. Since a harmonic diffeomorphism has the least total energy in its homotopy class we obtain the estimate $\mathcal{E}(h_i^{\tau'}) \leq \mathcal{E}(h^\tau \circ f_i^{-1})$. The inequality (10) now follows from (9).

□

4.2. **Prescribing the Hopf differential of a harmonic map.** We recall the following theorem proved independently (and using different means) by Hitchin [3], Wolf [17], and Wan [15] (see also [14]).

Theorem 4.2. *Let ψ be a holomorphic quadratic differential on S. There exists a Riemann surface M and a harmonic diffeomorphism $h : S \to (M, \sigma)$ with the property that $\mathbf{H}(h) = \psi$, where σ is the hyperbolic metric σ on M.*

Remark. The assumption that σ is the hyperbolic metric is essential in the previous theorem.

4.3. **The proof of Lemma 1.1.** Fix Riemann surfaces S, S', two mutually homotopic diffeomorphisms $f_1, f_2 : S \to S'$, and a holomorphic quadratic differential ϕ on S. It remains to prove the inequality (2).

Fix $t > 0$. Let (M_i, σ_i) be the hyperbolic Riemann surface, and $h_i : S \to (M_i, \sigma_i)$, $i = 1, 2$, the harmonic diffeomorphisms obtained from Theorem 4.2, such that $\mathbf{H}(h_1) = -\mathbf{H}(h_2) = t\phi$. From (10) we obtain the estimate

$$\mathcal{E}(h_1 \circ f_1^{-1}, h_1 \circ f_2^{-1}) \geq \mathcal{E}(h_1, h_2). \tag{11}$$

Combining this with (7) from Corollary 2.2, we get

$$\begin{aligned} 0 \leq \sum_{i=1}^{2} \left(\mathcal{E}(h_i \circ f_i^{-1}) - \mathcal{E}(h_i)\right) &\leq -4\operatorname{Re}\left(\int\limits_S t\phi \frac{\mu_{f_1}}{1-|\mu_{f_1}|^2}\right) + 4\int\limits_S t|\phi| \frac{|\mu_{f_1}|^2}{1-|\mu_{f_1}|^2} \\ &+ \left(-4\operatorname{Re}\left(\int\limits_S t(-\phi) \frac{\mu_{f_2}}{1-|\mu_{f_2}|^2}\right) + 4\int\limits_S t|\phi| \frac{|\mu_{f_2}|^2}{1-|\mu_{f_2}|^2}\right) \\ &+ 2\pi(2\mathbf{g}-2)\sum_{i=1}^{2}\left(\frac{||\mu_{f_i}||_\infty^2}{1-||\mu_{f_i}||_\infty^2}\right). \end{aligned} \tag{12}$$

Dividing all terms in (12) by $4t$ yields

$$
\begin{aligned}
\operatorname{Re}\left(\int\limits_S \phi\frac{\mu_{f_1}}{1-|\mu_{f_1}|^2}+\int\limits_S(-\phi)\frac{\mu_{f_2}}{1-|\mu_{f_2}|^2}\right) &\leq \int\limits_S |\phi|\frac{|\mu_{f_1}|^2}{1-|\mu_{f_1}|^2}+\int\limits_S |\phi|\frac{|\mu_{f_2}|^2}{1-|\mu_{f_2}|^2} \\
&+\frac{2\pi(2\mathbf{g}-2)}{4t}\sum_{i=1}^{2}\left(\frac{||\mu_{f_i}||_\infty^2}{1-||\mu_{f_i}||_\infty^2}\right).
\end{aligned} \tag{13}
$$

The last term on the right hand side of (13) tends to zero when $t\to\infty$. Thus, we get

$$
\operatorname{Re}\left(\int\limits_S \phi\frac{\mu_{f_1}}{1-|\mu_{f_1}|^2}+\int\limits_S(-\phi)\frac{\mu_{f_2}}{1-|\mu_{f_2}|^2}\right) \leq \int\limits_S |\phi|\frac{|\mu_{f_1}|^2}{1-|\mu_{f_1}|^2}+\int\limits_S |\phi|\frac{|\mu_{f_2}|^2}{1-|\mu_{f_2}|^2},
$$

which implies The Main New Inequality (2).

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