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NON-UNIQUENESS OF MINIMAL SURFACES IN A PRODUCT OF CLOSED RIEMANN SURFACES

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Abstract. We show that for every large enough $\mathbf{g}$ there exists a Fuchsian representation $\rho : \pi_1(\Sigma_{\mathbf{g}}) \to \prod_{i=1}^{3} \mathbf{PSL}(2,\mathbb{R})$ which yields multiple minimal surfaces in the corresponding product of closed Riemann surfaces.

1. Introduction

1.1. **Minimal surfaces in products.** Denote by $\Sigma_{\mathbf{g}}$ a surface of genus $\mathbf{g} \geq \mathbf{2}$, and let $\mathbf{T}_{\mathbf{g}}$ denote the Teichmüller space of marked complex structures on $\Sigma_{\mathbf{g}}$. Each Fuchsian representation $\rho : \pi_1(\Sigma_{\mathbf{g}}) \to \prod_{i=1}^{n} \mathbf{PSL}(2,\mathbb{R})$ yields the energy functional $\mathrm{E}_\rho : \mathbf{T}_{\mathbf{g}} \to (0,\infty)$, given as the sum of energies of the corresponding harmonic diffeomorphisms (see below). It is well known by the work of Schoen-Yau [13] that E_ρ is proper on $\mathbf{T}_{\mathbf{g}}$. Consequently, E_ρ achieves its global minimum and therefore it has at least one stationary point. Schoen [12] proved that this is the only stationary point of E_ρ providing $n = 2$ (the same trivially holds when $n = 1$). The purpose of this paper is to address the case $n > 2$.

Theorem 1.1. *For every large enough* $\mathbf{g} \geq 2$*, there exists a Fuchsian representation* $\rho : \pi_1(\Sigma_{\mathbf{g}}) \to \prod_{i=1}^{3} \mathbf{PSL}(2,\mathbb{R})$ *such that* $\mathrm{E}_\rho : \mathbf{T}_{\mathbf{g}} \to (0,\infty)$ *has at least two stationary points.*

Remark. Labourie conjectured that given a Hitchin representation of $\pi_1(\Sigma_{\mathbf{g}})$ in a split real Lie group, there exists a unique equivariant minimal surface in the corresponding symmetric space (in [6] he proved this conjecture when $n = 2$). Theorem 1.1 disproves the analogous conjecture for Fuchsian (Hitchin) representations of $\pi_1(\Sigma_{\mathbf{g}})$ to the semi-simple Lie groups $\prod_{i=1}^{3} \mathbf{PSL}(2,\mathbb{R})$. This shows that the theorem of Collier-Tholozan-Toulisse [3] about the maximal representations into Hermitian Lie groups can not be extended to Lie groups of rank greater than two. Another corollary of Theorem 1.1 is that there is no convex Riemannian metric on the Teichmüller space for which the energy functional is geodesically convex (this energy functional is defined at the beginning of Section 2).

1.2. **The New Main Inequality does not generalize.** Given a diffeomorphism $f : S \to S'$ between Riemann surfaces S and S', we let $\mathrm{Belt}(f)$ denote the Beltrami dilatation of f. Then $\mathrm{Belt}(f)$ is a $(-1,1)$ form on S which is expressed as $\mathrm{Belt}(f) = \frac{f_{\bar{z}}}{f_z}\frac{d\bar{z}}{dz}$, in local coordinates. A holomorphic quadratic differential ϕ on S is a $(2,0)$ form ($\phi = \phi\, dz^2$ in local coordinates).

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Lemma 1.1. *For every* $\mathbf{g}$ *sufficiently large, there exist*

- *closed Riemann surfaces* S *and* S' *of genus* $\mathbf{g}$,
- *mutually homotopic diffeomorphisms* $f_i : S \to S'$, $i = 1, 2, 3$,
- *holomorphic quadratic differentials* ϕ_i *on* S, *satisfying the condition*

$$\phi_1 + \phi_2 + \phi_3 = 0,$$

such that the following strict inequality

$$\operatorname{Re} \sum_{i=1}^{3} \int\limits_S \phi_i \frac{\mu_i}{1 - |\mu_i|^2} > \sum_{i=1}^{3} \int\limits_S |\phi_i| \frac{|\mu_i|^2}{1 - |\mu_i|^2} \tag{1}$$

holds, where $\mu_i = \operatorname{Belt}(f_i)$.

Remark. The lemma does not hold when $n = 2$. It was shown in [8] that in this case the right hand side of (1) is always greater or equal than the left hand side (we called this The New Main Inequality since it generalizes the classical Reich-Strebel Inequality). But this is as far as it goes, as Lemma 1.1 shows.

1.3. **Outline.** There are various proofs of the uniqueness of minimal diffeomorphisms between Riemann surfaces (see the works of Brendle [2], Micallef-Wolfson [10], and Lee [7]). However, the corresponding estimates for establishing the uniqueness of minimal surfaces in the product of three (or more) closed Riemann surfaces do not close (making this a supercritical case). In this paper we construct an explicit example of a minimal surface in the product of three closed hyperbolic Riemann surfaces which is not a global minimum of the energy functional. The strict inequality (1) conveys the failure of the energy functional to achieve its minimum at this minimal surface.

The content of Section 2 is to prove Theorem 1.1 assuming Lemma 1.1 (the argument closely follows [8]). We construct the minimal surface $h : S \to M_1 \times M_2 \times M_3$, where $h_i : S \to M_i$ is the harmonic diffeomorphism whose Hopf differential is equal to $t\phi_i$. Then S is a stationary point of the corresponding energy functional $E^t : \mathbf{T_g} \to (0, \infty)$. However, the strict inequality (1) implies $E^t(S') < E^t(S)$ providing that t is large enough. Therefore, energy functional does not achieve its minimum at S. The remainder of the paper is devoted to proving Lemma 1.1 (harmonic maps do not feature any longer).

In sections 3-6 we prove the version of Lemma 1.1 in which the Riemann surfaces S is replaced with the unit disc. In Section 3 we define the notion of an admissible path of Beltrami differentials $\mu(t) = t\dot{\mu} + t^2\ddot{\mu} + o(t^2)$, and formulate Lemma 3.1 which is an infinitesimal version of Lemma 1.1. This lemma claims the existence of suitable, mutually equivalent, admissible paths of Beltrami differentials on $\mathbb{D}$. Using the explicit formula for solving the Beltrami equation on $\mathbb{C}$, in Section 4 we explain how to build equivalent paths $\mu_i(t)$ of admissible Beltrami differentials only in terms of $\dot{\mu}_i$.

The proof of Lemma 3.1 (and therefore of Lemma 1.1) is explicit. We construct concrete holomorphic functions ϕ_i, and Beltrami dilatations μ_i, on $\mathbb{D}$ so that (1) holds. The functions ϕ_i are squares of quadratic polynomials, and μ_i's are then constructed so that (1) holds. In Section 5 we construct the corresponding Beltrami dilatations, and estimate the integrals from (1) for an arbitrary quadratic polynomial. In Section 6 we write down the three polynomials and complete the proof of Lemma 3.1. In Section 7 we extend the result from $\mathbb{D}$ to any closed Riemann surface of sufficiently large injectivity radius (using standard approximation).

Throughout the paper we only use basic and classical computations and tools, the extent of which is the formula for solving the Beltrami equation, and the occasional use of the Stokes theorem (everything follows easily from first principles). We use z to denote the complex variable on $\mathbb{C}$, or a local complex parameter on a Riemann surface. Also, we let $z = x + \mathbf{i}y$, where $\mathbf{i}$ is the imaginary unit.

1.4. **A few comments regarding the strategy for proving Theorem 1.1.** Let us say a few words about the strategy of the proof, and in particular about the connections between the New Main Inequality and the behavior of the energy functional at it's stationary points. Let $h : S \to M_1 \times M_2 \times M_3$ be the minimal surface where $h_i : S \to M_i$ is the harmonic diffeomorphism whose Hopf differential is equal to ϕ_i (in partucular, $\phi_1 + \phi_2 + \phi_3 = 0$). Then S is a stationary point of the corresponding energy functional $E : \mathbf{T_g} \to (0, \infty)$. After computing the second variation of the energy functional E, one verifies the following.

Observation. If S is a local minimum of E, then given any mutually equivalent admissible paths $\mu_i : [0, t_0] \to \mathrm{BD}(S)$ of Beltrami differentials, the inequality

$$\mathrm{Re} \sum_{i=1}^{3} \int\limits_{\mathbb{D}} \phi_i \ddot{\mu}_i \, dxdy \le \sum_{i=1}^{3} \int\limits_{\mathbb{D}} |\phi_i| |\dot{\mu}_i|^2 \, dxdy \qquad (\star)$$

holds (here $\mu_i(t) = t\dot{\mu}_i + t^2 \ddot{\mu}_i + o(t^2)$).

The inequality $(\star)$ does not feature the metric on the target surfaces which makes it robust and easier to work with (in fact, it holds for the energy functional defined with respect to any conformal Riemannian metric and not just the hyperbolic metric which is the one we work with here). Furthermore, if $(\star)$ does not hold for some choice of μ_i's and ϕ_i's, then S is a stationary point which is not a local minimum of E. This last statement motivates our strategy. Namely, in Lemma 3.1 we find examples when $(\star)$ fails to hold (the formula (7) is the negation of $(\star)$ in the case when S is replaced with $\mathbb{D}$).

Lemma 3.1 is proved in the case when S is replaced by the unit disc. So, the most direct way of showing that S may not be a local minimum of E would be to prove the version of Lemma 3.1 (and the inequality (7) in particular) in the case when $\mathbb{D}$ is replaced by S. But this is somewhat awkward to do (although not impossible). Instead, we first observe that (7) yields the formula (9), which is then routinely shown to imply (1) which holds on S. We then use (1) to prove that S is not a global minimum of E.

Remark. Technically we do not show that S fails to be a local minimum of E (we only show that the stationary point S is not a global minimum). This is the cost of not proving (7) in the case when $\mathbb{D}$ is replaced by S.

2. Harmonic diffeomorphisms between closed Riemann surfaces

Let $h : S \to M$ denote a diffeomorphism between Riemann surfaces S and M of genus $\mathbf{g}$, and let σ denote the density of the hyperbolic metric on M. Set

$$|\partial h|^2 = (\sigma^2 \circ h)|h_z|\, dxdy, \qquad |\bar{\partial} h|^2 = (\sigma^2 \circ h)|h_{\bar{z}}|^2\, dxdy.$$

Define the energy density form

$$\mathbf{e}(h) = |\partial h|^2 + |\bar{\partial} h|^2,$$

the (total) energy of h

$$\mathcal{E}(h) = \int\limits_S \mathbf{e}(h),$$

and set

$$\operatorname{Hopf}(h) = (\sigma^2 \circ h) h_z \overline{(h_{\bar{z}})}\, dz^2.$$

Then, h is harmonic if and only if $\operatorname{Hopf}(h)$ is a holomorphic quadratic differential on S (which we call the Hopf differential of h).

2.1. **Minimal surfaces in products.** For each $i = 1, 2, 3$, fix the following:

- Riemann surfaces M_i of genus $\mathbf{g}$ endowed with complete hyperbolic metrics with conformal densities σ_i,
- orientation preserving homeomorphisms $G_i : \Sigma_{\mathbf{g}} \to M_i$.

Set $M = M_1 \times M_2 \times M_3$, and $G : \Sigma_{\mathbf{g}} \to M$, where $G = (G_1, G_2, G_3)$. For each marked Riemann surface $S \in \mathbf{T}_{\mathbf{g}}$, let $h_i : S \to (M_i, \sigma_i)$ be the harmonic diffeomorphism homotopic to G_i. Denote by $h : S \to M$ the corresponding product map $h = (h_1, h_2, h_3)$, and let

$$\mathcal{E}(h) = \sum_{i=1}^{3} \mathcal{E}(h_i).$$

Definition 2.1. *Given a pair (M, G) we define the function $\mathrm{E}_{M,G} : \mathbf{T}_{\mathbf{g}} \to (0, \infty)$ by letting $\mathrm{E}_{M,G}(S) = \mathcal{E}(h)$.*

We say that $h : S \to M$ is a minimal surface if S is a stationary point of the function $\mathrm{E}_{M,G} : \Sigma_{\mathbf{g}} \to (0, \infty)$. In this section we use Lemma 1.1 to prove the following theorem.

Theorem 2.1. *For every large enough $\mathbf{g} \geq 2$, there exist Riemann surfaces M_i, and homeomorphisms $G_i : \Sigma_{\mathbf{g}} \to M_i$, $i = 1, 2, 3$, such that the corresponding energy functional $\mathrm{E}_{(M,G)} : \mathbf{T}_{\mathbf{g}} \to (0, \infty)$ has at least two critical points.*

2.2. **Proof of Theorem 1.1.** Theorem 2.1 is just a restatement of Theorem 1.1. Indeed, each homeomorphism $G_i : \Sigma_{\mathbf{g}} \to M_i$ yields the Fuchsian representation $\rho_i : \pi_1(\Sigma_{\mathbf{g}}) \to \mathbf{PSL}(2, \mathbb{R})$. Define the Fuchsian representation $\rho : \pi_1(\Sigma_{\mathbf{g}}) \to \prod_{i=1}^{3} \mathbf{PSL}(2, \mathbb{R})$, as the product of Fuchsian representations ρ_i. Furthermore, we have the identity $\mathrm{E}_{\rho} = \mathrm{E}_{M,G}$. Theorem 1.1 now follows from Theorem 2.1.

2.3. **Proof of Theorem 2.1.** Fix Riemann surfaces S, S', diffeomorphisms $f_i : S \to S'$, and holomorphic quadratic differentials ϕ_i satisfying the assumptions and conclusions of Lemma 1.1. For every $t > 0$, we let (M_i^t, σ_i^t) be the hyperbolic Riemann surface, and $h_i^t : S \to (M_i^t, \sigma_i^t)$ the harmonic diffeomorphism, such that $\operatorname{Hopf}(h_i^t) = t\phi_i$. The existence of such h_i^t was established independently (and by different means) by Hitchin [5], Wolf [16], and Wan [14] (see also Theorem 4.2 in [8]). Identify $\Sigma_{\mathbf{g}}$ with S, and set $G_i^t = h_i^t$. Below we consider the induced function $\mathrm{E}^t = \mathrm{E}_{(M^t, G^t)} : \mathbf{T_g} \to \mathbb{R}$.

Note that S is the critical point of each function E^t because

$$\operatorname{Hopf}(h^t) = \operatorname{Hopf}(h_1^t) + \operatorname{Hopf}(h_2^t) + \operatorname{Hopf}(h_3^t) = 0$$

where $h^t : S \to M^t$ is the corresponding product map. In other words, h^t is a minimal surface. The idea of this proof is to show that for t large enough E^t does not achieve its minimum at S. Thus, the global minimum of E^t provides another critical point of E^t which proves the theorem.

We begin by observing that the inequality

$$\mathrm{E}^t(S') \leq \sum_{i=1}^{3} \mathcal{E}(h_i^t \circ f_i^{-1})$$

holds because the total energy of the diffeomorphism $h_i^t \circ f_i^{-1} : S' \to M_i^t$ is greater or equal than the energy of the harmonic diffeomorphism in its homotopy class. Therefore, we derive

$$\mathrm{E}^t(S') - \mathrm{E}^t(S) \leq \sum_{i=1}^{3} \left(\mathcal{E}(h_i^t \circ f_i^{-1}) - \mathcal{E}(h_i^t) \right). \tag{2}$$

The following equality was proved in Proposition 2.3 in [8] (which is a restating of (1.1) in [11])

$$\mathcal{E}(h_i^t \circ f_i^{-1}) - \mathcal{E}(h_i^t) = -4 \operatorname{Re} \int_S t\phi_i \frac{\mu_i}{1 - |\mu_i|^2} + 2 \int_S \mathbf{e}(h_i^t) \frac{|\mu_i|^2}{1 - |\mu_i|^2},$$

where $\mu_i = \operatorname{Belt}(f_i)$ (recall that $\mathbf{e}(h_i^t)$ denotes the energy density form of h_i^t). Combining this with (2), we conclude that

$$\mathrm{E}^t(S') - \mathrm{E}^t(S) \leq -4 \operatorname{Re} \sum_{i=1}^{3} \int_S t\phi_i \frac{\mu_i}{1 - |\mu_i|^2} + 2 \sum_{i=1}^{3} \int_S \mathbf{e}(h_i^t) \frac{|\mu_i|^2}{1 - |\mu_i|^2}. \tag{3}$$

Dividing all terms in (3) by $4t$ yields

$$\frac{\mathrm{E}^t(S') - \mathrm{E}^t(S)}{4t} \leq -\operatorname{Re} \sum_{i=1}^{3} \int_S \phi_i \frac{\mu_i}{1 - |\mu_i|^2} + \sum_{i=1}^{3} \int_S \left(\frac{\mathbf{e}(h_i^t)}{2t} \right) \frac{|\mu_i|^2}{1 - |\mu_i|^2}. \tag{4}$$

Claim 1. *Let ρ denotes the density of the hyperbolic metrics on S. There exists a constant $C_i > 0$ such that*

$$\frac{\rho^{-2}\mathbf{e}(h_i^t)}{2t} \leq C_i, \quad \textit{for every} \quad t \geq 1.$$

Furthermore,

$$\frac{\rho^{-2}\mathbf{e}(h_i^t)}{2t} \ \to \ \rho^{-2}|\phi_i|, \qquad t \to \infty,$$

almost everywhere on S.

Proof. Let

$$a_i = \max_{p\in S} \rho^{-2}(p)|\phi_i(p)|.$$

It follows from Proposition 10 of Wan [14] that

$$\max_{p\in S} \rho^{-2}\mathbf{e}(h_i^t)(p) \leq 1 + \sqrt{1+4(ta_i)^2}.$$

Dividing both sides by $2t$ yields

$$\frac{\rho^{-2}\mathbf{e}(h_i^t)}{2t} \leq \frac{1}{2} + \sqrt{\frac{1}{4}+a_i^2} = C_i,$$

assuming $t \geq 1$. The second part of the claim follows from Lemma 2.2 in Wolf [15] (in fact, the convergence is uniform in t, and locally uniform on the surface away from the zeroes of ϕ_i). □

Using Claim 1, and applying the Dominated Convergence Theorem, we obtain

$$\sum_{i=1}^{3} \int_S \left(\frac{\mathbf{e}(h_i^t)}{2t}\right) \frac{|\mu_i|^2}{1-|\mu_i|^2} \ \to \ \sum_{i=1}^{3} \int_S |\phi_i| \frac{|\mu_i|^2}{1-|\mu_i|^2}, \qquad t \to \infty.$$

Thus, for every $\delta > 0$ there exists t_δ sufficiently large so that

$$\sum_{i=1}^{3} \int_S \left(\frac{\mathbf{e}(h_i^{t_\delta})}{2t_\delta}\right) \frac{|\mu_i|^2}{1-|\mu_i|^2} \leq \sum_{i=1}^{3} \int_S |\phi_i| \frac{|\mu_i|^2}{1-|\mu_i|^2} \ + \delta.$$

Replacing this into (4) yields the inequality

$$\frac{\mathrm{E}^t(S')-\mathrm{E}^t(S)}{4t_\delta} \leq -\operatorname{Re}\sum_{i=1}^{3} \int_S \phi_i \frac{\mu_i}{1-|\mu_i|^2} + \sum_{i=1}^{3} \int_S |\phi_i| \frac{|\mu_i|^2}{1-|\mu_i|^2} \ + \delta. \tag{5}$$

It remains to observe that for δ small enough the right hand side of inequality (5) is strictly negative. In particular, letting $t = t_\delta$ for any such δ, and using Lemma 1.1, yields

$$\mathrm{E}^t(S') - \mathrm{E}^t(S) < 0,$$

meaning that $\mathrm{E}^t = E_{(M^t,G^t)} : \mathbf{T_g} \to (0,\infty)$ does not achieve its global minimum at S. The proof is complete.

3. Admissible Paths Of Beltrami Differentials

In this section we prove Lemma 3.2 which is the version of Lemma 1.1 on the unit disc $\mathbb{D}$. The proof of Lemma 3.2 is based on Lemma 3.1 which can be seen as infinitesimal version of Lemma 1.1.

3.1. **The Normal Solution to the Beltrami equation.** The space of Beltrami differentials $L^\infty_1(\mathbb{D})$ is defined as the unit ball in the Banach space $L^\infty(\mathbb{D})$. Every Beltrami differential $\mu \in L^\infty_1(\mathbb{D})$ yields the unique quasiconformal homeomorphism $f^\mu : \mathbb{C} \to \mathbb{C}$ with the following properties:

(1) $\operatorname{Belt}(f^\mu) = \mu$ on $\mathbb{D}^*$ and $\operatorname{Belt}(f^\mu) = 0$ outside $\mathbb{D}^*$,

(2) $f^\mu(z) - z = O(1)$, when $|z|$ is large.

We refer to f^μ as the Normal Solution to the Beltrami equation. Note that the restriction of f^μ to $\mathbb{D}^* = \mathbb{C} \setminus \mathbb{D}$ is conformal. Recall the classical equivalence relation on $L^\infty_1(\mathbb{D})$.

Definition 3.1. *We say that $\mu, \nu \in L^\infty_1(\mathbb{D})$ are equivalent (and write $\mu \sim \nu$) if $f^\mu = f^\nu$ on $\mathbb{D}^*$.*

3.2. **Admissible paths.** A path in $L^\infty_1(\mathbb{D})$ is a continuous map $\mu : [0, t_0] \to L^\infty_1(\mathbb{D})$ where $t_0 > 0$ (here we used μ to denote both paths and elements in $L^\infty_1(\mathbb{D})$, but this should not cause any confusion). A collection of paths $\mu_i : [0, t_i] \to L^\infty_1(\mathbb{D})$, $t_i > 0$, are mutually equivalent if $\mu_i(t) \sim \mu_j(t)$, for every i, j, and every $t \le \min_i t_i$.

Definition 3.2. *A path $\mu : [0, t_0] \to L^\infty_1(\mathbb{D})$ is admissible if there exists $\dot{\mu}, \ddot{\mu} \in L^\infty(\mathbb{D})$ such that*

$$||\mu(t) - t\dot{\mu} - t^2\ddot{\mu}||_\infty = o(t^2). \tag{6}$$

We let $\mathcal{H}^1(\mathbb{D})$ denote the set of integrable holomorphic functions on $\mathbb{D}$. The purpose of the following three sections is to prove the following lemma.

Lemma 3.1. *There exist mutually equivalent admissible paths*

$$\mu_i : [0, t_0] \to L^\infty_1(\mathbb{D}) \cap C^\infty_0(\mathbb{D}),$$

and functions $\phi_i \in \mathcal{H}^1(\mathbb{D})$, $i = 1, 2, 3$, with the property

$$\phi_1 + \phi_2 + \phi_3 = 0,$$

such that the strict inequality

$$\operatorname{Re} \sum_{i=1}^{3} \int_{\mathbb{D}} \phi_i \ddot{\mu}_i \, dxdy > \sum_{i=1}^{3} \int_{\mathbb{D}} |\phi_i||\dot{\mu}_i|^2 \, dxdy \tag{7}$$

holds (here $C^\infty_0(\mathbb{D})$ denotes smooth functions with compact support in $\mathbb{D}$).

The following lemma is an immediate corollary. Before we state and prove it, we recall the definition:

Definition 3.3. *We say that $\dot{\mu}, \dot{\nu} \in L^\infty(\mathbb{D})$ are infinitesimally equivalent, and write $\dot{\mu} \approx \dot{\nu}$, if*

$$\int_{\mathbb{D}} \phi\dot{\mu} \, dxdy = \int_{\mathbb{D}} \phi\dot{\nu} \, dxdy, \quad \textit{for every} \quad \phi \in \mathcal{H}^1(\mathbb{D}).$$

Lemma 3.2. *There exist mutually equivalent Beltrami dilatation* $\mu_i \in L_1^\infty(\mathbb{D}) \cap C_0^\infty(\mathbb{D})$, *and holomorphic functions* $\phi_i \in \mathcal{H}^1(\mathbb{D})$, $i = 1, 2, 3$, *with the property*

$$\phi_1 + \phi_2 + \phi_3 = 0 \tag{8}$$

such that the strict inequality

$$\operatorname{Re} \sum_{i=1}^{3} \int_{\mathbb{D}} \phi_i \frac{\mu_i}{1 - |\mu_i|^2} \, dxdy > \sum_{i=1}^{3} \int_{\mathbb{D}} |\phi_i| \frac{|\mu_i|^2}{1 - |\mu_i|^2} \, dxdy \tag{9}$$

holds.

Proof. Let $\mu_i(t)$ be the admissible paths from Lemma 3.1. From the admissibility condition, we get

$$\left|\left| \frac{\mu_i(t)}{1 - |\mu_i(t)|^2} - (t\dot{\mu}_i + t^2 \ddot{\mu}_i) \right|\right|_\infty = o(t^2).$$

Using this, we obtain

$$\operatorname{Re} \sum_{i=1}^{3} \int_{\mathbb{D}} \phi_i \frac{\mu_i(t)}{1 - |\mu_i(t)|^2} \, dxdy = t \operatorname{Re} \sum_{i=1}^{3} \int_{\mathbb{D}} \phi_i \dot{\mu}_i \, dxdy + t^2 \operatorname{Re} \sum_{i=1}^{3} \int_{\mathbb{D}} \phi_i \ddot{\mu}_i \, dxdy \; + o(t^2), \tag{10}$$

and

$$\sum_{i=1}^{3} \int_{\mathbb{D}} |\phi_i| \frac{|\mu_i(t)|^2}{1 - |\mu_i(t)|^2} \, dxdy = t^2 \sum_{i=1}^{3} \int_{\mathbb{D}} |\phi_i| |\dot{\mu}_i|^2 \, dxdy \; + o(t^2). \tag{11}$$

Since $\mu_i(t) \sim \mu_j(t)$ for every t, it is well known that $\dot{\mu}_i \approx \dot{\mu}_j$ for every $i, j \in \{1, 2, 3\}$. Therefore, for every $\phi \in \mathcal{H}^1(\mathbb{D})$ the equality

$$\int_{\mathbb{D}} \phi \dot{\mu}_i \, dxdy = \int_{\mathbb{D}} \phi \dot{\mu}_j \, dxdy \qquad \forall i, j \in \{1, 2, 3\}$$

holds . Combining this with (8) yields

$$\operatorname{Re} \sum_{i=1}^{3} \int_{\mathbb{D}} \phi_i \dot{\mu}_i \, dxdy = 0.$$

Replacing this in (10) gives

$$\operatorname{Re} \sum_{i=1}^{3} \int_{\mathbb{D}} \phi_i \frac{\mu_i(t)}{1 - |\mu_i(t)|^2} \, dxdy = t^2 \operatorname{Re} \sum_{i=1}^{3} \int_{\mathbb{D}} \phi_i \ddot{\mu}_i(t) \, dxdy \; + o(t^2). \tag{12}$$

We now appeal to Lemma 3.1. Combining the strict inequality (7) with (12) produces the strict inequality

$$\operatorname{Re} \sum_{i=1}^{3} \int_{\mathbb{D}} \phi_i \frac{\mu_i(t)}{1 - |\mu_i(t)|^2} \, dxdy > t^2 \sum_{i=1}^{3} \int_{\mathbb{D}} |\phi_i| |\dot{\mu}_i|^2 \, dxdy \; + o(t^2).$$

Putting this together with (11) gives

$$\operatorname{Re}\sum_{i=1}^{3}\int_{\mathbb{D}}\phi_i\frac{\mu_i(t)}{1-|\mu_i(t)|^2}\,dxdy > \sum_{i=1}^{3}\int_{\mathbb{D}}|\phi_i|\frac{|\mu_i(t)|^2}{1-|\mu_i(t)|^2}\,dxdy\; + o(t^2).$$

Letting $\mu_i = \mu_i(t)$ for a small enough t proves the existence of mutually equivalent $\mu_1, \mu_2, \mu_3 \in L^\infty_1(\mathbb{D}) \cap C^\infty_0(\mathbb{D})$ satisfying the inequality (9) and we are done.

□

4. Equivalent paths of Beltrami differentials

To prove Lemma 3.1 we need to construct certain mutually equivalent admissible paths of Beltrami differentials. This construction is underpinned by classical methods in the theory of quasiconformal maps. In this section we state and prove what we need to carry out this task.

Given two infinitesimally equivalent $\dot{\mu}, \dot{\nu} \in L^\infty(\mathbb{D})$, we construct an admissible path of Beltrami differentials $\mu(t) = t\dot{\mu} + t^2\ddot{\mu} + o(t^2)$ which is equivalent with the path $t\dot{\nu}$, and which satisfies the property (2) from Proposition 4.2. This property expresses integration of the second derivative $\ddot{\mu}$ in terms of the integration of the first derivative $\dot{\mu}$ and $T(\dot{\mu})$ (here T is the Beurling transform).

4.1. Cauchy's and Beurling's transforms. We begin by recalling Cauchy's and Beurling's transforms (see [1], [4]). Suppose $h \in L^p(\mathbb{C})$ for some $p > 2$. The Cauchy transform P is defined by

$$P(h)(z) = -\frac{1}{\pi}\int_{\mathbb{C}}\frac{h(\zeta)}{\zeta - z}\,dxdy. \tag{13}$$

The Beurling transform T of $h \in C^2_0$ is the Cauchy principal value

$$T(h)(z) = \lim_{\epsilon\to 0}\left(-\frac{1}{\pi}\int_{|\zeta-z|>\epsilon}\frac{h(\zeta)}{(\zeta-z)^2}\,dxdy\right). \tag{14}$$

(Here C^2_0 denote the space of twice continuously differentiable functions with compact support in $\mathbb{C}$.) We recall (see [1], [4])

Proposition 4.1. *Assume $h \in L^\infty(\mathbb{C})$ has compact support. Then $P(h)$ and $T(h)$ are well defined, and $P(h)(z) \to 0$ when $|z| \to \infty$. Furthermore, the equalities $(P(h))_{\overline{z}} = h$ and $(P(h))_z = T(h)$ hold in the sense of distribution. If in addition h is smooth then $P(h)$ and $T(h)$ are smooth as well.*

4.2. The formula for the Normal Solution to the Beltrami equation. Suppose that $\mu \in L^\infty_1(\mathbb{D})$. Recall the Normal Solution $f^\mu : \mathbb{C} \to \mathbb{C}$ which solves the Beltrami equation $f^\mu_{\overline{z}} = \mu f^\mu_z$, and satisfies the normalization $f^\mu(z) = z + O(1)$ near ∞. We can express f^μ in terms of the singular operators (see [1], [4])

$$f^\mu = z + P(\mu) + P(\mu T(\mu)) + P(\mu T(\mu T(\mu))) + \cdots . \tag{15}$$

4.3. **Building mutually equivalent and admissible paths of Beltrami differentials.** Recall Definition 3.3. We note that $\dot\mu,\dot\nu \in L^\infty(\mathbb{D})$ are infinitesimally equivalent if and only if

$$P(\dot\mu)(z) = P(\dot\nu)(z), \qquad z \in \mathbb{D}^*.$$

Proposition 4.2. *Suppose $\dot\mu,\dot\nu \in L^\infty(\mathbb{D}) \cap C_0^\infty(\mathbb{D})$ are infinitesimally equivalent (that is, $\dot\mu \approx \dot\nu$). There exists an admissible path $\mu : [0,t_0] \to L_1^\infty(\mathbb{D}) \cap C_0^\infty(\mathbb{D})$ such that*

(1) *$\mu(t) \sim t\dot\nu$, for every $0 \le t \le t_0$,*
(2) *$\int\limits_{\mathbb{D}} \phi\ddot\mu \, dxdy = \int\limits_{\mathbb{D}} \phi\left(\dot\nu T(\dot\nu) - \dot\mu T(\dot\mu)\right) dxdy.$*

Proof. Assuming $t < (1/||\dot\nu||_\infty)$, we have

$$f^{t\dot\nu}(z) = z + tP(\dot\nu)(z) + t^2 P(\dot\nu T(\dot\nu))(z) + t^3 P(\dot\nu T(\dot\nu T(\dot\nu)))(z) + \cdots,$$

which we write as

$$f^{t\dot\nu}(z) = z + tP(\dot\nu)(z) + t^2 P(\dot\nu T(\dot\nu))(z) + R(t)(z),$$

where

$$R(t) = t^3 P(\dot\nu T(\dot\nu T(\dot\nu))) + t^4 P(\dot\nu T(\dot\nu T(\dot\nu T(\dot\nu)))) + \cdots.$$

Note that $f^{t\dot\nu}$ is smooth and $R_{\bar z} \in C_0^\infty(\mathbb{D})$. Set

$$\ddot\mu = \dot\nu T(\dot\nu) - \dot\mu T(\dot\mu),$$

and note that $\ddot\mu \in C_0^\infty(\mathbb{D})$. Define

$$g(t)(z) = z + tP(\dot\mu) + t^2(P(\ddot\mu) + P(\dot\mu T(\dot\mu))) + R(t)(z),$$

Since $P(\dot\mu) = P(\dot\nu)$ on $\mathbb{D}^*$, and from the choice of $\ddot\mu$, we conclude that

$$g(t)(z) = f^{t\dot\nu}(z), \quad \text{for} \quad z \in \mathbb{D}^*. \tag{16}$$

Observe that $\mathrm{Belt}\big(g(t)\big) \in C_0^\infty(\mathbb{D})$ because $\dot\mu,\ddot\mu,R_{\bar z} \in C_0^\infty(\mathbb{D})$. Moreover,

$$\mathrm{Belt}(g(t)) = t\dot\mu + t^2\ddot\mu + o(t^2). \tag{17}$$

Therefore, $g(t)$ is quasiconformal when t is small enough.

Let t_0 be small enough so that both $f^{t\dot\nu}$ and $g(t)$ are quasiconformal, and set $\mathrm{Belt}(g(t)) = \mu(t)$. Then $\mu : [0,t_0] \to L_1^\infty(\mathbb{D}) \cap C_0^\infty(\mathbb{D})$ is a well defined admissible path (the admissibility follows from (17)). Moreover, $g(t) = f^{\mu(t)}$ (because $f^{\mu(t)}$ is normalised by definition, and $g(t)$ by construction). From (16) we conclude that $\mu(t)$ and $t\dot\nu$ are equivalent (that is, $\mu(t) \sim t\dot\nu$) for every t. This proves the proposition. □

5. The Key Lemma

In the next two sections we prove Lemma 3.1. We choose certain quadratic polynomials to be the corresponding holomorphic functions $\phi_1, \phi_2, \phi_3 \in \mathcal{H}^1(\mathbb{D})$ from the statement of Lemma 3.1. We then use Proposition 4.2 to construct the corresponding equivalent and admissible paths of Beltrami differentials. Before we carry out this construction in the next section, we need to explore how Beltrami differentials pair up with quadratic polynomials.

In particular, we need to establish the inequality (7) from Lemma 3.1. In order to succinctly express this inequality, we introduce the following functional. For a piecewise smooth function $f : \mathbb{D} \to \mathbb{C}$, we define the functional

$$\mathcal{F}(f) = \int\limits_{\mathbb{D}} |f_{\bar{z}}|^2 \, dxdy \; + \; \mathrm{Re} \int\limits_{\mathbb{D}} f_z f_{\bar{z}} \, dxdy. \tag{18}$$

In the next section we explain how this functional is related to the inequality (7). In this section we compute the key inequality (20) involving $\mathcal{F}$ which will then be used to derive (7).

Lemma 5.1. *Suppose $\psi(z) = a + bz + cz^2$ is a quadratic polynomial which has no zeroes on the unit circle $\partial\mathbb{D}$. Then for every $\epsilon > 0$, there exists $\dot{\mu} \in L^\infty(\mathbb{D}) \cap C_0^\infty(\mathbb{D})$ such that*

$$P(\dot{\mu})(z) = \frac{1}{z}, \qquad z \in \mathbb{D}^*, \tag{19}$$

and

$$\left| \mathcal{F}\big(\psi P(\dot{\mu})\big) - \pi\big(|a|^2 + \mathrm{Re}(ac)\big) \right| \leq \epsilon. \tag{20}$$

5.1. An auxiliary proposition.

Proposition 5.1. *Let $Z \subset \mathbb{D}$ be a finite set of points, and $f : \overline{\mathbb{D}} \to \mathbb{C}$ a smooth function. Then for every $\epsilon > 0$, there exists a smooth function $g : \overline{\mathbb{D}} \to \mathbb{C}$ such that*

1. *$f(z) = g(z)$ for z in some neighbourhood of $\partial\mathbb{D}$,*
2. *$g(z) = 0$ for z in some neighbourhood of each $z_0 \in Z$,*
3. *$|\mathcal{F}(f) - \mathcal{F}(g)| \leq \epsilon/2$.*

Proof. Without loss of generality we may assume $Z = \{z_0\}$ (this will be evident from the proof). We split the proof of Proposition 5.1 into two claims.

Claim 2. *Suppose $f : \overline{\mathbb{D}} \to \mathbb{C}$ is a smooth function. For every $\epsilon > 0$, there exists a smooth function $h : \overline{\mathbb{D}} \to \mathbb{C}$ such that*

1. *$f(z) = h(z)$ for z in some neighbourhood of $\partial\mathbb{D}$,*
2. *$h(z) = f(z_0)$ for z in some neighbourhood of z_0,*
3. *$|\mathcal{F}(f) - \mathcal{F}(h)| \leq \epsilon/4$.*

Proof. Near z_0 we have $f(z) = f(z_0) + R(z)$, where $R(z) = (z - z_0)A(z) + \overline{(z - z_0)}B(z)$, and A and B are smooth functions near z_0. Therefore, there exists $K > 0$ (depending only on f) such that

$$|R(z)| \leq K|z-z_0|, \quad |R_{\bar{z}}|(z), |R_z|(z) \leq K, \quad \text{for} \quad |z-z_0| \quad \text{small.} \tag{21}$$

Let $\rho : \mathbb{D} \to [0,1]$ be such that

$$\rho(z) = \begin{cases} 1, & 2r \leq |z-z_0| \\ 0, & 0 \leq |z-z_0| \leq r, \end{cases}$$

and

$$||\rho_{\bar{z}}||_\infty = ||\rho_z||_\infty \leq \frac{10}{r}. \tag{22}$$

Set

$$h(z) = \begin{cases} f(z), & 2r \leq |z-z_0| \\ f(z_0) + \rho(z)R(z), & 0 \leq |z-z_0| \leq 2r. \end{cases}$$

Then h is smooth and satisfies the properties (1) and (2) from the statement of the claim assuming r is small enough. It remains to compute $\mathcal{F}(h)$. The following two identities (which hold when $|z-z_0| \leq 2r$)

$$h_{\bar{z}}(z) = \rho_{\bar{z}} R(z) + \rho(z) R_{\bar{z}}, \qquad h_z(z) = \rho_z R(z) + \rho(z) R_z,$$

together with (22) and (21), yield the estimate

$$|h_{\bar{z}}|(z), |h_z|(z) \leq 21K, \qquad \text{when } |z-z_0| \leq 2r.$$

Let D_{2r} be the disc of radius $2r$ centred at z_0. Since $h = f$ on $\mathbb{D} \setminus D_{2r}$, we get

$$|\mathcal{F}(f) - \mathcal{F}(h)| \leq \int\limits_{D_{2r}} \left(|f_{\bar{z}}|^2 + |h_{\bar{z}}|^2 + |f_z||f_{\bar{z}}| + |h_z||h_{\bar{z}}| \right) dxdy = O(r^2).$$

Choosing r small enough so that $O(r^2) \leq \epsilon/4$ proves the claim.

□

Claim 3. *Let $h : \overline{\mathbb{D}} \to \mathbb{C}$ be a smooth function which is constant in some neighborhood of z_0. For every $\epsilon > 0$, there exists a smooth function $g : \overline{\mathbb{D}} \to \mathbb{C}$ such that*

(1) $g(z) = h(z)$ *for z in some neighbourhood of $\partial\mathbb{D}$,*
(2) $g(z) = 0$ *for z in some neighbourhood of z_0,*
(3) $|\mathcal{F}(g) - \mathcal{F}(h)| \leq \epsilon/4$.

Proof. To simplify the notation we let $h(z_0) = V$. We may assume that $V \neq 0$ (otherwise we set $g = h$ and the claim is proved). Fix $s > 0$ so that the disc D_s is contained in $\mathbb{D}$. Let $r < s$ be small enough so that $h(z) = V, \quad \forall z \in D_r$. We define the function $\rho : \mathbb{D} \to \mathbb{C}$ by

$$\rho(z) = \begin{cases} 0, & s \leq |z-z_0| \\ \frac{V}{\log \frac{s}{r}} \log \frac{s}{|z-z_0|}, & r \leq |z-z_0| \leq s \\ V, & 0 \leq |z-z_0| \leq r. \end{cases}$$

Note that ρ is piecewise smooth. Set $g(z) = h(z) - \rho(z)$. Then g is piecewise smooth and it satisfies the properties (1) and (2) in the statement of the claim. We compute $\mathcal{F}(g)$. Since $g = h$ on $\mathbb{D} \setminus D_s$, and since both g and h are constant on D_r, we get

$$\mathcal{F}(g) - \mathcal{F}(h) = \int\limits_{D_s \setminus D_r} (|g_{\bar{z}}|^2 - |h_{\bar{z}}|^2)\, dxdy \, + \, \mathrm{Re} \int\limits_{D_s \setminus D_r} (g_z g_{\bar{z}} - h_z h_{\bar{z}})\, dxdy.$$

From here we obtain the estimate

$$|\mathcal{F}(g) - \mathcal{F}(h)| \leq \int\limits_{D_s \setminus D_r} \big(2|\rho_{\bar{z}}||h_{\bar{z}}| + |\rho_{\bar{z}}|^2 + |\rho_z||h_{\bar{z}}| + |\rho_{\bar{z}}||h_z| + |\rho_{\bar{z}}||\rho_z|\big)\; dxdy.$$

Choose $K > 0$ so that $||h_z||_\infty, ||h_{\bar{z}}||_\infty \leq K$. Then

$$|\mathcal{F}(g) - \mathcal{F}(h)| \leq 10(K+1) \int\limits_{D_s \setminus D_r} \big(|\rho_{\bar{z}}| + |\rho_{\bar{z}}|^2 + |\rho_z| + |\rho_{\bar{z}}||\rho_z|\big)\; dxdy. \tag{23}$$

Using the equalities

$$|\rho_z|(z) = |\rho_{\bar{z}}|(z) = \left(\frac{|V|}{2\log\frac{s}{r}}\right)\frac{1}{|z - z_0|}, \qquad \text{for} \quad r \leq |z - z_0| \leq s,$$

we compute

$$\int\limits_{D_s \setminus D_r} (|\rho_{\bar{z}}| + |\rho_z|)\, dxdy = 4\pi \left(\frac{|V|}{2\log\frac{s}{r}}\right) \int\limits_r^s dt = O\left(\frac{1}{\log\frac{1}{r}}\right),$$

and

$$\begin{aligned}\int\limits_{D_s \setminus D_r} (|\rho_{\bar{z}}|^2 + |\rho_{\bar{z}}||\rho_z|)\, dxdy &= 4\pi \left(\frac{|V|}{2\log\frac{s}{r}}\right)^2 \int\limits_r^s \frac{1}{t}\, dt \\ &= 4\pi \left(\frac{|V|}{2\log\frac{s}{r}}\right)^2 \left(\log\frac{s}{r}\right) = \frac{\pi|V|^2}{\log\frac{s}{r}} = O\left(\frac{1}{\log\frac{1}{r}}\right).\end{aligned}$$

Replacing this back into (23) shows that

$$|\mathcal{F}(g) - \mathcal{F}(h)| = O\left(\frac{1}{\log\frac{1}{r}}\right).$$

Choose r small enough so that

$$|\mathcal{F}(g) - \mathcal{F}(h)| \leq \frac{\epsilon}{8}.$$

Note that g is piecewise smooth with cracks at the circles ∂D_s and ∂D_r. It remains to smooth g over these cracks so it satisfies the properties (1), (2), and (3) from the lemma.

First, set $g_0 = g$ (observe that r and s that were used to define g_0 are now fixed). Let $\delta \geq 0$ be another parameter, and let $D_z(\delta)$ denote the disc of radius δ centered at z. Set

$$g_\delta(z) = \frac{1}{\operatorname{Area}\big(D_z(\delta) \cap \mathbb{D}\big)} \int\limits_{D_z(\delta)\cap\mathbb{D}} g(w)\, dudv.$$

For each small enough δ, the function g_δ satisfies the properties (1) and (2) from the lemma. Note that g_δ is smooth with respect to z, and continuous with respect to δ. Moreover, the first derivatives of g_δ are uniformly bounded for all δ's, and converge almost everywhere on $\mathbb{D}$ to the corresponding first derivatives of $g_0 = g$ (which are bounded, piecewise continuous functions). From the Dominated Convergence Theorem we conclude $\mathcal{F}(g_\delta) \to \mathcal{F}(g)$, when $\delta \to 0$. Therefore, we can find δ small enough so that g_δ also satisfies the property (3) from the lemma.

Remark. The second (and higher order) derivatives of g_δ do not stay bounded when $\delta \to 0$.

□

The proof of Proposition 5.1 follows now by applying Claim 2 and Claim 3. Namely, given a smooth function $f : \overline{\mathbb{D}} \to \mathbb{C}$, using Claim 2 we construct a smooth function h which is constant in some neighborhood of z_0. Then, using Claim 3 we find a smooth function g which is equal to zero in some neighborhood fo z_0. Both h and g agree with f on $\partial\mathbb{D}$. The property (3) in Proposition 5.1 follows by combining the corresponding properties (3) in the two claims.

5.2. **Proof of Lemma 5.1.** Set $\widehat{f}(z) = a\bar{z} + b + cz$. Since $\widehat{f}_{\bar{z}} \equiv a$, and $\widehat{f}_z \equiv c$, we easily compute

$$\mathcal{F}\big(\widehat{f}\,\big) = \pi\left(|a|^2 + \operatorname{Re}(ac)\right). \tag{24}$$

Note that $\widehat{f}(z) = \frac{a}{z} + b + cz$, when $|z| = 1$. In particular, $\widehat{f}$ can be extended holomorphically near $\partial\mathbb{D}$. Therefore, we can replace $\widehat{f}$ with a smooth $f : \overline{\mathbb{D}} \to \mathbb{C}$ so that $f = \widehat{f}$ on $\partial\mathbb{D}$, $f_{\bar{z}}$ has compact support in $\mathbb{D}$, and the inequality

$$\left|\mathcal{F}(f) - \pi\left(|a|^2 + \operatorname{Re}(ac)\right)\right| \leq \frac{\epsilon}{2}, \tag{25}$$

holds. Let $g : \overline{\mathbb{D}} \to \mathbb{C}$ be the function satisfying the properties (1), (2), and (3), from Proposition 5.1, where we take Z to be set of zeroes of ψ. Therefore

$$\left|\mathcal{F}(g) - \pi\left(|a|^2 + \operatorname{Re}(ac)\right)\right| \leq \epsilon, \tag{26}$$

and

$$g(z) = \frac{\psi(z)}{z}, \qquad z \in \partial\mathbb{D}. \tag{27}$$

We define $A : \mathbb{C} \to \mathbb{C}$ by

$$A(z) = \begin{cases} \frac{1}{z}, & 1 \leq |z| \\ \frac{g(z)}{\psi(z)}, & |z| \leq 1. \end{cases}$$

Let us show that A is smooth. This is clearly true away from $\partial\mathbb{D}$, and the zeroes of ψ. Since $g = 0$ near the zeroes of ψ, it follows that $A = 0$ near the zeroes of ψ. In particular, A is smooth near the zeroes of ψ. It remains to examine the situation near $\partial\mathbb{D}$. It follows from (27) that A is continuous near $\partial\mathbb{D}$. Since $f_{\bar{z}}$ is equal to zero near $\partial\mathbb{D}$, and $g = f$ there, it follows that g is holomorphic near $\partial\mathbb{D}$. Thus, A is holomorphic on $U \setminus \partial\mathbb{D}$, where $U \subset \mathbb{C}$ is some neighbourhood of $\partial\mathbb{D}$. Combining this with the fact that A is continuous on U shows that A is holomorphic on U. In particular, A is smooth on U.

Let $\dot{\mu}(z) = A_{\bar{z}}$. Note that $\dot{\mu} \in L_1^\infty(\mathbb{D}) \cap C_0^\infty(\mathbb{D})$ because $f_{\bar{z}} = 0$ near $\partial\mathbb{D}$, and $g = f$ near $\partial\mathbb{D}$. The function $(A - P(\dot{\mu}))$ is bounded and holomorphic on $\mathbb{C}$. Thus $(A - P(\dot{\mu})) \equiv const.$ Since both $A(z)$ and $P(\dot{\mu})(z)$ tend to zero when $z \to \infty$, the identity $A \equiv P(\dot{\mu})$ follows. Returning this into (26) shows

$$\left|\mathcal{F}\big(\psi P(\dot{\mu})\big) - \pi\left(|a|^2 + \operatorname{Re}(ac)\right)\right| \leq \epsilon,$$

and we are done.

$\square$

6. Proof of Lemma 3.1

We are now ready to prove Lemma 3.1. We begin by selecting the quadratic polynomials ψ_1, ψ_2, ψ_3. We then select the corresponding Beltrami differentials using Proposition 4.2. In particular, the property (2) from Proposition 4.2 enables us to reduce the computations involving $\ddot{\mu}$ to the computations involving $\dot{\mu}$. This is how we reduce the inequality (7) to the inequality (20) which was proved in the previous section.

6.1. **Three polynomials.** Define the following polynomials of order two

$$\psi_1 = \mathbf{i} - 5z + \mathbf{i}\frac{25}{4}z^2, \quad \psi_2 = \mathbf{i} + 5z + \mathbf{i}\frac{25}{4}z^2, \quad \psi_3 = -\sqrt{2} + \sqrt{2}\frac{25}{4}z^2.$$

The reader can easily verify the identity

$$\psi_1^2 + \psi_2^2 + \psi_3^2 \equiv 0, \tag{28}$$

and the equality

$$\pi\sum_{i=1}^{3}\left(|a_i|^2 + \operatorname{Re}(a_i c_i)\right) = -21\pi, \tag{29}$$

where $\psi_i(z) = a_i + b_i z + c_i z^2$, $i = 1, 2, 3$.

6.2. **Proof of Lemma 3.1.** Letting $\psi = \psi_i$, we apply Lemma 5.1 (taking ϵ to be equal to π) to obtain the corresponding $\dot{\mu}_i \in L^\infty(\mathbb{D}) \cap C_0^\infty(\mathbb{D})$ which satisfies the identity (19), and the inequality (20). Let $\dot{\nu} \in L^\infty(\mathbb{D}) \cap C_0^\infty(\mathbb{D})$ be any element such that

$$P(\dot{\nu})(z) = \frac{1}{z}, \quad z \in \mathbb{D}^*. \tag{30}$$

Using Proposition 4.2 we build admissible paths $\mu_i(t) : [0,t_0] \to L_1^\infty(\mathbb{D}) \cap C_0^\infty(\mathbb{D})$, each of them equivalent to $t\dot\nu$ for every t (that is, $\mu_t \sim t\dot\nu$), and such that

$$\int\limits_{\mathbb{D}} \psi_i^2 \ddot\mu_i \, dxdy = \int\limits_{\mathbb{D}} \psi_i^2 \left(\dot\nu T(\dot\nu) - \dot\mu_i T(\dot\mu_i)\right) \, dxdy. \tag{31}$$

Summing over $i = 1,2,3$, and using (28), we get

$$\sum_{i=1}^{3} \int\limits_{\mathbb{D}} \psi_i^2 \ddot\mu_i \, dxdy = -\sum_{i=1}^{3} \int\limits_{\mathbb{D}} \psi_i^2 \dot\mu_i T(\dot\mu_i) \, dxdy = -\sum_{i=1}^{3} \int\limits_{\mathbb{D}} \psi_i^2 P_{\bar z}(\dot\mu_i) P_z(\dot\mu_i) \, dxdy \tag{32}$$

(the second equality follows from Proposition 4.1). Observe the equality

$$\int\limits_{\mathbb{D}} \psi_i^2 P_{\bar z}(\dot\mu_i) P_z(\dot\mu_i) \, dxdy = \int\limits_{\mathbb{D}} \big(\psi_i P(\dot\mu_i)\big)_{\bar z} \big(\psi_i P(\dot\mu_i)\big)_z \, dxdy - \frac{1}{4} \int\limits_{\mathbb{D}} (\psi_i^2)_z (P^2(\dot\mu_i))_{\bar z} \, dxdy. \tag{33}$$

From the Stokes theorem (and using again that ψ_i is holomorphic), we find

$$\int\limits_{\mathbb{D}} (\psi_i^2)_z (P^2(\dot\mu_i))_{\bar z} \, dxdy = \frac{1}{2\mathbf{i}} \int\limits_{\mathbb{D}} \bar\partial \left((\psi_i^2)_z P^2(\dot\mu_i)\right) \, d\bar z \wedge dz = \frac{1}{2\mathbf{i}} \int\limits_{\partial\mathbb{D}} (\psi_i^2)_z P^2(\dot\mu_i) \, dz.$$

Since $P(\dot\mu_i) = P(\dot\mu_j)$ on $\partial\mathbb{D}$ for all i,j, and since $(\psi_1^2)_z + (\psi_2^2)_z + (\psi_3^2)_z \equiv 0$ (which follows from (28)), we conclude

$$\sum_{i=1}^{3} \int\limits_{\mathbb{D}} (\psi_i^2)_z (P^2(\dot\mu_i))_{\bar z} \, dxdy = 0.$$

Replacing this into (33) we obtain

$$\sum_{i=1}^{3} \int\limits_{\mathbb{D}} \psi_i^2 P_{\bar z}(\dot\mu_i) P_z(\dot\mu_i) \, dxdy = \sum_{i=1}^{3} \int\limits_{\mathbb{D}} \big(\psi_i P(\dot\mu_i)\big)_{\bar z} \big(\psi_i P(\dot\mu_i)\big)_z \, dxdy,$$

which together with (32) yields

$$\sum_{i=1}^{3} \int\limits_{\mathbb{D}} \psi_i^2 \ddot\mu_i \, dxdy = -\sum_{i=1}^{3} \int\limits_{\mathbb{D}} \big(\psi_i P(\dot\mu_i)\big)_{\bar z} \big(\psi_i P(\dot\mu_i)\big)_z \, dxdy. \tag{34}$$

Finally, using (20) and (29) we estimate from below the right hand side of (34)

$$\begin{aligned}
\sum_{i=1}^{3} \int\limits_{\mathbb{D}} \psi_i^2 \ddot{\mu}_i \, dxdy &= -\sum_{i=1}^{3} \int\limits_{\mathbb{D}} \big(\psi_i P(\dot{\mu}_i)\big)_{\bar{z}} \big(\psi_i P(\dot{\mu}_i)\big)_{z} \, dxdy \\
&\geq 20\pi + \sum_{i=1}^{3} \int\limits_{\mathbb{D}} |\big(\psi_i P(\dot{\mu}_i)\big)_{\bar{z}}|^2 \, dxdy \\
&\geq 20\pi + \sum_{i=1}^{3} \int\limits_{\mathbb{D}} \psi_i^2 |\dot{\mu}_i|^2 \, dxdy \\
&> \sum_{i=1}^{3} \int\limits_{\mathbb{D}} \psi_i^2 |\dot{\mu}_i|^2 \, dxdy.
\end{aligned}$$

This proves the strict inequality (7), and we are done.

7. Proof of Lemma 1.1

The proof of Lemma 1.1 is based on combining Lemma 3.2 together with the standard approximation method. In other words, we promote the strict inequality (9) which holds on $\mathbb{D}$ to the inequality (1) which holds on a closed Riemann surface of a sufficiently large injectivity radius.

Remark. Although the following observation is irrelevant for this paper, we note that this strategy is not reversible, meaning that it is not possible to promote (1) to (9) because the Theta projection is a strict contraction in L^1 norm as proved by McMullen [9].

7.1. **Theta Projection.** In this section Γ always denotes a Fuchsian group acting on $\mathbb{D}$ such that $\mathbb{D}/\Gamma \equiv S$ is a closed Riemann surface.

Definition 7.1. *By $L_1^{\infty}(\Gamma) \subset L_1^{\infty}(\mathbb{D})$ we denote the set of functions h satisfying the equivariance condition:*

$$h = (h \circ A)\frac{\overline{A'}}{A'},$$

for every $A \in \Gamma$.

Definition 7.2. *By $\mathrm{QD}(\Gamma)$ we denote the set of holomorphic functions h on $\mathbb{D}$ satisfying the equivariant condition:*

$$h = (h \circ A)(A')^2,$$

for every $A \in \Gamma$.

The Theta Projection $\Theta : \mathcal{H}^1(\mathbb{D}) \to \mathrm{QD}(\Gamma)$ is given by

$$\Theta(\phi) = \sum_{A \in \Gamma} (\phi \circ A)(A')^2.$$

Note that $\Theta(\phi)$ is a lift of a holomorphic quadratic differential from the Riemann surface S.

7.2. **Preliminary propositions.** We make a note of the following elementary proposition. The proof is left to the interested reader.

Proposition 7.1. *Let K be a compact subset of $\mathbb{D}$, and let $\phi \in \mathcal{H}^1(\mathbb{D})$. For every $\epsilon > 0$, there exists $d > 0$ such that whenever the injectivity radius of $S = \mathbb{D}/\Gamma$ is greater than d the estimate*

$$||\big(\Theta(\phi) - \phi\big)_K||_\infty \leq \epsilon, \tag{35}$$

holds. (Here $\big(\Theta(\phi) - \phi\big)_K$ is the restriction of $\Theta(\phi) - \phi$ to K).

Definition 7.3. *Suppose $\mu \in L^\infty_1(\mathbb{D})$. Let Γ be a Fuchsian group and fix $\nu \in L^\infty_1(\Gamma)$. We say that μ is ϵ-extended by ν, if there exists a fundamental domain $\Omega \subset \mathbb{D}$ for Γ such that $||\mu - \nu_\Omega||_\infty \leq \epsilon$ (here ν_Ω denotes the restriction of ν to Ω).*

Remark. Clearly, if $\mu \in L^\infty_1(\mathbb{D})$ is ϵ-extended by any $\nu \in L^\infty_1(\Gamma)$, then the support of μ must be contained in Ω.

Proposition 7.2. *Suppose $\mu_i \in L^\infty_1(\mathbb{D})$, $i = 1, 2, 3$, are compactly supported in $\mathbb{D}$, and mutually equivalent (that is, $\mu_i \sim \mu_j$ for all i, j). The for every $\epsilon > 0$, there exists $d > 0$ such that whenever the injectivity radius of $S = \mathbb{D}/\Gamma$ is greater than d, there exist $\nu_i \in L^\infty_1(\Gamma)$ which are mutually equivalent on S (that is, $\mu_i \sim \mu_j$), and such that μ_i is ϵ-extended by ν_i.*

Remark. The say $\nu_i \sim \nu_j$ if ν_i and ν_j are Beltrami dilatations of homotopic quasiconformal maps out of the Riemann surface S.

Proof. Let $K \subset \mathbb{D}$ be a compact set containing the supports of all μ_i's. If the injectivity radius of $\mathbb{D}/\Gamma$ is large enough, then Γ has a fundamental domain Ω containing K. Let $\eta_i \in L^\infty_1(\Gamma)$ be the Beltrami dilatation such that $(\eta_i)_\Omega = \mu_i$ on $\mathbb{D}$. Clearly, μ_i is 0-extended by η_i. However, η_i's may not be mutually equivalent. But when the injectivity radius of Γ is large enough, we can replace η_i by $\nu_i \in L^\infty_1(\Gamma)$ such that $\nu_i \sim \nu_j$ $\forall i, j$, and $||\eta_i - \nu_i|| \leq \epsilon$. Then μ_i is ϵ-extended by ν_i.

The existence of such ν_i's follows by a routine compactness argument which we sketch next (see [4]). Set $\nu_1 = \eta_1$. Suppose that for some i we can not find ν_i which is equivalent to ν_1, and such that $||\nu_i - \eta_i||_\infty \leq \epsilon$. Then there exist $\delta_0 > 0$, and a sequence of Fuchsian groups Γ_k, such that the injectivity radius of $\mathbb{D}/\Gamma_k$ tends to ∞, and such that the quasiconformal maps $g_k = f^{\eta_i} \circ (f^{\nu_1})^{-1}$ are not equivariantly homotopic to a $(1 + \delta_0)$-quasiconformal map. But then there exists a sequence of Möbius transformations A_k, B_k preserving $\partial\mathbb{D}$, such that the maps $h_k = A_k \circ g_k \circ B_k$ converge to a quasiconformal map $h_0 : \mathbb{D} \to \mathbb{D}$ whose restriction to $\partial\mathbb{D}$ is not a Möbius transformation. This can clearly be ruled out by computing the Beltrami dilatation of g_k (we leave the details to the reader). This contradiction proves the proposition. $\square$

7.3. **Proof of Lemma 1.1.** Suppose $\mu_i \in L_1^\infty(\mathbb{D})$, and $\phi_i \in \mathcal{H}^1(\mathbb{D})$, are as in Lemma 3.2. Since μ_i is compactly supported in $\mathbb{D}$ there exists a compact set $K \subset \mathbb{D}$ containing the supports of all μ_i's.

Claim 4. *Let $\delta > 0$. There exist Γ, and mutually equivalent $\nu_i \in L_1^\infty(\Gamma)$, such that*

$$\operatorname{Re}\sum_{i=1}^{3}\int_{\mathbb{D}} \phi_i \frac{\mu_i}{1-|\mu_i|^2}\,dxdy < \operatorname{Re}\sum_{i=1}^{3}\int_{\Omega} \Theta(\phi_i)\frac{\nu_i}{1-|\nu_i|^2}\,dxdy + \delta, \tag{36}$$

and

$$\operatorname{Re}\sum_{i=1}^{3}\int_{\mathbb{D}} |\phi_i| \frac{|\mu_i|^2}{1-|\mu_i|^2}\,dxdy \geq \operatorname{Re}\sum_{i=1}^{3}\int_{\Omega} |\Theta(\phi_i)|\frac{|\nu_i|^2}{1-|\nu_i|^2}\,dxdy - \delta. \tag{37}$$

Proof. For each $\epsilon > 0$, let Γ and $\nu_i \in L_1^\infty(\Gamma)$ be the Fuchsian group and corresponding Beltrami dilatations Proposition 7.2. We may also assume that Γ and K satisfy the conclusion of Proposition 7.1. Let Ω be the corresponding fundamental domain for Γ (in particular, $K \subset \Omega$). Combining the fact that μ_i is ϵ-extended by ν_i with the estimate (35) from Proposition 7.1, we see that for ϵ small enough the inequality

$$\left| \int_{K} \phi_i \frac{\mu_i}{1-|\mu_i|^2}\,dxdy - \int_{\Omega} \Theta(\phi_i)\frac{\nu_i}{1-|\nu_i|^2}\,dxdy \right| \leq \frac{\delta}{3},$$

holds. Summing over $i = 1, 2, 3$, gives

$$\left| \sum_{i=1}^{3}\int_{K} \phi_i \frac{\mu_i}{1-|\mu_i|^2}\,dxdy - \sum_{i=1}^{3}\int_{\Omega} \Theta(\phi_i)\frac{\nu_i}{1-|\nu_i|^2}\,dxdy \right| \leq \delta,$$

which implies (36) because the support of each μ_i is contained in K. The inequality (37) is proved in the same way.

$\square$

Now the endgame. Let $\delta > 0$ be such that

$$3\delta = \operatorname{Re}\sum_{i=1}^{3}\int_{\mathbb{D}} \phi_i \frac{\mu_i}{1-|\mu_i|^2}\,dxdy - \sum_{i=1}^{3}\int_{\mathbb{D}} |\phi_i| \frac{|\mu_i|^2}{1-|\mu_i|^2}\,dxdy.$$

Combining this with the inequalities (36) and (37) from Claim 4 yields the inequality

$$\operatorname{Re}\sum_{i=1}^{3}\int_{\Omega} \Theta(\phi_i)\frac{\nu_i}{1-|\nu_i|^2}\,dxdy - \sum_{i=1}^{3}\int_{\Omega} |\Theta(\phi_i)|\frac{|\nu_i|^2}{1-|\nu_i|^2}\,dxdy \geq \delta > 0. \tag{38}$$

Once again we observe that $\Theta(\phi_i)$ is a holomorphic quadratic differential on $S = \mathbb{D}/\Gamma$. Moreover,

$$\Theta(\phi_1) + \Theta(\phi_2) + \Theta(\phi_3) = \Theta(\phi_1 + \phi_2 + \phi_3) = 0.$$

Since ν_i's are the Beltrami dilatations of mutually homotopic diffeomorphisms mapping S to another Riemann surface S', the inequality (38) yields the proof of Lemma 1.1.

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