QUASISYMMETRIC GROUPS

VLADIMIR MARKOVIC

Abstract. One of the first problems in the theory of quasisymmetric and convergence groups (see Gehring-Palka [11] and Gehring-Martin [9]) was to investigate whether every quasisymmetric group that acts on the sphere $\mathbf{S}^n$, $n > 0$, is a quasisymmetric conjugate of a Möbius group that acts on $\mathbf{S}^n$. This was shown to be true for $n = 2$ by Sullivan [23] and Tukia [25] , and it was shown to be wrong for $n > 2$ by Tukia [26], and it also follows from the work of Martin [19] and Freedman-Skora [7]. In this paper we settle the case $n = 1$ by showing that any K-quasisymmetric group is K_1-quasisymmetrically conjugated to a Möbius group. The constant K_1 is a function K.

1. Introduction

1.1. **The main results.** Let $\mathbf{T}$ denote the unit circle and $\mathbf{D}$ the unit disc. Suppose that $f : \mathbf{T} \to \mathbf{T}$ is a homeomorphism. Let $\widehat{f} : \overline{\mathbf{D}} \to \overline{\mathbf{D}}$ be a homeomorphism too. We say that $\widehat{f}$ extends f, if $\widehat{f}$ and f agree on $\mathbf{T}$. All mappings in this paper are sense preserving (see the remark at the end of introduction).

Definition 1.1. *We say that a homeomorphism $f : \mathbf{T} \to \mathbf{T}$ is K-quasisymmetric if there exists a K-quasiconformal map $\widehat{f} : \mathbf{D} \to \mathbf{D}$ that extends f.*

This is one of a number of equivalent ways to define quasisymmetric maps of the unit circle $\mathbf{T}$. Let S be a Riemann surface and let f an element of the mapping class group of the surface S. We say that f is K-quasisymmetric if there exists a K-quasiconformal map $\widehat{f} : S \to S$ that represents f. If $S = \mathbf{D}$ then this agrees with the above definition.

Definition 1.2. *Let $\mathcal{G}$ be a subgroup of the group of homeomorphisms of $\mathbf{T}$. We say that $\mathcal{G}$ is a K-quasisymmetric group if every element of $\mathcal{G}$ is K-quasisymmetric.*

We will also say that a subgroup $\mathcal{G}$ of the mapping class group of a Riemann surface S is K-quasisymmetric if every element from $\mathcal{G}$ can be represented by a K-quasiconformal map of S onto itself.

Remark. Unless specified differently a quasiconformal map (or a quasiisometry, see Section 2) is assumed to be a selfmap of $\mathbf{D}$.

By $\mathcal{M}$ we denote the Lie group of Möbius transformations that preserve the unit disc $\mathbf{D}$ (therefore, those transformations preserve $\mathbf{T}$ as well). If $u \in \mathcal{M}$ we consider u as a homeomorphism of $\mathbf{T}$. The corresponding group that acts on $\mathbf{D}$ is denoted by $\widehat{\mathcal{M}}$, and the corresponding element by $\widehat{u}$. A subgroup $\mathcal{F}$ of $\mathcal{M}$ is called a Möbius group. If $\mathcal{F}$ is discrete we say that $\mathcal{F}$ is a Fuchsian group. The corresponding group that acts on $\mathbf{D}$ is denoted by $\widehat{\mathcal{F}}$. The following are the main results of this paper.

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Theorem 1.1. *Let $\mathcal{G}$ be a discrete K-quasisymmetric group. Then there exists a K_1-quasisymmetric map $\varphi : \mathbf{T} \to \mathbf{T}$, and a Fuchsian group $\mathcal{F}$, such that $\mathcal{G} = \varphi \mathcal{F} \varphi^{-1}$. The constant K_1 is a function of K, that is $K_1 = K_1(K)$.*

Hinkkanen proved (see [14], [16] and Proposition 1.2 below) that the same is true for quasisymmetric groups that are not discrete. This gives the next theorem.

Theorem 1.2. *Let $\mathcal{G}$ be a K-quasisymmetric group. Then there exists a K_2-quasisymmetric map $\varphi : \mathbf{T} \to \mathbf{T}$ and a Möbius group $\mathcal{F}$ such that $\mathcal{G} = \varphi \mathcal{F} \varphi^{-1}$. The constant K_2 is a function of K, that is $K_2 = K_2(K)$.*

It is a classical result of Sullivan and Tukia (see [23], [25]) that every quasiconformal group that is acting on the Riemann sphere $\mathbf{S}^2$ is quasiconformally conjugated to a Möbius group (subgroup of the group of Möbius transformations of $\mathbf{S}^2$). Tukia showed (see [26]) that this is no longer true for higher dimensional spheres. Martin [19], and Freedman-Skora [7] gave various important examples of this nature. Theorem 1.2 settles in positive the case of the one dimensional sphere $\mathbf{T}$.

The notion of convergence groups was introduced by Gehring and Martin (see the Gehring-Palka [11] for the origins of the theory of quasiconformal groups). The theory of quasiconformal groups is closely related to the theory of convergence groups (see [9], [10], [8], [27]). For example see [2] for connections with the geometric group theory and for further references. In particular, quasiconformal groups are convergence groups. One of the central results in geometric group theory is that every convergence group of circle homeomorphism is a conjugate of a Möbius group. This theorem was proved by Gabai [8]. Prior to that Tukia [27] has proved this result in many cases. Hinkkanen [15] proved the result for non-discrete groups. By different methods this theorem was independently proved by Casson and Jungreis [3] (see [3], [8], [27] for references to other important papers in this subject). We have

Proposition 1.1. *Let $\mathcal{G}$ be a K-quasisymmetric group. Then there exists a homeomorphism $\varphi : \mathbf{T} \to \mathbf{T}$ and a Möbius group $\mathcal{F}$ such that $\mathcal{G} = \varphi \mathcal{F} \varphi^{-1}$.*

To prove Theorem 1.2 one needs to show that in Proposition 1.1 one can find a Möbius group and a homeomorphism φ so that φ is quasiconformal. The methods used in [8], [25] to produce φ are constructive and explicit. Therefore, one can suspect that by repeating and modifying their construction, and keeping in mind that $\mathcal{G}$ is a quasisymmetric group, the resulting homeomorphism would be quasisymmetric. However, this does not appear to be the case. Nevertheless we will make a frequent use of Proposition 1.1.

It follows from the theorem on uniformly quasiconformal groups that Theorem 1.2 is true if and only if every K-quasisymmetric group can be extended to a K_1-quasiconformal group of $\mathbf{D}$, $K_1 = K_1(K)$. In [5] it was shown that there is no general algorithm that would produce such an extension. The proof of Theorem 1.1 is also explicit and for a given discrete quasisymmetric group we construct such a quasiconformal extension (this extension can be recovered easily from our proof).

We will assume that the reader is familiar with some elementary facts from the theory of convergence and quasiconformal groups. In particular, we will freely use the notion of hyperbolic, parabolic and elliptic elements of a quasisymmetric group. The order of an elliptic element $e \in \mathcal{G}$ is the smallest integer $n \in \mathbf{N}$, such that $e^n = id \in \mathcal{G}$.

1.2. **Elementary and non-discrete quasisymmetric groups.** Let $\varphi : \mathbf{T} \to \mathbf{T}$ be a homeomorphism. Let $a, b \in \mathbf{T}$, $a \neq b$, and let l be the geodesic that connects them. Let l' be the geodesic that connects the points $\varphi(a)$ and $\varphi(b)$. We say that l' is a push forward of the the geodesic l and write $\varphi_* l = l'$.

Hinkkanen proved Theorem 1.2 in various cases. In particular, the following proposition is a subcollection of results proved in [14], [16] that are going to be used in this paper.

Proposition 1.2. *Let $\mathcal{G}$ be a K-quasisymmetric group and suppose that $\mathcal{G}$ is either*

1. *A discrete elementary group.*
2. *A non-discrete group.*

Then there exist $\widetilde{K} = \widetilde{K}(K)$ and a $\widetilde{K}$-quasisymmetric map φ, such that φ conjugates $\mathcal{G}$ to a Fuchsian group.

In fact, Hinkkanen proves the above proposition for Abelian and non-discrete groups. But this readily implies the case of discrete elementary groups as follows. We know that every discrete elementary quasisymmetric group is conjugated to a Fuchsian group. The list of discrete elementary Fuchsian groups is short (see [17]). Up to a conjugacy (in $\mathcal{M}$) the only non Abelian discrete elementary group is generated by a hyperbolic element $u \in \mathcal{M}$ (with the fixed points $i, -i$) and the elliptic transformation $e_0 \in \mathcal{M}$, $e_0(z) = -z$, $z \in \mathbf{T}$. Note that u and e_0 satisfy the relation

$$u^{-1} = e_0 \circ u \circ e_0. \tag{1.1}$$

Every element in this group is either hyperbolic (from the same cyclic group generated by u) or it is an elliptic transformation of order two that permutes the two fixed points of u (there are infinitely many of these elliptic elements).

By Proposition 1.1 any elementary discrete quasisymmetric group $\mathcal{G}$ (that is not cyclic) is generated by a hyperbolic element h and the corresponding elliptic element e (that permutes the fixed points of h). By conjugating h by a suitable quasisymmetric map we may assume that $h = u \in \mathcal{M}$. So the group $\mathcal{G}$ is generated by u and e, where u is a Möbius map and e is some quasisymmetric map of order two. We can assume that u fixes the points $i, -i$. Since the group $\mathcal{G}$ is topologically conjugated to a Möbius group we have the following relation

$$u^{-1} = e \circ u \circ e. \tag{1.2}$$

Denote by R the interval $(i, -i)$ and by L the interval $(-i, i)$ (we take the standard counterclockwise orientation on $\mathbf{T}$). By replacing e if necessary by a map of the form $u^{-k} \circ e \circ u^k$, $k \in \mathbf{Z}$, we can assume that $e(-1)$ belongs to the interval $(u^{-1}(1), u(1)) \subset R$. Let $q : \mathbf{T} \to \mathbf{T}$ be the earthquake map that is the identity on L and such that $q(1) = e(-1)$ (such q is unique and it is quasisymmetric). The map q commutes with u (which means that it conjugates u to itself). By replacing e if necessary by $q^{-1} \circ e \circ q$ we can assume that $e(-1) = 1$.

Since the maps u, e, e_0 satisfy (1.1) and (1.2) we conclude that $e = e_0$ on $O(1)$ and $O(-1)$, where $O(1)$ is the orbit of the point 1 under the action of the cyclic group generated by u (similarly for $O(-1)$). Set $f(z) = (e_0 \circ e)(z)$ for $z \in R$, and $f(z) = z$ for $z \in L$. We have that f fixes every point from $O(1)$ and $O(-1)$. This readily implies that f is quasisymmetric (we already know that f is locally quasisymmetric on both R and L). It follows that f conjugates the group $\mathcal{G}$ to the Möbius group generated by u and e_0.

We also note the following elementary proposition.

Proposition 1.3. *Let $\mathcal{G}$ be a K-quasisymmetric group and suppose that every finitely generated subgroup of $\mathcal{G}$ is $\widetilde{K}$-quasiconformally conjugated to a Fuchsian group. Then so is $\mathcal{G}$.*

From now on in this paper we assume that every quasisymmetric group is discrete (unless specified otherwise).

1.3. **A brief outline.** The proof of Theorem 1.1 is divided into several steps. However, there are two main intermediate results which constitute the heart of this paper. The first is Theorem 1.3. This theorem "takes care" of elliptic elements of order three or more.

Remark. This case turned out to be combinatorially very complicated in the proof of Proposition 1.1 about convergence groups. Note that so called triangle groups must contain at least one elliptic element of order at least three.

The following Theorem 1.3 will be later stated and proved as Theorem 7.1 in Section 7.

Theorem 1.3. *For an arbitrary K-quasisymmetric group $\mathcal{G}$ there exists a K_1-quasisymmetric group $\mathcal{G}_1$, $K_1 = K_1(K)$, with the following properties.*

1. *$\mathcal{G}_1$ does not contain elliptic elements of order three or more.*
2. *If $\mathcal{G}_1$ is K'-quasisymmetrically conjugated conjugated to a Fuchsian group, $K' = K'(K)$, then there exists $K'' = K''(K)$ such that $\mathcal{G}$ is K''-quasisymmetrically conjugated to a Fuchsian group.*

Let $\mathcal{F}$ be a Fuchsian group and φ a homeomorphism such that $\varphi \mathcal{F} \varphi^{-1} = \mathcal{G}$. Denote by $E' \subset \mathbf{D}$ the set of fixed points of all elliptic elements of $\widehat{\mathcal{F}}$ that are of order three or more. Let $\widehat{\varphi}$ denote the barycentric extension of φ and set $E = \widehat{\varphi}(E')$; $S = \mathbf{D} - E$. Set $\mathcal{G}'_1 = \widehat{\varphi} \mathcal{F} \widehat{\varphi}^{-1}$. Then the group $\mathcal{G}'_1$ is a K_1-quasisymmetric group on S, $K_1 = K_1(K)$, which means that every element of $\mathcal{G}'_1$ is isotopic as a map of S ($rel\ \partial S$) to a K_1-quasiconformal map. By covering the surface S by the unit disc we can lift $\mathcal{G}'_1$ to the group $\mathcal{G}_1$ that is a K_1-quasisymmetric group on $\mathbf{D}$. The group $\mathcal{G}_1$ is not isomorphic to $\mathcal{G}'_1$ but it naturally projects to $\mathcal{G}'_1$. The kernel of this projection is the group of covering transformations of S. This is the outline of the proof of Theorem 1.3. Note that we put no restriction on the choice of the homeomorphism φ (in particular, it does not have to be quasisymmetric). The key to proving this theorem are certain analytical properties of the barycentric extension of Douady and Earle (see [4], [1]) that are of different nature than the standard conformal naturality requirement this extension satisfies (which is also important of course).

The second main intermediate result is Lemma 4.2. Once we eliminate elliptic elements of order three or more (by Theorem 1.3), we can prove Lemma 4.2. This lemma shows that the subgroup $\mathcal{G}_z$ of $\mathcal{G}$, that is generated by *small* elements with respect to a point $z \in \mathbf{D}$, is cyclic (see the definition in Section 4). Here by *small* elements we mean elements that are close to the identity (in C^0 topology) when seen from the point $z \in \mathbf{D}$. This is a quasisymmetric version of the corresponding results about small elements of Fuchsian groups. These results for the Fuchsian case are corollaries of theorems like Jorgensen inequality or Margulis lemma (which of course holds in the context of discrete lattices in Lie groups).

Remark. Note that in the Fuchsian case in order to prove Jorgensen inequality or Margulis lemma, one does not have to assume that a given Fuchsian group does not have elliptic elements of order three or more. It is possible to prove Lemma 4.2 (the quasisymmetric case) without that assumption as well, but that would require a more delicate proof that would involve the commutator trick (see chapter 4 in [24]) that is used to prove the Margulis lemma in the context of Lie groups. Although quasiconformal maps do not make a Lie group they still can be endowed with a manifold structure, and one can modify the ideas from the proof of the Margulis lemma to this case. It is interesting to try to investigate this for quasiconformal groups in higher dimensions, which are not necessarily conjugates of Möbius groups. In the general case, the corresponding group $\mathcal{G}_z$ should be almost Abelian.

We use the following rules in connection with the notation. If C and D stand for some abstract constants then the labeling $C = C(D)$ means that the constant C is a function of D. Sometimes C is a function, that is itself a function of D, that is we say that C depends on D. In other words, for a fixed D one can choose such a C. What is important here is that saying that $C = C(D)$ also says that the constant C does not depend on any other parameter. Typically, the constant C will depend on the constant D in a concrete way. Usually it will be bounded above or below in some way that depends on D. This will always be clear from the context but we will not necessarily make a note of it, since we do not need it.

From Section 4 onwards K will always stand for the quasisymmetry constant of a K-quasisymmetric group. Recall from Proposition 1.2 that $\widetilde{K}$ is the quasisymmetry constant of a map ψ that conjugates an elementary K-quasisymmetric group to a Fuchsian group. We allow $\widetilde{K}$ to be as large as necessary (but always as a function of K) so that we can choose appropriate $\widetilde{K}$-quasiconformal extensions of the map ψ. From Section 4 onwards, the constant $\widetilde{K}$ will have this meaning. All other constants will be valid throughout the subsection in which they were introduced. If we refer to a particular constant from a previous section, we will do this in a clear way and no confusion should arise.

In sections 2,3 we prove various technical lemmas about quasiisometries, quasiisometric groups, quasiconformal maps and their geometry in $\mathbf{D}$. Some of these results might be known. In Section 4 we study small elements and show how to remove them. In sections 5,6 we prove Theorem 1.1 for torsion-free groups. In Section 7 we show how to eliminate elliptic elements of order three or more. In Section 8 we deal with groups whose only elliptic elements are of order two.

After reading the introduction the reader can go straight to the very end of this paper, to consult the (very short) subsection where we give the summary of the proof of Theorem 1.1. This gives clear guidelines of what is the logical order of the proof.

Remark. It is a part of the definition of quasisymmetric (and quasiconformal maps in general) that they are sense-preserving. Originally, quasiconformal groups are defined to be sense-preserving (see [11], [14]). However, one can naturally extend this definition to sense-reversing maps. A sense-reversing map is quasiconformal, if its complex conjugate is quasiconformal in the ordinary sense. Hinkkanen (in [16]) considered these generalized quasisymmetric groups, and his results are valid in this case as well.

We do not state Theorem 1.2 for generalized quasisymmetric groups, but we make the following observation. It appears that all methods that we use go through for sense-reversing maps as well. The only place where we use that our maps are sense-preserving is when listing elementary Fuchsian groups. If one allows sense-reversing Möbius transformations, then this list would have a few more members. Consequently this implies that there are a few more cases of elementary discrete generalized quasisymmetric groups. Hinkkanen has dealt with this issue in [16], and it seems that his work covers all technical aspects that arise in dealing with these additional elementary groups.

2. Quasiisometries of the unit disc

2.1. **Quasiisometric continuation.** In this subsection we state several results about quasiisometries of the unit disc. Some of these results are classical. In this paper, $\mathbf{d}$ stands for the hyperbolic metric on $\mathbf{D}$, and $\mathbf{d}(z,w)$ always denotes the hyperbolic distance between the points $z,w \in \mathbf{D}$. By $\Delta(z,r)$ we denote the hyperbolic disc centered at $z \in \mathbf{D}$, and with the hyperbolic radius $r > 0$.

Definition 2.1. *Let $\widehat{f} : \mathbf{D} \to \mathbf{D}$ be a homeomorphism. We say that $\widehat{f}$ is a (L,a)-quasiisometry if*

$$L^{-1}\mathbf{d}(z,w) - a < \mathbf{d}(\widehat{f}(z),\widehat{f}(w)) < L\mathbf{d}(z,w) + a,$$

for some $L, a > 0$.

Remark. The assumption that $\widehat{f}$ is a homeomorphism of $\mathbf{D}$ is in the literature often weakened by assuming that $\widehat{f}$ is only a surjective map of $\mathbf{D}$ onto itself. To simplify the notation we will say that $\widehat{f}$ is a L-quasiisometry if $L = max\{L,a\}$.

It is well known that every L-quasiisometry has a continuous extension to $\overline{\mathbf{D}}$. Let $f : \mathbf{T} \to \mathbf{T}$ be the corresponding map (the restriction of the extended map $\widehat{f}$). There exists $K(L)$ such that f is a $K(L)$-quasisymmetric map. On the other hand, if $\widehat{f} : \mathbf{D} \to \mathbf{D}$ is a K-quasiconformal map, there is $a(K) > 0$ such that $\widehat{f}$ is a $(K, a(K))$-quasiisometry (see [6]).

Let l be a geodesic in $\mathbf{D}$ and let $\gamma : l \to \mathbf{D}$ be a map such that

$$L^{-1}\mathbf{d}(z,w) < \mathbf{d}(\gamma(z),\gamma(w)) < L\mathbf{d}(z,w),$$

for every $z,w \in l$ and some $L > 0$. We say that the map γ is a L bilipschitz quasigeodesic. Sometimes we will say that the corresponding curve $\gamma(l)$ is a bilipschitz quasigeodesic if it is clear what the mapping is. Let l_1 be the geodesic with the same endpoints as $\gamma(l)$. The main property of γ is that there is $D(L) > 0$ such that $\mathbf{d}(z,l_1) < D(L)$, for every $z \in \gamma(l)$.

Let $\widehat{f}$ be a L-quasiisometry and $l \subset \mathbf{D}$ a geodesic. The restriction of $\widehat{f}$ on l does not have to be a bilipschitz quasigeodesic. Nevertheless, it is easy to construct a $L'(L)$ bilipschitz quasigeodesic $\gamma : l \to \mathbf{D}$, such that for some $D_0 = D_0(L)$ we have $\mathbf{d}(\widehat{f}(z), \gamma(z)) < D_0$, for every $z \in l$. This implies that every point of $\widehat{f}(l)$ is within bounded hyperbolic distance of the corresponding geodesic with the same endpoints. This observation is the key ingredient of the proof of the following well known proposition.

Proposition 2.1. *Let $\widehat{f} : \overline{\mathbf{D}} \to \overline{\mathbf{D}}$ be a homeomorphism. Let $f : \mathbf{T} \to \mathbf{T}$ the restriction of $\widehat{f}$. We have.*

1. *Suppose that f is the identity, and that $\widehat{f}$ is a L-quasiisometry. There exists $D = D(L) > 0$, such that $\mathbf{d}(\widehat{f}(z), z) < D$, for every $z \in \mathbf{D}$.*
2. *Suppose that for every $z_0 \in \mathbf{D}$, there exists a L-quasiisometry $\widehat{f}_{z_0}$, which extends f, and such that $\mathbf{d}(\widehat{f}_{z_0}(z_0), \widehat{f}(z_0)) < D$, for some $D = D(L) > 0$. Then there exists $L' = L'(L)$ such that $\widehat{f}$ is L'-quasiisometry.*

Definition 2.2. *Let $(M, \mathbf{d}_1)$ and $(N, \mathbf{d}_2)$ be two metric spaces. Let $\delta : \mathbf{R}^+ \to \mathbf{R}^+$ be a function, $\epsilon \to \delta(\epsilon)$. We say that a map $F : M \to N$ is $\delta(\epsilon)$ continuous if $\mathbf{d}_2(F(z), F(w)) < \epsilon$, whenever $\mathbf{d}_1(z, w) < \delta(\epsilon)$, for every $z, w \in M$. We also say that such F is uniformly continuous. A homeomorphism F is $\delta(\epsilon)$ continuous if both F and F^{-1} are.*

If we say that a map defined on a subset of the unit disc is uniformly continuous that always refers (unless specified otherwise) to the corresponding hyperbolic metric on $\mathbf{D}$.

Let $x, y, z \in \mathbf{T}$, and let S be the geodesic triangle with vertices x, y, z. Denote the geodesics that represent the sides of S by $a_{x,y}, a_{y,z}, a_{z,x}$. Suppose that f is a K-quasisymmetric map. Suppose that there is $L_1 = L_1(K)$ with the following properties. There exist L_1-quasiisometries $\widehat{f}_{x,y}$, $\widehat{f}_{y,z}$, $\widehat{f}_{z,x}$ that extend f, and such that $\widehat{f}_{x,y}(a_{x,y}) = f_* a_{x,y}$, $\widehat{f}_{y,z}(a_{y,z}) = f_* a_{y,z}$, $\widehat{f}_{z,x}(a_{z,x}) = f_* a_{z,x}$.

Lemma 2.1. *With the assumptions as above the following holds. There exist $L'_1 = L'_1(K)$ and a L'_1-quasiisometry $\widehat{f}$ that extends f, such that the restriction of $\widehat{f}$ on $a_{x,y}$ $a_{y,z}$ and $a_{z,x}$ agrees with $\widehat{f}_{x,y}$, $\widehat{f}_{y,z}$ and $\widehat{f}_{z,x}$ respectively. Moreover, suppose that the restriction of $\widehat{f}$ is $\delta(\epsilon)$ continuous on the sides of the triangle S. Then there exists a function $\delta_1 : \mathbf{R}^+ \to \mathbf{R}^+$ ($\delta_1(\epsilon)$ depends only on $\delta(\epsilon)$), such that $\widehat{f}$ is $\delta_1(\epsilon)$ continuous on S.*

Proof. By pre composing and post composing f by Möbius transformations we may assume that $x = i, y = -1, z = 1$, and that $f(i) = i, f(-1) = -1, f(1) = 1$. On each halfspace determined by one of the geodesics $a_{x,y}$ $a_{y,z}$ and $a_{z,x}$, that does not contain the triangle S, set $\widehat{f}$ to be equal to the corresponding quasiisometry. It remains to define $\widehat{f}$ on S (see Figure 1).

Let m_1, m_2, m_3 be the middle points (in the Euclidean sense) of the geodesic $a_{x,y}$ $a_{y,z}$ and $a_{z,x}$ respectively. Denote by S_0 the corresponding geodesic triangle with vertices m_1, m_2, m_3, and denote by s_i, s_{-1}, s_1 the sides of S_0 that face the points $i, -1, 1$ respectively. Let s'_i, s'_{-1}, s'_1 denote the geodesic arcs which connect the points that are the images of the endpoints of s_i, s_{-1}, s_1 under the corresponding maps $\widehat{f}_{x,y}$, $\widehat{f}_{y,z}$, $\widehat{f}_{z,x}$. Let S_0' be the corresponding triangle (note that s_i', s_{-1}', s_1' also face the points $i, -1, 1$ respectively). We define $\widehat{f}$ on each of the sides of S_0 so that $\widehat{f}$ maps each side of S to the corresponding side of S_0', and so that $\widehat{f}$ is the restriction of the unique Möbius transformation (of $\mathbf{C}$) that maps the middle points of s_i, s_{-1}, s_1 onto the corresponding middle points of s_i', s_{-1}', s_1' respectively. We define $\widehat{f}$ inside S_0 to be any homeomorphism that extends the values of $\widehat{f}$ on ∂S_0 (this homeomorphism maps S_0 onto S_0'). We can arrange that this homeomorphism is $\delta_1(\epsilon)$ continuous, for some function δ_1 that is a function of δ (after supplying the regions S_0 and S_0' with their own Poincaré metrics, one can take this homeomorphism to be either the Euclidean harmonic extension, or the barycentric extension, or any other classical extension that is a homeomorphism). Note that the hyperbolic diameters of both S_0, S_0' are bounded above by a constant that is a function of K.

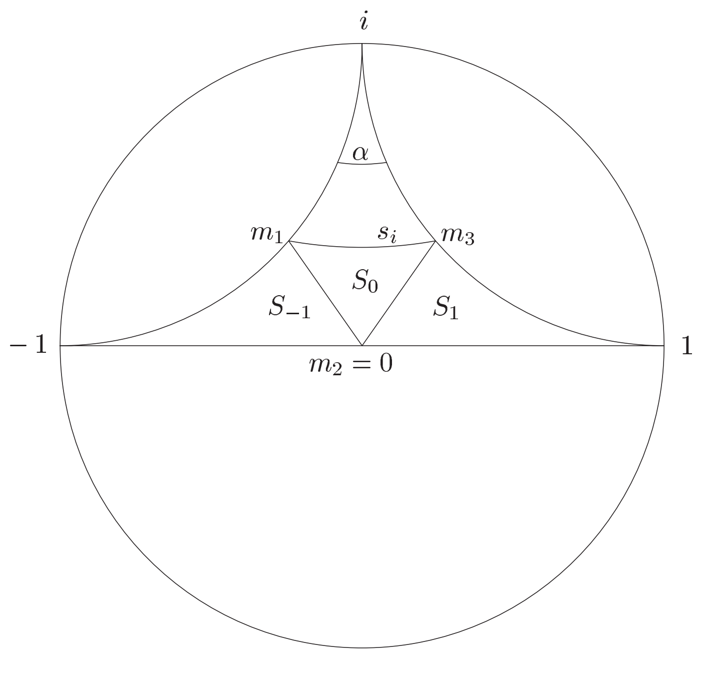

FIGURE 1

Denote by S_i, S_{-1}, S_1 the corresponding subtriangles of S such that S is a disjoint union of S_0, S_i,S_{-1} and S_1. For a point $w \in S_i$ let α be the geodesic arc that contains w, α is parallel to s_i, and α connects the two sides (other than s_i) of the triangle S_i. Two geodesic arcs are being parallel if they are subarcs of two concentric circles in $\mathbf{C}$. By α' we denote the geodesic arc that connects the points that are the images of the endpoints of α under the two of the corresponding maps $\widehat{f}_{x,y}$, $\widehat{f}_{y,z}$, $\widehat{f}_{z,x}$. We define $\widehat{f}$ on α so that $\widehat{f}$ maps α to α', and so that $\widehat{f}$ is the restriction of the unique Möbius transformation (of $\mathbf{C}$) that maps the (Euclidean) middle point of α onto the corresponding middle point of α'. We repeat the same process for the remaining points in S. It follows from this construction that $\widehat{f}$ is a L_1'-quasiisometry, for some $L_1' = L_1'(K)$. Also, it follows from the geometric nature of our extension that for a fixed function $\delta(\epsilon)$ there exists a function $\delta_1 : \mathbf{R}^+ \to \mathbf{R}^+$, such that $\widehat{f}$ is δ_1 continuous.

□

Let Λ be a countable collection of geodesics in $\mathbf{D}$ that is locally finite and so that no two geodesic intersect transversally in $\mathbf{D}$ (they may have a common endpoint in $\mathbf{T}$). Clearly Λ is a closed subset of $\mathbf{D}$. Such a collection Λ is also called a discrete geodesic lamination. Suppose that f is K-quasisymmetric and that $f_* l \in \Lambda$, $f_*^{-1} l \in \Lambda$, for each $l \in \Lambda$. In addition, suppose that for each $l \in \Lambda$, there exists a L_2-quasiisometry $\widehat{f_l}$ that extends f and such that $\widehat{f_l}(l) = f_* l \in \Lambda$. Here $L_2 = L_2(K)$.

Lemma 2.2. *With the assumptions as above the following holds. There exist $L_3 = L_3(K)$ and a L_3-quasiisometry $\widehat{f}$ that extends f, such that the restriction of $\widehat{f}$ on each $l \in \Lambda$ agrees with the restriction of $\widehat{f_l}$. Let β_i, $i \in [1, N]$, $N = N(K) \in \mathbf{N}$, be mutually disjoint geodesic arcs that do not intersect Λ transversally, but whose endpoints belong to certain geodesics from Λ. Then we can choose the above map $\widehat{f}$ so that $\widehat{f}(\beta_i)$ is a geodesic arc.*

In addition, assume that the above $\widehat{f}$ is $\delta(\epsilon)$ continuous on Λ (here $\widehat{f}$ agrees with $\widehat{f_l}$). Then $\widehat{f}$ is $\delta_1(\epsilon)$ continuous, where the function δ_1 is itself a function of $\delta(\epsilon)$ and K.

Proof. Each geodesic lamination can be completed to a maximal geodesic lamination. Let Λ' be a maximal geodesic lamination that contains Λ. Then $\mathbf{D}$ is a disjoint union of Λ' and a collection of open geodesic triangles whose sides are in Λ' (if Λ' foliates the whole unit disc than these triangles do not exist). Let s_n, $n \in \mathbf{N}$ be the sequence of geodesics from $\Lambda' - \Lambda$ so that the closure of the union of Λ and $\cup_{n \in \mathbf{N}} s_n$ is equal to Λ'.

We first define $\widehat{f}$ on s_1. For $z \in s_1$ let p be the geodesic that is orthogonal to s_1. Let α be the maximal geodesic subarc of p that contains z and that does not intersect Λ. There are three possibilities. The endpoints of α either belong to two different geodesics from Λ. One endpoint belongs to a geodesic from Λ and the other is in $\mathbf{T}$. Both endpoints belong to $\mathbf{T}$. In either case the value of $\widehat{f}$ is determined at these two endpoints by the maps $\widehat{f_l}$, $l \in \Lambda$. Denote by α' the corresponding geodesic arc whose endpoints are determined by the images of the endpoints of α. Then α' does not transversally intersect Λ. Define $\widehat{f}(z)$ to be the intersection between α' and $f_* s_1$. This is well defined because f is a homeomorphism and $f_* \Lambda = \Lambda$. Now, repeat this process for s_2, but instead of Λ we use $\Lambda \cup s_1$. And so on. By this we define $\widehat{f}$ on every s_n, and it follows from the construction that $\widehat{f}$ is well defined on Λ'. On the remaining geodesic triangles we define $\widehat{f}$ by using Lemma 2.1. It follows from Proposition 2.1 that $\widehat{f}$ is a L_2'-quasiisometry, $L_2' = L_2'(K)$. Because of the geometric construction we made it follows that $\widehat{f}$ is $\delta_1(\epsilon)$ continuous on $\mathbf{D}$. One can compute directly $\delta_1(\epsilon)$ in terms of $\delta(\epsilon)$ and the Hölder continuity constants of the map f (which are explicit functions of K by the theorem of Mori).

Let β_i, $i \in [1,N]$, be a geodesic arc from the statement of this lemma. Let β'_i be the geodesic arc with the same endpoints as the curve $\widehat{f}(\beta)$. We have that β'_i and $\widehat{f}(\beta)$ are finite hyperbolic distance apart (this upper bound depends only on K) since $\widehat{f}$ is a quasiisometry. Because of that one can find a L''_2-quasiisometry $I : \mathbf{D} \to \mathbf{D}$, $(L''_2 = L''_2(K))$, which extends the identity map of $\mathbf{T}$, so that I pointwise fixes each $l \in \Lambda$ and $I(\widehat{f}(\beta)) = \beta'$. The map $I \circ \widetilde{f}$ is a L_3-quasiisometry, $L_3 = L_3(K)$. Since $\widehat{f}$ is $\delta_1(\epsilon)$ continuous we see that if β is a very short geodesic arc then $\widehat{f}(\beta)$ is very close to the corresponding arc β'. This implies the existence of $\delta_1(\epsilon)$, such that $\widehat{f}$ is $\delta_1(\epsilon)$ continuous. δ_1 depends only on δ.

Repeat this process $N = N(K)$ times for all the arcs β_i to obtain the resulting quasiisometry.

□

2.2. **Quasiisometric groups.** Let $\widehat{\mathcal{G}}$ be a L-quasiisometric group on $\mathbf{D}$. This means that $\widehat{\mathcal{G}}$ is a group whose elements are homeomorphisms of $\overline{\mathbf{D}}$, each of them being L-quasiisometric. The restrictions of elements from $\widehat{\mathcal{G}}$ on the unit circle $\mathbf{T}$ make a group which we denote by $\mathcal{G}$. Clearly, $\mathcal{G}$ is a K-quasisymmetric group, $K = K(L)$. Our aim is to show that there is $K' = K'(L)$ such that $\mathcal{G}$ is K'-quasisymmetrically conjugated to a Fuchsian group. We will show this under extra assumptions. Suppose that there exist $\rho > 0$ and a function $\delta : \mathbf{R}^+ \to \mathbf{R}^+$ with the following properties. For every $\widehat{f} \in \widehat{\mathcal{G}} \setminus id$ we have that $\mathbf{d}(z, \widehat{f}(z)) > \rho$, for every $z \in \mathbf{D}$. Also, each $\widehat{f}$ is $\delta(\epsilon)$ continuous. These assumptions on the group $\widehat{\mathcal{G}}$ are valid throughout this subsection. Our aim is to show that there is a quasiconformal group $\widetilde{\mathcal{G}}$ that extends $\mathcal{G}$. Note that these assumptions imply that $\widehat{\mathcal{G}}$ (and therefore $\mathcal{G}$ as well) does not have any elliptic elements.

Let $\rho_0 > 0$. We define a discrete set $E = E(\rho_0) \subset \mathbf{D}$ as follows. For any point $z \in \mathbf{D}$ denote by $O(z)$ its orbit under $\widehat{\mathcal{G}}$. Let Σ denote the Fuchsian group that is the covering group of a closed Riemann surface of genus at least two. Let w_n, $n \in \mathbf{N}$, denote the orbit of the point $w_1 = 0$ under Σ. Set $z_1 = w_1 = 0$, and let $E_1 = O(z_1)$. Choose the smallest $n_0 > 1$ so that $\mathbf{d}(O(z_1), O(w_{n_0})) > \rho_0$. If such n_0 does not exists then set $E = E_1$. Otherwise, let $z_2 = w_{n_0}$ and set $E_2 = E_1 \cup O(z_2)$. Similarly, let n_1 be the smallest so that $\mathbf{d}(O(w_{n_1}), E_2) > \rho_0$. If such n_1 does not exists then set $E = E_2$. Otherwise, let $z_3 = w_{n_1}$ and $E_3 = E_2 \cup O(z_3)$. And so on. Let $E = \cup E_i$. The main properties of the set E are as follows.

1. E is invariant under $\widehat{\mathcal{G}}$.
2. There exists $\rho_1 = \rho_1(\rho_0, \rho, \delta(\epsilon))$ such that in every geodesic ball of radius ρ_1 there is it least one point from E.
3. Every geodesic ball of radius $\frac{\rho_0}{2}$ contains at most one point from E.

Remark. The one and only reason we choose Σ to be the covering group of a closed surface is to achieve that the sequence w_n is well distributed in $\mathbf{D}$, which then yields the condition (2) above.

If we let $\rho_0 \to 0$, then we can choose ρ_1 so that $\rho_1 \to 0$ (this comes from the fact that $\widehat{f}$ is $\delta(\epsilon)$ continuous). Because of that we can choose $\rho_0 = \rho_0(\rho, \delta(\epsilon))$ small enough so that the corresponding $\rho_1 < \frac{\rho}{100}$. We choose such ρ_0 and ρ_1 (see the remark below).

Set $S = \mathbf{D} - E$. Denote by $\mathbf{d}_S$ the hyperbolic metric on S. It follows from the assumptions that the metrics $\mathbf{d}$ and $\mathbf{d}_S$ are comparable, except on a micro scale very near the points from E. This means that if we take the corresponding densities $dens(\mathbf{d})$ and $dens(\mathbf{d}_S)$ that define the metrics $\mathbf{d}$ and $\mathbf{d}_S$ respectively, then there exists a constant $X > 0$ so that for $z \in S = \mathbf{D} - E$ we have

$$\frac{1}{X} < \frac{dens(\mathbf{d})(z)}{dens(\mathbf{d}_S)(z)} < X,$$

and the constant X depends only on the $\mathbf{d}(z, S)$. For $z, w \in E$ and a simple curve (simple arc) $\gamma \subset S$ with the endpoints z, w, let $\alpha(z, w, \gamma)$ denote the geodesic (with respect to $\mathbf{d}_S$) homotopic to γ (homotopic in S). We say that γ is $\delta_1(\epsilon)$ continuous if there is a $\delta_1(\epsilon)$ continuous homeomorphism between γ and $\alpha(z, w, \gamma)$ (here the uniform continuity is with respect to the metric $\mathbf{d}$). We assume that the inverse of this homeomorphism is also $\delta_1(\epsilon)$ continuous. Here $\delta_1 : (0, \infty) \to (0, \infty)$.

Let $z, w \in E$ and let $\gamma \subset S$ be a curve that connects them. We say that the homotopy class $[\gamma]$ (of γ) is admissible, if for every $\widehat{f} \in \widehat{\mathcal{G}}$ we have that the length of the curve $\alpha(\widehat{f}(z), \widehat{f}(w), \widehat{f}_* \gamma)$ is at most $\rho_2 = \frac{\rho}{10}$. Here we measure the length of $\alpha(\widehat{f}(z), \widehat{f}(w), \widehat{f}_* \gamma)$ in the hyperbolic metric $\mathbf{d}$ of the unit disc (the length of $\alpha(\widehat{f}(z), \widehat{f}(w), \widehat{f}_* \gamma)$ in the metric $\mathbf{d}_S$ is infinite).

Numerate points in E by z_n, $n \in \mathbf{N}$. Fix $i, j \in \mathbf{N}$. Suppose that $[\gamma]$ is an admissible homotopy class (with the corresponding endpoints z_i, z_j) and let $\alpha(z_i, z_j, \gamma)$ be the corresponding geodesic. Then $\alpha(z_i, z_j, \gamma)$ has the length at most ρ_2. Let $F_{i,j,\gamma}$ be the orbit of $\alpha(z, w, \gamma)$ under $\widehat{\mathcal{G}}$. By $F_{i,j}$ we denote the union of all $F_{i,j,\gamma}$, where $[\gamma]$ ranges over all corresponding admissible homotopy class (we note that there are only finitely many such homotopy classes for a fixed pair i, j). The union of all the sets $F_{i,j}$ is denoted by F.

Remark. By taking ρ_0 and ρ_1 a little bit smaller (but keeping ρ_2 as it is), we can arrange that every connected component of the set $\mathbf{D} - F$ is a polygon whose diameter is bounded above by a constant that is a function of $\delta(\epsilon)$ and ρ. This is the final choice of ρ_0 and ρ_1.

By definition each curve in F is $\delta(\epsilon)$ continuous ($\delta(\epsilon)$ is the function that was fixed at the beginning of the subsection). Each curve from F is homotopic to a geodesic (in metric $\mathbf{d}_S$), whose $\mathbf{d}$ length is less than ρ_2. From $\delta(\epsilon)$ continuity of curves from F, it follows that there can not be very many curves from F ending a given point of E. This implies that there is $N = N(\rho, \delta(\epsilon)) \in \mathbf{N}$, such that for each point $z \in E$ there are at most N curves from F that end at z. Because of that, the fact the each curve in F is uniformly continuous, and since each $\widehat{f} \in \widehat{\mathcal{G}}$ is uniformly continuous, we conclude that there exists $N_1 = N_1(\rho, \delta(\epsilon)) \in \mathbf{N}$, such that the in each hyperbolic disc of radius ρ_0 there are at most N_1 points that are intersection points between different curves from F. Now, we slightly deform the curves (within their homotopy classes in S) from F so that the hyperbolic distance (distance $\mathbf{d}$) between any two intersection points is greater than $\rho' = \rho'(\rho, \delta(\epsilon))$, and such that every intersection point is contained in exactly two curves from F. In addition, we can arrange that the new set of curves (which we also denote by F) is also invariant under $\widehat{\mathcal{G}}$. This is because ρ_2 was chosen to be small enough so that the distance between a curve from F, and any other curve in its orbit is bounded below by a positive constant that is a function of our original parameters ρ and $\delta(\epsilon)$.

Denote by E_1 the union of E and the set of all intersection points (after the deformation). It follows from the construction that each connected component of $\mathbf{D} - F$ is a polygon with at most $N_2 = N_2(\rho, \delta(\epsilon))$ sides. We can now add new curves (that connect points from E_1) to the set F (the new set is also called F), so that every component of $\mathbf{D} - F$ is a triangle. We can do so by retaining the uniform continuity properties of the curves from the original F.

Let $S_1 = \mathbf{D} - E_1$. Replace each curve from F by the corresponding geodesic (geodesic with respect to the hyperbolic metric on S_1). The new set of curves is denoted by F_1. Clearly, F_1 retains the same essential properties of F, except that F_1 is no longer invariant under $\widehat{\mathcal{G}}$. However, for each $\gamma \in F_1$ there is a unique curve in F_1 that is homotopic to $\widehat{f}_*\gamma$.

Remark. Note that if T is a triangle from this partition then $\widehat{f}_*T$ is well defined. If $\widehat{f}_*T = T$ then f is identity. This follows from the fact that $\widehat{f}$ has no fixed points inside the unit disc. We will not make any use of this fact.

Because the distance between any two points in E_1 is bounded below by ρ', there are constants $\rho_3, \rho_4 > 0$ that are functions of ρ and $\delta(\epsilon)$, such that the hyperbolic length $l_{\mathbf{d}}(\gamma)$ of $\gamma \in F_1$ satisfies $\rho_4 < l_{\mathbf{d}}(\gamma) < \rho_3$. Here l_d denotes the length with respect to $\mathbf{d}$.

We now define the new group $\widetilde{\mathcal{G}}$ of homeomorphisms of $\mathbf{D}$. For $f \in \mathcal{G}$ we denote by $\widetilde{f}$ the homeomorphism from $\widetilde{\mathcal{G}}$ that extends f. We define $\widetilde{f}$ to be equal to $\widehat{f}$ on the set E_1. On each curve γ from F_1 we define $\widetilde{f}$ to be the affine map with respect to the natural parameters on each of the two curves γ and $\widehat{f}_*\gamma = \widetilde{f}(\gamma)$. The natural parameter is taken with respect to the metric $\mathbf{d}$ (both curves have finite $\mathbf{d}$ length). It also follows that $\widetilde{f}$ respects the composition (composition of two affine maps is an affine map). It remains to define $\widetilde{f}$ on the interior of the corresponding triangles. We denote the set of all triangles by $\mathcal{T}$. Let T_i, $i \in \mathbf{I}$, be a list of all mutually non conjugated triangles from $\mathcal{T}$. Here $\mathbf{I}$ is either a finite set or $\mathbf{I} = \mathbf{N}$. The equivalence class of T_i is denoted by $[T_i]$. For each $f \in \mathcal{G}$ let $\overline{f} : T \to \widehat{f}_*T \in \mathcal{T}$ be a homeomorphism that extends the already defined values of $\widetilde{f}$ on ∂T. Let ρ_5 be minimal distance between points from E_1 divided by 10. Then, it does follow from our construction that this homeomorphism $\overline{f}$ can be chosen to be K_1-quasiconformal on $\mathbf{D}$, except at the points that are ρ_5 close (in the hyperbolic metric $\mathbf{d}$) to the points from E_1. Moreover, $K_1 = K_1(\rho, \delta(\epsilon), \rho_5)$. This is readily seen by passing onto the universal cover of S_1. Then the triangles T and $\widehat{f}_*T$ can be seen as geodesic (ideal) triangles. We can take $\overline{f}$ to be the barycentric extension (Douady-Earle extension and barycentric extension refer to the same thing) of the map already assigned on the boundary of these triangles. The only place where this map may fail to be uniformly quasiconformal is near the cusps (that correspond to points from E_1).

Fix a single representative, say T_i, from each class. Let $T'_i \in [T_i]$ be any triangle and let $\widehat{f}_*T'_i = T''_i$. Let $g', g'' \in \mathcal{G}$ be such that $\widehat{g}'_*T_i = T'_i$ and $\widehat{g}''_*T_i = T''_i$ (such

g', g'' are unique, since elements from $\widehat{\mathcal{G}}$ have no fixed points in $\mathbf{D}$). Set

$$\widetilde{f}(z) = \left(\overline{g}'' \circ \overline{g}'^{-1}\right)(z),$$

for $z \in T'_i$. $\overline{g}'$ and $\overline{g}''$ are the above defined maps on T_i that correspond to g' and g'' respectively. By repeating this process for every $i \in \mathbf{I}$, we have defined the extension $\widetilde{f}$ for every $f \in \mathcal{G}$. Denote the corresponding group by $\widetilde{\mathcal{G}}$. By the same arguments as above, for every $z \in E_1$ we can choose a small disc D_z (not necessarily geodesic), of the hyperbolic diameter $\sim \rho_5$, such that the set $S_2 = \mathbf{D} - \cup_{z \in E_1} D_z$ is a connected Riemann surface homotopic to S_1. Moreover, we can arrange that S_2 is invariant under the action of $\widetilde{\mathcal{G}}$. We now fix such $\rho_5 = \rho_5(\rho, \delta(\epsilon))$. The fact that $\widetilde{\mathcal{G}}$ is a group follows readily from the construction. Clearly, $\widetilde{\mathcal{G}}$ is a K_1-quasiconformal group on S_2, $K_1 = K_1(\rho, \delta(\epsilon))$. Therefore, there exist a Riemann surface S'_2, a K_2-quasiconformal map $\widetilde{\psi} : S_2 \to S'_2$, $K_2 = K_2(K_1) = K_2(\rho, \delta(\epsilon))$, and a discrete conformal group $\widetilde{\mathcal{F}}$ that acts on S'_2, such that $\widetilde{\mathcal{G}} = \widetilde{\psi}\widetilde{\mathcal{F}}\widetilde{\psi}^{-1}$. Note that the restriction of the map $\widetilde{\psi}$ to $\mathbf{T}$ is a quasisymmetric map (see the remark below).

Since one can realize S'_2 as the unit disc minus many (but countably many) small topological discs that lie inside $\mathbf{D}$, we can apply the results from [12] and conclude that there exists a conformal map $\pi : S'_2 \to S''_2$, where S''_2 is obtained as the unit disc minus countably many geometric discs (these discs are now hyperbolic disc in the unit disc). The map π extends continuously to the quasisymmetric map of the unit circle (see the remark below). Using [12] again, we know that every conformal map of S''_2 onto itself must be a restriction of a Möbius transformation. This shows that $\pi\widetilde{\mathcal{F}}\pi^{-1}$ acts as a Fuchsian group on the unit circle. This implies that the restriction of $\widetilde{\mathcal{F}}$ on the unit circle is a quasisymmetric conjugate of a Fuchsian group.

Remark. Here we use the following result of Kozlovski-Shen-Strien about quasiconformal mappings (see appendix in [18] for the formulation and proof). This theorem was inspired by work of Heinonen and Koskela [13] (also see [22]). Let F be a homeomorphism of $\mathbf{D}$, that is $\widehat{K}$-quasiconformal on the set M. The set M is of the form $M = \mathbf{D} - \cup D_i$, where D_i is a collection of topological discs in $\mathbf{D}$ with the following properties. The hyperbolic distance between D_i and D_j is bounded below by $C > 0$, for every $i \neq j$. The diameter of each D_i is bounded above by $C_1 > 0$. Then the restriction of F on $\mathbf{T}$ (which has also been proved to exist) is $K(C, C_1, \widehat{K})$-quasisymmetric.

Lemma 2.3. *With the assumptions and notation as above, we have that the group $\mathcal{G}$ is K_2-quasisymmetrically conjugated to a Fuchsian group. Here $K_2 = K_2(\rho, \delta(\epsilon))$.*

3. Quasiconformal continuation and barycentric extensions

3.1. **Quasiconformal continuation.** Suppose that $f : \mathbf{T} \to \mathbf{T}$ is a K-quasisymmetric map. Let $\mathcal{S} = \{S_i\}$, $i \in \mathbf{N}$, be a collection of subsets of $\overline{\mathbf{D}}$ with the following properties. Each S_i is either a Jordan region or a Jordan curve. Each S_i is a closed connected subset of $\overline{\mathbf{D}}$, such that there are no bounded (in $\mathbf{D}$) components of $\overline{\mathbf{D}} - S_i$. This implies that the interior (if any) of S_i is simply connected. Also, each S_i touches the unit circle at, at most two points. In addition, there exists $R > 0$ such that

$$\mathbf{d}(S_i, S_j) > R, i \neq j. \tag{3.1}$$

Let $K' = K'(K)$. Suppose that for each $i \in \mathbf{N}$, there is a K'-quasiconformal map $\widehat{f}_i$ that extends f, and such that $\widehat{f}_i(S_i) \in \mathcal{S}$.

Lemma 3.1. *With the assumption as above the following holds. There exists (large enough) $R = R(K)$, so that if* (3.1) *holds for R then there exist $K_1 = K_1(K)$ and a K_1-quasiconformal map $\widehat{f}$ that extends f, and such that $\widehat{f} = \widehat{f}_i$ on each S_i.*

Remark. By a more involved argument, one can prove that in the above lemma we can always take $R = 1$. Also, the assumption that each S_i touches the unit circle at, at most two points, is not essential. However, this will be the case in all applications of this lemma.

Proof. Let $\widetilde{f}$ be any K-quasiconformal extension of f. Let $\widehat{f}_i(S_i) = S_j$ for some $j \in \mathbf{N}$. The distance $\mathbf{d}(\widetilde{f}(z), \widehat{f}_i(z))$, $z \in S_i$, is bounded by a constant depending only on K. Choose $R = R(K)$ large enough so that the following holds. For each $i \in \mathbf{N}$, there exists an open connected set $U_i \subset \mathbf{D}$, that contains both S_j and $\widetilde{f}(S_i)$, and such that $U_i \cap U_j$ is an empty set, $i \neq j$. Moreover, every point in ∂U_i is at the hyperbolic distance at least 1 from any point from $S_j \cup \widetilde{f}_i(S_i)$. In addition, we can choose the set U_i so that U_i touches the unit circle at the same points as S_i does, and that the boundary of U_i is a Jordan curve. Note that the above assumptions imply that U_i is simply connected.

Now, one can construct a K''-quasiconformal map $I_i : \mathbf{D} \to \mathbf{D}$, $K'' = K''(K)$, so that $I_i \circ \widetilde{f} = \widehat{f}_i$ on S_i, and I_i is the identity on $\mathbf{D} - U_i$. To do this, let Ω_i be one of at most two components of the open set $U_i - \widetilde{f}(S_i)$. In either case, Ω_i has two boundary components (these two components meet at $\mathbf{T}$). One of them is a piece of the boundary of $\widetilde{f}(S_i)$, and another one is a piece of the boundary of U_i. Let Ω'_i be the corresponding component of the set $U_i - S_j$, so that $\partial\Omega'_i$ contains the same piece of the boundary of U_i as does $\partial\Omega_i$. Let $g : \partial\Omega_i \to \partial\Omega'_i$ be the homeomorphism that is the identity on the part of the boundary coming from U_i and $g = \widehat{f}_i \circ \widetilde{f}^{-1}$ on the other part. One can use the standard rescaling-compactness argument to show that there exists $K'_1 = K'_1(K)$, such that the map g is K'_1-quasisymmetric on $\partial\Omega_i$. Repeat the same process for the other component of $U_i - S_i$ (if it exists). Set I_i to be equal to g on $U_i - \widetilde{f}(S_i)$, and set $I_i = \widehat{f}_i \circ \widetilde{f}^{-1}$ on $\widetilde{f}(S_i)$. This shows the existence of the map I_i. Since the boundary of S_i is a Jordan curve (or S_i is a Jordan curve itself), we conclude that I_i is quasiconformal.

Set $\widehat{f} = \lim\limits_{i \to \infty} I_i \circ ... \circ I_1 \circ \widetilde{f}$. We have that for some constant $K_1 = K_1(K)$, $\widehat{f}$ is a K_1-quasiconformal map. $\square$

Definition 3.1. *Let $E \subset \mathbf{D}$ be a discrete set, and let $\rho > 0$. We say that E is ρ-discrete set if the hyperbolic distance between any two points in E is at least ρ.*

Lemma 3.2. *Let E be a ρ-discrete set. Let $C : E \to \mathbf{D}$, where $E' = C(E)$ is a ρ-discrete set, and $\mathbf{d}(C(z), z) < D$, for some $D > 0$, and every $z \in E$. Then, there exist $K_2 = K_2(\rho, D)$, and a K_2-quasiconformal map I that extends the identity, such that $I(z) = C(z)$, $z \in E$.*

Proof. Suppose first that the constant $D = D_0$ satisfies $0 < D_0 < \frac{\rho}{3}$. Let α_z be the geodesic arc that connects z and $C(z)$. Since $\mathbf{d}(C(z), z) < D_0$, no two such arcs would intersect. The positive orientation on α_z is from z toward $C(z)$. Let X_z be the vector field on α_z so that for $w \in \alpha_z$ we have that $X_z(w)$ is the positive tangent vector to α_z at w. We take $X_z(w)$ to have the length

$$\frac{l_{\mathbf{d}}(\alpha_z)}{D_0},$$

where $l_{\mathbf{d}}(\alpha_z)$ is the length with respect to the hyperbolic metric $\mathbf{d}$. Choose $D_0 = D_0(\rho)$ to be small enough such that we can define a vector field X on $\mathbf{D}$, which agrees with each X_z and satisfies the following. We can choose X so that there exists the one parameter family of diffeomorphism $I_t : \mathbf{D} \to \mathbf{D}$, $t \in [0, D_0]$, such that

$$\frac{\partial I_t}{\partial t} = X.$$

For this it is enough to arrange that X is a smooth vector field, and uniformly Lipschitz in $\mathbf{D}$ (which we can do since E is ρ-discrete and for D_0 small enough). Moreover, since E is ρ-discrete we can choose X uniformly (uniform on $\mathbf{D}$) smooth, so that the map I_{D_0} is K_2'-quasiconformal, for some $K_2' = K_2'(\rho)$. This conclusion comes from the standard estimates on the solutions of ODE's. Note that we can choose such X for some $D_0 > 0$, where D_0 does not depend on the particular set E. We have $D_0 = D_0(\rho)$. It follows from the construction that the time D_0 map I_{D_0} satisfies the properties stated in the lemma. Note that the family I_t is continuous in the variable t as well.

The general case goes as follows. We can join z and $C(z)$ by a C^∞ curve γ_z, whose length is equal to some fixed $L(\rho) > 0$ for each $z \in E$. We can also arrange the following. Let z_t be the point on γ_z, such that the length of the piece of γ_z between z and z_t is equal to $t \in [0, L(\rho)]$. Let E_t be the set of all z_t. We can choose the curves γ_z so that each set E_t is ρ_1-discrete, $\rho_1 = \rho_1(\rho)$. Now, chop up the interval $[0, L(\rho)]$ into N small intervals, $N = N(\rho, D) \in \mathbf{N}$, such that on each of them we can apply the above construction. This proves the lemma. ☐

Let E be a ρ-discrete set, and let $\widetilde{f}_t : \mathbf{D} \to \mathbf{D}$, $t \in [0, k_0]$, be a continuous family (in t) of homeomorphisms (of $\mathbf{D}$) with the following properties. For every t, $\widetilde{f}_t$ is the identity on $\mathbf{T}$. We have $\widetilde{f}_0 = id$. For every $\epsilon > 0$ there exists $\delta(\epsilon) = \delta > 0$, such that for every $t, s \in [0, k_0]$ and $z \in E$, we have that $\mathbf{d}(\widetilde{f}_t(z), \widetilde{f}_s(z)) < \epsilon$, whenever $|t - s| < \delta$. In addition, we assume that the set $E_t = \widetilde{f}_t(E)$ is ρ-discrete for every t.

Set $S_t = \mathbf{D} - E_t$.

Lemma 3.3. *With the above assumptions we have that there exists a K_3-quasiconformal map $\widehat{f}_t : S_0 \to S_t$, where $\widehat{f}_t$ is isotopic to $\widetilde{f}_t$ (isotopic in S_0) for every $t \in [0, k_0]$. The constant K_3 is a function of ρ, the function $\delta(\epsilon)$, and k_0.*

Proof. First, we chop up the interval $[0, k_0]$ into small enough intervals so that on each of these small intervals we use the construction from the first part of the proof of the previous lemma. Precisely, let $n \in \mathbf{N}$, $n = n(\rho, \delta(\epsilon), k_0)$, be such that for each interval $[s_i, s_{i+1}]$ we can construct a K_2-quasiconformal map $\widehat{f}_i$, $1 \leq i \leq n$, that maps E_{s_i} to $E_{s_{i+1}}$ (the map $\widehat{f}_i$ is from the first part of the proof of the previous lemma). Here $s_i = \frac{ik_0}{n}$. We can do that since $\widetilde{f}_t$ is $\delta(\epsilon)$ continuous. Also, the map $\widehat{f}_i$ is very close to the identity in C^0 sense uniformly in the metric $\mathbf{d}$ on $\mathbf{D}$.

In fact, we can choose n large enough so that the maps $\widehat{f}_i$ and $\widetilde{f}_{s_{i+1}} \circ \widetilde{f}_{s_i}^{-1}$ are isotopic. This can be proved by contradiction as follows. Suppose that for every $n \in \mathbf{N}$ we can produce an example where the above conclusion fails. Then because all the maps involved are uniformly continuous and the set E is ρ-discrete for some fixed $\rho > 0$, we can pass onto the limit. The limit of the maps $\widetilde{f}_{s_{i+1}} \circ \widetilde{f}_{s_i}^{-1} \circ \widehat{f}_i^{-1}$ is the identity map. This is a contradiction since we assumed that none of the maps in the sequence are homotopic to the identity.

By doing this, we produce a continuous family of K_3-quasiconformal mappings $\widehat{f}_t$, $t \in [0, k_0]$, such that for every $z \in E$, we have $\widehat{f}_t(z) = \widetilde{f}_t(z)$. Moreover, $\widehat{f}_t(z) \circ (\widetilde{f}_t)^{-1}$ is isotopic to the identity on S. $\square$

Lemma 3.4. *Let $E \subset \mathbf{D}$ be a ρ-discrete set and suppose that $0 \in E$. Set $S = \mathbf{D} - E$. For $x \in \mathbf{T}$ let s_x be the geodesic in $\mathbf{D}$ with the endpoints x and $-x$. Let $\widehat{f} : \mathbf{D} \to \mathbf{D}$ be a homeomorphism, that extends the identity map, and that fixes every point from E. Suppose that there is $L > 0$, such that the restriction of $\widehat{f}$ on s_x is homotopic (in S) to a L-bilipschitz quasigeodesic $\gamma_x : s_x \to \mathbf{D}$. Then there exists $K_4 = K_4(\rho, L)$, such that $\widehat{f}$ is isotopic (in S) to a K_4-quasiconformal map.*

Proof. Denote by $\mathbf{d}_S$ the hyperbolic distance on S. Since E is ρ-discrete we have that $\mathbf{d}_S$ is comparable with $\mathbf{d}$, except on a micro scale near the points from E. Let α_x be the geodesic in the metric $\mathbf{d}_S$ that is homotopic to s_x in S. α_x is really a union of geodesics in S that connect the corresponding points from E. By β_x we denote the geodesic (with respect to $\mathbf{d}_S$) that is homotopic to $\widehat{f}(s_x)$. From the assumption of the lemma, and the comparability between $\mathbf{d}$ and $\mathbf{d}_S$, we conclude that both curves α_x and β_x can be parametrized as L_1 bilipschitz quasigeodesics in $\mathbf{D}$, $L_1 = L_1(\rho, L)$. This implies that for every point $z \in \alpha_z$, there is a point $w \in \beta_x$, such that $\mathbf{d}_S(z, w) < M$, $M = M(\rho, L)$. Suppose that $w \in \beta_x$ is such that $\mathbf{d}_S(z, w) = \mathbf{d}_S(z, \beta_x)$. Define $\widetilde{f} : S \to S$ by setting $\widetilde{f}(z) = w$. The restriction of $\widetilde{f}$ on each α_x is the so called nearest point retraction map.

Fixing a (arbitrarily small) neighborhood of the set E, it is easy to show that there exists $L_1 = L_1(\rho, L)$, such that $\widetilde{f}$ is L_1 bilipschitz map (with respect to the metric $\mathbf{d}_S$) outside that neighborhood. We will not compute this but suggest how to do it. An easy way to see this is by passing onto the universal cover of S. There (meaning in the universal cover=unit disc), for each z and the corresponding $w = \widetilde{f}(z)$, α_x and β_x can be realized as two non intersecting geodesics and the distance between the corresponding lifts of z and w is less than M. Since those two geodesics do not intersect the bilipschitz constant of the nearest point retraction map (away from the cusps) depends only on the distance M. Since $\widetilde{f}$ is bilipschitz on the unit disc minus a fixed neighborhood of the set E, it is quasiconformal as well on the set. Now, one can change the map $\widetilde{f}$ in the neighborhood of the set E, so that the new map is isotopic to $\widetilde{f}$ and quasiconformal as well on $\mathbf{D}$. $\square$

The following lemma is elementary.

Lemma 3.5. *Let f be a K-quasisymmetric map and let l be a geodesic in $\mathbf{D}$. There exists a $\widehat{K}$-quasiconformal and L-bilipschitz map $\widehat{f}$ which extends f, $\widehat{K} = \widehat{K}(K)$, $L = L(K)$, such that $\widehat{f}(l) = f_* l$.*

Proof. Let $\widetilde{f}$ be the barycentric extension of f. Then $\widetilde{f}$ is L-bilipschitz. Let $\gamma = \widetilde{f}(l)$ and $l' = f_* l$. We have that γ is a bilipschitz quasigeodesic with the same endpoints as l'. Denote by Ω_1 and Ω_2 the two regions obtained by removing γ from $\mathbf{D}$. By H_1 and H_2 we denote the corresponding halfspaces obtained by removing l'. Let $p : \gamma \to l$ be any bilipschitz map such that $\mathbf{d}(p(z), z) < P$, $P = P(K)$. Denote by $\widehat{q_1}$ the barycentric extension of the map $q_1 : \partial\Omega_1 \to \partial H_1$ (here one has to map Ω_1 and H_1 onto the unit by the Riemann maps to define the barycentric extensions). The map q_1 is defined to be the identity on the $\mathbf{T}$ part of the boundary of Ω_1 (this part of the boundary also borders H_1). We set $\widehat{q_1} = p$ on γ. We do the analogous thing for Ω_2 to obtain the map $\widehat{q_2}$. Define $\widehat{g} : \mathbf{D} \to \mathbf{D}$ so that $\widehat{g} = \widehat{q_1}$ on Ω_1, and $\widehat{g} = \widehat{q_2}$ on Ω_2. It follows from standard estimates about the boundary behavior of conformal maps, and standard estimates on the barycentric extensions, that the map $\widehat{f} = \widehat{g} \circ \widetilde{f}$ satisfies the properties stated in the lemma. □

3.2. **The barycentric extension.** Let $\varphi : \mathbf{T} \to \mathbf{T}$ be a homeomorphism. The barycentric extension of Douady-Earle is a homeomorphism $\widehat{\varphi} : \mathbf{D} \to \mathbf{D}$, that extends φ, and which maps the point 0 to the barycenter $\widehat{\varphi}(0)$ of the corresponding probability measure $\varphi_*(\sigma_0)$. Here σ_0 is the normalized Lebesgue measure on $\mathbf{T}$. $\varphi_*(\sigma_0)$ means the push forward of σ_0 by φ. It is also customary to write $\widehat{\varphi}(0) = Bar(\varphi_* \sigma_0)$. By requiring that $\widehat{\varphi}$ is conformally natural the above uniquely determines this extension. See [4], [1], [21] for definitions and properties of the barycentric extension.

If φ is quasisymmetric then $\widehat{\varphi}$ is known to be quasiconformal. There are many conformally natural extensions that have this property. One important thing about this particular extension is that one can say what happens with $\widehat{\varphi}$ when φ degenerates. The theory of barycentric extensions can be developed not only for homeomorphisms of $\mathbf{T}$ but also for limits of homeomorphisms and for certain probability measures on $\mathbf{T}$. This case is of interest to us. In fact, if μ is a probability measure on $\mathbf{T}$, that does not contain strong atoms, then the barycenter $\widehat{\mu}(0) \in \mathbf{D}$ is well defined. An atom for a probability measure is said to be strong if it has the mass at least $\frac{1}{2}$.

However, the only results we need in this direction are known and they all follow from [1] (they were hinted in [4]). One of the results that will be used later is the following. Let μ_n be a sequence of probability measures on $\mathbf{T}$ such that none of these measures has atoms. Suppose that μ is a probability measure that has no atoms of mass $\geq \frac{1}{2}$. If $\mu_n \to \mu$ (in the sense of measures), then the barycenters $\widehat{\mu_n}(0) \to \widehat{\mu}(0) \in \mathbf{D}$. This observation was already stated in [4], but it was developed in details in sections 3,4 of [1].

Another result that we need is to show that under certain assumptions the barycentric extension $\widehat{\varphi}$ is quasiconformal in a neighborhood of 0, even though the homeomorphism (or a limit of homeomorphisms) φ is not. The following lemma follows from the so called four sines law, and both the statement and the proof of this lemma can be recovered from [1].

Lemma 3.6. *Let $\varphi : \mathbf{T} \to \mathbf{T}$ be a homeomorphism, and let $u : \mathbf{R} \to \mathbf{R}$ be an increasing function such that $\varphi(e^{it}) = e^{iu(t)}$, where $t \in \mathbf{R}$. Suppose that there are $0 < s_0 < \pi$ and $C > 0$, so that $u(t+s) - u(t) > C$, for every $0 < s_0 < s < \pi$*

and every $t \in \mathbf{R}$. *Then, there exist* $K = K(s_0, C)$ *and* $D = D(s_0, C)$*, such that the barycentric extension* $\widehat{\varphi}$ *is* K*-quasiconformal in the hyperbolic disc* $\Delta(0, D)$.

Proof. Following [21], we have

$$1 - |Belt(\widehat{\varphi})(0)|^2 > \frac{\delta^3}{16}. \tag{3.2}$$

Here $Belt(\widehat{\varphi})(0)$ is the complex dilatation of $\widehat{\varphi}$ at 0, and

$$\delta = \frac{1}{2\pi^2} \int\limits_0^\pi sin(s) \left(\int\limits_0^\pi v(t,s)dt \right) ds = \frac{1}{2\pi^2} \int\limits_{[0,\pi]^2} sin(s)v(t,s)dsdt. \tag{3.3}$$

Here

$$v(t,s) = sin(\theta_1(t,s)) + sin(\theta_2(t,s)) + sin(\theta_3(t,s)) + sin(\theta_4(t,s)),$$

for $\theta_1(t,s) = u(t+s) - u(t)$; $\theta_2(t,s) = u(t+2\pi) - u(t+s+\pi)$; $\theta_3(t,s) = u(t+s+\pi) - u(t+\pi)$; $\theta_4(t,s) = u(t+\pi) - u(t+s)$. Note that $\theta_1(t,s) + \theta_2(t,s) + \theta_3(t,s) + \theta_4(t,s) = 2\pi$, and $\theta_i(t,s) \geq 0$, $i = 1,2,3,4$.

It follows (see Lemma 4.4 in [1]) from basic trigonometry that

$$v(t,s) = 4sin\frac{(\theta_1+\theta_2)}{2} sin\frac{(\theta_1+\theta_4)}{2} sin\frac{(\theta_2+\theta_4)}{2} \geq 0. \tag{3.4}$$

The above formula holds for any choice of three out of the four θ_i's. Let $s_1 = \pi - s_0$. For $0 < s < s_1$, we have that both $\theta_2(t,s)$ and $\theta_4(t,s)$ are greater or equal to C. Let $P_0 = \{s, t : 0 < s < s_1, 0 < t < \pi\}$. Let $s_2 = \frac{s_1}{4}$. Since every interval of the length at least $\pi - s_1 = s_0$ is mapped by u onto a set that contains some interval of length C, we conclude that for some $t_0 \in (\frac{s_1}{3}, \pi - \frac{s_1}{3})$, we have

$$\theta_1(t_0, s_2) > \frac{C}{N},$$

where $N = N(s_0)$ is the minimal positive integer such that $s_2 > \frac{\pi}{N}$. Set $s_3 = \frac{s_2}{4}$. Then for every $(t,s) \in P_1 = \{(t,s) : 0 < t_0 - s_3 < t < t_0 \text{ } and \text{ } s_2 + s_3 < \frac{5\pi}{16} < s < \pi\}$, we have

$$\theta_1(t,s) > \frac{C}{N}. \tag{3.5}$$

Since the area of P_1 is more than $\frac{\pi}{2}s_3$, and since $\theta_2(t,s), \theta_4(t,s) > C$, we conclude from (3.3), (3.4) and (3.5) that there is $\delta_0 = \delta_0(s_0, C) > 0$, such that $\delta \geq \delta_0$. The rest follows from (3.2).

We need to show that $\widehat{\varphi}$ is quasiconformal at the points close to the origin. We do that by bringing them back to the origin (by Möbius transformations), and then using the same argument as above. Let $a_z \in \mathcal{M}$ be such that $\widehat{a}_z(0) = z$, and $\widehat{a}_z$ preserves the geodesic that contains both 0 and z. Let $\varphi_z = \varphi \circ a_z$. Then, one can choose $D = D(s_0, C) > 0$ small enough, such that for $z \in \Delta(0, D)$ the map φ_z satisfies the assumption of this lemma, where instead of s_0 we take $s_0 + \frac{\pi - s_0}{2}$. By repeating the same argument we conclude the proof of the lemma. $\square$

4. Small elements in discrete quasisymmetric groups

4.1. **Discrete groups generated by small elements are cyclic.** Let $\mathcal{G}$ be a K-quasisymmetric group. For each $f \in \mathcal{G}$, let ψ be a $\widetilde{K}$-quasisymmetric map and let $u \in \mathcal{M}$, such that $f = \psi \circ u \circ \psi^{-1}$. Let $\widehat{\psi}$ denote a $\widetilde{K}$-quasiconformal extension of ψ, $\widetilde{K} = \widetilde{K}(K)$. Set $\widehat{f} = \widehat{\psi} \circ \widehat{u} \circ \widehat{\psi}^{-1}$. The $\widehat{f}$ is a $\widetilde{K}^2$-quasiconformal map that extends f. For $z \in \mathbf{D}$ and for $f \in \mathcal{G}$, if f is not elliptic, let

$$P_f(z, \widehat{\psi}) = \mathbf{d}(\widehat{\psi}^{-1}(z), \widehat{u}(\widehat{\psi}^{-1}(z))).$$

Set

$$P_f(z) = sup_{\widehat{\psi}} P_f(z, \widehat{\psi}),$$

where supremum is being taken with respect to all such $\widetilde{K}$-quasiconformal maps that fix $1, i, -1$, and all corresponding $u \in \mathcal{M}$. Because of the compactness there are $u \in \mathcal{M}$ and $\widehat{\psi}$, such that the supremum above is attained. Since we only consider P_f for non-elliptic $f \in \mathcal{G}$, it is proper to say that $P_f(z)$ is the $\widetilde{K}$ distance (or just the distance) between f and the identity when seen from the point z.

Remark. Here we consider only non elliptic elements because of the nature of the subsequent applications. However, one can naturally define the notion of being small for elliptic elements.

Recall that if u is a hyperbolic Möbius transformation the we define its length as

$$\min_{z \in \mathbf{D}} \mathbf{d}(z, \widehat{u}(z)).$$

This minimum is attained for every point $z \in \mathbf{D}$ that belongs to the axis of the transformation $\widehat{u}$. Let $f \in \mathcal{G}$ be hyperbolic. Let $L_f(\widehat{\psi})$ be the length of the corresponding hyperbolic transformation u. Set $L_f = sup_{\widehat{\psi}} L_f(\widehat{\psi})$, where supremum is being taken with respect to all such $\widetilde{K}$-quasiconformal maps and all corresponding $u \in \mathcal{M}$. We say that L_f is the $\widetilde{K}$ length (or just the length). For $\epsilon > 0$ we say that f is ϵ-small, if $L_f \leq \epsilon$.

Remark. When we say that an element $f \in \mathcal{G}$ is ϵ-small that implies that we are talking about a hyperbolic element.

Let $z \in \mathbf{D}$. We say that z is moved by the hyperbolic distance $d \geq 0$ under $\widehat{u}$ if $\mathbf{d}(z, \widehat{u}(z)) = d$. Suppose that f is either hyperbolic or parabolic (the same is true for the corresponding $u \in \mathcal{M}$ that is a conjugate of f). Then the set of points in $\mathbf{D}$ that are moved for some fixed hyperbolic distance under $\widehat{u}$ is either the geodesic (or an equidistant line) that connects the fixed points of u in the case when u is hyperbolic, or a horocircle if u is parabolic. If $\widehat{\psi}$ is a fixed $\widetilde{K}$-quasiconformal extension of a map ψ that conjugates f to some $u \in \mathcal{M}$, then we will call the image (under $\widehat{\psi}$) of the corresponding set respectively the quasigeodesic, the quasiequidistant, the quasihorocircle, all of them with respect to $\widehat{\psi}$. Let R be a symmetric crescent around the axis of a hyperbolic $u \in \mathcal{M}$. Set $U = \widehat{\psi}(R)$. We say that U is a quasicrescent. If l is the quasiequidistant that borders U from either side, then we say that U is determined by l. Similarly, let H be a horoball that touches $\mathbf{T}$ at x, which is a fixed point of the parabolic transformation u. Set $U = \widehat{\psi}(H)$. We say that U us a quasihoroball. If l is the quasihorocircle that borders U, then we say that U is determined by l.

Lemma 4.1. *With the notation as above the following holds. Let $f \in \mathcal{G}$, such that f is non-elliptic.*

1. *For $v \in \mathcal{M}$ set $\mathcal{G}_v = v\mathcal{G}v^{-1}$. Then $P_f(z) = P_g(\widehat{v}(z))$ and $L_f = L_g$, where $g = v \circ f \circ v^{-1}$.*
2. *Let l be a quasiequidistant (quasihorocircle) and $z \in l$, such that $P_f(z) \leq \epsilon$. Then there is a function $c(\epsilon) = c(K)(\epsilon) > 0$, $c(\epsilon) \to 0$ when $\epsilon \to 0$, such that for any $w \in U$ (U is the quasicrescent (quasihoroball) that corresponds to l), we have $P_f(w) \leq c(\epsilon)$.*
3. *There exists $c_1(\epsilon) = c_1(K)(\epsilon) > 0$, such that $c_1(\epsilon) \to \infty$ when $\epsilon \to 0$, and such that if $P_f(z) \leq \epsilon$, then $P_f(w) \leq 2\epsilon$ for every $w \in \mathbf{D}$, with $\mathbf{d}(z,w) \leq c_1(\epsilon)$.*
4. *Let $f \in \mathcal{G}$, $L_f \leq \epsilon$. Then there is $c_2(\epsilon) = c_2(K)(\epsilon) > 0$, $c_2(\epsilon) \to 0$ when $\epsilon \to 0$, such that if g is a conjugate of f in $\mathcal{G}$, then $L_g \leq c_2(\epsilon)$.*
5. *Let $f \in \mathcal{G}$ be parabolic. Then for every $\epsilon > 0$ there is a quasihoroball U for f, such that $P_f(z) \leq \epsilon$, for $z \in U$. Also, for every $r > 0$, there exists a quasihoroball U, such that $P_f(z) > r$, $z \in \mathbf{D} - U$.*

Proof. The item (1) follows from the definitions of P_f and L_f. We prove (2). Let U be a quasicrescent with respect to a $\widetilde{K}$-quasiconformal map $\widehat{\psi}_1$, and let $u_1 \in \mathcal{M}$ be the corresponding hyperbolic transformation such that $U = \widehat{\psi}_1(R)$, for some symmetric crescent R that corresponds to u_1. Let $w \in U$. Let $u_0 \in \mathcal{M}$ and $\widehat{\psi}_0$ be such that $P_f(w, \widehat{\psi}_0) = \mathbf{d}(\widehat{\psi}_0^{-1}(w), \widehat{u}_0(\widehat{\psi}_0^{-1}(w))) = P_f(w)$. Let R' be the symmetric crescent with respect to u_0, such that w belongs to one of the two boundary equidistant lines of R'. Then, if $\widehat{\psi}_0^{-1}(z) \in R'$ we have that $\widehat{\psi}_0^{-1}(z)$ is a bounded hyperbolic distance away (the upper bound depends only on K because the distance between $\widehat{\psi}_1$ and $\widehat{\psi}_0$ depends only on K) from the equidistant that contains $\widehat{\psi}_0^{-1}(w)$. This shows

$$P_f(w) \leq qP_f(z, \widehat{\psi}_0) \leq qP_f(z),$$

for some $q = q(K) > 0$. If $\widehat{\psi}_0^{-1}(z)$ does not belong to R' we have $P_f(w) \leq P_f(z)$. Either way this yields (2).

Items $(3), (4), (5)$ are proved in a very similar way, and they, as does (2), all follow from the basic estimates on the distortion of the hyperbolic metric under $\widetilde{K}$-quasiconformal maps. We omit the details.

$\square$

For a given K-quasisymmetric group $\mathcal{G}$ and $\epsilon > 0$, let $\mathcal{G}_z(\epsilon)$ be the group generated by all $f \in \mathcal{G}$ (f is not elliptic) such that $P_f(z) < \epsilon$.

Lemma 4.2. *There exists $\epsilon(K) > 0$ with the following properties. Let $\mathcal{G}$ be a K-quasisymmetric group so that every elliptic element in $\mathcal{G}$ (if any) is of order two. Then the group $\mathcal{G}_z(\epsilon(K))$ is either cyclic or it contains only the identity.*

Remark. In fact, our proof works even if we assume that the order of any elliptic element in $\mathcal{G}$ is less than some fixed constant $C \geq 2$. However, then the constant $\epsilon(K)$ would depend on that constant C as well. Although $\mathcal{G}_z$ is generated by hyperbolic and/or parabolic elements, the group $\mathcal{G}_z$ could contain elliptic elements. In our case, the order of every such element is two.

Proof. We show that the group $\mathcal{G}_z$ is Abelian. Note that every Abelian Fuchsian group is cyclic. Proof by contradiction. Assume that there is a sequence of K-quasisymmetric groups $\mathcal{G}_n$, such that for some sequence $z_n \in \mathbf{D}$ we have that neither of the groups $\mathcal{G}_{z_n}$ is Abelian. Here $n \in \mathbf{N}$. Therefore, the group $\mathcal{G}_{z_n}$ has at least two generators $f_n, g_n \in \mathcal{G}$, that do not commute (here we used the fact that every Abelian Fuchsian group, and therefore every quasisymmetric group as well, must be cyclic). We have

$$P_{f_n}(z_n), P_{g_n}(z_n) \to 0,$$

as $n \to \infty$, and neither f_n nor g_n is elliptic. Set $g_n \circ f_n \circ g_n{}^{-1} = f'_n \in \mathcal{G}_n$. Since f_n and g_n do not commute we have that f'_n and f_n do not commute. Also, by Lemma 4.1 we have that $P_{f'_n}(z_n) \to 0$ as $n \to \infty$. By replacing g_n by f'_n if necessary, we can assume that both f_n and g_n are either hyperbolic, or both are parabolic. Denote by $G_n \subset \mathcal{G}_{z_n}$ the group generated by f_n, g_n. Since G_n is a sequence of K-quasisymmetric groups, we have that the geometric limit G (of the sequence G_n) is a K-quasiconformal group.

Remark. Here by the geometric limit we mean the following. A homeomorphism $h : \mathbf{T} \to \mathbf{T}$ belongs to G if for every $n \in \mathbf{N}$ we can choose $h_n \in G_n$ such that $h_n \to h$ in C^0 sense on $\mathbf{T}$. Clearly G is a K-quasisymmetric group.

Our aim is to show that the geometric limit of G_n is a non-elementary and non-discrete K-quasisymmetric group. However, since this limit can not contain elliptic elements of arbitrarily high order we will obtain a contradiction (see [17]). Here we use the fact that a non-discrete K-quasisymmetric group is a conjugate of a subgroup of $\mathcal{M}$. For each $n \in \mathbf{N}$ fix $\widetilde{K}$-quasisymmetric normalized maps ψ_n and ϕ_n, such that there are $u_n, v_n \in \mathcal{M}$, so that $f_n = \psi_n \circ u_n \circ \psi_n{}^{-1}$ and $g_n = \phi_n \circ v_n \circ \phi_n{}^{-1}$. By $\widehat{\psi}_n$ and $\widehat{\phi}_n$ we denote a choice of $\widetilde{K}$-quasiconformal extensions of ψ_n and ϕ_n respectively.

First we consider the case when both f_n, g_n are hyperbolic. By l_n and s_n we denote the corresponding quasigeodesics of f_n and g_n, with respect to $\widehat{\psi}_n$ and $\widehat{\phi}_n$ respectively. Suppose first that there is $d_0 > 0$, such that $0 \leq \mathbf{d}(l_n, s_n) < d_0$, for every n (this includes the case when l_n and s_n intersect transversally). By (x_n, y_n) and (x'_n, y'_n) we denote the ordered pairs of the repelling and the attracting fixed points of f_n and g_n respectively. Since they do not commute, we have that either x'_n or y'_n is disjoint from the set $\{x_n, y_n\}$.

Remark. In fact, since the group $\mathcal{G}$ is discrete it is known (see [20]) that non commuting hyperbolic elements can not fix the same point.

From Lemma 4.1 we have that the lengths of both f_n and g_n tend to 0 as $n \to \infty$. Conjugating the whole group G_n by a Möbius transformation (which does not change the $\widetilde{K}$ lengths of hyperbolic elements nor the $\widetilde{K}$ distance from the identity) we can assume that $x_n = -i$, $y_n = i$, and either that $x'_n = -1$ or $x'_n = 1$. Choose $x'_n = -1$. Since $\mathbf{d}(l_n, s_n) < d_0$, we have that y'_n, after passing to a subsequence if necessary, tends to a point $y \in \mathbf{T}$, and $y \neq x'_n = -1$. Denote this subsequence by m_n. Let G denote the geometric limit of the sequence G_{m_n}, and let F be a Fuchsian group that is conjugate to G. We have that the cyclic group generated by f_{m_n} tends to a one parameter, hyperbolic, K-quasisymmetric, group with the fixed points $-i, i$. We have the similar conclusion for g_{m_n}. Since F contains two one parameter, hyperbolic groups, that do not have the same fixed points, we have that F is neither discrete nor elementary. But the only elliptic elements F may contain are of order two, which is a contradiction.

Now, suppose that $\mathbf{d}(l_n, s_n) \to \infty$. Let p_n be the quasiequidistant (for f_n and with respect to $\widehat{\psi}_n$) that contains z_n. If the hyperbolic distance between p_n and s_n stays bounded (including the case when p_n and s_n intersect), as $n \to \infty$, then we choose a quasiequidistant p'_n (for f_n) with the following properties. p'_n is between p_n and l_n, the distances $\mathbf{d}(p'_n, l_n)$ and $\mathbf{d}(p'_n, s_n)$ both tend to ∞, and for every $z \in p'_n$ we have that $P_{g_n}(z) \to 0$, $n \to \infty$. We can make this choice because $\mathbf{d}(l_n, s_n) \to \infty$, and by Lemma 4.1. Note that $P_{f_n}(z) \to 0$ for every $z \in p'_n$, because p'_n is between p_n and l_n (this follows from Lemma 4.1). Now, let q_n be a quasiequidistant (for g_n and with respect to $\widehat{\phi}_n$) such that $p'_n \cap q_n$ is non-empty, and such that the interior of the quasicrescents that correspond to p'_n and q_n respectively, have empty intersection. Let w_n be any point in $p'_n \cap q_n$. We have that both $P_{f_n}(w_n)$ and $P_{g_n}(w_n)$ tend to 0. By Lemma 4.1 we can assume that $w_n = 0$.

Let $n \to \infty$. Since $\mathbf{d}(l_n, 0) \to \infty$, we conclude that both fixed points of f_n tend to a single point $x \in \mathbf{T}$. Similarly, the both fixed points for g_n tend to a single point $y \in \mathbf{T}$. We want to show that $x \neq y$. By passing to a subsequence if necessary, we may assume that $\widehat{\psi}_n$ and $\widehat{\phi}_n$ converge to $\widetilde{K}$-quasiconformal maps $\widehat{\psi}$ and $\widehat{\phi}$. The crescents that correspond to p'_n and q_n converge to quasihorocircles, with respect to $\widehat{\psi}$ and $\widehat{\phi}$ respectively. Note that both of these horocircles contains the origin 0. Since the interiors of the corresponding quasihoroballs have no points in common, we conclude that $x \neq y$. Similarly as above, after choosing a proper subsequence, we conclude that the geometric limit of G_n (or a properly chosen subsequence), contains two one parameter parabolic groups that do not have the same fixed point. This implies that this limit group is neither discrete nor elementary. But the only elliptic elements it may contain are of order two, which is a contradiction.

The case when both f_n and g_n are both parabolic is almost identical to the case when $\mathbf{d}(l_n, s_n) \to \infty$ above. The only difference is that instead of quasiequidistants we use quasihorocircles.

We have shown that there exists $\epsilon(K) > 0$ such that the corresponding group $\mathcal{G}_z$ is cyclic. $\square$

The above results are analogues of the corresponding results for Fuchsian groups which state that two non-commuting (hyperbolic or parabolic) elements of a Fuchsian group can not both move a given point for a very small distance. Some of the main results of this sort for Fuchsian groups are the Jorgensen inequality and the Margulis lemma. This lemma is one of the central results in the theory of Lie groups, and it generalizes these results for Fuchsian groups to discrete latices in Lie groups (see [24]).

Lemma 4.3. *Let $\mathcal{F}$ be a Fuchsian group. Suppose that $u, v \in \mathcal{F}$, are elliptic elements of order at least three. Then there is $\epsilon > 0$ such that for $z, w \in \mathbf{D}$, $z \neq w$, that are the fixed points of $\widehat{u}$ and $\widehat{v}$ respectively, we have $\mathbf{d}(z, w) > \epsilon$. The constant ϵ is universal (it does not depend on the choice of $\mathcal{F}$, or the elements $u, v \in \mathcal{F}$).*

Proof. This proof follows the same line of argument as the proof of the fact that every Fuchsian group that contains only elliptic elements must be cyclic. Suppose that $z = 0$ (z is the fixed point of $\widehat{u}$), and denote by $g = u^{-1} \circ v^{-1} \circ u \circ v$ the commutator of u and v. Assume first that the trace of g is greater than two. Then g is a hyperbolic transformation. Moreover, the length of g is less than δ, where $\delta \to 0$ when $\epsilon \to 0$. Set $f = u^{-1} \circ g \circ u$. Since u is not of order two, we have that f and g are hyperbolic elements that do not commute. Taking ϵ small enough we obtain a contradiction from the Jorgensen inequality (or any other result of that sort).

Therefore, we have that the trace of g is equal to two. But this yields that $z = w = 0$ which is a contradiction. $\square$

4.2. **Removing small hyperbolic elements.**

Theorem 4.1. *There exists $\epsilon(K) > 0$ with the following properties. For an arbitrary K-quasisymmetric group $\mathcal{G}$, that does not contain any elliptic elements of order three or more, there exist K_1-quasisymmetric groups $\mathcal{G}_i$, $i \in \mathbf{N}$, $K_1 = K_1(K)$, such that*

(1) *None of the groups $\mathcal{G}_i$ contain any $\epsilon(K)$-small hyperbolic elements, nor any elliptic elements of order three or more.*
(2) *If every $\mathcal{G}_i$ is K'-quasisymmetrically conjugated to a Fuchsian group, $K' = K'(K)$, then there exists $K'' = K''(K)$, such that $\mathcal{G}$ is K''-quasisymmetrically conjugated to a Fuchsian group.*

Proof. Let $\mathcal{G}$ be an arbitrary K-quasisymmetric group. We will construct the groups $\mathcal{G}_i$. Providing that these groups are quasisymmetrically conjugated to the corresponding Fuchsian groups, we construct a homomorphism E of $\mathcal{G}$ into the group of quasiconformal selfmaps of $\mathbf{D}$, such that E is an isomorphism onto its image.

Let $\epsilon > 0$, and let $h_1, .., h_{i,...} \in \mathcal{G}$, $i \in \mathbf{I}$, be the list of all primitive, mutually non conjugate, ϵ-small hyperbolic elements. Here $\mathbf{I}$ is either the set $1, 2, ..., n$, for some $n \in \mathbf{N}$, or $\mathbf{I} = \mathbf{N}$, depending on whether there are finitely or infinitely many h_i. Recall that h_i is primitive element means that h_i is not a power of another element in $\mathcal{G}$. Set $[h_i] = \cup_{k \in \mathbf{Z}} (\cup_{f \in \mathcal{G}} f^{-1} \circ h_i^k \circ f)$. In each class $[h_i]$ fix one representative that is primitive, say p_i. Let x_i, y_i denote respectively the repelling and the attracting fixed points of h_i. Let $Stab_i$ be the subgroup of $\mathcal{G}$ whose elements fix the set $\{x_i, y_i\}$. This is an elementary group and therefore there exist an elementary Fuchsian group $Stab'_i$ and a $\widetilde{K}$-quasisymmetric map $\psi_i : \mathbf{T} \to \mathbf{T}$, which conjugates $Stab'_i$ and $Stab_i$.

We can choose ψ_i so that $\psi_i(x_i) = x_i$, $\psi_i(y_i) = y_i$. Let $\widehat{\psi_i}$ be some $\widetilde{K}$-quasiconformal extension of ψ_i. Let $E_i(1)$ be the symmetric crescent of the hyperbolic width 1, around the geodesic that connects x_i and y_i, and set $U_i(1) = \widehat{\psi_i}(E_i(1))$.

Now fix $i \in \mathbf{I}$. For every $h \in Stab_i$ put $\widehat{h} = \widehat{\psi_i} \circ \widehat{u}_i \circ \widehat{\psi}_i^{-1}$, where $u_i \in Stab'_i$ such that $h = \psi_i \circ u_i \circ \psi_i{}^{-1}$. For $f, g \in \mathcal{G}$, we say that $f \sim g$, with respect to the pair (x_i, y_i), if $f(x_i) = g(x_i)$ and $f(y_i) = g(y_i)$. Denote the corresponding equivalence class by $[f]_i$. Note that for $f \sim h$, where $h \in Stab_i$, we have already defined the extension $\widehat{f}$ because each such f must be in the corresponding stabilizer $Stab_i$. In every other equivalence class choose one representative, say f, and let $\widehat{f}$ be a fixed K-quasiconformal extension. For every other $g \in [f]_i$, there exists

$h \in Stab_i$, such that $f^{-1} \circ g = h$. Set $\widehat{g} = \widehat{f} \circ \widehat{h}$. Therefore, for $f \in \mathcal{G}$, and for fixed $i \in \mathbf{I}$, we have defined the extension $\widehat{f}$, which is $\widetilde{K}^3$-quasiconformal. Set $[U_i(1)] = \cup_{f \in \mathcal{G}} \widehat{f}(U_i(1))$. Now repeat this process for each i. Note that formally we should denote the extension $\widehat{f}$ by $\widehat{f}_i$, because a separate extension is defined for every $i \in \mathbf{I}$. We avoid doing this to simplify the notation, and no confusion should arise.

By Lemma 4.1 and Lemma 4.2 there exists $\epsilon_1(K) > 0$ small enough, so that if we set $\epsilon = \epsilon_1(K)$, then for every $f \in \mathcal{G}$ we have

$$\mathbf{d}(\widehat{f}(U_j(1)), \widehat{g}(U_k(1))) > 1,$$

whenever $\widehat{f}(U_j(1))$ and $\widehat{g}(U_k(1))$ do not touch the unit circle $\mathbf{T}$ at the same points (otherwise, for every $\epsilon > 0$, there would be $z \in \mathbf{D}$ such that $\mathcal{G}_z$ is not cyclic). Note that this choice of ϵ yields that $f(x_i) = g(x_i)$ implies $f(y_i) = g(y_i)$, which we already know to be true.

Let $R = R(K)$ be the constant from the Lemma 3.1. By the same argument as above, we can choose $\epsilon_2(K) > 0$, and the symmetric crescent E_i (E_i contains the crescent $E_i(1)$), with the following properties. Set $U_i = \widehat{\psi}_i(E_i)$, and $[U_i] = \cup_{f \in \mathcal{G}} \widehat{f}(U_i)$. If $f \in Stab_i$, then

$$\epsilon_2(K) < \mathbf{d}(z, \widehat{f}(z)), \tag{4.1}$$

for any $z \in \partial U_i$. If $f, g \in \mathcal{G}$ are such that $f(x_j) \neq g(x_k)$ (or equivalently $f(y_j) \neq g(y_k)$) , where $j, k \in \mathbf{I}$ are any two numbers (j, k may be equal), then

$$\mathbf{d}(\widehat{f}(U_j), \widehat{g}(U_k)) > R. \tag{4.2}$$

Set $\Omega = \mathbf{D} - \cup[U_i]$. We have that Ω is a disconnected set, but any connected component of Ω is simply connected. This follows from (4.2). Since $\mathcal{G}$ is not an elementary group, there are infinitely many connected components of Ω. If Ω_0 is a connected component of Ω, then Ω_0 is uniquely determined by its boundary points that lie in $\mathbf{T}$. Therefore, if $f \in \mathcal{G}$, we can properly define the connected component $f_* \Omega_0$ of Ω. We denote by Ω_j, $j \in \mathbf{I}'$, mutually non conjugated components, such that every other component is in one of the classes $[\Omega_j] = \cup_{f \in \mathcal{G}} f_*(\Omega_j)$. Here $\mathbf{I}'$ is the range for j, and it may be either a finite set or $\mathbf{I}' = \mathbf{N}$ (see Figure 2).

For $f \in \mathcal{G}$, we first define $E(f)$ on each $[U_i]$. Let U_i' be a component of $[U_i]$ (U_i' touches $\mathbf{T}$ at x_i', y_i'). Set $x_i'' = f(x_i')$, $y_i'' = f(y_i')$. Let $g', g'' \in \mathcal{G}$ be such that $g'(x_i) = x_i'$, $g'(y_i) = y_i'$, and $g''(x_i) = x_i''$, $g''(y_i) = y_i''$ (such $g', g'' \in \mathcal{G}$ are not uniquely determined). Then, there exists $h \in Stab_i$, such that $g'' \circ h \circ {g'}^{-1} = f$. Set

$$\left(\widehat{g''} \circ \widehat{h} \circ \widehat{g'}^{-1} \right)(z) = E(f)(z),$$

for $z \in U_i'$. This defines $E(f)$ on $\cup[U_i] = \mathbf{D} - \Omega$. It follows from the definition of the corresponding extensions of g', g'', that $E(f)$ is well defined.

Lemma 4.4. *$E(f)$ is a homeomorphism of $\cup[U_i]$. Also,*

1. *$E(f) \circ E(g) = E(f \circ g)$; $E(id) = id$.*
2. *For every $f \in \mathcal{G}$, there exist $K_1 = K_1(K)$ and a K_1-quasiconformal map $\widetilde{f}$ which extends f, such that $\widetilde{f} = E(f)$ on $\cup[U_i] = \mathbf{D} - \Omega$.*

Proof. The item (1), and the fact that $E(f)$ is a homeomorphism, follows from the definition of E and the discussion above. On each component of $[U_i]$, the map $E(f)$ is a restriction of a $\widetilde{K}^8$-quasiconformal map $\widehat{g''} \circ \widehat{h} \circ \widehat{g'}^{-1}$. Combining this with (4.2), it follows from Lemma 3.1 that there exists the map $\widetilde{f}$ with the stated properties. $\square$

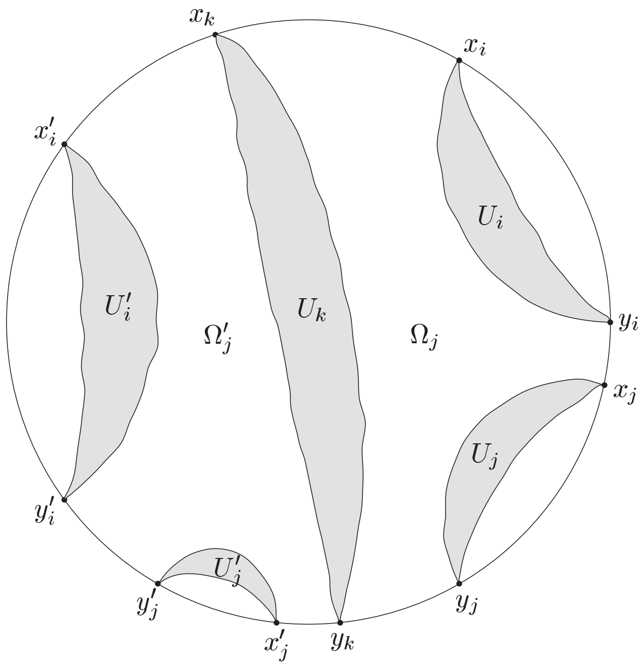

FIGURE 2

Next, we define E on the rest of $\mathbf{D}$. In each class $[\Omega_j]$, $j \in \mathbf{I}'$, fix one representative, say Ω_j. For $f \in \mathcal{G}$, $f_*\Omega_j = \Omega_j$, let $e(f)$ be the restriction of the map $E(f)$ on $\partial\Omega_j$. Let St_j be the group of all maps $e(f)$. The group St_j is isomorphic to the subgroup of $\mathcal{G}$ that contains all f so that $f_*\Omega_j = \Omega_j$. From Lemma 4.4 it follows that St_j is K_1-quasisymmetric group on Ω_j. By conjugating the group St_j via the boundary values of the Riemann map that maps Ω_j to the unit disc, we obtain the K_1-quasisymmetric group $\mathcal{G}_j$. We get from (4.1), from Lemma 4.1, and from the definition of Ω, that there is $\epsilon_3(K) > 0$, such that $\mathcal{G}_j$ does not contain $\epsilon_3(K)$-small hyperbolic elements, for some $\epsilon_3(K) > 0$. Note that for a hyperbolic $f \in \mathcal{G}$, such that $e(f) \in St_j$, we have that $e(f)$ is hyperbolic. It follows from the definition that the length of f is greater or equal than the corresponding length of $e(f)$. This follows from the fact that a conformal map of $\mathbf{D}$ to a subset of $\mathbf{D}$, is a contraction with respect to the hyperbolic metric on $\mathbf{D}$. By repeating this for every j we obtain the collection of groups from the statement of this theorem. It follows from the construction that none of them has elliptic elements of order three or more (because $\mathcal{G}$ does not have any such elliptic elements).

If we assume that $\mathcal{G}_j$ is K'-quasiconformally conjugated to a Fuchsian group, then there exist $K_2 = K_2(K)$, a Fuchsian group St'_i (St'_j acts on $\mathbf{D}$ of course), and a K_2-quasiconformal map $\widehat{\psi_j} : \mathbf{D} \to \Omega_j$, such that the induced map $\psi_j : \mathbf{T} \to \partial\Omega_j$ conjugates the groups St'_i and St_i.

Now fix $i \in \mathbf{I}$. For every $e(f) \in St_j$, we put $\widehat{f} = \widehat{\psi}_j \circ \widehat{u}_i \circ \widehat{\psi}_i^{-1}$, where $u \in St'_j$, such that $e(f) = \psi_j \circ u \circ {\psi_j}^{-1}$. For $f, g \in \mathcal{G}$, we say that $f \sim g$ with respect Ω_j, if $f_*(\Omega_j) = g_*(\Omega_j)$. Denote the corresponding equivalence class by $[f]_i$. Note that for $f \sim h$, where $e(h) \in St_i$, we have already defined the extension $\widehat{f}$, because for each such f, the corresponding map $e(f)$ must be in the stabilizer St_i. In every other equivalence class, choose one representative, say f, and let $\widehat{f}$ be the K_1-quasiconformal map from Lemma 4.4 (note that $\widehat{f}$ agrees with $E(f)$ on $\cup[U_i]$). For every other $g \in [f]_i$ there exists $h \in St_i$ such that $f^{-1} \circ g = h$. Set $\widehat{g} = \widehat{f} \circ \widehat{h}$. Therefore, for $f \in \mathcal{G}$, we have defined the extension $\widehat{f}$ which is K_3-quasiconformal, $K_3 = K_3(K)$.

Let Ω'_j be a component of $[\Omega_j]$. Set $\Omega''_j = f_*\Omega'_j$. Let $g', g'' \in \mathcal{G}$ be such that $g'_*\Omega_j = \Omega'_j$ and $g''_*\Omega_j = \Omega''_j$ (such $g', g'' \in \mathcal{G}$ are not uniquely determined). Then, there is a $h \in \mathcal{G}$, $h_*\Omega_j = \Omega_j$, such that $g'' \circ h \circ {g'}^{-1} = f$. Set

$$\left(\widehat{g''} \circ \widehat{h} \circ \widehat{g'}^{-1}\right)(z) = E(f)(z),$$

for $z \in \Omega'_j$. This defines $E(f)$ on Ω. It is clear from the above discussion that E satisfies all the required properties. In particular there exists $K'_1 = K'_1(K)$, such that $E(\mathcal{G})$ is a K'_1-quasiconformal group, which implies that $\mathcal{G}$ is K''-quasisymmetrically conjugated to a Fuchsian group, for some $K'' = K''(K)$. This completes the proof of Theorem 4.2. □

5. Hyperbolic quasisymmetric groups

Throughout this section $\mathcal{G}$ denotes a K-quasisymmetric group whose all elements are hyperbolic. We also assume that for some $0 < \widetilde{\epsilon}(K) < \epsilon(K)$, $\mathcal{G}$ does not contain any $\widetilde{\epsilon}(K)$-small elements. The constant $\epsilon(K)$ is from Theorem 4.1. We show that such a group is a quasisymmetric conjugate of a Fuchsian group

5.1. **The pants-annuli decomposition of a quasisymmetric group.** Recall that if $\mathcal{F}$ is a Fuchsian group then $\mathcal{F}$ acts on $\mathbf{T}$, and the corresponding group that acts on $\mathbf{D}$ is denoted by $\widehat{\mathcal{F}}$. We first assume that $\mathcal{G}$ is finitely generated. Let $\mathcal{F}$ be a Fuchsian group and $\varphi : \mathbf{T} \to \mathbf{T}$ a homeomorphism such that $\mathcal{G} = \varphi\mathcal{F}\varphi^{-1}$. Then $\mathcal{F}$ is finitely generated, and therefore $\mathbf{D}/\widehat{\mathcal{F}} = S$ is a topologically finite Riemann surface. This means that S is biholomorphic to a closed Riemann surface of finite genus with at most finitely many discs removed.

Remark. The only reason we temporarily assumed that $\mathcal{G}$ is finitely generated is because we can more clearly describe the pants-annuli decomposition of the surface S (see [24]). A similar decomposition is valid for all Riemann surfaces, but we choose to work with the finitely generated case and when needed apply Proposition 1.3.

Consider the induced hyperbolic metric on S (this metric agrees with the hyperbolic metric that S inherits from $\mathbf{D}/\widehat{\mathcal{F}} = S$). We recall the standard decomposition of a topologically finite Riemann surface and the corresponding lift into the universal cover. Let S_0 be the convex core of S (if S is closed then $S_0 = S$). S is a union of S_0, the simple closed geodesics (in the future just geodesics) that represent the boundary of S_0, and the annuli (each annulus is bounded by a geodesic which is also a boundary component of S_0, and a boundary component of S). Now, cut up the surface S_0 into pairs of pants (recall that a pair of pants is an open Riemann surface which is biholomorphic to the Riemann sphere minus three discs). We have that S is a disjoint union of the pairs of pants, the annuli, and the geodesics that border the pairs of pants (some of them also border the annuli if they exist). Denote by $\widetilde{P}_i$, $1 \leq i \leq n_1$, the pairs of pants; by $\widetilde{A}_i$; $1 \leq i \leq n_2$ the annuli; and by $\widetilde{l}_i$, $1 \leq i \leq n_3$, the geodesics that border the pairs of pants in this decomposition of S. Here, $n_1, n_2, n_3 \in \mathbf{N}$ depend on the Euler number of S and the number of free boundary components of S. We denote by $[\widetilde{P}_i]$, $[\widetilde{A}_i]$ and $[\widetilde{l}_i]$ the totality of the corresponding lifts (the $i's$ run throughout the corresponding ranges) in $\mathbf{D}$.

Each connected component in $\mathbf{D}$ of a fixed $[\widetilde{P}_i]$ is a convex subset of $\mathbf{D}$, whose relative boundary (in $\mathbf{D}$) is contained in the collection of geodesics which are the lifts of geodesics on S that border $\widetilde{P}_i$ (this is a subset of the union of $[\widetilde{l}_i]$, $1 \leq i \leq n_3$ which we denote by $\cup[\widetilde{l}_i]$). We also denote by $\widetilde{P}_i$ the closure of a fixed single lift of a given pair of pants $\widetilde{P}_i$ in S. $\widetilde{P}_i$ is the closure of a chosen fundamental region for the corresponding pair of pants. We choose it so that $\widetilde{P}_i$ is a right-angled octagon. Four sides of this octagon are contained in four geodesics from $\cup[\widetilde{l}_i]$, and the other four sides are geodesic arcs that are orthogonal to the first four sides, so that all together they complete the boundary of the right-angled octagon $\widetilde{P}_i$. Each $[\widetilde{P}_i]$ is equal to the union $\cup\widehat{u}(\widetilde{P}_i)$, where $\widehat{u} \in \widehat{\mathcal{F}}$ (this union is disjoint except that the points which belong to certain sides of $\widetilde{P}_i$ may occur more than once in this union). Note that a fixed (connected) component of $[\widetilde{P}_i]$ contains many copies of $\widetilde{P}_i$ (see Figure 3).

A connected component of a fixed $[\widetilde{A}_i]$ is a hyperbolic halfspace bounded by a lift of the geodesic in S that borders $\widetilde{A}_i$. We also denote by $\widetilde{A}_i$ the closure a fixed single lift of a given annulus $\widetilde{A}_i \subset S$. Again, $\widetilde{A}_i$ is the closure of a chosen fundamental domain of the corresponding annulus. We choose it so that $\widetilde{A}_i$ is a right-angled parallelogram. One of its sides is contained in a geodesic from $\cup[\widetilde{l}_i]$, its opposite side is an arc contained in $\mathbf{T}$, and the remaining two sides are geodesic rays that connect the first two sides to complete the boundary of the right-angled parallelogram $\widetilde{A}_i$. Note that each $[\widetilde{A}_i]$ is the union of $\widehat{u}(\widetilde{A}_i)$, where $\widehat{u} \in \widehat{\mathcal{F}}$ (this union is disjoint except that points in certain sides of $\widetilde{A}_i$ appear twice). Note that a fixed (connected) component of $[\widetilde{A}_i]$ contains many copies of $\widetilde{A}_i$.

A connected component of a fixed $[\widetilde{l}_i]$ is a geodesic in $\mathbf{D}$. We also denote by $\widetilde{l}_i$ a fixed connected component (or just a component) of $[\widetilde{l}_i]$ (note that this is much more than a fundamental domain of the corresponding closed geodesic on S). In this case we also have that $[\widetilde{l}_i]$ is equal to the union $\cup_{\widehat{u}\in\widehat{\mathcal{F}}}\widehat{u}(\widetilde{l}_i) = \cup_{u\in\mathcal{F}}u_*(\widetilde{l}_i)$, but this union is not disjoint. Note that to each $\widetilde{l}_i$ corresponds an infinite cyclic subgroup of the group $\mathcal{F}$, which consists of the elements of $\mathcal{F}$ that fix its endpoints. We denote the generator of this subgroup by $h_{\widetilde{l}_i}$. Also, if u is an element of $\mathcal{F}$ which fixes the endpoints of $\widetilde{l}_i$, then u belongs to this cyclic subgroup. $\mathcal{F}$ has only hyperbolic elements, therefore no element of $\mathcal{F}$ can permute the endpoints of $\widetilde{l}_i$. Note that no two geodesics from $\cup[\widetilde{l}_i]$ intersect.

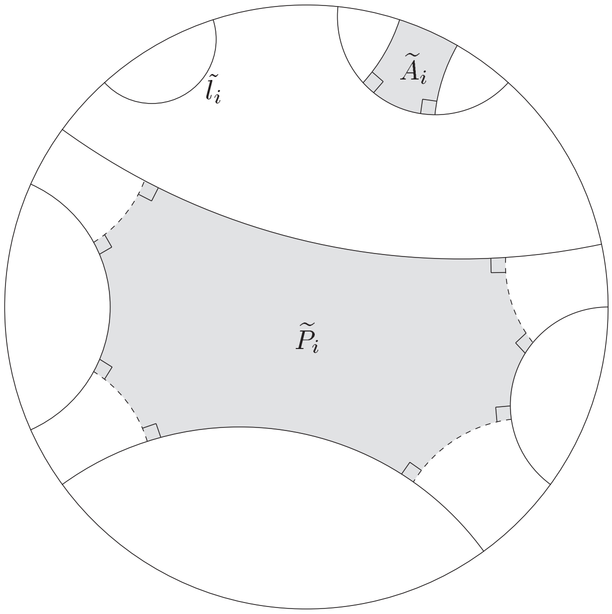

FIGURE 3. Dotted lines represent sides of $\widetilde{P}_i$ that are not subsets of $\cup[\widetilde{l}_i]$.

In the remainder of this section $\widetilde{P}_i, \widetilde{A}_i$ will always refer to a choice of single lifts (described above) of the corresponding pair of pants and annulus, while $\widetilde{l}_i$ will stand for a fixed component in $[\widetilde{l}_i]$. Now, fix $\widetilde{l}_i$. Since φ is a homeomorphism we define the geodesic l_i, by setting $l_i = \varphi_*(\widetilde{l}_i)$. We set $[l_i] = \cup_{f \in \mathcal{G}} f_*(l_i) = \cup_{u \in \mathcal{F}} \varphi_*(u_*(\widetilde{l}_i))$. Also, to each l_i corresponds an infinite cyclic subgroup of $\mathcal{G}$ whose elements fix its endpoints. We denote the generator of this subgroup by h_{l_i}. We have

$$h_{l_i} = \varphi \circ \widetilde{h}_{\widetilde{l}_i} \circ \varphi^{-1}.$$

Note that if $f(l_i) = l_i$, for some $f \in \mathcal{G}$, then because $\mathcal{G}$ contains only hyperbolic elements, we have that f belongs to the cyclic subgroup generated by h_{l_i}. Similarly as above, no element of $\mathcal{G}$ can permute the endpoints of l_i. Again, no two geodesics in $\cup[l_i]$ intersect.

5.2. **Extending the action of $\mathcal{G}$ to the unit disc.** Our aim is to extend the action of $\mathcal{G}$ to $\mathbf{D}$. We will define a homomorphism E (which is an isomorphism onto its image) of the group $\mathcal{G}$ into the group of quasiisometries of $\mathbf{D}$, so that for each $f \in \mathcal{G}$, $E(f)$ extends f to $\mathbf{D}$. The image of the group $\mathcal{G}$ under E is denoted simply by $E(\mathcal{G})$. We do this in steps. First we define $E(f)$ for every $f \in \mathcal{G}$, on every $[l_i]$, $1 \leq i \leq n_3$, and then on every $[P_i]$ and $[A_i]$ (see below for the definition of $[P_i]$ and $[A_i]$).

Fix $i \in [1, n_3]$, and let l_i be a fixed component of the set $[l_i]$. Recall that h_{l_i} is the generator of the cyclic group that fixes the endpoints of l_i. Let ψ be a $\widetilde{K}$-quasisymmetric map and $u \in \mathcal{M}$, such that $\psi \circ u \circ \psi^{-1} = h_{l_i}$. Let $\widehat{\psi}$ be a $\widetilde{K}$-quasiconformal and bilipschitz extension of ψ. Set $\widehat{\psi} \circ \widehat{u} \circ \widehat{\psi}^{-1} = \widehat{h_{l_i}}$. Since $\widehat{\psi}$ is bilipschitz we conclude that $\widehat{h_{l_i}}$ is bilipschitz, where the corresponding constant is a function of K. For $f, g \in \mathcal{G}$, we say $f \sim g$ if $f_*(l_i) = g_*(l_i)$. Denote by $[f]_i$ the corresponding equivalence class. Fix one $f \in [f]_i$. Let $\widehat{f}$ be a K_1'-quasiconformal map, $K_1' = K_1'(K)$, which extends f, and which maps l_i onto the geodesic $f_* l_i$ (such exists by Lemma 3.5). We also assume that $\widehat{f}$ is bilipschitz, where the corresponding constant is a function of K. For an arbitrary $g \in [f]_i$, there is $k \in \mathbf{Z}$, such that $f^{-1} \circ g = h^k_{l_i}$. Set $\widehat{g} = \widehat{f} \circ \widehat{h_{l_i}}^k$. Therefore, for each $f \in \mathcal{G}$, we have defined a K_1-quasiconformal extension, $K_1 = K_1(K)$. Now, repeat this for every i.

Let f be an arbitrary element of $\mathcal{G}$, and let l_i' be a component from $[l_i]$ (l_i' is not necessarily equal to l_i which is fixed). Let $l_i'' = f_*(l_i')$. Choose $g', g'' \in \mathcal{G}$, such that $g'(l_i) = l_i'$ and $g''(l_i) = l_i''$ (such g', g'' are not unique). Since $({g''}^{-1} \circ f \circ g')_*(l_i) = l_i$, there exists $k \in \mathbf{Z}$, such that ${g''}^{-1} \circ f \circ g' = h^k_{l_i}$. We set

$$E_1(f)(z) = \left(\widehat{g''} \circ \widehat{h}^k_{l_i} \circ \widehat{g'}^{-1} \right)(z) \tag{5.1}$$

for $z \in l_i'$. Repeat this for every i. In this way we have defined the mapping $E_1(f)$ for every point in every geodesic from any of the $[l_i]$, $1 \le i \le n_3$. Since no two geodesics form $\cup [l_i]$ intersect, $E_1(f)$ is well defined. Also, this definition does not depend on the choice of g', g'' because of the way we have defined the corresponding extensions.

We need to slightly modify $E_1(f)$ (to obtain $E(f)$), $f \in \mathcal{G}$, so that $E(f)$ is $\delta(\epsilon)$ continuous on $\cup[l_i]$, where $\delta : \mathbf{R}^+ \to \mathbf{R}^+$ is a function that is itself a function of K. We have already shown that the restriction of $E_1(f)$ on every geodesic from $\cup[l_i]$ is bilipschitz. Let $\epsilon_0, \epsilon_1 > 0$, such that $0 < \epsilon_1 < \epsilon_0 < \frac{1}{2}$. Let l, l' be two geodesics from $\cup[l_i]$ such that $\mathbf{d}(l, l') < \epsilon_0$. Let $\alpha \subset l$ be a subarc of l, such that for $z \in (l - \alpha)$ we have that the distance between z and l' is greater than ϵ_1. We similarly define $\alpha' \subset l'$. Denote by $h, h' \in \mathcal{G}$ the corresponding elements that fix the endpoints of l and l' respectively (h, h' generate the corresponding cyclic groups). Then there exists $L_0 = L_0(\epsilon_0, K)$, such that the lengths of h and h' are greater than L_0. Moreover, we have that $L_0 \to \infty$, for $\epsilon_0 \to 0$ (if not, we would have that the geodesic $h_* l' \in \cup[l_i]$ intersects the geodesic l'). This implies that we can can choose $0 < \epsilon_1 < \epsilon_0$, both of them being functions of K, such that $E_1(h)(\alpha)$ is disjoint from α. Similarly, $E_1(h')(\alpha')$ is disjoint from α'. For ϵ_1 small enough, the hyperbolic length of both α and α' is greater than 1. Now fix ϵ_0, ϵ_1 with the above properties.

For $f \in \mathcal{G}$, let $\alpha_f = E_1(f)(\alpha)$, $\alpha'_f = E_1(f)(\alpha')$. Since f is K-quasisymmetric, and since the value of $E_1(f)$ on l is a restriction of a bilipschitz map (see (5.1)) of $\mathbf{D}$ onto itself, it follows that there exists $\epsilon_2 = \epsilon_2(K)$, such that for any $z \in (f_* l - \alpha_f)$ we have that $\mathbf{d}(z, f_* l') > \epsilon_2$. We have the analogous conclusion for α'_f.

We now define a new map $E(f)$ that agrees with $E_1(f)$ on $\cup[l_i] - \cup_{f \in \mathcal{G}} \alpha'_f$. Let $I_f : \alpha_f \to \alpha'_f$ be the affine map, that maps each endpoint of α_f to the closer of the two endpoints of α'_f. If α_f is very long, then so is α'_f, and I_f is nearly an isometry.

In any case I_f is bilipschitz (with the constant that is a function of K) regardless of the length of α_f (since the hyperbolic length of α is at least one we conclude that the length of α_f is not too small). For $f \in \mathcal{G}$ and $z \in \alpha' = \alpha'_{id}$ set

$$E(f)(z) = (I_f \circ E_1(f) \circ I_{id}^{-1})(z).$$

Let $g \in \mathcal{G}$, and let $z \in \alpha'_f$, for some $f \in \mathcal{G}$. Set

$$E(g)(z) = E(g \circ f) \circ E(f)^{-1}.$$

Since $E_1(h)(\alpha) \cap \alpha$ is an empty set, it follows that $E(f)$ is well defined. Let $l_1, l'_1 \in \mathcal{G}$ be a pair of geodesics, that is not in the orbit of the pair l, l', and such that $\mathbf{d}(l_1, l'_1) < \epsilon_0$. Repeat the same process. The operator $E(f)$ is conjugated to $E_1(f)$, and therefore it respects the group structure.

By this, we have defined the extension $E(f)$, $f \in \mathcal{G}$, which is a modification of $E_1(f)$. It follows from the construction that $E(f)$ is $\delta(\epsilon)$ continuous, where $\delta(\epsilon)$ is a function of K.

Lemma 5.1. *Let $f, g \in \mathcal{G}$ and let id denote the identity mapping. There exist $L = L(K)$, $\delta : \mathbf{R}^+ \to \mathbf{R}^+$, $\rho = \rho(K)$, and a L-quasiisometry $\widetilde{f}$ which extends f, such that*

1. *E(f) is a homeomorphism of every geodesic from $\cup[l_i]$.*
2. *$E(f)$ is $\delta(\epsilon)$ continuous on $\cup[l_i]$, where $\delta(\epsilon)$ is a function of K.*
3. *For every $z \in \cup[l_i]$, $f \in \mathcal{G}$, we have $\mathbf{d}(E(f)(z), z) > \rho$.*
4. *$E(f \circ g) = E(f) \circ E(g)$, and $E(id) = id$.*
5. *$\widetilde{f}$ agrees with $E(f)$ on $\cup[l_i]$.*
6. *$\widetilde{f}$ is is $\delta(\epsilon)$ continuous on $\mathbf{D}$.*

Proof. It follows from (5.1) that $E_1(f \circ g)) = E_1(f) \circ E_1(g)$ and $E_1(id) = id$. The modification preserves this property. We have already proved that $E(f)$ is a uniformly continuous homeomorphisms of $\cup[l_i]$. The existence of the quasiisometry $\widetilde{f}$ follows from Lemma 2.2.

Let $l \in \cup[l_i]$, and $h \in \mathcal{G}$ the corresponding hyperbolic element. Let $z \in l$. If $E(f)(z)$ is very close to z, then z can not be in l since h can not be $\epsilon(K)$-small. This implies that f does not preserve l. Let $l' = f_* l$ be the geodesic from $\cup[l_i]$ that contains $E(z)$. Since l and l' are disjoint, we have that the corresponding (repelling and attracting respectively) endpoints of l and l' are very close to each other (because z and $E(z)$ are very close). But this implies again that the length of f is very small, which is a contradiction (recall that we assume that $\mathcal{G}$ does not contain any $\widetilde{\epsilon}(K)$-small hyperbolic elements). This shows the existence of ρ. □

Next, we define $E(f)$ on the rest of the unit disc. To do this we first need to define an appropriate extension $\widehat{\varphi} : \mathbf{D} \to \mathbf{D}$ of the map φ. Since no two geodesics from $\cup[\widetilde{l}_i]$ intersect and since $\varphi_*(\widetilde{l}_i) = l_i$, we can choose a homeomorphism $\widehat{\varphi}$ so that $\widehat{\varphi}(\widetilde{l}_i) = l_i$ and so that the equality

$$\widehat{\varphi} \circ \widetilde{h}_{\widetilde{l}_i} = h_{l_i} \circ \widehat{\varphi} \tag{5.2}$$

holds on $\widetilde{l}_i$, for every $\widetilde{l}_i$. Because of (5.2) one can arrange that for every geodesic arc α, that is a side of some octagon $\widetilde{P'}_i \in [\widetilde{P}_i]$, we have that $\widehat{\varphi}(\alpha)$ is also a geodesic arc (if α is contained in one of the geodesics from $\cup[\widetilde{l}_i]$ then this is already the case).

This directly follows from the fact that the endpoints of α belong to two geodesics from $\cup[\widetilde{l_i}]$, and no two such sides can intersect (except that they can meet at the endpoints which are always in $\cup[\widetilde{l_i}]$). Similarly, if α is a side in the parallelogram $\widetilde{A'}_i \in [\widetilde{A}_i]$ we can arrange that $\widehat{\varphi}(\alpha)$ is a geodesic arc, geodesic ray or a circular arc in $\mathbf{T}$, depending on which one of these is α.

We set $\widehat{\varphi}(\widetilde{P}_i) = P_i$ and $\widehat{\varphi}(\widetilde{A}_i) = A_i$, for every $\widetilde{P}_i$ ($i \in [1, n_1]$) and $\widetilde{A}_i$ ($i \in [1, n_2]$). Note that each P_i is an octagon and each A_i is a parallelogram (P_i and A_i are not necessarily right-angled anymore). Also, $\widehat{\varphi}([\widetilde{P}_i]) = [P_i]$ and $\widehat{\varphi}([\widetilde{A}_i)] = [A_i]$. Since $\mathbf{D}$ is a union of $\cup[\widetilde{P}_i]$ and $\cup[\widetilde{A}_i]$, we have that the same is true for $[P_i]$ and $[A_i]$.

Lemma 5.2. *Let $f \in \mathcal{G}$ and fix P_i, for some $i \in [1, n_1]$. Then we can choose $L_1 = L_1(K)$, $\rho_1 = \rho_1(K)$, and a L_1-quasiisometry $\widehat{f}$, which extends f, such that*

1. *$\widehat{f}$ agrees with $E(f)$ on $\cup[l_i]$.*
2. *$\widehat{f}(P_i) = \widehat{\varphi}(\widehat{u}(\widetilde{P}_i))$, where $u = \varphi^{-1} \circ f \circ \varphi$.*
3. *Let $f \in \mathcal{G}$ be such that for the above defined extension $\widehat{f}$ we have $\widehat{f}(P_i) \cap P_i = \beta$, where β is a side of P_i. Then, $\left(\widehat{f}^{-1} \circ \widehat{(f^{-1})}\right)(z) = z$, for $z \in \beta$.*
4. *$\widehat{f}$ is $\delta_1(\epsilon)$ continuous, where $\delta_1 : \mathbf{R}^+ \to \mathbf{R}^+$ is a function of K.*
5. *For every $z \in \mathbf{D}$, we have $\mathbf{d}(\widehat{f}(z), z) > \rho_1$.*

Also, for a fixed A_i, $i \in [1, n_2]$, we can choose $L_1 = L_1(K)$, $\rho_1 = \rho_1(K)$, and a L_1-quasiisometry $\widehat{f}$, which extends f, such that

1. *$\widehat{f}$ agrees with $E(f)$ on $\cup[l_i]$.*
2. *$\widehat{f}(A_j) = \widehat{\varphi}(\widehat{u}(\widetilde{A}_i))$, where $u = \varphi^{-1} \circ f \circ \varphi$.*
3. *Let $f \in \mathcal{G}$ be such that for the above defined extension $\widehat{f}$ we have $\widehat{f}(A_i) \cap A_i = \beta$, where β is a side of A_i. Then, $\left(\widehat{f}^{-1} \circ \widehat{(f^{-1})}\right)(z) = z$, for $z \in \beta$.*
4. *$\widehat{f}$ is $\delta_1(\epsilon)$ continuous, where $\delta_1 : \mathbf{R}^+ \to \mathbf{R}^+$ is a function of K.*
5. *For every $z \in \mathbf{D}$, we have $\mathbf{d}(\widehat{f}(z), z) > \rho_1$.*

Proof. Let $\widetilde{f}$ be the L-quasiisometry from Lemma 5.1. $\widetilde{f}$ already agrees with $E(f)$ on $\cup[l_i]$. This implies that the octagons $\widetilde{f}(P_i)$ and $\widehat{\varphi}(\widehat{u}(\widetilde{P}_i))$ have the corresponding four sides in common. Now, by Lemma 2.2 we can choose a L_1-quasiisometry $\widehat{f}$, which agrees with $\widetilde{f}$ on $\cup[l_i]$, such that for a fixed P_i we have $\widehat{f}(P_i) = \widehat{\varphi}(\widehat{u}(\widetilde{P}_i))$ and such that $\widehat{f}$ is $\delta_1(\epsilon)$ continuous.

Let $f \in \mathcal{G}$ be such that for the above defined extension $\widehat{f}$ we have $\widehat{f}(P_i) \cap P_i = \beta$, where β is a side of P_i. First we define the extension H_1 of f^{-1} which satisfies (1), (2) and (4) of this lemma. Then we post-compose H_1 with a homeomorphism H_2 of $\mathbf{D}$ which pointwise fixes the set $\cup[l_i]$, and all the sides of P_i except β. Since the hyperbolic distance between H_1 and $(\widehat{f})^{-1}$ is bounded above (by a bound which is a function of K), we can choose H_2 to be a quasiisometry and uniformly continuous, so that $H_2 \circ H_1 = \widehat{f^{-1}}$ satisfies (3).

The item (5) follows from Lemma 5.1. We proceed similarly to define $\widehat{f}$ that corresponds to a fixed A_i.

∎

Let $f \in \mathcal{G}$ and let $z \in \mathbf{D}$. If $z \in [l_i]$ then we have already defined $E(f)(z)$. Suppose $z \in [P_i]$. Then there exists $P_i' \subset [P_i]$ that contains z. If z belongs to a side of the octagon P_i' then there will exist at least two, and at most three (this can happen only in the case when z is in some $[l_i]$), octagons from this partition that contain z. Recall that four sides of the octagon P_i' lie in $\cup[l_i]$. Since $E(f)$ is defined there, $E(f)$ uniquely determines the octagon $P_i'' = E(f)_*(P_i')$ whose four sides are the images of the four sides of P_i' under $E(f)$. In the same way we can find the elements $g', g'' \in \mathcal{G}$ such that $g'_*(P_i) = P_i'$ and $g''_*(P_i) = P_i''$ (such g', g'' are unique). We define $E(f)(z)$ by

$$E(f)(z) = \left(\widehat{g''} \circ \widehat{g'}^{-1}\right)(z), \tag{5.3}$$

where $\widehat{g'}$ and $\widehat{g''}$ are the extensions from Lemma 5.2 of g' and g'' respectively (these extensions correspond to the fixed P_i). It follows from Lemma 5.2, that if z belongs to a side of P_i', than the definition of $E(f)(z)$ does not depend on the choice of the octagon which contains z. This, together with Lemma 5.1, shows that $E(f)$ is continuous and therefore a homeomorphism on every $[P_i]$. We similarly define $E(f)$ on every $[A_i]$.

Lemma 5.3. *Let $f \in \mathcal{G}$. Then $E(f)$ is a L_2-quasiisometry of $\mathbf{D}$, $L_2 = L_2(K)$, and $E : \mathcal{G} \to E(\mathcal{G})$ is an isomorphism. $E(f)$ is $\delta_2(\epsilon)$ continuous, where $\delta_2(\epsilon)$ is a function of K. Also, there exist $\rho_1 = \rho_1(K)$ such that $\mathbf{d}(E(f)(z), z) > \rho_1$.*

Proof. The fact that $E(f)$ is an isomorphism and $\delta_2(\epsilon)$ continuous follows from the definition of E. It also follows from definition of E that there exists $L_2' = L_2'(K)$ with the following properties. For every $z \in \mathbf{D}$, there is a L_2'-quasiisometry $\bar{f}$, that extends f, such that $E(f)(z) = \bar{f}(z)$ (at every step of the way we have always defined E to be a restriction of some quasiisometric map). The rest follows Proposition 2.1, Lemma 5.1 and Lemma 5.2.

□

Theorem 5.1. *Let $\mathcal{G}$ be a K-quasisymmetric group that does not contain any $\widetilde{\epsilon}(K)$-small elements for some $0 < \widetilde{\epsilon}(K) < \epsilon(K)$ ($\epsilon(K)$ is the constant from Theorem 4.1). Then there exist a Fuchsian group $\mathcal{F}$, $K_3 = K_3(K)$, and a K_3-quasisymmetric mapping $\varphi : \mathbf{T} \to \mathbf{T}$ such that $\mathcal{G} = \varphi \mathcal{F} \varphi^{-1}$.*

Proof. If $\mathcal{G}$ is finitely generated we have established above that $E(\mathcal{G})$ is a L_2-quasiisometric group (this means that every element of $E(\mathcal{G})$ is a L_2-quasiisometry) so that $\mathbf{d}(E(f)(z), z) > \rho_1$ and $E(f)$ is $\delta_2(\epsilon)$ continuous. By Lemma 2.3 there exists a K_3-quasisymmetric mapping $\varphi : \mathbf{T} \to \mathbf{T}$ such that $\mathcal{G} = \varphi \mathcal{F} \varphi^{-1}$, for some Fuchsian group $\mathcal{F}$. The case of infinitely generated $\mathcal{G}$ follows from Proposition 1.3. □

6. Torsion-free quasisymmetric groups

Let $\mathcal{G}$ be a torsion-free K-quasisymmetric group and assume that $\mathcal{G}$ does not contain any $\widehat{\epsilon}(K)$-small elements, for some $0 < \widehat{\epsilon}(K) < \epsilon(K)$. Here $\epsilon(K)$ is the constant from Theorem 4.1. We show in this section that such a group is a quasisymmetric conjugate of a Fuchsian group. Our aim is to construct a homomorphism E (which is an isomorphism onto its image) of $\mathcal{G}$ into the group of quasiconformal self mappings of $\mathbf{D}$. We first assume that $\mathcal{G}$ is finitely generated. Since $\mathcal{G}$ is topologically conjugate to a Fuchsian group we conclude that there are $p_1, .., p_n \in \mathcal{G}$, $n \in \mathbf{N}$, mutually non conjugate parabolic elements, such that every other parabolic element of $\mathcal{G}$ is contained in some conjugacy class $[p_i] = \cup_{k\in\mathbf{Z}}(\cup_{f\in\mathcal{G}} f^{-1}\circ p_i^k \circ f)$. Note that this implies that each p_i is primitive, that is, it is not a power of another element from $\mathcal{G}$. Also, if $f\in\mathcal{G}$ fixes the fixed point of some p_i then f belongs to the cyclic group generated by p_i. In each class $[p_i]$ fix one representative that is primitive, say p_i, and let $x_i\in\mathbf{T}$ be the point fixed by p_i. There exists a parabolic Möbius transformation u_i and a $\widetilde{K}$-quasisymmetric map $\psi_i:\mathbf{T}\to\mathbf{T}$, such that $\psi_i\circ u_i = p_i\circ\psi_i$ (also $\psi_i(x_i)=x_i$). Let $\widehat{\psi}_i$ be some $\widetilde{K}$-quasiconformal extension of ψ_i. Let H_i be a horoball that touches $\mathbf{T}$ at x_i and set $U_i=\widehat{\psi}_i(H_i)$.

Now, fix $i\in[1,n]$. Let $f=p_i^k$, for some $k\in\mathbf{Z}$. Let $\widehat{f}$ be a $\widetilde{K}^2$-quasiconformal extension of f defined by $\widehat{f}=\widehat{p}_i^k=\widehat{\psi}_i\circ\widehat{u}_i^k\circ\widehat{\psi}_i^{-1}$. For $f,g\in\mathcal{G}$, we say that $f\sim g$ with respect to x_i, if $f(x_i)=g(x_i)$. Denote the corresponding equivalence class by $[f]_i$. Note that for $f\sim p_i$ we have already defined the extension $\widehat{f}$ because each such f must be in the corresponding cyclic group. In every other equivalence class $[f]_i$ choose one representative, say f, and let $\widehat{f}$ be a fixed K-quasiconformal extension. For every other $g\in[f]_i$ there exist $k\in\mathbf{Z}$ such that $f^{-1}\circ g={p_i}^k$. Set $\widehat{g}=\widehat{f}\circ\widehat{p}_i^k$. Therefore, for $f\in\mathcal{G}$ we have defined the extension $\widehat{f}$ which is $\widetilde{K}^3$-quasiconformal. Set $[U_i]=\cup_{f\in\mathcal{G}}\widehat{f}(U_i)$. Now, repeat this process for each i (see Figure 4).

Let $R=R(K)$ be the constant from Lemma 3.1. By Lemma 4.1 and Lemma 4.2, we can choose the horoballs H_i, $i\in[1,n]$, so that if $f,g\in\mathcal{G}$ are such that $f(x_j)\neq g(x_k)$, where $j,k\in[1,n]$ are any two numbers (j,k may be equal), then

$$\mathbf{d}(\widehat{f}(U_j),\widehat{g}(U_k)) > R. \tag{6.1}$$

Let U' be a component of $\cup[U_i]$, and let $f\in\mathcal{G}$ be the parabolic element that fixes U' (such exists by construction). Since R is fixed and by choosing the horoballs properly, we can also arrange (see Lemma 4.1 item (5)) that

$$\mathbf{d}(z,\widehat{f}(z)) > \epsilon_1(K) > 0, \tag{6.2}$$

for z from the quasihorocircle $\partial U'$, and some constant $\epsilon_1(K)>0$. Set $\Omega=\mathbf{D}-\cup[U_i]$. It follows from (6.1) that Ω is a simply connected region whose boundary consists of $\mathbf{T}$ and $\cup\partial[U_i]$.

For $f\in\mathcal{G}$ we first define $E(f)$ on each $[U_i]$. Let U_i' be a component of $[U_i]$ (U_i' touches $\mathbf{T}$ at x_i'). Set $x_i''=f(x_i')$. Let $g',g''\in\mathcal{G}$ be such that $g'(x_i)=x_i'$ and $g''(x_i)=x_i''$ (such $g',g''\in\mathcal{G}$ are not uniquely determined). Then, there is $k\in\mathbf{Z}$, such that $g''\circ p_i^k\circ {g'}^{-1}=f$. Set

$$\left(\widehat{g''}\circ\widehat{p}_i^k\circ{\widehat{g'}}^{-1}\right)(z)=E(f)(z),$$

for $z\in U_i'$. Repeat this construction for every $i\in[1,n]$.

This defines $E(f)$ on $\cup[U_i]=\mathbf{D}-\Omega$. From the definition of the extensions $\widehat{g'}$ and $\widehat{g''}$, it follows that $E(f)(z)$ is well defined, that is, it does not depend on the choice of g',g''.

Lemma 6.1. *$E(f)$ is a homeomorphism of $\cup[U_i]$ such that*

(1) *$E(f)\circ E(g)=E(f\circ g)$; $E(id)=id$.*

(2) *For every $f \in \mathcal{G}$ there exist $K' = K'(K)$ and a K'-quasiconformal map $\widetilde{f} : \mathbf{D} \to \mathbf{D}$, which extends f, such that $\widetilde{f} = E(f)$ on $\cup[U_i] = \mathbf{D} - \Omega$.*

Proof. The item (1) and the fact that $E(f)$ is a homeomorphism follow from the definition of E and the discussion above. On each component of a given $[U_i]$, the map $E(f)$ is a restriction of a $\widetilde{K}^8$-quasiconformal map $\widehat{g''} \circ \widehat{p}_i^k \circ \widehat{g'}^{-1}$. Combining this with (6.1), it follows from Lemma 3.1 that there exists a map $\widetilde{f}$ with the stated properties.

□

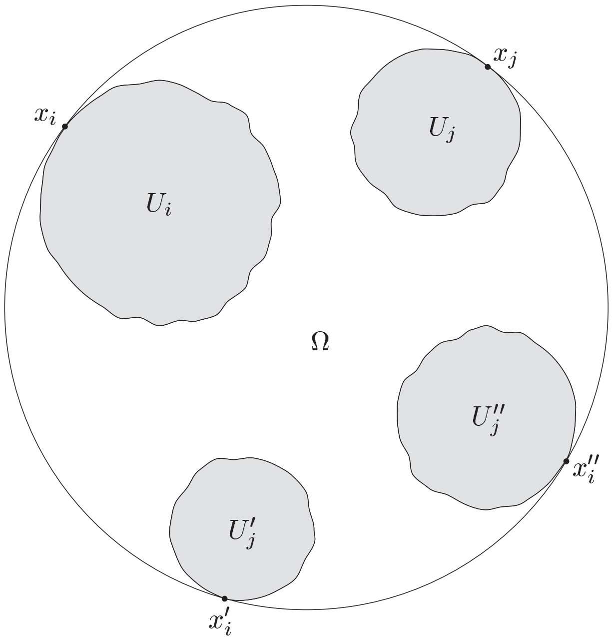

FIGURE 4

Let $\mathcal{G}_1$ be the group (isomorphic to the group $\mathcal{G}$) whose elements are the mappings $e(f) : \partial\Omega \to \partial\Omega$, where $e(f)$ is the restriction of $E(f)$. Let $\phi : \Omega \to \mathbf{D}$ be the Riemann map. Then $\phi\mathcal{G}_1\phi^{-1}$ is a K'-quasisymmetric group, where K' is from Lemma 6.1. Note that the group $\phi\mathcal{G}_1\phi^{-1}$ contains only hyperbolic elements. Moreover, we have assumed that $\mathcal{G}$ does not contain any $\widehat{\epsilon}(K)$-small elements. Then from (6.2) it follows that there is $\widetilde{\epsilon}(K) > 0$, such that $\phi\mathcal{G}_1\phi^{-1}$ does not contain any $\widetilde{\epsilon}(K)$-small elements. From Theorem 5.1 we conclude that the group $\phi\mathcal{G}_1\phi^{-1}$ is K_1-quasisymmetrically conjugated to a Fuchsian group, $K_1 = K_1(K)$. Conjugating the action of this Fuchsian group by the Riemann map, we get that for each $f \in \mathcal{G}$ there is a K_1-quasiconformal map $\widehat{e(f)} : \Omega \to \Omega$, which extends $e(f)$ to Ω, and such that the map $e(f) \to \widehat{e(f)}$ is an isomorphism of the group $\mathcal{G}_1$ onto a subgroup of the group of quasiconformal self maps of Ω. Set $E(f) = \widehat{e(f)}$ on Ω, for each $f \in \mathcal{G}$. Therefore, we have constructed an isomorphism of $\mathcal{G}$ onto a subgroup of the group of quasiconformal self maps of $\mathbf{D}$. Moreover, each $E(f)$ is K_1'-quasiconformal, where $K_1' = K_1'(K)$ is the maximum of the set $\{K_1, \widetilde{K}^8\}$.

Theorem 6.1. *Let $\mathcal{G}$ be a torsion-free K-quasisymmetric group that does not contain any $\widehat{\epsilon}(K)$-small elements, for some $0 < \widehat{\epsilon}(K) < \epsilon(K)$. Then there exist $K_2 = K_2(K)$, and a K_2-quasisymmetric map $\varphi : \mathbf{T} \to \mathbf{T}$, such that $\mathcal{G} = \varphi \mathcal{F} \varphi^{-1}$ for some Fuchsian group $\mathcal{F}$.*

Proof. If $\mathcal{G}$ is finitely generated then the existence of a K_2-quasisymmetric map φ and a Fuchsian group $\mathcal{F}$ follows from the fact that $E(\mathcal{G})$ is a K_1'-quasiconformal group. The infinitely generated case follows from Proposition 1.3. □

7. Elliptic elements of order greater than 2 and the barycentric extension

7.1. **Quasisymmetric groups with elliptic elements of order at least three.** For integer $m \geq 3$ let $u(z) = exp(\frac{2\pi i}{m})z$ be the rotation, $z \in \mathbf{T}$ (exp will sometime stand for the exponential function). Denote by O_u the orbit of the point $z = 1$. Then, irrespectively of the order m we can choose three points $x_i, x_2, x_3 \in O_u$, such that $\frac{\pi}{13} < \sigma(x_i, x_j) < \pi - \frac{\pi}{13}$, for every $i \neq j$ ($\frac{\pi}{13}$ is not the best bound). Here σ denotes the standard spherical metric on $\mathbf{T}$. By l_σ we will denote the spherical length.

Let $\varphi : \mathbf{T} \to \mathbf{T}$ be a homeomorphism, normalized by $\varphi(x_i) = x_i$, $i = 1, 2, 3$, such that the cyclic group generated by $\varphi \circ u \circ \varphi^{-1}$ is a K-quasisymmetric group (in this section the symbol φ always refers to a homeomorphism with these properties). From Proposition 1.2 we have that $\varphi = \psi \circ \phi$, where ϕ fixes pointwise the set O_u and ϕ commutes with u (or equivalently, ϕ conjugates u to itself), and ψ is an $\widetilde{K}$-quasisymmetric map. Clearly $\psi(x_i) = x_i$, $i = 1, 2, 3$.

Set $l_1 = l_1(K) = \sup l_\sigma(\psi(\alpha))$, where the supremum is taken with respect to all arcs $\alpha \subset \mathbf{T}$ of length at most $2\pi - \frac{\pi}{13}$, and all $\widetilde{K}$-quasisymmetric maps $\psi : \mathbf{T} \to \mathbf{T}$ that fix three points, say x_1, x_2, x_3, such that $\frac{\pi}{13} < \sigma(x_i, x_j) < \pi - \frac{\pi}{13}$, $i \neq j$. Since this family of $\widetilde{K}$-quasisymmetric maps is a normal family we have $0 < l_1 < 2\pi$. On the other hand, let $\beta \subset \mathbf{T}$ be an arc of the length at least $2\pi - \frac{\pi}{13}$. Let $l_2 = \inf_{m \geq 3} l_\sigma(\phi(\beta))$, where the infimum is taken with respect to all such β, and all homeomorphisms ϕ which commute with the rotation $u(z) = exp(\frac{2\pi i}{m})z$. Clearly $l_2 > \pi$. This proves the next lemma.

Lemma 7.1. *With the notation as above we have the following. Let α be an arc of the spherical length at least l_1. Then the spherical length $l_\sigma(\varphi^{-1}(\alpha))$ is at least l_2. Here l_2 is a fixed constant (does not depend on K).*

Set $l_2' = l_2 - (\frac{l_2 - \pi}{2})$. Recall that for $z \in \mathbf{D}$, a_z is the hyperbolic Möbius transformation such that $\widehat{a_z}(0) = z$, and which preserves the geodesic that contains both 0 and z. Let $d_0 > 0$ be small enough such that for every $z \in \Delta(0, d_0)$ we have $l_\sigma(\widehat{a}_z(\alpha)) > l_2'$, for every arc α of the length at least l_2.

Denote by $\widehat{\varphi}$ the barycentric extension of the homeomorphism φ defined above.

Lemma 7.2. *With the notation as above the following holds.*

(1) *There exists* $r_0 = r_0(K) > 0$ *so that for every* $z \in \Delta(0, d_0)$ *we have* $\mathbf{d}(\widehat{\varphi}(z), 0) < r_0$.
(2) $\widehat{\varphi}$ *is* K_1*-quasiconformal in* $\Delta(0, d_1)$. *Here,* $d_1 = d_1(K)$, $K_1 = K_1(K)$.

Proof. Let $r_0 > 0$ be such that for every $z \in \mathbf{D}$, $\mathbf{d}(z, 0) > r_0$, there exists an arc $\alpha \subset \mathbf{T}$ of the spherical length at least l_1, such that $0 < l_\sigma(a_z(\alpha)) < 1 - \frac{\pi}{l_2'}$. Clearly $r_0 < \infty$. Let $w = \widehat{\varphi}(z)$ for $z \in \Delta(0, d_0)$. Then the barycentric extension of the map ${a_w}^{-1} \circ \varphi \circ a_z$ is equal to $\widehat{a}_w^{-1} \circ \widehat{\varphi} \circ \widehat{a}_z$. Moreover $\left(\widehat{a}_w^{-1} \circ \widehat{\varphi} \circ \widehat{a}_z\right)(0) = 0$. This implies (see [4],[1])

$$\int_{\mathbf{T}} \left({a_w}^{-1} \circ \varphi \circ a_z\right)(\zeta)|d\zeta| = 0. \tag{7.1}$$

Suppose that there exists $z \in \Delta(0, d_0)$ such that $\mathbf{d}(\widehat{\varphi}(z), 0) = \mathbf{d}(w, 0) > r_0$. There exists an arc $\alpha \subset \mathbf{T}$, $l_\sigma(\alpha) \geq l_1$, with the property that $l_\sigma(a_w^{-1}(\alpha)) < 1 - \frac{\pi}{l_2'}$. Let y be any point in $a_w^{-1}(\alpha)$. Then for $\beta = (\varphi \circ a_z)^{-1}(\alpha)$ we have

$$\left| y \int_\beta |d\zeta| - \int_\beta \left({a_w}^{-1} \circ \varphi \circ a_z\right)(\zeta)|d\zeta| \right| < l_\sigma(\beta)\left(1 - \frac{\pi}{l_2'}\right) < 2(l_2' - \pi). \tag{7.2}$$

Here we used that $l_\sigma(\beta) > l_2'$. This follows from Lemma 7.1, and the definition of l_2'. But then (7.2) contradicts (7.1). This proves (1).

Let $v : \mathbf{R} \to \mathbf{R}$ is an increasing function such that $\varphi(z) = \varphi(e^{it}) = e^{iv(t)}$. Since ϕ commutes with a rotation u of order at least three, and ψ is normalized $\widetilde{K}$-quasiconformal map (fixed the three points x_1, x_2, x_3), we conclude that there exists $C = C(K)$ such that $v(t+s) - v(t) > C$, for $\frac{2\pi}{3} < s < \pi$, and every $t \in \mathbf{R}$. Now (2) follows from Lemma 3.5.

□

Let μ be an element of the unit ball of the Banach space $L^\infty(\mathbf{D})$ of essentially bounded measurable functions on $\mathbf{D}$. Set $k_0 = \frac{K-1}{K+1}$. For $t \in [0, k_0]$, let $\widetilde{\eta}_t : \mathbf{D} \to \mathbf{D}$ be the quasiconformal map with the complex dilatation $t\mu$ and which fixes the points x_1, x_2, x_3 as above. Let $\eta_t : \mathbf{T} \to \mathbf{T}$ be the map, such that $\widetilde{\eta}_t$ extends η_t. Set $\varphi_t = \eta_t \circ \varphi$. By $\widehat{\varphi}_t$ we denote the barycentric extension of φ_t.

Lemma 7.3. *With the notation as above we have the following. Let* $h : [0, k_0] \to \mathbf{D}$ *be the curve* $h(t) = (\widetilde{\eta}_t^{-1} \circ \widehat{\varphi}_t)(0)$, $t \in [0, k_0]$. *Then for every* $\epsilon > 0$ *there exists* $\delta(K, \epsilon) > 0$, *such that* $\mathbf{d}(h(t), h(s)) < \epsilon$, *for* $|t - s| < \delta(K, \epsilon)$. *In other words,* $h(t)$ *is* $\delta(K, \epsilon)$ *continuous.*

Proof. It is enough to prove the statement of the lemma for the curve $e(t) = \widehat{\varphi}_t(0)$. This follows from the fact that the family $\widetilde{\eta}_t$ is uniformly Hölder continuous with respect to both $t \in [0, k_0]$, and the variable in any fixed compact set in $\mathbf{D}$ and that bound depends only on k_0, and that fixed compact set in $\mathbf{D}$. It follows from the proof below that $\widehat{\varphi}_t(0)$ does not leave the disc $\Delta(0, r_0)$

Proof by contradiction. Suppose that for some fixed $\epsilon > 0$ there exist sequences $\{\varphi_n\}$ and $\{\mu_n\}$, $n \in \mathbf{N}$, with the above properties, and such that for each n there are $t_n, s_n \in [0, k_0]$, $|t_n - s_n| < \frac{1}{n}$, so that

$$\mathbf{d}(e_n(t_n), e_n(s_n)) > \epsilon. \tag{7.3}$$

Here, $e_n : [0, k_0] \to \mathbf{D}$ denotes the curve $e_n(t) = \widehat{\varphi}_{n,t}(0)$, where $\varphi_{n,t} = \eta_{n,t} \circ \varphi_n$, and $\eta_{n,t}$ is the normalized (fixing x_1, x_2, x_3) quasisymmetric map that can be extended to the normalized quasiconformal map with the complex dilatation $t\mu_n$. Let $n \to \infty$. Then we can assume that t_n, s_n tend to $t_0 \in [0, k_0]$. After passing to a subsequence if necessary, the mappings η_{n,t_n}, η_{n,s_n} converge to a K-quasisymmetric map η_∞ which fixes x_1, x_2, x_3. Write $\varphi_n = \psi_n \circ \phi_n$. Here ϕ_n fixes pointwise the set O_{u_n} and it commutes with u_n, and ψ_n is an $\widetilde{K}$-quasisymmetric map which fixes the three points $x_1, x_2, x_3 \in O_u$ satisfying $\frac{\pi}{13} < \sigma(x_i, x_j) < \pi - \frac{\pi}{13}$, for every $i \neq j$. Here, for every $n \in \mathbf{N}$, u_n is the rotation of order $m_n \geq 3$. After passing to a subsequence if necessary, ψ_n converges to a normalized $\widetilde{K}$-quasisymmetric map ψ_∞.

Consider the sequence $\phi_{n*}\sigma_0$ of the corresponding probability measures on $\mathbf{T}$. Here σ_0 denotes the normalized Lebesgue measure on $\mathbf{T}$ (we have already used σ to denote the ordinary (non normalized) Lebesgue measure on $\mathbf{T}$). We have the induced sequences ${\varphi_{n,t_n}}_*\sigma_0$ and ${\varphi_{n,s_n}}_*\sigma_0$ of probability measures. After passing to a subsequence if necessary, $\phi_{n*}\sigma_0$ converges to a probability measure ϕ_∞ on $\mathbf{T}$. Since ϕ_n commutes with a standard rotation of order at least three we conclude that if ϕ_∞ has atoms, then ϕ_∞ has at least three atoms of the same mass, and therefore the mass of every atom is at most $\frac{1}{3}$.

Remark. The above observation is the key observation of this section. It is not valid if the order of the rotation is allowed to be two, because then ϕ_∞ can contain strong atoms.

This implies that both ${\varphi_{n,t_n}}_*\sigma_0$ and ${\varphi_{n,s_n}}_*\sigma_0$ converge to the probability measure $\varphi_\infty = (\eta_\infty \circ \psi_\infty)_* \phi_\infty$. We have seen that every atom (if any) of φ_∞ has the mass at most $\frac{1}{3}$. Therefore, the sequences $\widehat{\varphi}_{n,t_n}(0) = Bar({\varphi_{n,t_n}}_*\sigma_0)$ and $\widehat{\varphi}_{n,s_n}(0) = Bar({\varphi_{n,s_n}}_*\sigma_0)$, both converge to the point $\widetilde{\varphi}_\infty(0)$. Here Bar stands for the barycenter of the corresponding measure. Note that since the measure φ_∞ has no strong atoms we have that $\widetilde{\varphi}_\infty(0) \in \mathbf{D}$. On the other hand, from Lemma 7.2 we know that the points $\widehat{\varphi}_{n,t_n}(0)$ and $\widehat{\varphi}_{n,s_n}(0)$ are contained in $\Delta(0, r_1)$, $r_1 = r_1(K)$. Therefore the sequences $\{\widehat{\varphi}_{n,t_n}(0)\}$, $\{\widehat{\varphi}_{n,s_n}(0)\}$, after passing to a subsequence, converge to the same point in the closure $\Delta(0, r_1)$. But this contradicts (7.3).

□

7.2. **Removing elliptic elements of order at least three.** We are in the position to prove the following theorem.

Theorem 7.1. *For an arbitrary K-quasisymmetric group $\mathcal{G}$ there exists a K_1-quasisymmetric group $\mathcal{G}_1$, $K_1 = K_1(K)$, with the following properties.*

1. *$\mathcal{G}_1$ does not contain elliptic elements of order three or more.*
2. *If $\mathcal{G}_1$ is K'-quasisymmetrically conjugated conjugated to a Fuchsian group, $K' = K'(K)$, then there exists $K'' = K''(K)$, such that $\mathcal{G}$ is K''-quasisymmetrically conjugated to a Fuchsian group.*

Proof. Let $\mathcal{G}$ be an arbitrary K-quasisymmetric group. Let $\mathcal{F}$ be a Fuchsian group and let $\varphi : \mathbf{T} \to \mathbf{T}$ be a homeomorphism, so that $\varphi \mathcal{F} \varphi^{-1} = \mathcal{G}$. Denote by E' the set of all points in $\mathbf{D}$, such that $z \in E'$ if $\widehat{u}(z) = z$, for some elliptic element $\widehat{u} \in \widehat{\mathcal{F}}$ of order at least three. We have by Lemma 4.3, that E' is a ρ_0-discrete set for some universal constant $\rho_0 > 0$. Let $\widehat{\varphi}$ be the barycentric extension of φ. Set $E = \widehat{\varphi}(E')$ and $S = \mathbf{D} - E$. Clearly, E is a discrete subset of $\mathbf{D}$, and S is a Riemann surface.

The group of homeomorphisms $\widehat{\mathcal{G}} = \widehat{\varphi}\widehat{\mathcal{F}}\widehat{\varphi}^{-1}$ naturally acts on S. Denote by $\mathcal{G}'$ the subgroup of the mapping class group of S induced by $\widehat{\mathcal{G}}$. We have that $\mathcal{G}'$ is isomorphic to $\mathcal{G}$ (because $\widehat{\mathcal{G}}$ is). We show that $\mathcal{G}'$ is a K_1-quasisymmetric group.

Fix $f \in \mathcal{G}$ and let $\widehat{f} = \widehat{\varphi} \circ \widehat{v} \circ \widehat{\varphi}^{-1}$ where $v \in \mathcal{F}$, such that $\varphi \circ v \circ \varphi^{-1} = f$. We need to show that $\widehat{f}$ is isotopic (as a selfmap of S) to a K_1-quasiconformal map. Let $\eta : \mathbf{T} \to \mathbf{T}$ be a $\widetilde{K}$-quasisymmetric map such that $\eta \circ f \circ \eta^{-1} = \omega$ for some $\omega \in \mathcal{M}$. Because of the conformal naturallity of the barycentric extension, we conclude that $\widehat{f}$ is isotopic $(rel\ \partial S)$ to the map $A^{-1} \circ \widehat{\omega} \circ A$. Here $A = B \circ \widehat{\varphi}^{-1}$, where B is the barycentric extension of $\eta \circ \varphi$. So, it is enough to show that A is K_1'-quasisymmetric map of S, $K_1' = K_1'(K)$, (note that A does not have to map S onto itself). Let $\mu' \in L^{\infty}(\mathbf{D})$ be the complex dilatation of η, and set $\mu = \frac{\mu'}{|\mu'|}$. For $t \in [0, k_0]$, let $\widetilde{\eta}_t : \mathbf{D} \to \mathbf{D}$ be the quasiconformal map with the complex dilatation $t\mu$, and which fixes the points $1, i, -1$. Let $\eta_t : \mathbf{T} \to \mathbf{T}$ be the map such that $\widetilde{\eta}_t$ extends η_t. Here $t \in [0, k_0]$, $k_0 = \frac{K-1}{K+1}$. Set $\varphi_t = \eta_t \circ \varphi$, and let $\widehat{\varphi}_t$ be the barycentric extension of φ_t. Finally, let $A_t = \widehat{\varphi}_t \circ \widehat{\varphi}^{-1}$ and $E_t = A_t(E)$. Note $A = A_{k_0}$. From Lemma 7.2 (item (2)) and Lemma 4.3 we have that E_t is $\rho(K)$-discrete set, $t \in [0, k_0]$. Consider the map $\widetilde{\eta}_t^{-1} \circ A_t$. Note that the restriction of $\widetilde{\eta}_t^{-1} \circ A_t$ on $\mathbf{T}$ is the identity. After properly pre composing and post composing map $\widehat{\eta}_t^{-1} \circ A_t$ by Möbius transformations, it follows from Lemma 7.3 that this map satisfies the assumptions of Lemma 3.3. This proves that f is isotopic $(rel\ \partial S)$ to a K_1-quasiconformal map.

Now, we cover the surface S by the unit disc and lift the action of $\mathcal{G}'$ to the unit disc. This lifted group $\mathcal{G}''$ is a K_2-quasisymmetric group of $\mathbf{T}$. Moreover, $\mathcal{G}''$ does not have any elliptic elements of order three or more. Note that there is a natural homomorphism $N : \mathcal{G}'' \to \mathcal{G}'$, and the kernel Ker_N of this homomorphism is the group of covering transformation from the covering of S. Define $\mathcal{G}_1 = \mathcal{G}''$ ($\mathcal{G}_1$ is the group from the statement of this theorem). We have that $\mathcal{G}_1$ satisfies (1).

Assume that $\mathcal{G}_1 = \mathcal{G}''$ is K'-quasisymmetrically conjugated to a Fuchsian group $\mathcal{F}''$. Let M be the normal subgroup of $\mathcal{F}''$ which corresponds to Ker_N under this conjugation. Then $\mathbf{D}/M$ is a Riemann surface which is biholomorphic to the disc $\mathbf{D}$ minus a discrete set of points. Let $S' = \mathbf{D}/M$. Note that there is a homomorphism of $\mathcal{F}''$ onto a conformal group $\mathcal{F}'$, and the kernel of this homomorphism is M. $\widehat{\mathcal{F}'}$ acts on S', and if we see S' as a subset of $\mathbf{D}$ (which we can because of the biholomorphic type of S'), then $\widehat{\mathcal{F}'}$ acts on $\mathbf{D}$ as well. Therefore, $\widehat{\mathcal{F}'}$ is a Fuchsian group. It follows that the above conjugation induces a homeomorphism between S and S', such that the restriction of this homeomorphism on $\mathbf{T}$, $\varphi : \mathbf{T} \to \mathbf{T}$, is a K''-quasisymmetric map, $K'' = K''(K)$, which conjugates the group $\mathcal{G}$ to the Fuchsian group $\mathcal{F}'$. This proves (2).

□

8. Elliptic elements of order 2 and the proof of Theorem 1.1

Throughout next two subsections we assume that $\mathcal{G}$ is a K-quasisymmetric group that has no $\epsilon(K)$-small hyperbolic elements, and that $\mathcal{G}$ does not contain any elliptic elements of order three or more. Here $\epsilon(K)$ is the constant from Theorem 4.1.

8.1. **Quasicenter of an elliptic element of order two.** Denote by $\mathcal{E}$ the set of elliptic elements of $\mathcal{G}$ of order two. We adopt a few notions introduced by Gabai (see [8]). Let l be an oriented geodesic in $\mathbf{D}$ with the endpoints a and b (positive orientation is from a to b). H^r and H^l denote respectively the right and the left halfspaces determined by l. This is chosen so that the arc $[a,b] \subset \mathbf{T}$ borders H^r (we use the standard counterclockwise orientation on $\mathbf{T}$). Let $e \in \mathcal{E}$. A pair of points $x, y \in \mathbf{T}$ is said to be an orbit of e if $e(x) = y$. We say that e is to the left of l if there is a pair of points that is an orbit of e and that belong to the open arc $(b,a) \subset \mathbf{T}$. Similarly we define the notion to the right. We say that e is on l if the pair a, b is an orbit of e (it is clear that every $e \in \mathcal{E}$ has to be in one and only one category). If e was a Möbius transformations, then the center of the map $\widehat{e}$ would have been in H^r; in H^l; on l, depending on whether e is to the right of l; to the left of l; on l. Let $h \in \mathcal{G}$ be a hyperbolic element, and let a_h, b_h denote its repelling and attracting fixed points respectively. Let l_h be the oriented geodesic with endpoints a_h, b_h (the positive orientation is from a_h to b_h). H^r_h and H^l_h denote respectively the right and the left halfspaces determined by l_h. We say that e is to the left of h; to the right of h; on h, if e is to the left of l_h; to the right of l_h; on l_h.

Let $L \geq K$ and $e \in \mathcal{E}$. We say that a point $z \in \mathbf{D}$ is an L quasicenter of e if there exists a L-quasiconformal map $\widehat{e} : \mathbf{D} \to \mathbf{D}$, which extends e and such that $\widehat{e} \circ \widehat{e}$ is the identity map. Since $\mathcal{G}$ is K-quasisymmetric there exists at least one $\widetilde{K}^2$ quasicenter for every $e \in \mathcal{E}$.

Lemma 8.1. *With the notation as above, there exists $D = D(K) > 0$ such that the following holds. Let l be an oriented geodesic and $e \in \mathcal{E}$. If $z \in \mathbf{D}$ is an L quasicenter of e, the following holds.*

1. *If e is to the right (left) of l, then z either belongs to H^r (H^l), or $\mathbf{d}(l,z) < D$.*
2. *If e is on l, then $\mathbf{d}(l,z) < D$.*
3. *For any $r > 0$ there exists $L' = L'(L,r) \geq L$, so that if $z \in \mathbf{D}$ is an L quasicenter for e, then every point in $\Delta(z,r)$ is an L' quasicenter for e.*
4. *There exists $r(L) > 0$ such that every L quasicenter of $e \in \mathcal{E}$ is contained in a fixed hyperbolic disc of radius $r(L)$.*

Proof. Let $\widehat{e} : \mathbf{D} \to \mathbf{D}$ be as above, that is $\widehat{e}$ extends e, and $\widehat{e} \circ \widehat{e} = id$. Then we can choose a $\widetilde{K}$-quasiconformal map $\widehat{f}$, such that $\widehat{e} = \widehat{f}^{-1} \circ \widehat{e_0} \circ \widehat{f}$, where $\widehat{e_0}(z) = -z$, for $z \in \mathbf{D}$, is the standard order two rotation. Note that $\widehat{f}(z_0) = 0$, where z_0 is the L quasicenter that corresponds to $\widehat{e}$. Let $\gamma = \widehat{f}(l)$, and let l' be the geodesic with the same endpoints as γ. Then 0 is to the left of l'; to the right of l'; on l', if and only if e is to the left of l; to the right of l; on l. Therefore z_0 is to the left of $\widehat{f}^{-1}(l')$; to the right of $\widehat{f}^{-1}(l')$; on $\widehat{f}^{-1}(l')$, if and only if e is to the left of l; to the right of l; on l. Since $\widehat{f}$ is $\widetilde{K}$-quasiconformal, (1) and (2) follow.

If $z \in \Delta(z_0, r)$, let $\widehat{g} : \mathbf{D} \to \mathbf{D}$ be a K'-quasiconformal map, $K' = K'(L,r)$, which maps z_0 to z, and which is the identity on $\mathbf{T}$. Then, z is the fixed point of the map $\widehat{e'} = \widehat{g}^{-1} \circ \widehat{e} \circ \widehat{g}$. This proves (3).

Let z, z' be two L quasicenters of e, and $\widehat{f}$ and $\widehat{f}'$ the corresponding $\widetilde{K}$-quasiconformal maps that conjugate e to e_0, where z, z' are respectively the fixed points of

$\widehat{f}^{-1} \circ \widehat{e_0} \circ \widehat{f}$, and $\widehat{f'}^{-1} \circ \widehat{e_0} \circ \widehat{f'}$. Therefore, there is a $\widetilde{K}^2$-quasiconformal map which maps z to z'. This proves the last part. $\square$

Recall that we assume that $\mathcal{G}$ has no $\epsilon(K)$-small hyperbolic elements, nor elliptic elements of order three or more.

Lemma 8.2. *Let $\mathcal{G}$ be a K-quasisymmetric group. Let $c : \mathcal{E} \to \mathbf{D}$ be the map that to each $e \in \mathcal{E}$ associates an L quasicenter $c(e)$, $L = L(K)$. There exists $N = N(K) \in \mathbf{N}$, such that in each geodesic ball of radius 1 there are at most N points from $c(\mathcal{E})$. In addition, if we assume that E is a $\rho(K)$- discrete set , $\rho(K) > 0$, then for each $f \in \mathcal{G}$, there exists a $\widehat{K} = \widehat{K}(K)$-quasiconformal map $\widehat{f}$, which extends f, and such that $\widehat{f}(c(e)) = c(f \circ e \circ f^{-1})$, for every $e \in \mathcal{E}$.*

Proof. We prove the first part by contradiction. Assume that $\mathcal{G}_n$ is a sequence of K-quasisymmetric groups such that for each $n \in \mathbf{N}$, we have at least n mutually different $e^n_1, ..., e^n_n \in \mathcal{E}$, and the corresponding L quasicenters $c_n(e_1), ..., c_n(e_n)$ are in the hyperbolic disc $\Delta(0,1)$. For each $1 \leq i \leq n$, by f^n_i we denote a $\widetilde{K}$-quasiconformal map, that conjugates e^n_i to the standard order two rotation e_0, and such that there is a $\widetilde{K}$-quasiconformal extension $\widehat{f^n_i}$, with $\widehat{f_i}^n(0) = c(e^n_i)$. The family of $\widetilde{K}$-quasiconformal maps which map 0 into $\Delta(0,1)$ is a normal family. We conclude that for every $\epsilon > 0$, there is n large enough, and a choice two maps $\widehat{f^n_i}$ and $\widehat{f^n_j}$, $i \neq j$, such that $\widehat{f^n_i} \circ \widehat{f^n_j}^{-1}$ is ϵ close to the identity map in the C^0 topology. In particular, we have that $e^n_i \circ (e^n_j)^{-1}$ is ϵ close to the identity in C^0 topology. The composition of two non identical elements of order two is always a non identity hyperbolic element, that is $e^n_i \circ {e^n_j}^{-1} \neq id$. By choosing ϵ small enough we obtain a contradiction, because none of the groups $\mathcal{G}_n$ has $\epsilon(K)$-small hyperbolic elements.

If $f \in \mathcal{G}$ and $\widetilde{f}$ a K-quasiconformal extension of f, then it follows from the proof of the previous lemma that $\mathbf{d}(\widetilde{f}(c(e)), c(f \circ e \circ f^{-1})) < r(K)$. Set $\widetilde{E} = \cup_{f \in \mathcal{G}} \widetilde{f}^{-1}(c(f \circ e \circ f^{-1})$. Since E is $\rho(K)$-discrete, and $\widetilde{f}$ is K-quasiconformal, it follows that $\widetilde{E}$ is $\rho_1(K)$-discrete, for some $\rho_1(K) > 0$. The proof of this lemma follows by applying Lemma 3.2. $\square$

8.2. **Removing the elliptic elements of order two.** Let $e \in \mathcal{E}$. We can find a $\widetilde{K}$-quasisymmetric map, that conjugates e to the map $e_0(z) = -z$. By conjugating the whole group $\mathcal{G}$ by this quasisymmetric map, we may assume that $\mathcal{G}$, if $\mathcal{E}$ is not empty, always contains the transformation e_0.

Now, fix $x \in \mathbf{T}$. Let G_x be the subgroup of $\mathcal{G}$ which contains all elements from $\mathcal{G}$ fixing the set $\{x, -x\}$. Each G_x contains at least e_0, and it is an elementary group. There is a $\widetilde{K}$-quasisymmetric map $\psi_x : \mathbf{T} \to \mathbf{T}$ with the following properties. ψ_x conjugates G_x to a Fuchsian group F_x which is an elementary group that fixes the same set $\{x, -x\}$. Denote by $\widehat{\psi}_x$ its $\widetilde{K}$-quasiconformal extension. This Fuchsian group may be assumed to contain e_0 and that the conjugation map conjugates e_0 to itself. Also, by Lemma 3.5 we may assume that the conjugation map preserves the origin and the geodesic s_x that connects x and $-x$. An element of F_x is either a hyperbolic transformation which preserves the points $x, -x$, or an elliptic element of order two that permutes x and $-x$. Denote by E''_x the set of fixed points (in $\mathbf{D}$) of all elliptic transformations from $\widehat{F_x}$ (note that $E''_x \subset s_x$). Set $E'_x = \widehat{\psi}^{-1}_x(E''_x)$. Let

$c : E'_x \to s_x$ be the induced map that associates to each $e \in \mathcal{G}_x$, the corresponding $\widetilde{K}^2$ quasicenter $c(e)$.

Repeat this process for every $x \in \mathbf{T}$ (by choosing the appropriate quasisymmetric map ψ_x for every fixed $x \in \mathbf{T}$). Set $E' = \cup_{x\in\mathbf{T}} E'_x$. Note that the origin 0 belongs to all $E'_x \subset s_x$ (and all $E'_x \subset s_x$). Let $c : E' \to \mathbf{D}$ be the induced map that associates to each $e \in \mathcal{E}$, and in particular $e \in G_x$, the corresponding $\widetilde{K}^2$ quasicenter $c(e) \in E'$.

Remark. Here we use the fact that every $e \in \mathcal{E}$ must be contained in E'_x, for some $x \in \mathbf{T}$. That is, for every $e \in \mathcal{E}$, there exists $x \in \mathbf{T}$ such that $x, -x$ is an orbit of e. In particular, if $h = e \circ e_0$ is the corresponding hyperbolic transformation then $x, -x$ are the fixed points of h. This is obviously true for Fuchsian groups, and Proposition 1.1 implies it for quasisymmetric groups.

If $z, w \in E'$ both belong to a fixed E'_x, for some $x \in \mathbf{T}$, then there exists $\rho'(K) > 0$, such that $\mathbf{d}(z, w) \geq \rho'(K)$. This follows from the fact that $\mathcal{G}$ contains no $\epsilon(K)$-small hyperbolic elements, and that $\widehat{\psi_x}$ is $\widetilde{K}$-quasiconformal. For each $x \in \mathbf{T}$, let T_x be a diffeomorphism of s_x onto itself, which preserves each x, $-x$ and 0, and such that

$$\mathbf{d}(t_1, t_2) - \frac{\rho'(K)}{3} < \mathbf{d}(T_x(t_1), T_x(t_2)) < \mathbf{d}(t_1, t_2) + \frac{\rho'(K)}{3} \tag{8.1}$$

It follows, with the aid of Lemma 8.2 (since E' is the set of $\widetilde{K}^2$ quasicenters of $\mathcal{E}$, it follows from Lemma 8.2 that E' is not too dense), that for each x we can choose T_x which satisfies (8.1), and such that the set

$$E = \cup_{x\in\mathbf{T}} T_x(E'_x) \tag{8.2}$$

is $\rho(K)$-discrete, for some $\rho(K) > 0$. We use the same notation for the induced map $c : E \to \mathbf{D}$ which associates to each $e \in \mathcal{E}$ the corresponding L'' quasicenter (since we moved points from E' only a finite distance, we have by Lemma 8.2 that $L'' = L''(K)$. By E_x we denote the subset of E that is contained in s_x.

For a given group $\mathcal{G}$, we have constructed the set E of L'' quasicenters. This set is fixed from now on. Let $\mathcal{F}$ be a Fuchsian group, and let $\varphi : \mathbf{T} \to \mathbf{T}$ be a homeomorphism such that $\mathcal{G} = \varphi\mathcal{F}\varphi^{-1}$. We may assume that $\mathcal{F}$ contains the transformation e_0 and that φ conjugates e_0 to itself. For each $y \in \mathbf{T}$, let F_y be the subgroup of $\mathcal{F}$ containing elements that fix the set $\{y, -y\}$. The set of centers of elliptic elements (which are all of order two) that lie on s_y is denoted by $\widetilde{E}_y$. Set $\widetilde{E} = \cup_{y\in\mathbf{T}} \widetilde{E}_y$. Let $\widetilde{c} : \widetilde{E} \to \mathbf{D}$ be that map that associates the center to each elliptic element from $\mathcal{F}$. Let $\varphi(y) = x$. Note that φ induces the map from $\widetilde{E}_y$ onto E_x by $c \circ \circ\varphi \circ \widetilde{c}^{-1}$. It follows from (8.2) (the definition of E) that this map is orientation preserving, where the orientation for sets $\widetilde{E}_y$ and E_x comes from the way they lie in s_y and s_x respectively.

We define an appropriate extension $\widehat{\varphi}$ of the above homeomorphism φ. For each $y \in \mathbf{T}$, the restriction of $\widehat{\varphi}$ to s_y is a homeomorphism which maps s_y onto s_x, where $x = \varphi(y)$ ($\widehat{\varphi}$ maps a radius onto a radius). Also $\widehat{\varphi}(0) = 0$, $\widehat{\varphi}(y) = x$, $\widehat{\varphi}(-y) = -x$. Furthermore, we can choose $\widehat{\varphi}$ with the following property. If $w = \widetilde{c}(u)$, for some elliptic $u \in \mathcal{F}$ of order two, then $\widehat{\varphi}(w) = z$, where $z = c(\varphi \circ u \circ (\varphi)^{-1})$.

Set $S=\mathbf{D}-E$, and let $\mathcal{G}_1$ be the subgroup of the mapping class group of S defined as follows. To $f\in\mathcal{G}$ we assign the isotopy class of the map $\widehat{\varphi}\circ\widehat{u}\circ\widehat{\varphi}^{-1}:S\to S$, where $u=\varphi^{-1}\circ f\circ\varphi$. The group of the corresponding isotopy classes is $\mathcal{G}_1$. The map from $\mathcal{G}$ onto $\mathcal{G}_1$ is an isomorphism.

Lemma 8.3. *There exists $K_2=K_2(K)$, such that the group $\mathcal{G}_1$ is K_2-quasisymmetric.*

Proof. Fix $f\in\mathcal{G}$. Let $\widetilde{f}=\widehat{\varphi}\circ\widehat{u}\circ\widehat{\varphi}^{-1}$, where $u=\varphi^{-1}\circ f\circ\varphi$. Our aim is to show that $\widetilde{f}$ is isotopic in S to a $K_2(K)$-quasiconformal map.

E is $\rho(K)$-discrete and E is the set of L'' quasicenters. Let $\widehat{f}$ be a $\widetilde{K}$-quasiconformal map from Lemma 8.2, that is, for every $z\in E$, $z=c(e)$, we have $\widehat{f}(z)=c(f\circ e\circ f^{-1})$.

Fix $x\in\mathbf{T}$, and set $\gamma_x=\widetilde{f}(s_x)$. Note that $\widehat{\varphi}^{-1}(\gamma_x)=\gamma'_y$ is a geodesic in $\mathbf{D}$ and $\widehat{u}(s_y)=\gamma'_y$, where $\varphi(y)=x$ and φ conjugates u to f. Let $v\in\mathcal{F}$ be an elliptic element of order two. Since the position of $\widetilde{c}(v)$ (the center of $\widehat{v}$) with respect to γ'_y respects whether v is to the right of , to the left of , or on γ'_y, we have that the same is true for the curve γ_x and the corresponding quasicenter from E. Also, if the endpoints of γ_x are not an orbit of e_0 then γ_x intersects each geodesic s_y at most once (this is because the corresponding statement is true for γ'_y) (see Figure 5).

Denote by $[\gamma_x]$ the homotopy class of γ_x in S. For $y\in\mathbf{T}$ let $a_y=\gamma_x\cap s_y$ if this intersection is not empty. For s_y, where this intersection is non empty, denote by $I_y\subset s_y$ the maximal open geodesic arc which contains $\gamma_x\cap s_y=a_y$, and such that I_y does not contain any points from E (except for a_y, if $a_y\in E$). If s_y contains no points from E other than 0, then then I_y is one of the two geodesic rays that end at 0 and which constitute s_y .

Let $\alpha:s_x\to\mathbf{D}$ be a curve in $\mathbf{D}$ that intersects each s_x, $x\in\mathbf{T}$, at most once, and set $b_y=\alpha\cap s_y$. We have that $\alpha\in[\gamma_x]$ if and only if the map α is isotopic to the restriction of $\widetilde{f}$ on s_x, which is equivalent to the following two conditions being satisfied.

(1) If $a_y\in E$ then $a_y=b_y$
(2) If a_y does not belong to E then $b_y\in I_y$.

Now we show that for each $x\in\mathbf{T}$ we can choose a L_1 bilipschitz quasigeodesic $\alpha:s_x\to\mathbf{D}$, $\alpha\in[\gamma_x]$, $L_1=L_1(K)$. If γ_x is one of the geodesics that contain the origin then $\alpha=\gamma_x$. If not, then γ_x intersects each s_y at most once. Let l be the geodesic with the same endpoints as the curve γ_x. Then, by Lemma 8.1 we have that

$$\mathbf{d}(l,I_y)<r(K),$$

and

$$\mathbf{d}(l,a_y)<r(K),a_y\in E, \tag{8.3}$$

for each I_y and for each $a_y\in E$. Since the set E is $\rho(K)$-discrete, it follows from (8.3) that we can choose a L'_1 bilipschitz quasigeodesic $\beta:l\to\mathbf{D}$, $L'_1=L'_1(K)$, such that the curve β is in $[\gamma_x]$.

Let $\alpha':s_x\to\beta$ be defined as follows (here $\beta=\beta(l)$). For $z\in s_x$, let $\alpha'(z)$ be the point $\beta\cap s_y$, where $y\in\mathbf{T}$ is the unique point such that s_y contains the point

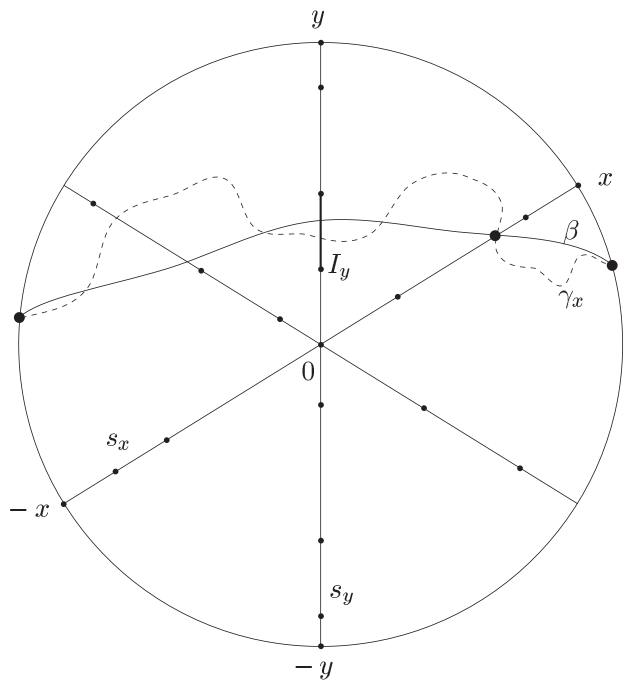

FIGURE 5

$\widetilde{f}(z)$. Either $\widetilde{f}(z) = \alpha'(z)$, or $\widetilde{f}(z)$ and $\alpha'(z)$ belong to the same I_y. The map α' does not have to be a quasiisometry but we have $\alpha'(z) = \widehat{f}$ for $z \in E_x$.

Since $\widehat{f}$ is $\widehat{K}$-quasiconformal (recall that $\widehat{f}$ is the map from Lemma 8.2 that corresponds to the set E), and since $\beta = \alpha'(s_x)$ is a bilipschitz quasigeodesic, we can construct $\alpha : s_x \to \beta = \alpha'(s_x)$, so that α is L_1 bilipschitz quasigeodesic.

Consider the map $\widehat{f}^{-1} \circ \widetilde{f} : S \to S$. This map is the identity on $\mathbf{T}$, and it fixes every point in E. Also, for each s_x, the restriction of the map $\widehat{f}^{-1} \circ \widetilde{f}$ on s_x is isotopic (rel ∂S) to a L_1 bilipschitz quasigeodesic. It follows from Lemma 3.4 that $\widetilde{f}$ is isotopic (rel ∂S) to a K_2-quasiconformal map, for every $f \in \mathcal{G}$, and the group $\mathcal{G}_1$ is K_2-quasisymmetric. □

Note that $\mathcal{G}_1$, as a group that acts on the Riemann surface S, does not have any elliptic elements, that is $\mathcal{G}_1$ is a torsion-free group.

8.3. **Proof of Theorem 1.1.** Let $\mathcal{G}$ be an arbitrary K-quasisymmetric discrete group. We want to prove that such a group is a quasisymmetric conjugate of a Fuchsian group.

By Theorem 7.1 we can assume that $\mathcal{G}$ has no elliptic elements of order three or more. Then, we can apply Theorem 4.1 to the group $\mathcal{G}$ and it follows from Theorem 4.1 that we can further assume that $\mathcal{G}$ does not contain any $\epsilon(K)$-small hyperbolic elements, where $\epsilon(K) > 0$ is the constant from Theorem 4.1.

Since $\mathcal{G}$ has no $\epsilon(K)$-small hyperbolic elements, and no elliptic elements of order three, it follows from Lemma 8.3 that the corresponding group $\mathcal{G}_1$ is a K_2-quasisymmetric, $K_2 = K_2(K)$, torsion-free group. But $\mathcal{G}_1$ acts on the Riemann surface $S = \mathbf{D} - E$. We now cover the surface S by the unit disc, and in the same way as in the proof of Theorem 7.1 we produce a new K_2-quasisymmetric, torsion-free group $\mathcal{G}_2$ (that acts on $\mathbf{D}$), such that $\mathcal{G}_2$ naturally projects to $\mathcal{G}_1$. Moreover, since $\mathcal{G}$ does not have any $\epsilon(K)$-small hyperbolic elements, and since the set E is $\rho_1(K)$ discrete (this implies that there are no very short closed geodesics on S), we can find $\widehat{\epsilon}(K) > 0$, such that $\mathcal{G}_2$ does not contain any $\widehat{\epsilon}(K)$-small hyperbolic elements. Same as in the proof of Theorem 7.1, we have that $\mathcal{G}_2$ is quasisymmetrically conjugated to a Fuchsian group if and only if $\mathcal{G}$ is.

So, $\mathcal{G}_2$ is K_2-quasisymmetric, $K_2 = K_2(K)$, torsion-free group, that does not contain any $\widehat{\epsilon}$-small hyperbolic elements. We can now apply Theorem 6.1. This concludes the proof of Theorem 1.1.

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University of Warwick, Institute of Mathematics, Coventry, CV4 7AL, UK
E-mail address: `markovic@maths.warwick.ac.uk`