MATEMATIQKI VESNIK
48 (1996), 107–112

UDK 517.54
originalni nauqni rad
research paper

THE UNIQUE EXTREMAL QC MAPPING AND UNIQUENESS OF HAHN-BANACH EXTENSIONS

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Abstract. Let χ be an essentialy bounded complex valued measurable function defined on the unit dise Δ, and let Λ_χ be the corrensponding linear functional on the space $\mathcal{B}$ of analytic L^1-integrable functions.

An outline of proof of main steps of the following is given: If $|\chi|$ is a constant function in Δ, then the uniqueness of Hahn-Banach extension of Λ_χ from B to L^1, when $\|\Lambda_\chi\| = \|\chi\|_\infty$, implies that χ is the unique complex dilatation.

We give a short review of some related results.

1. Introduction

Let $\Delta = \{\, |z| < 1 \,\}$, $\Gamma = \partial\Delta$, $\partial = (\partial_x - i\partial_y)/2$ and $\bar\partial = (\partial_x + i\partial_y)/2$.

Like many authors, we shall use "qc mapping" as an abbreviation for "quasi-conformal mapping".

For a qc mapping F on Δ, denote by $\mu = \mu[F]$ the complex dilatation $\mu[F] = \bar\partial F/\partial F$.

We let $L^\infty = L^\infty(\Delta)$ be the space of essentially bounded complex-valued measurable functions on Δ, and let M be the open unit ball in L^∞. For any χ in M there exists a solution $f\colon \Delta \to \Delta$ of the Beltrami equation

$$\bar\partial f = \chi\, \partial f \tag{1}$$

unique up to a postcomposition by a Möbius transformation.

We let f^χ be the solution f of (1) normalized by $f(i) = i$, $f(1) = 1$ and $f(-1) = -1$.

Two elements μ_0 and μ_1 in M are equivalent if f^{μ_0} and f^{μ_1} coincide on $\partial\Delta$.

For given $\mu \in M$ the equivalence class $[\mu]$ contains at least one element μ_0 such that

$$\|\mu_0\|_\infty = \inf\{\, \|\tau\|_\infty \;:\; \tau \in [\mu] \,\}.$$

AMS Subject Classification: Primary 32 G 15; Secondary 30 C 60; 30 C 70; 30 C 75.

Supported by Ministry of Science and technology RS, grant number 04M01

Such a μ_0 is referred to as an extremal complex dilatation and $f_0 = f^{\mu_0}$ as an extremal *qc* mapping (abbreviated EQC mapping)

Let $\mathcal{B}$ be the Banach space consisting of holomorphic functions φ, belonging to $L^1 = L^1(\Delta)$, with norm

$$\|\varphi\| = \iint_\Delta |\varphi(z)|\,dx\,dy < \infty, \quad \varphi \in \mathcal{B}.$$

For $\chi \in L^\infty$ we consider the linear functional

$$\Lambda_\chi(\varphi) = (\chi, \varphi), \quad \varphi \in \mathcal{B},$$

where

$$(\chi, \varphi) = \iint_\Delta \chi(z)\,\varphi(z)\,dx\,dy,$$

and denote by

$$\|\chi\|_* = \|\Lambda_\chi\|$$

the norm of χ as an element of the dual space of $\mathcal{B}$.

For a measurable subset D of $\mathbb{C}$ we denote by $|D|$ the Lebesque measure of D.

For $\chi \in L^\infty$ we say that it satisfies the Hamilton-Krushkal condition if $\|\chi\|_* = \|\chi\|_\infty$.

We are now ready to state the main result about extremal complex dilatations.

Hamilton-Krushkal and Reich-Strebel Theorem. *Let $\chi \in M$. A necessary and sufficient condition that f^χ is an EQC mapping is that*

$$\|\chi\|_* = \|\chi\|_\infty.$$

The interested reader can see more about this in [Ga]. Here we will mention only the following.

The proof of the necessity of the Hamilton-Krushkal condition is due to Hamilton and, independently, to Krushkal. The sufficiency of Hamilton-Krushkal condition was first proved by Reich and Strebel.

We shall write $\chi \in HBU$ if Λ_χ has the unique Hahn-Banach extension from $\mathcal{B}$ to L^1 and if $\|\chi\|_* = \|\chi\|_\infty < 1$.

In several papers Reich mentioned the following question concerning the uniqueness of EQC mapping (cf. [Re 1–5]).

Question A. *Is $\chi \in HBU$ necessary and/or sufficient for f^χ to be the unique EQC (abbreviated by UQC)?*

Reich proved that the answer is yes in some special cases and got several related results. (On Reich's contribution it will also be spoken later).

At this moment, we believe that answer is yes in general. We proved that the condition $\chi \in HBU$ is a sufficient condition for f^χ to be an UQC mapping.

For background concerning this subject and further references we refer the interested reader to Reich's papers [Re 1–5]. In particular, Reich's paper [Re 5] is very relevant.

Our main tool in this research is the procedure of "raising" $|\chi|$ (described in Lemma L) which enables us to make use of the uniqueness part of our assumption that $\chi \in HBU$.

Also, we use a corollary of "main inequality" obtained by Reich if $|\chi|$ is constant a.e. in Δ.

2. Proof of Theorem M.

Throughout this paper we will use notation $\|\chi\|_\infty = k$. In this section we will prove the following result.

THEOREM M. *Suppose that $\chi \in HBU$ and that $|\chi(x)| = k$ a.e. in Δ. Then $f = f^\chi$ is an UQC mapping.*

In [Re 3] E. Reich used some estimates obtained by "main inequality" to study the question when χ satisfies HBU condition. Before that, we state Reich's estimate, which was the starting point in our proof, and we need some notations.

Following [Re 3], for $\varphi \in \mathcal{B}$ set

$$\delta\{\varphi\} = k\|\varphi\| - \mathrm{Re}(\chi, \varphi), \quad \varphi \in \mathcal{B}.$$

Let f and g be qc self-mapping of Δ agreeing on Γ and let

$$\chi = \mu[f], \quad \alpha = \mu[f^{-1}] \circ f, \quad \beta = \mu[g^{-1}] \circ f$$

and $\varrho(z) = |\alpha(z) - \beta(z)|^2$.

In [Re 3], Reich proved the following result.

LEMMA RE. *Let $|\chi(z)| = k$ and $|\beta(z)| \leqslant k$ a.e. in Δ. Then there is a constant C which depends only on k such that*

$$(\varrho, |\varphi|) \leqslant C\,\delta\{\varphi\}$$

for every $\varphi \in \mathcal{B}$.

Proof of Theorem M. Recall that we suppose that $\chi \in HBU$. Let $f = f^\chi$ and let g be qc self mapping of Δ agreeing with f on Γ. By the above notation we must show that $\alpha = \beta$ a.e. in Δ.

If this is not true then there is $p > 0$ and a compact set $K \subset \Delta$ of positive measure such that $\varrho(z) \geqslant p$ in K. Next, we construct χ_r by means of "lifting" $|\chi|$ on K, which enables us to make use of the uniqueness part of the condition HBU (this idea is due to V. Marković). For convenience of the reader we state the following result.

Lemma L. *Let $\chi \in HBU$ and let K be a compact subset of Δ of positive measure and let $|\chi(z)| = k = \|\chi\|_\infty$ a.e. in K, and $r > 0$. Then there is a $\varphi = \varphi_r \in \mathcal{B}$, which is not identically zero, such that*

$$\delta_r = \delta\{\varphi_r\} \leqslant kr \iint_K |\varphi_r| \, dx \, dy \tag{1}$$

First, we finish the proof of Theorem M and then we will give an outline of proof of Lemma L.

Proof. By Lemma Re and Lemma L, we obtain that there is $\varphi_r \in \mathcal{B}$ such that (1) holds and that

$$(\varrho, |\varphi_r|) \leqslant C\ \delta\{\varphi_r\},$$

where K is the compact set on which $\rho(z) \geqslant p$.

Hence by letting r to approach 0, we get a contradiction.

Outline of the proof of Lemma L.

Let χ_r, $r > 0$, is defined by $\chi_r = \chi$ in K^c and $\chi_r = (1+r)\chi$ in K. By Hahn-Banach theorem there exists $\tau \in L^\infty$ such that $\chi_r - \tau \in \mathcal{N}$ and $\|\chi_r\|_* = \|\tau\|_\infty$, where $\mathcal{N}$ denote the annihilator of $\mathcal{B}$ in L^∞. Using the uniqueness part of condition HBU, we can conclude that $\|\tau\|_\infty > \|\chi\|_\infty = k$. Applying a normal family argument to a Hamilton sequence for τ, we can show that τ is a Teichmüller diferential represented by $\varphi_r \in \mathcal{B}$.

3. An open question and the Annihilators

If $\chi \in HBU$ then

$$\operatorname*{ess}_{z \in G} |\chi(z)| = k \quad \text{for each open set } G, \quad G \subset \Delta,$$

as has been observed by Reich [Re 4]. Therefore, the following question (cf. also [Re 4]) is natural.

Question B. *Does $\chi \in HBU$ actually imply that*

$$|\chi(z)| = k \quad \text{a.e.?}$$

If the answer to this question was yes then the following result would follow immediately from Theorem M.

Theorem 1. *$\chi \in HBU$ implies f^χ is UQC mapping.*

In order to give positive answer to Question B one can suppose the contrary— that $|A| > 0$ and try to construct infinitesimally trivial function η with support in A, which is not zero a.e. in Δ. It will be shown that this is not possible in general setting.

Let $\mathcal{N}$ denote the subset of L^∞ which is orthogonal to $\mathcal{B}$, i.e. for whose element η,

$$(\eta, \varphi) = 0,$$

for every $\varphi \in \mathcal{B}$. Differentials which belong to $\mathcal{N}$ are called infinitesimally trivial. The set $\mathcal{N}$ is called the annihilator of $\mathcal{B}$ in L^∞.

For given $\nu \in L^\infty$, let $T\nu$ denote the Cauchy operator defined by

$$(T\nu)(z) = -\frac{1}{\pi}\iint_\Delta \nu(\zeta)\,\frac{d\xi\,d\eta}{\zeta - z}, \quad \zeta = \xi + i\eta.$$

The function $u = T\nu$ is well defined in complex plane $\mathbb{C}$ and holomorphic in $\mathbb{C}\setminus\Delta$. Moreover, it is known that u is uniformly continuous in $\mathbb{C}$, with modulus of continuity that is $O(\delta\,\log 1/\delta)$. Also $\bar\partial u = \nu$ in the sense of distribution in Δ:

Suppose, in addition, that $\nu \in \mathcal{N}$ and that $F = \overline{S}$, where $S = \{\, z \in \Delta \;:\; \nu(z) \neq 0 \,\}$ is the support of ν, and that F is nowhere dense set of positive measure, that does not disconnect Δ.

Since u is analytic in $\mathbb{C}\setminus F$, we therefore conclude that $u = 0$ in $\Delta \setminus F$.

Hence, it follows from continuity of ν in Δ, that $u = 0$ in Δ and consequently $\nu = \bar\partial u = 0$ a.e. in Δ.

Thus we have the following result.

Lemma 1. *There exist sets $F \subset \Delta$, $|F| > 0$, with the property that if $\nu \in \mathcal{N}$ and if* $\operatorname{supp}\nu \subset F$ *then ν must be* 0 *a.e. in Δ.*

We can give the following construction of sets F.

Let $\{p_n \geqslant 2\}$ be the sequence of all points in Δ such that both coordinates of p_n are rational and $p_1 = (1,0) = 1$. Let γ be the polygonal line which connects the points of the sequence in given order and begins with p_1.

Let ε be an arbitrary small number. It is easy to construct domain $G \subset \Delta$, of the measure less than ε, which looks like a channel and contains the curve γ without point 1. Let $F = \Delta \setminus G$.

After we had announced the above lemma, E. Reich kindly informed us that he proved previously this Lemma in [Re 4], Lemma 3.1. In the same paper E. Reich indicates that the problem: whether $\chi \in HBU$ implies $|\chi| = k$ a.e., is still open, contrary to an assertion in the literature.

At this moment, we can show that the answer to Question B is yes if $\chi(z) \geqslant 0$ a.e.

Acknowledgements and Remarks.

We would like to thank E. Reich, for his kind assistance and advices concerning some details and related literature.

Also we would like to thank N. Lakić for usefull discussions concerning these problems.

This note is an extended version of our communication: Bergman and Teichmüler spaces, which has been held at the International Conference *"Generalized Functions—Linear and Nonlinear Problems"*, Novi Sad, August 31–September 04, 1996.

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(received 04.11.1996.)

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