A NEW VERSION OF THE MAIN INEQUALITY AND THE UNIQUENESS OF HARMONIC MAPS

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A. Introduction and statement of main results

A1. In this paper we state a new version of an inequality of Reich and Strebel, namely their so called Main Inequality, and use it to study the uniqueness property of harmonic mappings. To state the main inequality we need the following notation.

Let D and G be domains in $\mathbb{C}$ and let $f : D \mapsto G$ be a mapping. We use the notation

$$df = p\,dz + q\,d\overline{z}, \quad \text{where} \quad p = \partial f \quad \text{and} \quad q = \overline{\partial} f.$$

Also, we introduce the complex (Beltrami) dilatation

$$\mu_f = \text{Belt}\,[f] = \frac{q}{p}$$

and the dilatation

$$D_f = \frac{|\,p\,| + |\,q\,|}{|\,p\,| - |\,q\,|}.$$

It is convenient to suppose that our mappings are sense preserving.

We say that a homeomorphism $f : D \mapsto G$ is K quasiconformal if f is ACL (absolutely continuous on lines) on D and $D_f(z) \leq K$ a.e. on D.

Let Δ denote the unit disk and

$$T_\mu \varphi(z) = \frac{\left|\,1 - \mu(z)\frac{\varphi(z)}{|\varphi(z)|}\,\right|^2}{1 - |\,\mu(z)\,|^2}.$$

We will refer to the following result as the Reich-Strebel inequality or the Main Inequality.

Theorem RS (Reich and Strebel). *Suppose that f is a quasiconformal homeomorphism of Δ onto itself which is the identity on $\partial\Delta$. Then, with $\mu = \mu_f$*

$$\text{(A1)} \qquad \iint\limits_{\Delta} |\,\varphi(z)\,|\; dxdy \leq \iint\limits_{\Delta} |\,\varphi(z)\,|\, T_\mu \varphi(z)\, dxdy,$$

for every analytic integrable function φ on Δ.

Various forms of this result play a major role in the theory of quasiconformal mappings and have many applications.

For application to extremal and uniquely extremal quasiconformal mappings, we refer the interested reader to the book by Gardiner ([**G**]), and for some recent results to [**MM1**], [**BMM**], [**BLMM**] and [**Re3**].

It is convenient to explain our results first in the setting of the unit disk. Suppose that

(a) f is a homeomorphism of $\overline{\Delta}$ onto itself
(b) f has first partial derivatives on Δ
(c) f is the identity on $\partial\Delta$.

Theorem 1. *With the hypothesis and notation stated above the inequality* (A1) *holds for every analytic integrable function* φ *on* Δ.

This theorem gives a new version of the Main Inequality which is applicable to mappings which are not quasiconformal mappings.

A2. Let M and N be two Riemann surfaces with local conformal metrics $\sigma \mid dz \mid^2$ and $\rho \mid dw \mid^2$ and let $f : M \mapsto N$.

It is convenient to use the notation $df = p\,dz + q\,\overline{dz}$ in local coordinates, where $p = \partial f$ and $q = \overline{\partial} f$.

The energy integral of f is

$$E(f,\rho) = \int\limits_M \rho \circ f(\mid p \mid^2 + \mid q \mid^2)\, dxdy.$$

A critical point of the energy functional is called a harmonic mapping. The Euler-Lagrange equation for the energy functional is

$$f_{z\overline{z}} + \partial(\log \rho) \circ f\; pq = 0. \tag{A2}$$

It seems that the classical theory of elliptic equations does not suffice for a study of the Euler-Lagrange equation without the use of additional tools. In [MM2] (see also [MM3]) the current authors initieted the study of the uniqueness of solutions of equation (A2) with help of the Main Inequality. With ordinary harmonic function in mind, it was natural to consider the question of uniqueness of harmonic mappings, without assumption that the mappings are of bounded dilatation, and to try to extend the range of applications of the Main Inequality. This led us to Theorem 1 and its applications.

For basic properties of harmonic mappings and for further information on the literature we refer to Jost [**J**], Schoen-Yau [**SY**] and Schoen [**Sc**].

This general notion of harmonic mappings (see Section E1 below) (see section E1 below), allows for the following:

A3. If f is a harmonic mapping then $\varphi = \rho \circ f\, p\, \overline{q}\, dz^2$ is a holomorphic quadratic differential, which is called the Hopf differential of f and which is denoted by $Hopf(f)$.

For example if M and N are subsets of the complex plane $\mathbb{C}$, this simply means that the function $\rho \circ f\, p\, \overline{q}$ is a holomorphic function.

In the next item we are going to announce some applications of the Theorem 1.

A4. The following is an application of Theorem 1.

Theorem 2 (The uniqueness property of the identity). *Under the hypothesis of Theorem 1, if (in addition) we suppose that* f *is harmonic and that the Hopf differential of* f *is nonzero and integrable on* Δ, *then* f *is the identity on* $\overline{\Delta}$.

To state the Theorem 3 we need the notation of locally-quasiconformal mapping.

Let M and N be two Riemann surfaces and $f : M \mapsto N$. We say that f is locally quasiconformal if for every point $p \in M$ there is a neighbourhood V such that the restriction of f on V is a quasiconformal mapping.

Theorem 3 (First uniqueness property). *Suppose that f and g are harmonic homeomorphisms from the closure of the unit disk onto itself, which are locally quasiconformal on Δ and suppose that $f = g$ on the boundary of the unit disk. If, in addition, we suppose that the Hopf differentials of f and g are integrable on Δ, then f and g are identical.*

Moreover, we shall prove a uniqueness result for homotopic mappings between compact Riemann surface (see Theorem 4, the second uniqueness property, and part E4 of the section E).

The idea of the proof of Theorem 3, and Theorem 4, is to apply Theorem 1, the Main Inequality, to the functions $\varphi = \rho \circ f\, p\overline{q}$ and $\psi = \rho \circ g\, A\overline{B}$, where $A = \partial g$ and $B = \overline{\partial} g$.

For further generalizations of Theorem 3 and Theorem 4 see the part F5 of Section F.

Note that when we work with mappings which have the first generalized derivatives (without the assumption of quasiconformality) we have new features (see below B3, and sections C, E1, E3, F4 and F5 for more details).

We start now with a short review of some previously obtained results, which are related to Theorem 3 and Theorem 4.

Finally, in A6, we describe the structure of this paper.

A5. After writing a previous version E. Reich pointed out to us that H. Wei [**We**] studied uniqueness property of harmonic mappings with the help of the Main Inequality. Also, we became aware of the Coron-Helein paper [**CH**].

By using the formula for the energy of variation of a mapping (see [**ReS2**]) and the Reich-Strebel inequality, H. Wei proved a weaker version of Theorem 2 concerning quasiconformal mapping under additional hypothesis that:

(c) f and g are quasiconformal mappings on the unit disk Δ onto itself
(d) the metric density ρ is an integrable function on Δ.

Note that the hypothesis (c) and (d) provide that the energy integral of f and g are finite.

In [**CH**], Coron-Helein used an approach, which is completely different from H. Wei in [**We**], to study minimizing harmonic mappings. Their approach was based on the decomposition of a given metric g on Δ as the sum of two metrics c and h such that c is a conformal metric w.r.t the euclidean metric e, h has non-positive Gaussian curvature and the identity mapping, Id, is a harmonic mapping between (Δ, e) and (Δ, h).

Theorem CH (Coron-Helein). *Let (M, h) and (N, g) be two Riemannian compact surfaces of class C^∞ possibly with boundary. Then any smooth harmonic diffeomorphism between (M, h) and (N, g) is minimizing in its homotopy class. Moreover, if ∂M is non-empty or if the genus of M is strictly larger than one, then such a diffeomorphism is the unique minimizing map in its homotopy class.*

An inspection of the proofs of Theorem 2, [**We**], Theorem 3 and Theorem 3', [**CH**], shows that the assumption that harmonic maps are of finite energy is essential for both proofs.

Note that the assumption that f and g have finite energy integrals provides that Hopf differentials of f and g are integrable functions on Δ. Thus we prove our Theorem 3 under weaker conditions. Namely, we require only that the Hopf differentials of corresponding mappings are integrable.

We refer the interested reader to [**J**] for the global uniqueness theorem of Al'ber and Hartman, for the result of Jäger and Kaul and to the Schoen-Yau book ([**SY**]) for uniqueness theorems concerning harmonic maps into non-positive curved metric spaces.

A6. The structure of this paper is as follows. In Section B we prove a version of Grötzsch principle concerning mappings with L^1- derivatives. In section C we will give proofs of Lemma 1 and Lemma 2, which we use in the proof of Theorem 1. In section D the proof of a new version of the Main Inequality (Theorem 1) is given. In section E we use Theorem 1 to prove the uniqueness results (Theorems 2-4). In section F we will give further results, comments and some applications of Theorem 2 to the case when the energy integrals are infinite.

B. The Problem of Grötzsch

B1. In order to motivate the statement and proof of our version of the Main Inequality we will emphasize the main points in the proof of Grötzsch's principle.

We will follow [**MM2**], where we announced the results of this section.

If Q is a square and R is a rectangle, not a square, there is no conformal mapping of Q on R which maps vertices on vertices. Instead, Grötzsch asked for the most nearly conformal mapping of this kind and took the first step toward the creation of a theory of quasiconformal mappings.

Let $w = f(z)$ be a mapping from one region to another. Recall that we use notation

$$df = p\,dz + q\,d\overline{z}, \quad \text{where} \quad p = \partial f \quad \text{and} \quad q = \overline{\partial} f.$$

The complex (Beltrami) dilatataion is:

$$\mu_f = \text{Belt}\,[f] = \frac{q}{p}$$

The dilatation of f dilatation

$$D_f = \frac{|\, p\,| + |\, q\,|}{|\, p\,| - |\, q\,|}$$

We pass to the Grötzsch problem and give it a precise meaning by saying that f is most nearely conformal if sup D_f is as small as possible.

Let R, R' be two rectangles with sides a, b and a', b'. We may assume that $K = \frac{b'}{a'} : \frac{b}{a} \geq 1$. The mapping f is supposed to be C^1-homeomorphism from $\overline{R}$ onto $\overline{R'}$, which takes a-sides into a-sides and b-sides into b-sides.

Next, let Γ_x be the vertical segment which is the intersection of the line Re$z = x$ with $\overline{R}$ and γ_x the curve which is image of Γ_x under f.

The starting point of Grötzsch's approach is the geometric obvious inequality

$$b' \leq \text{length}(\gamma_x) = \int_0^b |\, p - q\,| \; dy. \tag{B1}$$

Using:

$$\iint_R J_f \, dxdy = a'b', \tag{B2}$$

where J_f denotes the Jacobian of f, and the Cauchy-Schwarz inequality one get

$$K \le \sup D_f. \tag{B3}$$

The minimum is attained for the affine mapping.

We note the following connections to with Grötzsch's problem.

The restriction to C^1-mapping is not essential.

The inequality (B3) holds for quasiconformal mapping (see, for example, [**Ah**]).

In order to give a version of Grötzsch principle concerning mappings with L^1- derivatives, we need the following definition.

B2. Definition of L^p- derivatives

Let D be a domain in C. We say that a function $f : D \mapsto C$ has L^p- derivatives, $p \ge 1$, if it statisfies the following two conditions:

(a) f is absolutely countinuous on lines in D

(b) The partial derivatives f_x and f_y belong to L^p in every compact subset of D.

When we say that f has first generalized derivatives in D this means that f has L^1- derivatives in D. For various characterizations of functions with L^p- derivatives and their important role in the theory of quasiconformal mappings we refer to Chapters III to VI of the book by Lehto Virtanen ([**LV**]).

B3. A version of Grötzsch's principle

Before, we give further exstension of Grötzsch's principle, it is useful to consider the following example when (B1) and (B3) does not hold.

Example 1. Let $\alpha : I \mapsto I$, where $I = [0, 1]$, be Cantor function and let $f(z) = x + i(y + \alpha(y))$.

Note that this function does not satisfy ACL property and that the known formula for the length of curve by means of first partial derivatives does not hold.

Suppose that

(a) f is a homeomorphism of closed rectangle $\overline{R}$ onto the closed rectangle $\overline{R'}$ which maps a-sides onto a'-sides and b-sides onto b'-sides.

(b) f has the first generalized derivatives on R.

In order to get some conclusion we can follow the outline of the proof of Grötzsch principle from the part B1 of this section.

We need the following definition below.

At the point z where $\mu(z)$ is defined and $|\mu(z)| \ne 1$ we define $T_\mu(z)$ by

$$T_\mu(z) = \frac{|1 - \mu(z)|^2}{1 - |\mu(z)|^2}$$

Also, at point z where $|p(z)| = |q(z)|$ we define $T_\mu(z)$ to be zero if $p(z) = q(z)$ and $+\infty$ if $p(z) \ne q(z)$.

Now, we can give the precise meaning of $T_\mu \varphi$ by means of T_χ, where

$$\chi = \mu\varphi / |\varphi| .$$

Since f satisfies the ACL-property, inequality (B1) holds for a.a. $x \in [0, a]$.

In order to prove inequality (B4) (see below), we can suppose that T_μ is defined and finite a.e. on R, because otherwise the right-hand side of (B4) is infinite.

Next, we can integrate w.r.t. dx over $[0,a]$ and use the fact that the Jacobian

$$J_f =| p |^2 \,(1- | \mu |^2) \quad \text{a.e. on } R.$$

Instead of (B2), we have

$$\int_R J_f \, dxdy \leq \text{area}(R') = a'b'.$$

Now, an application of Cauchy-Schwarz inequality gives

$$\text{area}(R)\frac{b'}{b} \leq [\text{area}(R')]^{1/2}\Big[\iint_R T_\mu\Big]^{1/2}. \tag{B4}$$

Further development of the ideas outlined above leads us to Lemma 1 (see bellow), which will be used in the proof of the new version of the main inequality.

C. Proofs of Lemma 1 and Lemma 2

Let D be a vertically convex domain of finite area in the complex plane $\mathbb{C}$ and let F be a mapping from the domain D onto the domain G.

Suppose that we have metric $ds = \rho(w) \mid dw \mid$ on G.

Let Γ_x be the interval which is the intersection of D by the straight line $\text{Re}z = x$ and let γ_x be the curve which is the image of Γ_x under F. Let $p(x,y) = x$ be the projection and let $(\alpha,\beta) = p(D)$.

Lemma 1. *With the notation and hypothesis just stated, suppose (in addition) that the mapping F is homeomorphism which has first generalized derivatives and that*

$$\text{length}(\Gamma_x) \leq \int_{\gamma_x} \rho(w) \mid dw \mid \quad \text{a.e. in} \quad (\alpha,\beta). \tag{C1}$$

Then

$$\text{area}(D) \leq \Big[\iint_G \rho^2(w)\, dudv\Big]^{1/2}\Big[\iint_D T_\nu \, d\xi d\eta\Big]^{1/2}, \tag{C2}$$

where

$$\nu = \text{Belt}[F].$$

Proof. We will use the notation $dF = P\, d\zeta + Q\, d\overline{\zeta}$, where $P = \partial F$ and $Q = \overline{\partial} F$.

We can suppose that T_ν is defined and finite a.e. on D, because otherwise the right-hand side of (C2) is infinite. With definition of T_ν in mind, this means that $P = Q$ a.e. on A, where A is the set on which Jacobian J_F equals zero.

Since F is absolutely countinuous on Γ_x for a.e. $x \in (\alpha,\beta)$, we find

$$\rho - \text{length}(\gamma_x) = \int_{\Gamma_x} (\rho \circ F)(\zeta) \mid P \mid\mid 1-\nu \mid \, d\eta.$$

By Fubini's theorem and assumption (C1),

$$(area)(D) \leq \iint\limits_D (\rho \circ F)(\zeta) \mid P \mid\mid 1 - \nu \mid \, d\xi d\eta.$$

Since

$$J_F = \mid P \mid^2 (1 - \mid \nu \mid^2) \text{ a.e. on } D,$$

the term on the right can be written in the form

$$\tau = \iint\limits_D (\rho \circ F)(\zeta) J_F^{1/2} T_\nu^{1/2} \, d\xi d\eta.$$

Next, using the Cauchy-Schwarz inequality we conclude that

$$\tau \leq A^{1/2} \cdot B^{1/2},$$

where

$$A = \iint\limits_D (\rho^2 \circ F)(z) \, J_F(z) \, dxdy \quad \text{and} \quad B = \Big[\iint\limits_D T_\nu \, d\xi d\eta \Big].$$

Let

$$C = \iint\limits_G \rho^2(w) \, dudv.$$

We need the following lemma to finish the proof.

Lemma 2. $A \leq C$.

Proof. Let the measure μ be defined by

$$\mu(E) = \int\limits_E J_F(z) \, dxdy$$

and

$$\mu_F(E) = m(F(E)),$$

for every Lebesgue measurable set E.

Since F is a homeomorphism which possesses finite partial derivatives a.e. in D, by Lemma 3.3 ([**LV**], p.131)

$$\mu(E) \leq \mu_F(E)$$

and therefore we have the desired result.

D. Proof of new version of the Main Inequality

There are several papers of Reich and Strebel which concern various forms of the Main Inequality. Our proof is based on their ideas.

Here, we will give a complete proof of the Theorem 1, because we need to be careful when we work with mappings whose dilatation is not bounded. For convenience of the reader let us recall the statement of Theorem 1.

Theorem 1. *Suppose that*

(a) f *is a homeomorphism of* $\overline{\Delta}$ *onto itself*
(b) f *has first generalized derivatives on* Δ
(c) f *is the identity on* $\partial\Delta$.

Then the inequality

$$\iint\limits_{\Delta} \mid \varphi(z) \mid \; dxdy \leq \iint\limits_{\Delta} \mid \varphi(z) \mid T_\mu \varphi(z)\, dxdy$$

holds for every analytic integrable function φ *on* Δ.

D1. First, we observe that Theorem 1 can be reduced to the case when φ is also analytic on $\partial\Delta$. Namely, let φ_r, $0 < r < 1$, be the function defined by $\varphi_r(z) = \varphi(rz)$, $z \in \Delta$. If Theorem 1 holds for every φ_r, $0 < r < 1$, then, when r approaches 1 we conclude that the theorem holds for φ, by Lebesgue dominated convergence theorem.

Suppose that φ is an analytic function in $\overline{\Delta}$.

The following decomposition is possible (see [**S**] and [**S1**]). Up to a set of Lebesgue 2-dimensional measure zero, $\Delta = \cup_{k=1}^n \Sigma_k$, where $\{\Sigma_k\}$ are disjoint simple connected "strip" domains. Each Σ_k is swept out by a family of vertical trajectories of the holomorphic quadratic differential $\varphi(z)\, dz^2$ and in each Σ_k there exists a single valued schlicht branch $\zeta = \Phi_k(z)$ of $\int \sqrt{\varphi(z)}\, dz$. Each region $D_k = \Phi_k(\Sigma_k)$ is verticaly convex.

In [**S**] it is merely assumed that φ is analytic on Δ, instead of $\partial\Delta$, so that countably many, instead of merely finitely many $\sum_k$, can occur. Actually, in our use of the strip domains, the advantage of limiting ourselves to finitely many is purely didactic.

For the local and global behaviour of the trajectories of holomorphic quadratic differentials we refer reader to Strebel's book ([**S2**]).

The following fact is important in the proof of Theorem 1.

D2. The vertical trajectories of holomorphic quadratic differential are globally geodesics in Teichmüller's metric

$$ds^2 = \mid \varphi(z) \mid \mid dz \mid^2 .$$

Note that $\zeta = \Phi_k(z)$ is a single-valued branch of $\int \sqrt{\varphi}\, dz$ in Σ_k and that $D_k = \Phi_k(\Sigma_k)$ and $\Gamma_x = \Gamma_x^k$ is the interval which is intersection of D_k by straight line $Re\, z = x$.

Let $\theta_x = \Phi_k^{-1}(\Gamma_x)$ and $G_k = f(\Sigma_k)$. Thus θ_x is trajectory of holomorphic quadratic differential $\varphi(z)\, dz^2$. Let $\gamma_x = f(\theta_x)$.

Since θ_x is a global geodesic in Teichmüller metric

$$ds^2 = \mid \varphi(z) \mid \mid dz \mid^2,$$

$$\text{length} \quad (\Gamma_x) = \int\limits_{\theta_x} \mid \varphi(z) \mid^{1/2} \mid dz \mid \leq \int_{\gamma_x} \mid \varphi(w) \mid^{1/2} \mid dw \mid .$$

Thus we can apply Lemma 1 to the function $F_k = f \circ \Phi_k^{-1}$ and
$\rho(w) = \mid \varphi(w) \mid^{1/2}$. Hence, by Lemma 1,

$$\text{(D1)} \qquad \iint\limits_{\Sigma_k} \mid \varphi(z) \mid \; dxdy = \text{area}(D_k) \leq A_k B_k,$$

where

$$A_k = \Big[\iint\limits_{G_k} \mid \varphi(w) \mid \, dudv\Big]^{1/2}$$

and

$$B_k = \Big[\iint\limits_{D_k} T_\nu \, d\xi d\eta\Big]^{1/2}, \quad \nu = \nu_k = \mathrm{Belt}(F_k).$$

Using the change of variables $z = \Phi_k^{-1}(\zeta)$, we get

$$B_k = \iint\limits_{\Sigma_k} \mid \varphi(z) \mid \; T_\mu \varphi(z) \, dxdy.$$

Further application of the Cauchy-Schwarz lemma and (D1) give

$$\sum_{k=1}^{n} \Big(\varphi - \mathrm{area}(\Sigma_k)\Big) \le \sum_{k=1}^{n} A_k B_k \le A \cdot B,$$

where

$$A = (\sum_{k=1}^{n} A_k^2)^{1/2} \quad \text{and} \quad B = (\sum_{k=1}^{n} B_k^2)^{1/2}$$

Now, Theorem 1 follows from the fact that

$$A = \Big[\iint\limits_{\Delta} \mid \varphi(z) \mid \; dxdy\Big]^{1/2}$$

and

$$B = \Big[\int\limits_{\Delta} \mid \varphi \mid T_\mu \varphi \, dxdy]^{1/2},$$

where $\mu = \mathrm{Belt}[f]$.

E. The uniqueness results

E1. The following example may serve to explain the definition of harmonic mappings (see below).

Example 2. Let $0 < \epsilon < 1$ and let J be an open set in $(0,1)$ of measure ϵ, which is dense in $(0,1)$. Define the function v by

$$v(y) \; = \; \int_0^y K_G(t) \, dt, \; y \ge 0,$$

where $G = \cup_{n=0}^{\infty}(J+n)$ and K_G is the characteristic function of G.

Put $f(z) = x + iv(y)$ and $F = [0,\infty) \setminus G$. We leave to the reader to check the following facts

(a) f is a homeomorphism of the upper half plane H onto itself

(b) f is a conformal mapping on $R \times G$

(c) the set $R \times F$ has infinite Lebesgue 2-dimensional measure and its image under f has measure zero.

Let M and N be two Riemann surfaces and let ρ be a measurable function on N which statisfies $\rho > 0$ a.e. on N. A function $f : M \mapsto N$ is called harmonic, with respect to ρ (metric density), if the expression $\varphi(z)\, dz^2$, where $\varphi(z) = \tau \circ f \, p \overline{q}$, is equal a.e. to a holomorphic quadratic differential on M, where $\tau = \rho$ a.e. on N.

For a similar definition see [**Re1**], [**Re2**] and [**ReS2**].

Note that we do not require that ρ is an integrable function over N. Also, observe that the definition does not require any smoothness of the metric density ρ image surface N, while this is required in order to write the Euler-Lagrange equation (see (A2)). From now on, when we say harmonic it means harmonic w.r.t. some metric density ρ in the sense of the last definition.

Let τ be an arbitrary Borel mesurable function on H, which is positive a.e. on H, and let φ be the conformal mapping of Δ onto H and let ψ be the inverse function of φ. Define the metric density ρ on Δ by $\rho(w) = \tau(\varphi(w)) \mid \varphi'(w) \mid$ and let $g = \psi \circ f \circ \varphi$. It is easy to verify that g is harmonic homeomorphism of Δ onto itself, w.r.t. the metric density ρ, which is the identity on $\partial\Delta$ but is not the identity on Δ.

Note that this example shows why we need the Hopf differential of f to be nonzero in the condition (e) of Theorem 2 (see below).

In order to give motivation for the proofs of Theorem 2 and Theorem 3 the following considerations is useful.

E2. The case when $f = Id$ on $\partial\Delta$

Suppose that f is a harmonic difeomorphism of $\overline{\Delta}$ onto itself and that the Hopf differential of f is integrable on Δ. Then the Beltrami dilatation μ of f has the form

$$\mu(z) = s(z)\frac{\overline{\varphi(z)}}{\mid \varphi(z) \mid},$$

where s is a non-negative measurable function and $\varphi = \rho \circ f\ p\overline{q}$ is an analytic integrable function on Δ. Thus we see that the expression $\mu\varphi/ \mid \varphi \mid$ which appears in the main inequality is equal to $\mid \mu \mid$ and an application of the Main Inequality to the analytic function φ gives

$$\int\limits_{\Delta} \mid \varphi \mid \, dxdy \leq \int\limits_{\Delta} \mid \varphi \mid \frac{1- \mid \mu \mid}{1+ \mid \mu \mid} \, dxdy.$$

This inequality implies that if φ is not identicaly 0 that $\mu = 0$ a.e. in Δ. Hence we conclude that f is a conformal mapping and since $f = Id$ on $\partial\Delta$, that $f = Id$ on Δ.

E3. Proof of Theorem 2

For convenience of the reader let us recall the statement of Theorem 2 and a few facts concerning harmonic mappings.

Suppose that f has L^1- derivatives in a domain D and that f is a harmonic mapping w.r.t. metric ρ on D. Let μ be Beltrami dilatation and φ Hopf differential of f. Then

(a) $J_f = 0$ iff $\mid \varphi \mid p = \varphi q$
(b) $\mu = \mid \mu \mid\mid \varphi \mid / \varphi$.

Note that this expression for the Beltrami dilatation μ is a key for the application of the Main Inequality to harmonic mappings.

Theorem 2. *Suppose that*

(a) f is a homeomorphism of $\overline{\Delta}$ onto itself

(b) f has first generalized derivatives on Δ
(c) f is the identity on $\partial\Delta$.
(d) f is harmonic w.r.t. the metric density ρ
(e) The Hopf differential of f is nonzero and integrable on Δ.

Then f is the identity on $\overline{\Delta}$.

Remark. If $Hopf(f) = 0$ on Δ Theorem 2 is also true under the additional hypothesis that f maps sets of measure zero onto sets of measure zero.

Proof of Remark. Assumption that f has the first generalized derivatives implies that f has finite partial derivatives a.e. in Δ. Since f is a homeomorphism then f is differentiable a.e. in Δ by Gehring-Lehto lemma. Hence, we conclude that $J_f \geq 0$ a.e. in Δ, because f is sense preserving mapping.

Let A be the set on which q is defined and is equal to zero. The assumption $\varphi = 0$ on Δ implies that $p = 0$ a.e. on $A^c = \Delta \setminus A$. Since $\mid p \mid \geq \mid q \mid$ a.e. in Δ, we conclude that $q = 0$ a.e. in Δ. Next, the assumption that f has first generalized derivatives and Weyl's lemma, imply that f is equal to an analytic function a.e. in Δ. Using the boundary condition and continuity of f, it is easy to verify that f is the identity on Δ.

Proof of Theorem 2. Let B denote the set on which J_f is defined and equals zero.

The expression $T_\mu \varphi$ which appears in the statement of Theorem 1 must be defined appropriately on the set B.

Since f is harmonic $\mu\varphi / \mid \varphi \mid$ equals $\mid \mu \mid$ a.e. on Δ. Using this as motivation (see the definition of $T_\mu \varphi$), we can define $T_\mu \varphi$ to be zero on B. Hence, $T_\mu\varphi$ is defined a.e. on Δ and

$$T_\mu\varphi = \frac{1- \mid \mu \mid}{1+ \mid \mu \mid}.$$

Now, an application of Theorem 1 to the analytic function φ gives

$$\int_\Delta \mid \varphi \mid \, dxdy \leq \int_\Delta \mid \varphi \mid \frac{1- \mid \mu \mid}{1+ \mid \mu \mid} dxdy.$$

In a similar way as in E1 one can conclude that $\mu = 0$ a.e. on Δ, and hence f is the identity on Δ.

E4. Corollaries of the Main Inequality

In order to prove Theorem 3 we need a few lemmas.

First, let us recall the hypothesis of Theorem 3:

(a) f and g are homeomorphisms of $\overline{\Delta}$ onto itself and $f = g$ on $\partial\Delta$
(b) f and g are locally quasiconformal on Δ
(c) f and g are harmonic mappings w.r.t. some metric density ρ on Δ
(d) The Hopf differentials of f and g are integrable on Δ.

We will use the notation $\mu = \mathrm{Belt}[f]$, $\tilde{\mu} = \mathrm{Belt}[f^{-1}]$ and $\tilde{\chi} = \mathrm{Belt}[g^{-1}]$.

Also, it is convenient to use the short notation, $\zeta = f(z)$, $k = k(\zeta) = \mid \mu(z) \mid = \mid \tilde{\mu}(\zeta) \mid$ and $s = s(\zeta) = \mid \tilde{\chi}(\zeta) \mid$.

The following result is a corollary of Theorem 1.

Lemma 1. *With the above notation and the hypothesis stated for Theorem* 3,

$$\int\limits_{\Delta} |\varphi|\, dxdy \le \int\limits_{\Delta} |\varphi| \frac{1-|\mu(z)|}{1+|\mu(z)|}\frac{1+s(\zeta)}{1-s(\zeta)}\, dxdy, \tag{E1}$$

where $\varphi = \rho \circ f\, p\overline{q}$.

Proof. By the hypothesis $g^{-1}\circ f$ is the identity on $\partial\Delta$ and $g^{-1}\circ f$ is locally quasiconformal on Δ and hence has L^2- derivatives.

An application of new version of the Main Inequality to $g^{-1}\circ f$ and φ gives

$$\int\limits_{\Delta} |\varphi| \le \int\limits_{\Delta} |\varphi|\, T_\mu\varphi \frac{|1-\tilde{\chi}(\zeta)\theta\varphi/|\varphi||^2}{1-|\tilde{\chi}(\zeta)|^2}\, dxdy \tag{E2}$$

where $\theta = \frac{\overline{p}}{p}\overline{(1-\mu\varphi/|\varphi|)}(1-\mu\varphi/|\varphi|)^{-1}$ and $p = \partial f$.

Recall that $\mu\varphi/|\varphi| = |\mu|$ and therefore $\theta = \frac{\overline{p}}{p}$. By the triangle inequality

$$|1-\tilde{\chi}\theta\varphi/|\varphi|| \le 1+|\tilde{\chi}(\zeta)| \tag{E3}$$

and hence we obtain (E1).

For our purpose it is convenient to use the change of variables $\zeta = f(z)$ in (E1) and express the inequality (E1) in terms of metric density ρ instead of φ.

Using the change of variables $\zeta = f(z)$ we obtain

$$|\varphi|\, dxdy = |\varphi\circ f^{-1}||J_{f^{-1}}(\zeta)|\, d\xi d\eta.$$

Let $\tilde{\varphi}(\zeta) = |\varphi(z)|/J_f(z)$. Since $|\varphi(z)| = \rho(\zeta)|p||q|$, we have

$$\tilde{\varphi}(\zeta) = \rho(\zeta)\frac{k(\zeta)}{1-k^2(\zeta)}.$$

This gives the following result.

Lemma 2. *With the above notation and the hypothesis stated in Theorem* 3,

$$\int\limits_{\Delta} \rho\frac{k}{1-k^2}\, d\xi d\eta \le \int\limits_{\Delta} \rho\frac{k}{1-k^2}\frac{1-k}{1+k}\frac{1+s}{1-s}\, d\xi d\eta, \tag{E4}$$

where ρ, k *and* s *are the function of* ζ *defined at begining of the item* E3 *of this section.*

Note that Lemma 1 and Lemma 2 are valid without the assumption that g is a harmonic mapping.

E5. Proof of Theorem 3 and Theorem 4

Note that the inequality (E4) holds if k and s change roles. If we sum the corresponding inequalities, we obtain

$$\int\limits_{\Delta} \alpha\rho \le \int\limits_{\Delta} \beta\rho, \tag{E5}$$

where

$$\alpha = \frac{k}{1-k^2} + \frac{s}{1-s^2}, \quad \beta = \frac{k}{(1+k)^2}\frac{1+s}{1-s} + \frac{s}{(1+s)^2}\frac{1+k}{1-k}.$$

We need the following lemma to finish the proof.

Lemma 3. $\alpha \ge \beta$.

The proof of this lemma is elementary and follows from the fact that

$$\alpha - \beta = 2(k-s)(f(k) - f(s))[(1-k)(1-s)]^{-1},$$

where $f(x) = x(1+x)^{-2}$ is increasing function in $x \in [0,1)$.

Lemma 3 with (E5) actually show that $\alpha = \beta$, i.e. $k = s$.

Thus we have equality in (E1) and therefore equality in (E3). This means that $\tilde{\chi}\theta\varphi / \mid \varphi \mid$ is non-positive a.e. in Δ. Since $-\tilde{\mu}(\zeta) \cdot \frac{\overline{p}}{p} = \mu(z)$, we conclude that $\frac{\tilde{\chi}}{\tilde{\mu}}$ is positive a.e. in Δ, i.e. $\tilde{\chi} = \tilde{\mu}$ a.e. in Δ. This, finaly, gives $f = g$ in $\overline{\Delta}$.

Since the Reich-Strebel inequality holds on Riemann surfaces of finite analytic type, using the same method as in the proof of Theorem 3 we obtaine the following result.

Theorem 4 (Second uniqueness property). *Let M and N be two compact Riemann surface of the same genus $g \geq 2$. Suppose that f and g are harmonic quasiconformal mappings from M onto N, which are homotopic each other. If, in addition we suppose that the Hopf differential of f and g are integrable on Δ, then they are identical.*

Proof. Using the same procedure and notation as in the proof of Theorem 3, we can conclude that $\tilde{\mu} = \tilde{\chi}$ a.e. on R'. Hence, $h = f^{-1} \circ g$ is a conformal mapping of R onto itself, which is homotopic to the identity. If genus $g \geq 2$, then by Theorem A and B (see below) h must be the identity!

We would like to call attention to the difference between this proof and that of Theorem 1 at this point.

To finish the proof we can use two interesting results (see [**L**]).

Theorem A (Theorem 5.2 [**L**], p.157). *The covering group G of the upper half-plane over compact Riemann surface is finitely generated and of first kind.*

This means that the limit set L of G is whole real axis and, in particular, than covering froup G is non-elementary, i.e. the limit set L contains more than two points.

Theorem B (Theorem 1.3 [**L**], p.179). *Let R be a Riemann surface with a non-elementary covering group. If $f : R \mapsto R$ is a conformal mapping homotopic to the identity, then f is the identity mapping.*

Combining these theorems, we conclude that h is the identity and therefore $f \equiv g$ on R.

F. Further results and comments

We give an application of Theorem 3 in a case when the energy integral is infinite.

F1. Suppose that

(a) f and g are harmonic diffeomorphisms from the Δ onto itself w.r.t. the Poincaré metric.

(b) The Hopf differentials φ = Hopf (f) and ψ = Hopf (g) are integrable on Δ.

Since φ and ψ belong to the Bers space (see, for example, [**Ah**], [**W**] and [**AMM**] for definition and properties of Bers space) a result of Wan [**W**] shows that f and g are quasiconformal mappings of Δ onto itself.

If, in addition, we suppose that $f = g$ on the boundary of the unit disk, an application of Theorem 3 shows that f and g are identical.

Note that every harmonic diffeomorphism of Δ onto itself w.r.t. Poincaré metric has infinite energy integral.

The following example shows that without the assumption that the Hopf differentials are integrable Theorem 2 is not valid.

F2. Let φ be the conformal mapping of the unit disk Δ onto the upper half-plane H and let $\rho(w) =| \varphi'(w) |$. Next, let $g = \psi \circ h \circ \varphi$, where ψ is the inverse function of φ and h is given by $h(z) = x + iky$, $k > 0$. We leave to the reader to verify that g is a quasiconformal harmonic mapping (w.r.t. ρ) of the unit disk Δ onto itself and that $g = Id$ on the boundary of Δ.

Although, the metric defined by the density ρ is flat on the complex plane C except at one point, Theorem 2 is not valid.

F3. In connection with the parts (F1) and (F2) of this section, we will give a short discussion (we follow Schoen [**Sc**]).

There is an interesting conjecture which is due to Schoen (see also [**Sc**]).

Conjecture. The quasiconformal harmonic homeomorphisms from the unit disk Δ onto itself, w.r.t. the Poincaré metric, are parametrized by the boundary values of quasiconformal maps of the disk.

This is a question which involves proving both an existence and a uniqueness theorem. The existence result for this ideal boundary value problem has been shown by Li and Tam ([**LT1**]) under the additional hypothesis that the boundary map is sufficiently differentiable. They have also obtained counterexamples to uniqueness without the quasiconformal hypothesis (but with continuity) and then proved the uniqueness part of Schoen's conjecture (see [**LT2**]).

A result of Wan ([**W**]) gives a parameterization of the quasiconformal harmonic homeomorphisms of Δ in terms of bounded holomorphic quadratic differentials on Δ. Wan has shown that if f is quasiconformal mapping then the Hopf differential of f is bounded w.r.t. the Poincaré metric on Δ. Conversely, he has shown that for any bounded holomorphic quadratic differential Φ on Δ there is a unique quasiconformal harmonic homeomorphism $f : \Delta \mapsto \Delta$ such that Hopf $(f) = \Phi$.

F4. In this section we will an example of a harmonic homeomorphism, whose Jacobian is zero on a set of positive measure.

Let $\varepsilon > 0$ be arbitrary and let G be an open set in R of measure ε, which contains all rational numbers. Next, let h be the characteristic function of G and let v be the function defined by

$$v(x) = \int_{-\infty}^{x} h(t)\, dt\,, \qquad x \in R.$$

Put $f(z) = 2x + iv(y)$ and let ρ be the metric density defined by $\rho(f(z)) = 4$ for $y \in G$ and $\rho(f(z)) = 3$ for $y \in K$, where K is the complement of G.
We leave to the reader to check the following facts.

(a) f is a homeomorphism of C onto $R \times (0, \varepsilon)$, which has $L^\infty-$ derivatives
(b) f is a harmonic mapping w.r.t. metric density ρ on C
(c) The Jacobian J_f is equal zero a.e. on $R \times K$.

Finally, we are going to state some results communicated by the second author on the Symposiums Contemporary Mathematics (see [**M**]).

F5. Theorem 3 remains valid if condition (b) (in the hypothesis of Theorem 3, the item E3) is replaced by the following.

(e) f, g and their inverse mapping have L^2- derivatives.

The idea of the proof is as follows. If the condition (e) holds then one can conclude that $f\circ g^{-1}$ and $g\circ f^{-1}$ have L^1- derivatives and its partial derivatives statisfy the chain rule (for a details see Lemma 6.4 [**LV**], p.151). Hence, we can apply Theorem 1 to these functions in a similar way as in the proof of Lemma 1 of item E3.

It is well-known that the condition (b) implies the condition (e) (see, for example, [**LV**]).

Also, Theorem 4 is valid under weaker conditions. Instead of the assumption that f and g are quasiconformal mappings from M onto N, it is sufficient to suppose that f and g are homeomorphisms from M onto N, which statisfy the condition (e).

For a development of theory of harmonic mappings by means of Sobolev spaces, we refer to ([**SY**]).

F6. Harmonic maps and extremal QC mapping

Before we state the results, we need some notations.

Suppose that f is a quasiconformal mapping of the unit disk Δ onto itself.

Let

$$k[f] = ess\,sup\{|\mu_f(z)| : z\in\Delta\}$$

and let $Q(f)$ denote the collection of all quasiconformal mappings of Δ whose pointwise boundary values on $\partial\Delta$ agree with those of f. We call f extremal (in its Teichmüller class) if $k[f] \le k[g]$ for every $g\in Q(f)$. An extremal quasiconformal mapping f is uniquely extremal (in its Teichmüller class) if $k[f] < k[g]$ for every other g in $Q(f)$.

Theorem 5 (First removable singularity theorem). *Suppose that*

(a) f is a quasiconformal mapping from Δ onto Δ

(b) f is a harmonic function with respect to the metric density ρ on $\Delta\backslash K$, where K is compact subset of Δ

(c) f is extremal in its Teichmüller class

(d) there are two positive constant m and M such that $m\le|\varphi(z)|\le M$ for each $z\in\Delta\backslash K$, where φ is Hopf differential of f.

Then φ has an analytic extension $\tilde\varphi$ from $\Delta\backslash K$ to Δ and

$$\mu(z) = k\ |\tilde\varphi(z)|/\tilde\varphi(z)\quad a.e.\quad in\ \Delta,$$

where k is a constant.

Theorem 6 (Second removable singularity theorem). *Suppose that*

(a) f is a uniquely extremal quasiconformal mapping, in its class, from Δ onto Δ

(b) f is a harmonic function with respect to the metric density ρ on $\Delta\backslash K$, where K is compact subset of Δ.

Then we have the same conclusion as in the previous theorem.

During our work with Božin on the problems related to uniquely extremal quasiconformal mapping (see [**BMM**]), we also obtained some results of this type.

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