

# THE TEICHMÜLLER DISTANCE BETWEEN FINITE INDEX SUBGROUPS OF $PSL_2(\mathbb{Z})$

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ABSTRACT. For a given  $\epsilon > 0$ , we show that there exist two finite index subgroups of  $PSL_2(\mathbb{Z})$  which are  $(1 + \epsilon)$ -quasisymmetrically conjugated and the conjugation homeomorphism is not conformal. This implies that for any  $\epsilon > 0$  there are two finite regular covers of the Modular once punctured torus  $T_0$  (or just the Modular torus) and a  $(1 + \epsilon)$ -quasiconformal between them that is not homotopic to a conformal map. As an application of the above results, we show that the orbit of the basepoint in the Teichmüller space  $T(\mathcal{S}^p)$  of the punctured solenoid  $\mathcal{S}^p$  under the action of the corresponding Modular group (which is the mapping class group of  $\mathcal{S}^p$  [5], [6]) has the closure in  $T(\mathcal{S}^p)$  strictly larger than the orbit and that the closure is necessarily uncountable.

## 1. INTRODUCTION

Let  $F$  be a quasiconformal map between two Riemann surfaces. By

$$\mu(F) = \frac{\bar{\partial}F}{\partial F},$$

we denote the Beltrami dilatation (or just the dilatation) of  $F$ . The function

$$K(F) = \frac{1 + |\mu|}{1 - |\mu|},$$

is called the distortion function of  $F$ . If  $K \geq 1$  is such that  $1 \leq K(F) \leq K$  a.e. we say that  $F$  is  $K$ -quasiconformal. If  $F$  is homeomorphism of the unit disc onto itself that is 1-quasiconformal then  $F$  is a Möbius transformations.

Let  $f : \mathbf{S}^1 \rightarrow \mathbf{S}^1$  be a homeomorphism of the unit circle onto itself. We say that  $f$  is a  $K$ -quasisymmetric map if there exists a  $K$ -quasiconformal map  $F : \mathbf{D} \rightarrow \mathbf{D}$  of the unit disc onto itself so that the continuous extension of  $F$  on  $\mathbf{S}^1$  agrees with  $f$  (recall that every quasiconformal maps of the unit disc onto itself extends continuously to a homeomorphism of the unit circle). If  $f$  is homeomorphism of the unit circle onto itself that is 1-quasisymmetric then  $f$  is conformal, that is  $f$  is a Möbius transformations.

The first main result in this paper is

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**Theorem 1.** *For every  $\epsilon > 0$  there exist two finite index subgroups of  $PSL_2(\mathbb{Z})$  which are conjugated by a  $(1 + \epsilon)$ -quasisymmetric homeomorphism of the unit circle and this conjugation homeomorphism is not conformal.*

Unless stated otherwise by a Riemann surface we always mean a Riemann surface of finite type. Every such Riemann surface is obtained by deleting at most finitely many points from a closed Riemann surface. If  $M$  and  $N$  are Riemann surfaces we say that a map  $\pi : N \rightarrow M$  is a finite degree, regular covering if  $\pi$  is holomorphic, of finite degree and locally univalent (we also say that  $N$  is a cover of  $M$ ). (Some people prefer the term unbranched covering instead of regular covering.) Unless stated otherwise all coverings are assumed to be regular and of finite degree. Note that if  $M$  and  $N$  have punctures then the regularity assumption does not imply that  $\pi$  is locally univalent in a neighborhood of a puncture (which is only natural since the punctures are not part of the corresponding surface).

Given two Riemann surfaces  $M$  and  $N$  the Ehrenpreis conjecture asks if for every  $\epsilon > 0$  there are coverings  $M_\epsilon \rightarrow M$  and  $N_\epsilon \rightarrow N$  such that there exists a  $(1 + \epsilon)$ -quasiconformal map  $F : M_\epsilon \rightarrow N_\epsilon$ . In this case we say that  $M_\epsilon$  and  $N_\epsilon$  are  $\epsilon$ -close. It is easy to see that this conjecture is true if  $M$  and  $N$  are tori.

Recall that the notion of quasiconformal  $\epsilon$ -closeness between hyperbolic Riemann surfaces is in fact a geometric property. After endowing the Riemann surfaces  $M$  and  $N$  with the corresponding hyperbolic metrics it is well known that a  $(1 + \epsilon)$ -quasiconformal map  $F : M_\epsilon \rightarrow N_\epsilon$  is isotopic to a  $(1 + \delta)$ -biLipschitz homeomorphism such that  $\delta \rightarrow 0$  when  $\epsilon \rightarrow 0$  (the function  $\delta = \delta(\epsilon)$  does not depend on the choice of surfaces  $M$  and  $N$ ). One such biLipschitz map is obtained by taking the barycentric extension [1] of the boundary map of the lift to the universal covering of  $f : M_\epsilon \rightarrow N_\epsilon$  (this observation was made in [2]).

There are no known examples of hyperbolic Riemann surfaces  $M$  and  $N$ , such that for every  $\epsilon > 0$  there are coverings  $M_\epsilon \rightarrow M$  and  $N_\epsilon \rightarrow N$  which are  $\epsilon$ -close, unless  $M$  and  $N$  are commensurate. We say that  $M$  and  $N$  are commensurate if they have a common cover (recall that we assume throughout the paper that all coverings are regular). If  $M$  and  $N$  are commensurate one can say that  $M$  and  $N$  have coverings that are 0-close.

It is not difficult to see (see the last section for the proof) that if the Ehrenpreis conjecture had a positive answer then for any Riemann surface  $M$  and for any  $\epsilon > 0$ , there would exist two coverings  $M_1, M_2 \rightarrow M$  and a  $(1 + \epsilon)$ -quasiconformal homeomorphism  $F : M_1 \rightarrow M_2$  that is not homotopic to a conformal map. In general, for a given Riemann surface  $M$  the problem of constructing two such covers  $M_1$  and  $M_2$  and the corresponding  $(1 + \epsilon)$ -quasiconformal map  $F$  (where  $F$  is not homotopic to a conformal map) seems to have a similar degree of difficulty as the Ehrenpreis conjecture.

**Problem.** *Let  $M$  be a hyperbolic Riemann surface. Is it true that for every  $\epsilon > 0$  there exist two coverings  $M_1, M_2 \rightarrow M$  and a  $(1 + \epsilon)$ -quasiconformal homeomorphism  $F : M_1 \rightarrow M_2$  that is not homotopic to a conformal map?*

If such covering surfaces  $M_1$  and  $M_2$  exist they may be conformally equivalent. In that case the homeomorphism  $F$  is not allowed to be homotopic to any such conformal equivalence. If  $M_1$  and  $M_2$  are not conformally equivalent then we could say that the coverings  $M_1$  and  $M_2$  are  $\epsilon$ -close but not 0-close.

In this paper we show that this problem has a positive answer for the Modular torus (and any other Riemann surface commensurate with it).

**Corollary 1.** *Let  $T_0$  denote the Modular torus. Then for every  $\epsilon > 0$  there are finite degree, regular coverings  $\pi_1 : M_1 \rightarrow T_0$  and  $\pi_2 : M_2 \rightarrow T_0$ , and a  $(1 + \epsilon)$ -quasiconformal homeomorphism  $F : M_1 \rightarrow M_2$  that is not homotopic to a conformal map.*

**Remark.** This result of this corollary should not be confused with the result from [7] which states that on a topological closed surface, of a large enough genus, one can put a complex structure so that certain pseudo-Anosov quasiconformal homeomorphisms of that surface onto itself have small dilatations.

*Proof.* This follows directly from Theorem 1. Assume that  $G_1, G_2 < PSL_2(\mathbb{Z})$  are two finite index subgroups that are conjugated by  $(1 + \epsilon)$ -quasisymmetric map of the unit circle that is not conformal. Let  $M_i$ ,  $i = 1, 2$ , be the Riemann surface that is conformally equivalent to the quotient  $\mathbf{D}/G_i$ . Then  $M_1$  and  $M_2$  satisfy the assumptions in the statement of this corollary.  $\square$

Consider the coverings  $\pi_1 : M_1 \rightarrow T_0$  and  $\pi_2 : M_2 \rightarrow T_0$ , where the surfaces  $M_1$  and  $M_2$  are from Corollary 1. Then for  $\epsilon$  small enough the  $(1 + \epsilon)$ -quasiconformal map  $F : M_1 \rightarrow M_2$  can not be a lift of a self homeomorphism of  $T_0$ . That is, there is no homeomorphism  $\hat{F} : T_0 \rightarrow T_0$  so that  $\hat{F} \circ \pi_1 = \pi_2 \circ F$ . The non-existence of such a map  $\hat{F}$  follows from the discreteness of the action of the Modular group on the Teichmüller space of the surface  $T_0$ . This illustrates what is difficult about proving Corollary 1. An important ingredient in the proof of Theorem 1. is the fact that  $PSL_2(\mathbb{Z})$  is an arithmetic lattice. At the moment we can not prove this result for other punctured surfaces (or for any closed surface, not even those whose covering groups are arithmetic). However, already from Corollary 1. one can make strong conclusions about the Teichmüller space of the punctured solenoid.

**Remark.** In [3] Long and Reid defined the notion of pseudo-modular surfaces and have shown their existence. As an important special case, it would be interesting to examine whether one can prove the above corollary for a pseudo-modular surface instead of the Modular torus.

Recall that the inverse limit  $\mathcal{S}$  of the family of all pointed regular finite covers of a closed hyperbolic Riemann surface is called *the Universal hyperbolic solenoid* (see [9]). It is well known that the commensurator group of the fundamental group of a closed Riemann surface acts naturally on the Teichmüller space  $T(\mathcal{S})$  of the solenoid  $\mathcal{S}$  [5]. Sullivan has observed that the Ehrenpreis conjecture is equivalent to the question whether the orbits of this action are dense in  $T(\mathcal{S})$  with respect to the corresponding Teichmüller metric.

In [9] mainly closed Riemann surface have been considered (as a model how such holomorphic inverse limits should be constructed). We consider the family of all

pointed coverings of some fixed once punctured torus  $T$ . The punctured solenoid  $\mathcal{S}^p$  is the inverse limit of the above family [8]. The covers of the punctured torus  $T$  are regular, but as we already pointed out, the coverings can be naturally extended to the punctures in the boundary and are allowed to be branched over those boundary punctures. The punctured solenoid  $\mathcal{S}^p$  is an analog, in the presence of the punctures, of the universal hyperbolic solenoid  $\mathcal{S}$ . The (peripheral preserving) commensurator group  $Comm_{per}(\pi_1(T))$  of the fundamental group  $\pi_1(T)$  of  $T$  acts naturally on the Teichmüller space  $T(\mathcal{S}^p)$  of the punctured solenoid  $\mathcal{S}^p$ . We consider the orbit space  $T(\mathcal{S}^p)/Comm_{per}(\pi_1(T))$ .

The corollary below is a significant improvement from [4] of our understanding of  $T(\mathcal{S}^p)/Comm_{per}(\pi_1(T))$ . Namely, we showed in [4] that  $T(\mathcal{S}^p)/Comm_{per}(\pi_1(T))$  is non-Hausdorff by showing that orbits under  $PSL_2(\mathbb{Z})$  of marked hyperbolic metrics on  $\mathcal{S}^p$  which are not lifts of hyperbolic metrics on finite surfaces have accumulation points in  $T(\mathcal{S}^p)$ . In this paper we start with the basepoint in  $T(\mathcal{S}^p)$ , i.e. a marked hyperbolic metric from the Modular torus, and find an explicit sequence of elements in  $Comm_{per}(\pi_1(T))$  such that the image of the basepoint under these elements accumulates onto itself. Moreover, we establish that the orbit of the basepoint has closure strictly larger than the orbit itself.

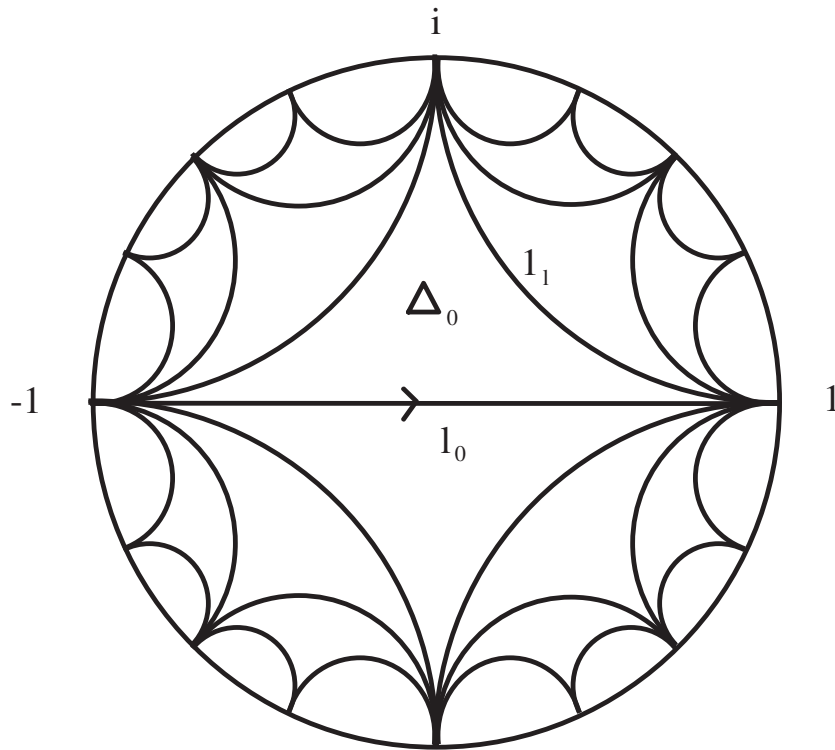
**Corollary 2.** *The closure in the Teichmüller metric of the orbit (under the base leaf preserving mapping class group  $Comm_{per}(\pi_1(T))$ ) of the basepoint in  $T(\mathcal{S}^p)$  is strictly larger than the orbit. Moreover, the closure of this orbit is uncountable.*

The above Corollary is proved using the Baire category theorem and Theorem 3.3 (see Section 3). However, we are also able to find an explicit sequence in  $Comm_{per}(\pi_1(T))$  whose limit point in  $T(\mathcal{S}^p)$  is not an element of  $Comm_{per}(\pi_1(T))$  (see Corollary 4.2 in Section 4).

## 2. THE FAREY TESSELLATION

We define the Farey tessellation  $\mathcal{F}$  of the unit disk  $\mathbf{D}$  as follows (see Figure 1). Let  $\Delta_0$  be the ideal triangle in  $\mathbf{D}$  with vertices  $-1$ ,  $1$  and  $i$ . We invert  $\Delta_0$  by applying the three hyperbolic involutions, each of the three preserves setwise one boundary side of  $\Delta_0$  (but it changes the orientation on the corresponding geodesic). By this, we obtain three more ideal triangles each sharing one boundary side with  $\Delta_0$ . We continue the inversions with respect to the new triangles indefinitely. As a result, we obtain a locally finite ideal triangulation of  $\mathbf{D}$  called *the Farey tessellation*  $\mathcal{F}$ . The set of the vertices in  $S^1$  of the ideal triangles from  $\mathcal{F}$  is denoted by  $\bar{\mathbb{Q}}$ . A hyperbolic geodesic that is a side of a triangle from  $\mathcal{F}$  is also called an *edge* in  $\mathcal{F}$ . Denote by  $l_0$  the edge with the endpoints  $-1$  and  $1$ , and fix an orientation on  $l_0$  such that  $-1$  is the initial point and  $1$  is the terminal point. We call this edge the *distinguished oriented edge* of  $\mathcal{F}$ . Also, denote by  $l_1$  the oriented edge of  $\mathcal{F}$  with the endpoints  $1$  and  $i$  (and in that order).

Let  $f : S^1 \rightarrow S^1$  be a homeomorphism. Then  $f(\mathcal{F})$  is a well defined ideal triangulation of  $\mathbf{D}$ . We say that  $\mathcal{F}$  is invariant under  $f$  if  $f(\mathcal{F}) = \mathcal{F}$  as the ideal triangulations. The Farey tessellation  $\mathcal{F}$  is invariant under the action of the group  $PSL_2(\mathbb{Z})$ . If a homeomorphism of  $S^1$  preserves  $\mathcal{F}$ , then it is necessarily in  $PSL_2(\mathbb{Z})$ . This easy but important observation was proved in [8].



Farey tessellation  $\mathcal{F}$  ;  $l_0 = (-1, 1)$  ;  $l_1 = (1, i)$

FIGURE 1

Consider two arbitrary locally finite ideal triangulations  $\mathcal{F}^1$  and  $\mathcal{F}^2$  of  $\mathbf{D}$ . Fix two oriented edges  $e_1$  and  $e_2$  from  $\mathcal{F}^1$  and  $\mathcal{F}^2$ , respectively. Then  $e_1$  and  $e_2$  are called the *distinguished oriented edges* of tessellations  $\mathcal{F}^1$  and  $\mathcal{F}^2$ , respectively. There exists a unique homeomorphism  $h : S^1 \rightarrow S^1$  which maps  $\mathcal{F}^1$  onto  $\mathcal{F}^2$  such that  $e_1$  is mapped onto  $e_2$  in the orientation preserving manner [7]. We call such  $f$  the *characteristic map* of  $\mathcal{F}^1$  and  $\mathcal{F}^2$ . (The characteristic maps between the Farey tessellation and an arbitrary tessellation of  $\mathbf{D}$  were used in [7] to study the space of homeomorphisms of  $S^1$ . In this paper, we use the notion of a characteristic map in a slightly broader sense that its domain is not only the Farey tessellation, but we allow an arbitrary tessellation.)

We recall the construction of Whitehead homeomorphisms of  $S^1$  (the construction below has been developed in [8]). Throughout this paper  $G_0 < PSL_2(\mathbb{Z})$  denotes the finite index subgroup such that  $\mathbf{D}/G_0$  is the Modular torus  $T_0$ . An ideal triangulation of  $\mathbf{D}$  is said to be an *invariant tessellation* if it is invariant under the action of a finite index subgroup  $K < G_0$ . Equivalently, an invariant tessellation is an ideal triangulation of  $\mathbf{D}$  that is the lift of a finite, ideal triangulation

of some finite Riemann surface that covers  $T_0$ . In particular, the Farey tessellation is an invariant tessellation.

We use the following result:

**Theorem 2.1.** ([8]) *Let  $\mathcal{F}^1$  and  $\mathcal{F}^2$  be two invariant tessellations with the distinguished oriented edges  $e_1$  and  $e_2$ , respectively. The characteristic map of  $\mathcal{F}^1$  onto  $\mathcal{F}^2$  which sends the distinguished oriented edge  $e_1$  onto the distinguished oriented edge  $e_2$  conjugates a finite index subgroup of  $G_0$  onto another (possibly different) finite index subgroup of  $G_0$ .*

Let  $\mathcal{F}^1$  be an invariant tessellation (with the distinguished oriented edge  $e_1$ ) that is invariant under the action of a finite index subgroup  $K < G_0$  and let  $e$  be an edge of  $\mathcal{F}^1$ . Let  $K_1 < K$  be a finite index subgroup. For the simplicity of the exposition, we assume that the distinguished oriented edge  $e_0$  does not belong to the orbit  $K_1\{e\}$  of the edge  $e$  of  $\mathcal{F}^1$ .

**Definition 2.2.** A *Whitehead move* on  $\mathcal{F}_1$  along the orbit  $K_1\{e\}$  is the operation of replacing the orbit of edges  $K_1\{e\}$  by the new orbit of edges  $K_1\{f\}$ , where  $f$  is other diagonal of the ideal quadrilateral in  $(\mathbf{D} \setminus \mathcal{F}_1) \cup \{e\}$  (see Figure 2). As the result of this operation we obtain a new ideal triangulation of the unit disc  $\mathbf{D}$  that is in fact an invariant tessellation. This new tessellation is denoted by  $\mathcal{F}_{K_1,e}^1$ , it is invariant under the action of the group  $K_1$  and its distinguished oriented edge is  $e_0$ .

Consider the homeomorphisms  $h$  of  $S^1$  which fixes  $e_0$  and which maps  $\mathcal{F}^1$  onto  $\mathcal{F}_{K_1,e}^1$ . By Theorem 2.1,  $h$  conjugates a finite index subgroup of  $G_0$  onto another finite index subgroup of  $G_0$ .

**Definition 2.3.** ([8]) The *Whitehead homeomorphism* corresponding to the Whitehead move along the orbit  $K_1\{e\}$  is the characteristic map  $h : S^1 \rightarrow S^1$  of  $\mathcal{F}^1$  and  $\mathcal{F}_{K_1,e}^1$  that fixes the common distinguished oriented edge  $e_0$  of  $\mathcal{F}^1$  and  $\mathcal{F}_{K_1,e}^1$ . It follows directly that the Whitehead homeomorphism  $h$  depends only on  $\mathcal{F}^1$  and  $\mathcal{F}_{K_1,e}^1$ , and we already noted that it conjugates a finite index subgroup of  $G_0$  onto another (possibly different) finite index subgroup of  $G_0$ .

### 3. MAPPING CLASSES WITH SMALL DILATATIONS

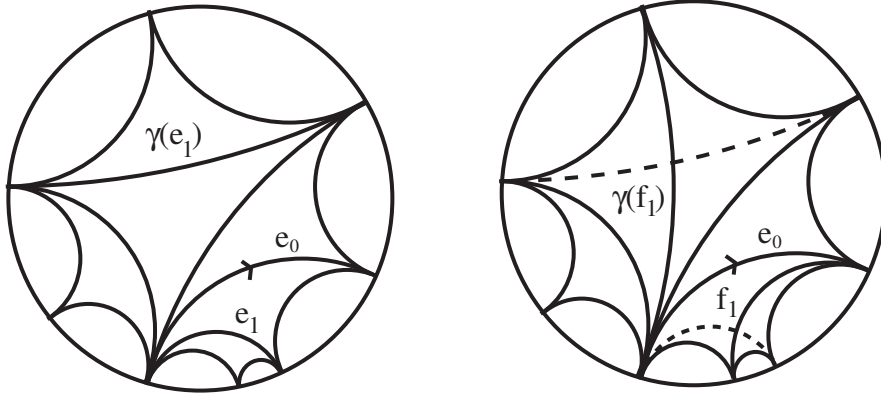
The following lemma is the first step in finding quasiconformal maps of  $\mathbf{D}$  which conjugate two finite index subgroups of  $PSL_2(\mathbb{Z})$  that are not conformally conjugated to each other. We first show that the barycentric extensions [1] of the Whitehead homeomorphisms (which we have defined in the previous section) have dilatations essentially supported in a neighborhood of the diagonal exchange for the corresponding Whitehead move.

We will use the following notation below.

**Definition 3.1.** Let  $F : \mathbf{D} \rightarrow \mathbf{D}$  be a quasiconformal map and let  $N \in \mathbb{N}$ . Then

$$V(F, N) := \{z \in \mathbf{D} : |\mu(F)(z)| \geq \frac{1}{N}\}.$$

Let  $f$  be a homeomorphism of the circle. By  $E(f) : \mathbf{D} \rightarrow \mathbf{D}$  we always denote the barycentric extension of  $f$  (see [1]).



Whitehead move ;  $e_1 \mapsto f_1$ ;  $\gamma(e_1) \mapsto \gamma(f_1)$ ;  $\gamma \in K_1 < G$

FIGURE 2

Recall that  $\mathcal{F}$  is the Farey tessellation with the distinguished oriented edge  $l_0$  (which is an oriented geodesic with endpoints  $-1$  and  $1$ ). We keep the notation  $l_1$  for the edge of  $\mathcal{F}$  whose endpoints are  $1$  and  $i$ . Let  $A \in PSL_2(\mathbb{Q})$  be a hyperbolic translation with the oriented axis  $l_0$ . Let  $\mathcal{F}^A$  denote the image  $A(\mathcal{F})$  of the Farey tessellation  $\mathcal{F}$  under  $A$ . Then  $\mathcal{F}^A$  is invariant under the group  $AG_0A^{-1}$ . Define  $G_A := G_0 \cap AG_0A^{-1}$ . Since  $A$  is in the commensurator of  $PSL_2(\mathbb{Z})$  and since intersections of finitely many finite index subgroups of  $PSL_2(\mathbb{Z})$  is a finite index subgroup of  $PSL_2(\mathbb{Z})$ , we conclude that the group  $G_A$  is a subgroup of finite index in  $PSL_2(\mathbb{Z})$ . It follows that  $\mathcal{F}^A$  is an invariant tessellation of  $\mathbf{D}$  which is invariant under the finite index subgroup  $G_A < G_0$  (note that the relation  $G_A < G_0$  follows from the definition of  $G_A$ ).

**Lemma 3.2.** *Let  $\mathcal{F}^A$  be an invariant tessellation of  $\mathbf{D}$  which is the image of the Farey tessellation  $\mathcal{F}$  under a hyperbolic translation  $A \in PSL_2(\mathbb{Q})$  with the oriented axis  $l_0$ . Let  $\mathcal{F}_{G,A(l_1)}^A$  be the image of  $\mathcal{F}^A$  under the Whitehead move along the orbit  $G\{A(l_1)\}$ , where  $G < G_A$  is any subgroup of finite index. Let  $f_A$  be the Whitehead homeomorphism which maps  $\mathcal{F}_{G,A(l_1)}^A$  onto  $\mathcal{F}^A$  fixing the common distinguished oriented edge  $l_0$  and let  $E(f_A)$  be its barycentric extension. Then, for each  $N \in \mathbb{N}$  there exists  $K_N = K_N(z_0, \mathcal{F}^A) > 0$  such that  $V(E(f_A), N)$  is a subset of the  $K_N$ -neighborhood of the orbit  $G\{z_0\}$ , where  $z_0 \in l_0$  is an arbitrary point. The constant  $K_N$  is independent of  $G$ .*

**Remark.** According to the definition of the Whitehead homeomorphisms, the characteristic map between  $\mathcal{F}^A$  and  $\mathcal{F}_{G,A(l_1)}^A$  is the Whitehead homeomorphism, and  $f_A$  is its inverse. However, the Whitehead move on  $\mathcal{F}_{G,A(l_1)}^A$  along the orbit  $G\{A(l'_1)\}$  gives  $\mathcal{F}^A$ , where  $l'_1$  is the other diagonal of the ideal quadrilateral in  $(\mathbf{D} \setminus \mathcal{F}) \cup \{l_1\}$ . Therefore,  $f_A$  is also a Whitehead homeomorphism corresponding to this “inverse” Whitehead move. Although the notation  $f_A$  does not suggest that the map  $f_A$

depends on the group  $G < G_A$ , it is important to remember that it does. It will always be clear from the context what is the corresponding group  $G$ .

**Remark.** The above lemma includes the possibility that  $A = id$ . In this case the barycentric extension of the Whitehead homeomorphism  $f_{id}$  between  $\mathcal{F}_{G,l_1}$  and the Farey tessellation  $\mathcal{F}$  is supported on the  $G$  orbit of a neighborhood of  $z_0 \in l_0$ .

**Remark.** The inverse  $E(f_A)^{-1}$  of the barycentric extension of the Whitehead homeomorphism  $f_A$  which maps  $\mathcal{F}^A$  onto  $\mathcal{F}_{G,A(l_1)}^A$  has Beltrami dilatation “essentially” supported in  $K'_N = K'_N(K_N)$ -neighborhood of  $E(f_A)(G\{z_0\})$ . If  $A = id$  then  $E(f_{id})(G\{z_0\}) = H\{E(f_{id})(z_0)\}$ , where  $H < PSL_2(\mathbb{Z})$  is conjugated to  $G$  by  $f_{id}$  (see Theorem 2.1 and its proof in [8]).

**Proof.** Fix  $N \in \mathbb{N}$ . The proof is by contradiction. That is we assume that there exists a sequence of subgroups  $G_n < G_A$  of finite index (every element in every group  $G_n$  necessarily preserves  $\mathcal{F}^A$ ) and a sequence of points  $w_n \in \mathbf{D}$  such that

$$dist(w_n, G_n\{z_0\}) \rightarrow \infty$$

and

$$|\mu(E(f_n))(w_n)| \geq \frac{1}{N},$$

where  $f_n$  is the Whitehead homeomorphism which maps  $\mathcal{F}_{G_n,A(l_1)}^A$  onto  $\mathcal{F}^A$  fixing the common distinguished oriented edge  $l_0$ .

Let  $\mathcal{Z}_0 := G_A\{z_0\}$  be the full orbit of  $z_0$  under  $G_A$ . After passing onto a subsequence if necessary, there are two cases that we have to consider:

- (1) There exists  $C > 0$  so that  $dist(w_n, \mathcal{Z}_0) \leq C$  for all  $n \in \mathbb{N}$ .
- (2) We have that  $dist(w_n, \mathcal{Z}_0) \rightarrow \infty$  as  $n \rightarrow \infty$ .

We denote by  $\mathcal{F}_n^A := \mathcal{F}_{G_n,A(l_1)}^A$  the image of the invariant tessellation  $\mathcal{F}^A$  under the Whitehead move along the orbit  $G_n\{A(l_1)\}$  of  $A(l_1)$ .

We first settle the first case, that is we assume that  $dist(w_n, \mathcal{Z}_0) \leq C$  for all  $n$ . Since  $l_1$  is within the bounded distance from  $z_0$  it follows that  $w_n$  is within the bounded distance from  $G_A\{l_1\}$ . From the assumptions that  $dist(w_n, G_n\{z_0\}) \rightarrow \infty$  as  $n \rightarrow \infty$ , and that  $dist(w_n, \mathcal{Z}_0) \leq C$  for all  $n$ , we get that  $dist(w_n, G_n\{l_1\}) \rightarrow \infty$  as  $n \rightarrow \infty$ .

Recall that  $f_n$  is the Whitehead homeomorphism which maps  $\mathcal{F}_n^A$  onto  $\mathcal{F}^A$  and which fixes the common distinguished oriented edge  $l_0$ . The barycentric extension of  $f_n$  is denoted by  $E(f_n)$ .

Choose  $\gamma_n \in G_A$  such that  $dist(w_n, \gamma_n(z_0)) \leq C$  for all  $n \in \mathbb{N}$ . Since  $\mathcal{F}^A$  is invariant under  $G_A$ , there exists a fundamental polygon for  $G_A$  which is a union of finitely many adjacent triangles from  $\mathcal{F}^A$ . Moreover, we can choose such a fundamental polygon  $\omega$  with the following properties

- (1) The boundary of  $\omega$  contains the distinguished oriented edge  $l_0$ .
- (2) The polygon  $\omega$  is to the left of  $l_0$ .
- (3) We have  $l_1 \subset \omega^\circ$ , where  $\omega^\circ$  is the interior of  $\omega$ .

The union of translates of  $\omega$  under the group  $G_A$  tiles the unit disk  $\mathbf{D}$ .



It is important to note that the tessellations  $\mathcal{F}^A$  and  $\mathcal{F}_n^A$  agree on the orbit  $(G_A \setminus G_n)\{\omega\}$  of the fundamental polygon  $\omega$  (they differ inside the orbit  $G_n\{\omega\}$ ). Let  $\alpha_n \in G_A$  be such so that  $w_n \in \alpha_n(\omega)$ . Also, let  $T_n \subset \alpha_n(\omega)$  be a triangle in  $\mathcal{F}_n^A$  which contains  $w_n$ . Then the triangles  $\alpha_n^{-1}(T_n)$  are contained in  $\omega$  for each  $n$ . After passing onto a subsequence if necessary we may assume that  $\alpha_n^{-1}(T_n)$  is the same triangle  $T$  in  $\omega$  for each  $n$ .

Since  $\text{dist}(w_n, G_n\{l_1\}) \rightarrow \infty$  as  $n \rightarrow \infty$  and  $\text{dist}(w_n, G_A\{z_0\}) \leq C$  for all  $n$ , we conclude that the tessellations  $\alpha_n^{-1}(\mathcal{F}_n^A)$  and  $\mathcal{F}^A$  agree on the edges intersecting a hyperbolic disk with the center  $z_0$  and the hyperbolic radius  $r_n$ , where  $r_n \rightarrow \infty$  as  $n \rightarrow \infty$ . In particular, the triangle  $T$  is in  $\mathcal{F}^A$ . We already know that the triangle  $T'_n = f_n(T_n)$  is in  $\mathcal{F}^A$  because  $f_n$  maps  $\mathcal{F}_n^A$  onto  $\mathcal{F}^A$ . Therefore, there exists a unique  $\beta_n \in A \circ PSL_2(\mathbb{Z}) \circ A^{-1}$  such that  $\beta_n(T'_n) = T$  and such that  $\beta_n \circ f_n \circ \alpha_n$  fixes each vertex of  $T$  (the fact that  $PSL_2(\mathbb{Z})$  is transitive on the oriented edges of the Farey tessellation  $\mathcal{F}$  implies that  $A \circ PSL_2(\mathbb{Z}) \circ A^{-1}$  is transitive on the oriented edges of the invariant tessellation  $\mathcal{F}^A$  which implies the existence of such  $\beta_n$ ).

The circle homeomorphism  $\beta_n \circ f_n \circ \alpha_n$  maps  $\alpha_n^{-1}(\mathcal{F}_n^A)$  onto  $\beta_n(\mathcal{F}^A) = \mathcal{F}^A$ . Its barycentric extension is  $\beta_n \circ E(f_n) \circ \alpha_n$  (the barycentric extension is conformally natural, see [1]). As we have already shown, given any neighborhood of the origin in the unit disc, we can find  $n \in \mathbb{N}$ , so that the tessellations  $\alpha_n^{-1}(\mathcal{F}_n^A)$  and  $\mathcal{F}^A$  agree on that neighborhood. Since  $\beta_n \circ f_n \circ \alpha_n$  fixes every vertex of the triangle  $T$  it follows that  $\beta_n \circ f_n \circ \alpha_n \rightarrow \text{id}$  on the circle as  $n \rightarrow \infty$ . This implies that the Beltrami dilatation  $\mu(\beta_n \circ E(f_n) \circ \alpha_n)$  converges to zero uniformly on compact subsets of  $\mathbf{D}$ . Since  $w_n$  is on the bounded distance from  $Z_0$  it follows that  $\alpha_n^{-1}(w_n)$  is in a compact subset of  $\mathbf{D}$ . This implies that

$$(1) \quad |\mu(\beta_n \circ E(f_n) \circ \alpha_n)(\alpha_n^{-1}(w_n))| \rightarrow 0$$

as  $n \rightarrow \infty$ . Since  $|\mu(\beta_n \circ E(f_n) \circ \alpha_n)| = |\mu(E(f_n)) \circ \alpha_n|$ , we derive a contradiction to the assumption that  $|\mu(E(f_n))(w_n)| \geq \frac{1}{N}$  for all  $n \in \mathbb{N}$ .

It remains to consider the case when  $\text{dist}(w_n, Z_0) \rightarrow \infty$  as  $n \rightarrow \infty$ . We keep the notation  $f_n$  for the Whitehead homeomorphism which maps  $\mathcal{F}_n^A := \mathcal{F}_{G_n, A(l_1)}^A$  onto  $\mathcal{F}^A$  and which fixes  $l_0$ .

**Remark.** Note that the condition  $\text{dist}(w_n, Z_0) \rightarrow \infty$ , as  $n \rightarrow \infty$ , means that the projection of the sequence  $w_n$  onto the surfaces obtained as the quotient of the unit disc by finite index subgroups  $G_n$  of  $G_0$  converges to the punctures in the boundary of that surfaces. The fact (that we prove in detail below) that the Beltrami dilatation of  $E(f_n)$  tends to zero along the sequence  $w_n$  is actually a corollary of the fact that the circle homeomorphism  $f_n$  is differentiable at every ‘‘rational’’ point on the circle (these are the fixed points of parabolic transformations from  $G_0$ ).

We fix a fundamental polygon  $\omega$  for  $G_A$  as above. That is,  $\omega$  is the union of adjacent triangles in  $\mathcal{F}^A$  such that

- (1)  $l_0$  is on the boundary of the fundamental polygon  $\omega$
- (2)  $\omega$  is to the left of  $l_0$

(3)  $l_1 \subset \omega^\circ$ , where  $\omega^\circ$  is the interior of  $\omega$

Let  $T_n$  be a triangle from  $\mathcal{F}_n^A$  which contains  $w_n$  and let  $T'_n = f_n(T_n)$  be the image triangle in  $\mathcal{F}^A$  as before. Let  $\alpha_n \in G_A$  be such that  $T_n \subset \alpha_n(\omega)$  and let  $\beta_n \in G_A$  be such that  $\beta_n(T'_n) \subset \omega$ . After passing onto a subsequence if necessary, we can assume that  $\alpha_n^{-1}(T_n)$  and  $\beta_n(T'_n)$  are fixed triangles  $T$  and  $T'$  in  $\omega$ , respectively. We note that  $T'$  is a triangle in  $\mathcal{F}^A$ . On the other hand,  $T$  is in  $\alpha_n^{-1}(\mathcal{F}_n^A)$  for each  $n$ , which implies that  $T$  is either in  $\mathcal{F}^A$  (if  $\alpha_n \in G_A \setminus G_n$  or if  $\alpha_n \in G_n$  and  $T_n \subset \mathcal{F}^A \setminus \mathcal{F}_n^A$ ) or in  $\mathcal{F}_n^A$  (if  $\alpha_n \in G_n$  and  $T_n \subset \mathcal{F}_n^A \setminus \mathcal{F}^A$ ).

The map  $\tilde{f}_n := \beta_n \circ f_n \circ \alpha_n^{-1}$  maps  $T$  onto  $T'$ . After passing onto a subsequence if necessary, the points  $\alpha_n^{-1}(w_n) \in T$  converge to a single ideal vertex  $y$  of  $T$ . Let  $y' := \tilde{f}_n(y) \in T'$  which is a fixed point for all  $n$  after possibly passing onto a subsequence if necessary. Let  $l$  be a boundary side of  $T$  with  $y$  its ideal endpoint such that  $l \in \mathcal{F}^A$  (at least one of the two boundary sides of  $T$  with  $y$  their ideal endpoint is in  $\mathcal{F}^A$  because the tessellation  $\alpha_n^{-1}(\mathcal{F}_n^A)$  is obtained by a Whitehead move on  $\mathcal{F}^A$  along the orbit  $\alpha_n^{-1}G_n\{A(l_1)\}$  which implies that no two adjacent geodesics can be changed by the definition of a Whitehead move). Let  $l' = \tilde{f}_n(l) \in \mathcal{F}^A$  be a boundary side of  $T'$  with an ideal endpoint  $y'$  (since  $\tilde{f}_n(T)$  is a fixed triangle  $T'$  for each  $n$ , then after passing onto a subsequence if necessary the side  $\tilde{f}_n(l)$  is the same boundary side  $l'$  of  $T'$ ).

Let  $\gamma \in G_A$  be a primitive parabolic element which fixes  $y$  and let  $\gamma' \in G_A$  be a primitive parabolic element in  $G_A$  which fixes  $y'$ . Then the set of edges in  $\mathcal{F}^A$  with one ideal endpoint  $y$  is invariant under the action of  $\gamma$  and a fundamental set for the action of a cyclic group  $\langle \gamma \rangle$  generated by  $\gamma$  consists of finitely many adjacent geodesics of  $\mathcal{F}^A$  with one endpoint  $y$ . Similarly, the set of edges in  $\mathcal{F}^A$  with one ideal endpoint  $y'$  is invariant under the action of  $\gamma'$  and a fundamental set for the action of  $\langle \gamma' \rangle$  consists of finitely many adjacent geodesics of  $\mathcal{F}^A$  with one endpoint  $y'$ .

The group  $G_n$  is a finite index subgroup of  $G_A$  and it follows that  $\alpha_n^{-1}G_n\alpha_n$  is also a finite index subgroup of  $G_A$ . Therefore, the isotropy subgroup of  $y$  in  $\alpha_n^{-1}G_n\alpha_n$  is of finite index in the isotropy group  $\langle \gamma \rangle$  of  $y$  in  $G_A$ . Thus a generator  $\gamma_n \in \alpha_n^{-1}G_n\alpha_n$  of the isotropy group of  $y$  is equal to a finite, non-zero, integer power of  $\gamma$ . After possibly replacing  $\gamma_n$  by its inverse if necessary, we have  $\gamma_n = \gamma^{p_n}$ , for some  $p_n \in \mathbb{N}$ . A fundamental set for the action of  $\langle \gamma_n \rangle$  on the geodesics of  $\mathcal{F}^A$  with one endpoint  $y$  is obtained by taking  $p_n$  translates by  $\gamma$  of a fixed fundamental set for  $\langle \gamma \rangle$ .

Note that the tessellation  $\alpha_n^{-1}(\mathcal{F}_n^A)$  is obtained by a Whitehead move on  $\mathcal{F}^A$  along the orbit  $\alpha_n^{-1}G_n\alpha_n\{A(l_1)\}$ . Let  $k_y$  be the number of geodesics in a fundamental set for the action of  $\langle \gamma \rangle$  on the set of edges of  $\mathcal{F}^A$  which have one endpoint  $y$ . If  $l$  is a fixed edge of  $\mathcal{F}^A$  with one endpoint  $y$  then  $k_y$  is the number of edges in  $\mathcal{F}^A$  with one endpoint  $y$  which lie in between  $l$  and  $\gamma(l)$ , where we count  $l$  but do not count  $\gamma(l)$ . Let  $k_{y'}$  be the number of geodesics in a fundamental set for the action of  $\langle \gamma' \rangle$  on the edges of  $\mathcal{F}^A$  with one endpoint  $y'$ . Equivalently,  $k_{y'}$  is the number of geodesics in  $\mathcal{F}^A$  with one ideal endpoint  $y'$  in between  $l'$  and  $\gamma'(l')$  counting  $l'$  but not counting  $\gamma'(l')$ , where  $l'$  is a fixed edge of  $\mathcal{F}^A$  with one endpoint

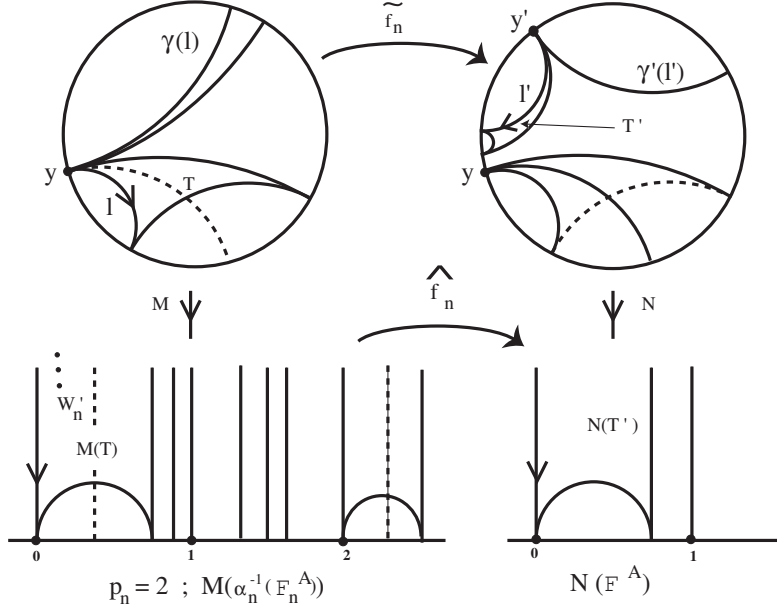


FIGURE 3

$y'$ . The number of geodesics in a fundamental set for the action of  $\langle \gamma'_n \rangle$  on the set of edges of  $\mathcal{F}^A$  with one endpoint  $y$  is  $k_y p_n$ .

Recall that  $\alpha_n^{-1}(\mathcal{F}_n^A)$  is obtained by a Whitehead move on  $\mathcal{F}^A$  along the orbit  $\alpha_n^{-1}G_n\alpha_n\{A(l_1)\}$ . The Whitehead move can either add edges at  $y$ , erase edges at  $y$ , or do not change edges at  $y$ . We further assume that the choice of the edge  $l$  in  $\mathcal{F}^A$  with one endpoint  $y$  is such that the Whitehead move does not erase  $l$ . The number of geodesics in  $\alpha_n^{-1}(\mathcal{F}_n^A)$  with an ideal endpoint at  $y$  in between  $l$  and  $\gamma_n(l)$  (including  $l$  but not including  $\gamma_n(l)$ ) is  $k_y p_n + a$ , where  $a = 0$  if the Whitehead move does not change any edge at  $y$ ,  $a = 1$  if the Whitehead move adds edges at  $y$  or  $a = -1$  if the Whitehead move erases edges at  $y$ . (In the top part of Figure 3, we illustrate the case when the Whitehead move adds geodesics at  $y$ ;  $k_y = 3$ ;  $k_{y'} = 2$ .)

Let  $M : \mathbf{D} \rightarrow \mathbb{H}$  be a Möbius map which sends  $y$  to  $\infty$ ,  $l$  to a geodesic with endpoints 0 and  $\infty$ , and  $\gamma(l)$  to a geodesic with endpoints 1 and  $\infty$ . Let  $N : \mathbf{D} \rightarrow \mathbb{H}$  be a Möbius map which sends  $y'$  to  $\infty$ ,  $l'$  to a geodesic with endpoints 0 and  $\infty$ , and  $\gamma'(l')$  to a geodesic with endpoints 1 and  $\infty$ . Define  $\hat{f}_n := N \circ \tilde{f}_n \circ M^{-1}$ . Let  $w'_n := M(\alpha_n^{-1}(w_n)) \in \mathbb{H}$  (see Figure 3). Then  $w_n \rightarrow \infty$  as  $n \rightarrow \infty$  inside the triangle  $M(T)$ . Namely,  $b_n := \text{Im}(w'_n) \rightarrow \infty$  as  $n \rightarrow \infty$  and  $0 \leq \text{Re}(w'_n) < 1$  for all  $n \in \mathbb{N}$ . This implies that  $\frac{1}{b_n}w'_n$  stays in a compact subset of  $\mathbb{H}$ .

We consider the pointwise limit of  $\frac{1}{b_n}\hat{f}_n(b_n x)$  as  $n \rightarrow \infty$  for all  $x \in \mathbb{R}$ . Our goal is to show that it is a linear map. There are two possibilities (after passing onto a subsequence if necessary)

- (1)  $p_n \rightarrow \infty$  as  $n \rightarrow \infty$
- (2)  $p_n = p$  is fixed, for all  $n \in \mathbb{N}$

We assume that  $p_n \rightarrow \infty$  as  $n \rightarrow \infty$ . We obtain an upper bound

$$(2) \quad \widehat{f}_n(b_n x) < \widehat{f}_n([b_n x] + 1) \leq ([b_n x] + 1)k_y \frac{1}{k_{y'}} + \left( \frac{[b_n x] + 1}{p_n} + 1 \right) \frac{1}{k_{y'}} + 2,$$

where  $[b_n x]$  is the integer part of  $b_n x$ . The first inequality in (2) follows because  $b_n x < [b_n x] + 1$  and  $\widehat{f}_n$  is an increasing function. The second inequality in (2) is obtained as follows. By the choice of  $M$  and  $N$  above, we have  $(M \circ \gamma \circ M^{-1})(z) = z + 1$ ,  $(M \circ \gamma_n \circ M^{-1})(z) = z + p_n$  and  $(N \circ \gamma' \circ N^{-1})(z) = z + 1$  (see Figure 3). In between 0 and  $[b_n x] + 1$  there is  $\left\lceil \frac{[b_n x] + 1}{p_n} \right\rceil$  of adjacent intervals of length  $p_n$ . For each interval of length  $p_n$ , the number of geodesics in  $M(\alpha_n^{-1}(\mathcal{F}_n^A))$  with one endpoint  $\infty$  and the other endpoint in the interval is at most the number of geodesics in  $M(\mathcal{F}^A)$  with one endpoint  $\infty$  and the other point in the interval plus one extra geodesic (because the Whitehead move adds at most one geodesic in such an interval). The map  $\widehat{f}_n$  fixes 0 and  $\infty$ , and it maps the geodesics of  $M(\alpha_n^{-1}(\mathcal{F}_n^A))$  onto the geodesics of  $N(\mathcal{F}^A)$ . Therefore, we need to estimate the number of geodesics in  $M(\alpha_n^{-1}(\mathcal{F}_n^A))$  with one endpoint  $\infty$  and the other endpoint in the interval  $[0, [b_n x] + 1]$ . The second inequality in (2) is obtained by noting that  $([b_n x] + 1)k_y$  is the number of geodesics in  $M(\mathcal{F}^A)$  with one endpoint  $\infty$  and the other point in the interval  $[0, [b_n x] + 1]$  and that we add at most  $\frac{[b_n x] + 1}{p_n} + 1$  geodesics to get the corresponding geodesics of  $M(\alpha_n^{-1}(\mathcal{F}_n^A))$ . We need to divide the number of geodesics by  $k_{y'}$  because  $N(\mathcal{F}^A)$  has  $k_{y'}$  geodesics with one endpoint  $\infty$  and the other endpoint in a fixed interval of length 1. Since the quantities  $([b_n x] + 1)k_y$  and  $\frac{[b_n x] + 1}{p_n} + 1$  are not necessarily divisible with  $k_{y'}$ , we add 2 to ensure that we have an upper bound in (2).

In a similar fashion, we obtain a lower bound

$$(3) \quad \widehat{f}_n(b_n x) \geq \widehat{f}_n([b_n x]) \geq [b_n x]k_y \frac{1}{k_{y'}} - \left( \frac{[b_n x]}{p_n} + 1 \right) \frac{1}{k_{y'}} - 2.$$

Since  $p_n \rightarrow \infty$  and  $\frac{[b_n x]}{b_n} \rightarrow x$  as  $n \rightarrow \infty$ , the inequalities (2) and (3) imply that

$$\frac{1}{b_n} \widehat{f}_n(b_n x) \rightarrow \frac{k_y}{k_{y'}} x,$$

as  $n \rightarrow \infty$ . Thus  $\frac{1}{b_n} \widehat{f}_n(b_n x)$  converges pointwise to a linear map in the case when  $p_n \rightarrow \infty$  as  $n \rightarrow \infty$ .

We assume that  $p_n = p$  for all  $n \in \mathbb{N}$ . Then we obtain the following upper bound

$$(4) \quad \widehat{f}_n(b_n x) < \widehat{f}_n([b_n x] + 1) \leq \left\lceil \frac{[b_n x] + 1}{p} \right\rceil (k_y p + a) \frac{1}{k_{y'}} + k_y p \frac{1}{k_{y'}}.$$

The second inequality in (4) is obtained by noting that there is  $\left\lceil \frac{[b_n x] + 1}{p} \right\rceil$  adjacent disjoint intervals of length  $p$  from 0 to  $[b_n x] + 1$  each of which contains endpoints of  $k_y p + a$  geodesics of  $M(\alpha_n^{-1}(\mathcal{F}_n^A))$  with the other endpoint at  $\infty$ . Since each interval of length 1 contains  $k_{y'}$  endpoints of geodesics of  $N(\mathcal{F}^A)$  with the other endpoint  $\infty$ , we obtain the first summand on the right of (4). We add  $k_y p \frac{1}{k_{y'}}$  to the

right of (4) because  $\frac{[b_n x]+1}{p}$  might not be an integer and, in this case, the interval  $[[\frac{[b_n x]+1}{p}]p, [b_n x] + 1]$  is of the length strictly smaller than  $p$ . Our upper estimate of this part uses an interval of length  $p$ .

The following lower bound

$$(5) \quad \widehat{f}_n(b_n x) \geq \widehat{f}_n([b_n x]) \geq \left[ \frac{[b_n x]}{p} \right] (k_y p + a) \frac{1}{k_{y'}}$$

is obtained similarly to the above upper bound.

The inequalities (4) and (5) together with the facts that  $[\frac{[b_n x]+1}{p}]/b_n \rightarrow \frac{1}{p}x$  and  $[\frac{[b_n x]}{p}]/b_n \rightarrow \frac{1}{p}x$  as  $n \rightarrow \infty$  imply that

$$\frac{1}{b_n} \widehat{f}_n(b_n x) \rightarrow \left( \frac{k_y}{k_{y'}} + \frac{a}{pk_{y'}} \right) x$$

as  $n \rightarrow \infty$ . Thus  $\frac{1}{b_n} \widehat{f}_n(b_n x)$  converges to a linear map in the case  $p_n = p$  as well.

We showed above that  $\frac{1}{b_n} \widehat{f}_n(b_n x)$  converges pointwise to a linear map in both cases which implies that  $|\mu(E(B_n^{-1} \circ \widehat{f}_n \circ B_n))| \rightarrow 0$  uniformly on compact subsets, where  $B_n(z) := b_n z$ . Since  $\frac{1}{b_n} w'_n$  stays in a compact subset of  $\mathbb{H}$ , we get (by the conformal naturality of the barycentric extension [1]) that  $|\mu(E(B_n^{-1} \circ \widehat{f}_n \circ B_n))(\frac{1}{b_n} w'_n)| = |\mu(E(\widehat{f}_n))(w'_n)| \rightarrow 0$  as  $n \rightarrow \infty$ . But this is in contradiction with the starting assumption that  $|\mu(E(f_n))(w_n)| \geq \frac{1}{N}$  which proves the lemma.  $\square$

Recall that  $A \in PSL_2(\mathbb{Q})$  is a hyperbolic translation with the oriented axis  $l_0$  and that  $G_A = AG_0A^{-1} \cap G_0$ . Then  $G_A$  is a finite index subgroup of  $G_0$ . We keep the notation  $\mathcal{F}$  for the Farey tessellation and the notation  $\mathcal{F}^A$  for the image of  $\mathcal{F}$  under  $A$ . Then  $\mathcal{F}^A$  is a tessellation of  $\mathbf{D}$  invariant under  $G_A$ .

If  $G$  is a finite index subgroup of  $G_0$ , recall that  $\mathcal{F}_{G,l_1}$  is the image of  $\mathcal{F}$  under the Whitehead move along the orbit  $G\{l_1\}$ . If  $G$  is a finite index subgroup of  $G_A$ , recall that  $\mathcal{F}_{G,A(l_1)}^A$  is the image of  $\mathcal{F}^A$  under the Whitehead move along the orbit  $G\{A(l_1)\}$ .

We say that a sequence  $f_n$  of quasymmetric maps of  $S^1$  converges in the Teichmüller metric to a quasymmetric map  $f$  of  $S^1$  if there exists a sequence of quasiconformal extensions  $F_n : \mathbf{D} \rightarrow \mathbf{D}$  of  $f_n$  and a quasiconformal extension  $F : \mathbf{D} \rightarrow \mathbf{D}$  of  $f$  such that  $\|\mu(F_n) - \mu(F)\|_\infty \rightarrow 0$  as  $n \rightarrow \infty$ . Note that the Teichmüller metric on the space of quasymmetric maps of  $S^1$  is a pseudometric. The Teichmüller metric projects to a proper metric on the quotient of the space of quasymmetric maps of  $S^1$  by the action of  $PSL_2(\mathbb{R})$  (where the action is given by the post-composition of quasymmetric maps with  $PSL_2(\mathbb{R})$ ).

We show below that the Whitehead homeomorphism from  $\mathcal{F}_{G,A(l_1)}^A$  to  $\mathcal{F}^A$  followed by the Whitehead homeomorphism from  $\mathcal{F}$  to  $\mathcal{F}_{G,l_1}$  converges to the identity in the Teichmüller metric as  $A$  converges to the identity.

**Theorem 3.3.** *Let  $A \in PSL_2(\mathbb{Q})$  be a hyperbolic translation with the oriented axis  $l_0$ . Let  $G_A$  and  $\mathcal{F}^A$  be as above, and let  $G$  be a finite index subgroup of  $G_A$ . Let  $f_{id}$*

be the Whitehead homeomorphism fixing  $l_0$  which maps  $\mathcal{F}_{G,l_1}$  onto  $\mathcal{F}$ , and let  $g_A$  be the Whitehead homeomorphism fixing  $l_0$  which maps  $\mathcal{F}_{G,A(l_1)}^A$  onto  $\mathcal{F}^A$ . Then

$$g_A \circ f_{id}^{-1} \rightarrow id$$

in the Teichmüller metric as  $A \rightarrow id$ .

**Proof.** We denote by  $E(f_{id})$  and  $E(g_A)$  the barycentric extensions of  $f_{id}$  and  $g_A$ , respectively. It is enough to show that

$$\|\mu(E(f_{id})) - \mu(E(g_A))\|_\infty \rightarrow 0$$

as  $A \rightarrow id$ . (It is important to note that  $\|\mu(E(f_{id})) - \mu(E(g_A))\|_\infty$  is equal to  $\sup_{z \in \mathbf{D}} |\mu(E(f_{id}))(z) - \mu(E(g_A))(z)|$  because the barycentric extensions of quasymmetric maps are analytic diffeomorphisms which implies that their Beltrami dilatations are continuous maps.)

Assume on the contrary that there exist sequences  $w_n \in \mathbf{D}$ ,  $A_n \in PSL_2(\mathbb{Q})$  and  $G_n < G_{A_n}$  such that  $A_n$  is a hyperbolic translation with the oriented axis  $l_0$ ,  $A_n \rightarrow id$  as  $n \rightarrow \infty$ ,  $[G_{A_n} : G_n] < \infty$  and

$$(6) \quad |\mu(E(f_{id}))(w_n) - \mu(E(g_{A_n}))(w_n)| \geq \frac{1}{N}$$

for all  $n \in \mathbb{N}$  and for a fixed  $N \in \mathbb{N}$ . This implies that either  $|\mu(E(f_{id}))(w_n)|$  or  $|\mu(E(g_{A_n}))(w_n)|$  is at least  $1/N$ . By Lemma 3.2, there exists  $K_N(z_0, \mathcal{F}) > 0$  such that  $V(E(f_{id}))$  is a subset of the  $K_N(z_0, \mathcal{F})$ -neighborhood of the orbit  $G_n\{z_0\}$ . Again by Lemma 3.2, there exists  $K_N(z_0, \mathcal{F}^{A_n}) > 0$  such that  $V(E(g_{A_n}))$  is a subset of the  $K_N(z_0, \mathcal{F}^{A_n})$ -neighborhood of the orbit  $G_n\{z_0\}$ .

**Remark.** Let  $l'_1$  be the diagonal of the ideal quadrilateral in  $(\mathbf{D} \setminus \mathcal{F}) \cup \{l_1\}$  different from  $l_1$ . We note that the Whitehead homeomorphism  $f_{id}^{-1}$  from  $\mathcal{F}$  to  $\mathcal{F}_{G_n,l_1}$  does not necessarily map the orbit  $G_n\{l_1\}$  in  $\mathcal{F}$  onto the orbit  $G_n\{l'_1\}$ ; the Whitehead homeomorphism  $g_{A_n}^{-1}$  from  $\mathcal{F}^{A_n}$  to  $\mathcal{F}_{G_n,A_n(l_1)}^{A_n}$  does not necessarily map the orbit  $G_n\{A_n(l_1)\}$  in  $\mathcal{F}^{A_n}$  onto the orbit  $G_n\{A_n(l'_1)\}$  in  $\mathcal{F}_{G_n,A_n(l_1)}^A$ . On the other hand,  $\mathcal{F}$  and  $\mathcal{F}^{A_n}$  are both obtained by infinite number of inversions in any of their triangles. Thus, it is better to consider inverse Whitehead homeomorphisms  $f_{id}$  and  $g_{A_n}$  because the image tessellations of  $\mathcal{F}_{G_n,l_1}$  and  $\mathcal{F}_{G_n,A_n(l_1)}^{A_n}$  are geometrically well-behaved. This was utilized in the proof of Lemma 3.2 to claim that the support of the barycentric extension of the Whitehead homeomorphism is “essentially” at the place where the Whitehead move exchanges diagonals.

**Remark.** Note that  $A_n \rightarrow id$  does not imply that  $l$  and  $A_n(l)$  are close uniformly for all edges  $l$  of  $\mathcal{F}$ . In fact, if the distance from  $l$  to  $l_0$  goes to infinity then the distance between  $l$  and  $A_n(l)$  goes to infinity for each  $n$  fixed. It is essential that we choose Whitehead moves along  $G_n\{l_1\}$  and  $G_n\{A_n(l_1)\}$  with  $l_1$  close to  $l_0$  and fixed. Since  $l_1$  and  $A_n(l_1)$  are close, then their images under  $G_n$  are close which allows us to compare the two maps along the orbits  $G_n\{l_1\}$  and  $G_n\{A_n(l_1)\}$  whose corresponding elements are close. The crucial fact that allows our method to work is that maps  $E(f_{id})$  and  $E(g_{A_n})$  have small Beltrami dilatations far away from the place where the Whitehead moves exchange the diagonals because we do not have a uniform geometric control over the maps away from the places where the diagonals are exchanged, see above remark.

Let  $K_N = \max\{K_N(z_0, \mathcal{F}), K_N(z_0, \mathcal{F}^A)\}$ . Then, by (6) and by the choice of the above neighborhoods of the  $G_n$ -orbit of  $z_0$ ,  $w_n$  belongs to the  $K_N$ -neighborhood of the orbit  $G_n\{z_0\}$ . Thus there exists  $\gamma_n \in G_n$  such that  $w_n$  is in the  $K_N$ -neighborhood of  $\gamma_n(z_0)$  for each  $n \in \mathbb{N}$ . By the transitivity of  $PSL_2(\mathbb{Z})$  on the oriented edges of  $\mathcal{F}$  and by the transitivity of  $A_n \circ PSL_2(\mathbb{Z}) \circ A_n^{-1}$  on the oriented edges of  $\mathcal{F}^{A_n}$ , there exist  $\delta_n \in PSL_2(\mathbb{Z})$  and  $\delta'_n \in A_n \circ PSL_2(\mathbb{Z}) \circ A_n^{-1}$  such that  $\delta_n \circ f_{id} \circ \gamma_n(l_0) = l_0$  and  $\delta'_n \circ g_{A_n} \circ \gamma_n(l_0) = l_0$ .

Since  $A_n \rightarrow id$  as  $n \rightarrow \infty$ , it follows that  $[G_0 : G_{A_n}] \rightarrow \infty$  as  $n \rightarrow \infty$ . This implies that  $[G_0 : G_n] \rightarrow \infty$  as  $n \rightarrow \infty$ . The homeomorphism  $\delta_n \circ f_{id} \circ \gamma_n$  maps  $\mathcal{F}_{G_n, l_1}$  onto  $\mathcal{F}$ , and the homeomorphism  $\delta'_n \circ g_{A_n} \circ \gamma_n$  maps  $\mathcal{F}_{G_n, A_n(l_1)}^{A_n}$  onto  $\mathcal{F}^{A_n}$ . The sequence of tessellations  $\mathcal{F}_{G_n, l_1}$  converges to the tessellation  $\mathcal{F}_{l_1}$  which differs from the Farey tessellation  $\mathcal{F}$  by the Whitehead move on the single edge  $l_1$ ; the sequence of tessellations  $\mathcal{F}_{G_n, A_n(l_1)}^{A_n}$  converges to the tessellation  $\mathcal{F}_{l_1}$  as well; and  $\mathcal{F}^{A_n}$  converges to the Farey tessellation  $\mathcal{F}$  (the convergence is in the Hausdorff topology on compact subsets of the space of geodesics in  $\mathbf{D}$ ). The above convergence of the tessellations and the normalizations of  $\delta_n \circ f_{id} \circ \gamma_n$  and of  $\delta'_n \circ g_{A_n} \circ \gamma_n$  implies that both maps pointwise converge to the Whitehead homeomorphism  $f_{l_1}$  which maps  $\mathcal{F}_{l_1}$  onto the Farey tessellation  $\mathcal{F}$  and which fixes  $l_0$  (see Figure 4).

Since  $\delta_n \circ f_{id} \circ \gamma_n \rightarrow f_{l_1}$  and  $\delta'_n \circ g_{A_n} \circ \gamma_n \rightarrow f_{l_1}$  pointwise as  $n \rightarrow \infty$ , it follows that the Beltrami dilatations  $\mu(\delta_n \circ E(f_{id}) \circ \gamma_n) = \mu(E(f_{id}) \circ \gamma_n)$  and  $\mu(\delta'_n \circ E(g_{A_n}) \circ \gamma_n) = \mu(E(g_{A_n}) \circ \gamma_n)$  converge uniformly on compact subsets of  $\mathbf{D}$  to the Beltrami dilatation  $\mu(E(f_{l_1}))$  of the barycentric extension  $E(f_{l_1})$  of  $f_{l_1}$  (see [1]). Since  $\gamma_n^{-1}(w_n)$  belongs to the  $K_N$ -neighborhood of  $z_0$ , it follows that

$$|\mu(E(f_{id}) \circ \gamma_n)(\gamma_n^{-1}(w_n)) - \mu(E(g_{A_n}) \circ \gamma_n)(\gamma_n^{-1}(w_n))| \rightarrow 0$$

as  $n \rightarrow \infty$ . This is the same as

$$|\mu(E(f_{id}))(w_n) - \mu(E(g_{A_n}))(w_n)| \rightarrow 0$$

as  $n \rightarrow \infty$ . But this is in the contradiction with (6). The contradiction proves the theorem.  $\square$

To finish the proof of Theorem 1 it remains to establish that the circle homeomorphisms  $g_A \circ f_{id}^{-1}$  are not conformal maps. In fact, the proof below shows this for  $A$  close enough to the identity and when the corresponding group  $G$  (that determines the map  $f_A$ ) has a sufficiently large index.

**Theorem 1.** *For every  $\epsilon > 0$  there exist two finite index subgroups of  $PSL_2(\mathbb{Z})$  which are conjugated by a  $(1+\epsilon)$ -quasisymmetric homeomorphism of the unit circle and this conjugation homeomorphism is not conformal.*

**Proof.** Recall that the Whitehead homeomorphism  $f_{id}$  maps  $\mathcal{F}_{G, l_1}$  onto the Farey tessellation  $\mathcal{F}$ , and that the Whitehead homeomorphism  $g_A$  maps  $\mathcal{F}_{G, A(l_1)}^A$  onto  $\mathcal{F}^A$ , where  $G < G_A$  is any subgroup of finite index. By Theorem 3.3, there exists a neighborhood  $\mathcal{U}_{id}$  of the identity in  $PSL_2(\mathbb{Q})$  such that for any hyperbolic translation  $A \in \mathcal{U}_{id}$  whose oriented axis is  $l_0$  the composition  $E(g_A) \circ E(f_{id})^{-1}$  of the barycentric extension of  $g_A$  and  $f_{id}$  has the quasiconformal constant less than  $1+\epsilon$ . It is enough to show that  $g_A \circ f_{id}^{-1}$  conjugates a finite index subgroup of  $G_A$  onto

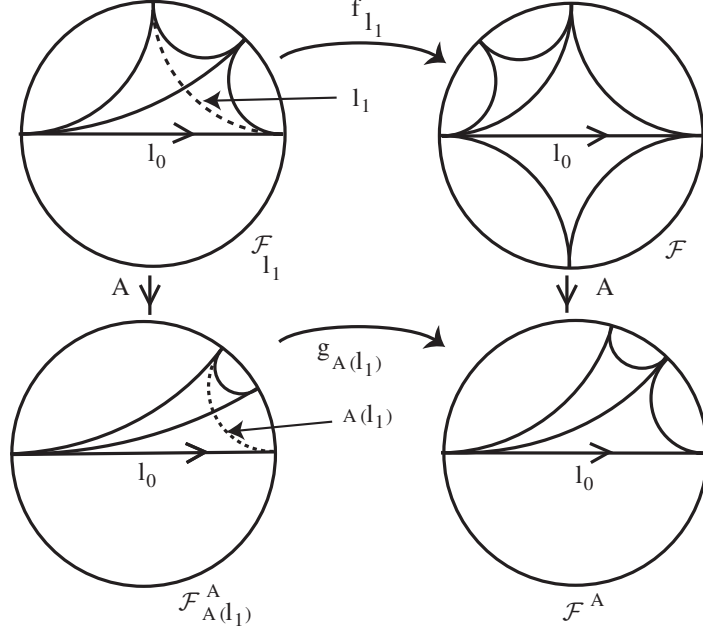


FIGURE 4

another finite index subgroup of  $PSL_2(\mathbb{Z})$  and that  $g_A \circ f_{id}^{-1}$  is not conformal (that is, the homeomorphism  $g_A \circ f_{id}^{-1}$  is not a Möbius transformation).

Let  $f_n$  and  $g_n$  be two Whitehead homeomorphisms corresponding to the Whitehead moves along the orbits  $G_n\{l_1\}$  and  $G_n\{A(l_1)\}$  on  $\mathcal{F}$  and  $\mathcal{F}^A$ , where  $G_n < G_A$  is a sequence of finite index subgroups with  $\bigcap_{n=1}^{\infty} G_n = \{id\}$ . In this case, the sequence  $f_n$  converges pointwise to the Whitehead homeomorphism  $f_{l_1}$  which maps the tessellation  $\mathcal{F}_{l_1}$  onto the Farey tessellation  $\mathcal{F}$ , where  $\mathcal{F}_{l_1}$  is the image of the Farey tessellation  $\mathcal{F}$  under the Whitehead move on a single edge  $l_1$ . The sequence  $g_n$  pointwise converges to the Whitehead homeomorphism  $g_{A(l_1)}$  which maps  $\mathcal{F}_{A(l_1)}^A$  onto  $\mathcal{F}^A$ , where  $\mathcal{F}_{A(l_1)}^A$  is the image of  $\mathcal{F}^A$  under the Whitehead move on a single edge  $A(l_1)$  (see Figure 4).

To see that  $g_n \circ (f_n)^{-1}$  is not a Möbius map for  $n$  large enough, it is enough to show that  $g_{A(l_1)} \circ f_{l_1}^{-1}$  is not a Möbius map. We note that the Whitehead homeomorphisms  $f_{l_1}$  is given by  $f_{l_1}^{-1} = id$  on  $[-1, 1] \subset S^1$ , where

$$[-1, 1] = \{z \in S^1; -1, z, 1 \text{ are in the counterclockwise order}\}.$$

The restriction  $f_{l_1}^{-1}|_{[x_0, -1]}$  is the unique element of  $PSL_2(\mathbb{Z})$  which maps the oriented geodesic  $(-1, x_0)$  onto the oriented geodesic  $(-1, i)$ , where  $x_0$  is the third vertex of the complementary triangle of  $\mathcal{F}$  to the left of the oriented geodesic  $(-1, i)$  with  $(-1, i)$  on its boundary. Also, the restriction  $f_{l_1}^{-1}|_{[i, x_0]}$  is the unique element of  $PSL_2(\mathbb{Z})$  which maps the oriented geodesic  $(x_0, i)$  onto the oriented geodesic  $(i, y_0)$ , where  $y_0$  is the third vertex of the complementary triangle of  $\mathcal{F}$  to the



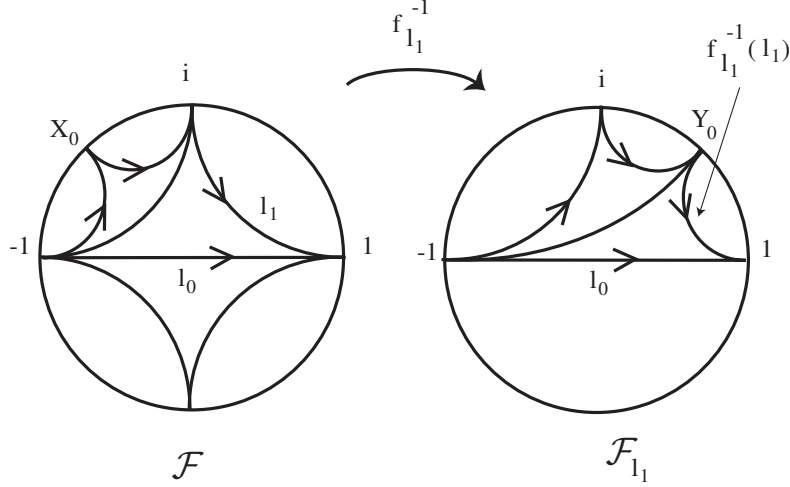


FIGURE 5

left of  $(i, 1)$  with  $(i, 1)$  on its boundary. Finally,  $f_{l_1}^{-1}|_{[1, i]}$  is the unique element of  $PSL_2(\mathbb{Z})$  which maps  $(i, 1)$  onto  $(y_0, 1)$  (see Figure 5). Thus, the homeomorphism  $f_{l_1}^{-1}$  is a piecewise  $PSL_2(\mathbb{Z})$  with four singular points  $-1, 1, i$  and  $x_0$ . At these points the map  $f_{l_1}^{-1}$  changes its definition from one to another element of  $PSL_2(\mathbb{Z})$  (see Figure 5). It is interesting to note (although we do not use this fact) that the homeomorphism  $f_{l_1}^{-1}$  is differentiable at every point on the circle.

Similarly, the singular points where  $g_{A(l_1)}$  changes its definition from one to another  $PSL_2(\mathbb{Z})$  element are  $-1, 1, A(y_0)$  and  $A(i)$ . Then  $g_{A(l_1)} \circ f_{l_1}^{-1}$  is the identity on  $[-1, 1]$ , but at the point  $i$  we have that  $f_{l_1}^{-1}$  changes its definition from one to another element of  $PSL_2(\mathbb{Z})$ , while the restriction of  $g_{A(l_1)}$  to a neighborhood of  $f_{l_1}^{-1}(i) = y_0$  equals a single element of  $PSL_2(\mathbb{Z})$  (because  $A(y_0) \neq y_0$ ). This implies that  $g_{A(l_1)} \circ f_{l_1}^{-1}$  is not the identity in a neighborhood of  $i$ . Thus  $g_{A(l_1)} \circ f_{l_1}^{-1}$  is not a Möbius map on  $S^1$ . Consequently,  $g_n \circ (f_n)^{-1}$  is not a Möbius map for all  $n$  large enough. This completes the proof of Theorem 1.  $\square$

#### 4. THE PUNCTURED SOLENOID $\mathcal{S}^p$

Ehrenpreis conjecture asks whether any two compact Riemann surfaces have finite regular covers which are close to being conformal, i.e. if there exists a quasiconformal map between the covers which has quasiconformal constant arbitrary close to 1. Instead of taking two arbitrary compact Riemann surfaces at a time and studying their covers, an idea of Sullivan is to take all compact Riemann surfaces at one time (i.e. in a single space) and keep track of the lifts via the action of a Modular group. The same idea can be used for punctured surfaces. We give more details below.

Let  $T_0$  be the (once-punctured) Modular torus and let  $G_0 < PSL_2(\mathbb{Z})$  be its universal covering group, i.e.  $T_0 \equiv \mathbf{D}/G_0$ . Let  $S \rightarrow T_0$  be any finite regular covering of  $T_0$ . Then there exists a natural isometric embedding  $T(T_0) \hookrightarrow T(S)$  of the Teichmüller space  $T(T_0)$  of the Modular torus  $T_0$  into the Teichmüller space  $T(S)$  of the covering surface  $S$ . Moreover, if for a finite regular covering  $S_1 \rightarrow T_0$  there exist finite regular coverings  $S_2 \rightarrow T_0$  and  $S_1 \rightarrow S_2$  such that the composition  $S_1 \rightarrow S_2 \rightarrow T_0$  is equal to the original covering  $S_1 \rightarrow T_0$  then there is a natural embedding  $T(S_2) \hookrightarrow T(S_1)$  such that the image of  $T(T_0)$  in  $T(S_2)$  is mapped onto the image of  $T(T_0)$  in  $T(S_1)$ . The inverse system of the finite regular coverings of  $T_0$  induces a direct system of Teichmüller spaces of the covering surfaces. We denote the direct limit of the system of Teichmüller spaces of all finite regular coverings of  $T_0$  by  $T_\infty$  (see [5] for more details). The peripheral preserving commensurator group  $Comm_{per}(G_0)$  of the Modular torus group  $\pi_1(T_0) = G_0$  keeps track of different lifts of the complex structure on the Modular torus. Thus, the Ehrenpreis conjecture is equivalent to the statement whether  $Comm_{per}(G_0)$  has dense orbit in  $T_\infty$ .

The Teichmüller space  $T(S)$  of any finite regular covering  $S \rightarrow T_0$  embeds in the universal Teichmüller space  $T(\mathbf{D})$  (i.e. the Teichmüller space of the unit disk  $\mathbf{D}$ ) as follows. Let  $G < G_0$  be such that  $\mathbf{D}/G \equiv S$  and that the covering  $\mathbf{D}/G \rightarrow \mathbf{D}/G_0$  is conformally equivalent to  $S \rightarrow T_0$ . Then the image of  $T(S)$  in  $T(\mathbf{D})$  consists, up to an equivalence, of all Beltrami dilatations  $\mu$  on  $\mathbf{D}$  such that

$$(7) \quad \mu(A(z)) \frac{\overline{A'(z)}}{A'(z)} = \mu(z)$$

for all  $A \in G$  and  $z \in \mathbf{D}$ . Two Beltrami dilatations  $\mu$  and  $\nu$  are equivalent if there is a quasiconformal map of  $\mathbf{D}$  whose Beltrami dilatation is  $\mu - \nu$  and which extends to the identity on  $\mathbf{S}^1$ .

Thus the image of the embedding  $T_\infty \hookrightarrow T(\mathbf{D})$  consists of all Beltrami dilatations  $\mu$  on  $\mathbf{D}$  which satisfy (7) for some finite index subgroup  $G$  of  $G_0$ . The image of  $T(S)$  under the embedding  $T(S) \hookrightarrow T(\mathbf{D})$  is a finite-dimensional complex submanifold of  $T(\mathbf{D})$  but the embedding is not an isometry for the Teichmüller metric (in fact, it is a bi-biLipschitz map with the constant  $1/3$  [?]). The image of  $T_\infty$  in  $T(\mathbf{D})$  is not a closed subspace. The completion  $\overline{T_\infty}$  of the image of  $T_\infty$  is a separable, complex Banach submanifold of  $T(\mathbf{D})$  [5]. The completion  $\overline{T_\infty}$  consists of all Beltrami coefficients  $\mu$  on  $\mathbf{D}$  which are *almost invariant* under  $G_0$  (modulo the equivalence relation), i.e.  $\overline{T_\infty}$  consists of all  $\mu$  which satisfy

$$\sup_{A \in G_n} \|\mu \circ A \frac{\overline{A'}}{A'} - \mu\|_\infty \rightarrow 0$$

as  $n \rightarrow \infty$  where  $G_n$  is the intersection of all subgroups of  $G_0$  of index at most  $n$ . (Note that each  $G_n$  is a finite index subgroup of  $G_0$  and that  $\bigcap_{n=1}^\infty G_n = \{id\}$ .) The Ehrenpreis conjecture is also equivalent to the question whether  $Comm_{per}(\mathcal{S}^p)$  has dense orbits in  $\overline{T_\infty}$ .

The points in  $\overline{T_\infty} \setminus T_\infty$  are obtained as limits of quasiconformal maps between finite Riemann surfaces. These points are represented by Beltrami coefficients on  $\mathbf{D}$  with the additional property of being almost invariant. Sullivan [9] introduced a new object, called the universal hyperbolic solenoid, on which these limit points

appear in a geometrically natural fashion as quasiconformal maps between the universal hyperbolic solenoids. (Note that the quasiconformal maps between finite surfaces lift to quasiconformal maps between the universal hyperbolic solenoids as well.) We study the punctured solenoid  $\mathcal{S}^p$  which is the counter part of the universal hyperbolic solenoid in the presence of punctures. We give the details below. An important feature is that the Teichmüller space  $T(\mathcal{S}^p)$  of the punctured solenoid is naturally isometrically and bi-holomorphically equivalent to  $\overline{T_\infty}$ .

We recall the definition and basic properties of the punctured solenoid  $\mathcal{S}^p$  [8], which is an analogue in the presence of the punctures of the universal hyperbolic solenoid introduced by Sullivan [9]. We keep the notation  $T_0$  for the Modular once punctured torus. Then  $T_0$  is conformally identified with  $\mathbf{D}/G_0$ , where  $\mathbf{D}$  is the unit disk and  $G_0 < PSL_2(\mathbb{Z})$  is the unique uniformizing subgroup. Consider the family of all finite degree, regular coverings of  $T_0 \equiv \mathbf{D}/G_0$ . The family is inverse directed and the inverse limit  $\mathcal{S}^p$  is called the *punctured solenoid* (see [8]). The punctured solenoid  $\mathcal{S}^p$  is a non-compact space which is locally homeomorphic to a 2-disk times a Cantor set ( $\equiv$  the transverse set); each path component, called a *leaf*, is a simple connected 2-manifold which is dense in  $\mathcal{S}^p$ .  $\mathcal{S}^p$  has one topological end which is homeomorphic to the product of a horoball and the transverse set of  $\mathcal{S}^p$  modulo continuous action by a countable group (see [8]). A fixed leaf of  $\mathcal{S}^p$  is called the *baseleaf*. The punctured solenoid  $\mathcal{S}^p$  has a natural projection  $\Pi : \mathcal{S}^p \rightarrow T_0$  such that the restriction to each leaf is the universal covering. The hyperbolic metric on  $T_0$  lifts to a hyperbolic metric on each leaf of  $\mathcal{S}^p$  and the lifted leafwise hyperbolic metric on  $\mathcal{S}^p$  is locally constant in the transverse direction. The punctured solenoid has a unique holonomy invariant transverse measure (see [6]). When the transverse measure is coupled with the leafwise measure given by the hyperbolic area on leaves, the resulting product measure is finite on  $\mathcal{S}^p$ .

We define an arbitrary *marked hyperbolic punctured solenoid*  $X$  to be a topological space locally homeomorphic to a 2-disk times a Cantor set with transversely continuous leafwise hyperbolic metrics together with a homeomorphism  $f : \mathcal{S}^p \rightarrow X$  which is quasiconformal when restricted to each leaf and whose leafwise Beltrami coefficients are continuous in the essential supremum norm over the global leaves for the transverse variation (for more details see [8]). A hyperbolic metric on any finite sheeted, unbranched cover of  $T_0$  gives a marked hyperbolic punctured solenoid whose hyperbolic metric is transversely locally constant for a choice of local charts, and any transversely locally constant punctured solenoid arises as a lift of a hyperbolic metric on a finite area punctured surface. We define the *Teichmüller space*  $T(\mathcal{S}^p)$  of the punctured solenoid  $\mathcal{S}^p$  to be the space of all marked hyperbolic punctured solenoids modulo an equivalence relation. Two marked hyperbolic punctured solenoids  $f_1 : \mathcal{S}^p \rightarrow X_1$  and  $f_2 : \mathcal{S}^p \rightarrow X_2$  are equivalent if there exist an isometry  $c : X_1 \rightarrow X_2$  such that the map  $f_2^{-1} \circ c \circ f_1 : \mathcal{S}^p \rightarrow \mathcal{S}^p$  is isotopic to the identity; the equivalence class of  $f_1 : \mathcal{S}^p \rightarrow X$  is denoted by  $[f_1]$ . The set of all marked transversely locally constant hyperbolic punctured solenoids is dense in  $T(\mathcal{S}^p)$  (see [9], [8]). The *basepoint* of  $T(\mathcal{S}^p)$  is the equivalence class  $[id : \mathcal{S}^p \rightarrow \mathcal{S}^p]$  of the identity map.

The *modular group*  $Mod(\mathcal{S}^p)$  (also called the *baseleaf preserving mapping class group*  $MCG_{BLP}(\mathcal{S}^p)$  in the literature [6], [8]) of the punctured solenoid  $\mathcal{S}^p$  consists

of homotopy classes of quasiconformal self-maps of  $\mathcal{S}^p$  which preserve the baseleaf. The restriction to the baseleaf of  $Mod(\mathcal{S}^p)$  gives an injective representation of  $Mod(\mathcal{S}^p)$  into the group of the quasisymmetric maps of  $S^1$  (see [6]). From now on, we identify  $Mod(\mathcal{S}^p)$  with this representation without further mentioning. Then  $Mod(\mathcal{S}^p)$  consists of all quasisymmetric maps of  $S^1$  which conjugate a finite index subgroup of  $G_0$  onto (a possibly different) finite index subgroup of  $G_0$  such that parabolic (peripheral) elements are conjugated onto parabolic (peripheral) elements (see [6], [8]). In other words,  $Mod(\mathcal{S}^p)$  is isomorphic to the subgroup  $Comm_{per}(G_0)$  of the abstract commensurator of  $G_0$  consisting of all elements which preserve parabolics. In particular,  $Mod(\mathcal{S}^p)$  contains  $PSL_2(\mathbb{Q})$  and all lifts to the unit disk  $\mathbf{D}$  of the mapping class groups of the surfaces  $\mathbf{D}/K$ , where  $K < G_0$  ranges over all finite index subgroups. Recall that the Teichmüller space  $T(\mathcal{S}^p)$  embeds into the universal Teichmüller space  $T(\mathbf{D})$  by restricting the leafwise quasiconformal homeomorphisms of  $\mathcal{S}^p$  onto variable solenoids to the baseleaf. From now on, we identify  $T(\mathcal{S}^p)$  with its image in  $T(\mathbf{D})$  under this embedding. Then the Ehrenpreis conjecture is equivalent to the question whether  $Mod(\mathcal{S}^p)$  has dense orbits in  $T(\mathcal{S}^p)$ .

If the Ehrenpreis conjecture is correct then we show that for any  $\epsilon > 0$  and for any finite Riemann surface there exist two finite degree, regular covers and a  $(1+\epsilon)$ -quasiconformal map between the covers which is not homotopic to a conformal map. We remark that Theorem 1 establishes the existence of such covers for the Modular punctured torus  $T_0$  and any of its finite regular covers (without the assumption that the Ehrenpreis conjecture is correct) but it seems a difficult question to establish the existence of such covers for an arbitrary punctured surface.

**Lemma 4.1.** *Assume that the Ehrenpreis conjecture is correct. Then for any  $\epsilon > 0$  and for any finite Riemann surface there exist two finite degree, regular covers and a  $(1+\epsilon)$ -quasiconformal map between the covers which is not homotopic to a conformal map.*

**Proof.** Since we assumed that the Ehrenpreis conjecture is correct, we get that the orbits of  $Mod(\mathcal{S}^p)$  are dense in  $T(\mathcal{S}^p)$ . Let  $S$  be an arbitrary finite area punctured hyperbolic surface and let  $f : S_0 \rightarrow S$  be a quasiconformal map from a finite, unbranched covering surface  $S_0$  of the Modular punctured torus  $T_0$  to the surface  $S$ . We note that the map  $f : S_0 \rightarrow S$  lifts to a map  $\tilde{f} : \mathcal{S}^p \rightarrow X$  and that the equivalence class  $[\tilde{f}]$  is an element of the Teichmüller space  $T(\mathcal{S}^p)$ . Then the orbit under  $Mod(\mathcal{S}^p)$  of  $[\tilde{f}] \in T(\mathcal{S}^p)$  is dense and, in particular, it accumulates onto  $[\tilde{f}]$ . Let  $g_n \in Mod(\mathcal{S}^p)$  be a sequence such that  $[\tilde{f} \circ g_n^{-1}] \rightarrow [\tilde{f}]$  as  $n \rightarrow \infty$ . This implies that the Beltrami dilatation of  $\tilde{f} \circ g_n^{-1} \circ \tilde{f}^{-1}$  is converging to zero as  $n \rightarrow \infty$  but the dilatation of any quasiconformal map homotopic to  $\tilde{f} \circ g_n^{-1} \circ \tilde{f}^{-1}$  is not equal to zero. The map  $\tilde{f} \circ g_n^{-1} \circ \tilde{f}^{-1}$  conjugates a finite index subgroup of  $\pi_1(S)$  to a different subgroup of  $\pi_1(S)$ . The two subgroups of  $\pi_1(S)$  are conjugated by a quasiconformal map with a small Beltrami dilatation and therefore they establish the lemma.  $\square$

In our previous work [4], we find an infinite family of orbits with accumulation points outside the orbits. In particular,  $T(\mathcal{S}^p)/Mod(\mathcal{S}^p)$  is not a Hausdorff space. The points of the orbits are non-transversely locally constant points in  $T(\mathcal{S}^p)$  (i.e.

they correspond to points in  $\overline{T_\infty} \setminus T_\infty$ ) and elements of  $Mod(\mathcal{S}^p)$  which give accumulation points are in  $G_0$ . In this paper, we find accumulation points outside the orbit of a transversely locally constant point in  $T(\mathcal{S}^p)$  (i.e. a point in  $T_\infty$ ) corresponding to the basepoint  $[id : \mathcal{S}^p \rightarrow \mathcal{S}^p]$  for the hyperbolic metric on  $\mathcal{S}^p$  obtained by the lift of the hyperbolic metric on  $T_0 \equiv \mathbf{D}/G_0$ .

We show that the closure of the orbit under the modular group of the basepoint  $[id] \in T(\mathcal{S}^p)$  is strictly larger than the orbit and that the closure is uncountable. We use Baire category theorem and Theorem 3.3 together with fact that elements in Theorem 3.3 are not Möbius which is established in the course of the proof of Theorem 1.

**Corollary 2.** *The closure in the Teichmüller metric of the orbit under the modular group  $Mod(\mathcal{S}^p)$  of the basepoint in  $T(\mathcal{S}^p)$  is strictly larger than the orbit. Moreover, the closure of the orbit is an uncountable set without isolated points.*

**Remark.** We showed in [4] that there is a set of points in  $T(\mathcal{S}^p)$  such that the closures of their orbits under the modular group  $Mod(\mathcal{S}^p)$  are strictly larger than the orbits. These points were all non-transversely locally constant points in  $T(\mathcal{S}^p)$ . The above corollary establishes that the orbit of the basepoint, which is a transversely locally constant point, under the modular group  $Mod(\mathcal{S}^p)$  contains points outside the orbit. However, it is still unknown whether any of the accumulation points of the orbit of the basepoint is a transversely locally constant point in  $T(\mathcal{S}^p)$ . This is equivalent to the question whether we can find an example of two non-commensurate surfaces for which the Ehrenpreis conjecture is correct.

**Proof.** We use Baire category theorem. Assume on the contrary that the closure of the orbit under  $Mod(\mathcal{S}^p)$  of the basepoint in  $T(\mathcal{S}^p)$  is equal to the orbit.

Thus the orbit is a closed subset in  $T(\mathcal{S}^p)$ , hence it is of the second kind in itself (in the sense of Baire). We claim that there exists a point of the orbit which is an isolated point. If not, then each point of the orbit is nowhere dense. Since a single point in a metric space is always a closed subset, it follows that the orbit can be written as a countable union of its singletons (which are nowhere dense closed sets). This contradicts the Baire theorem.

Therefore, at least one point  $[f] \in T(\mathcal{S}^p)$  where  $f \in Mod(\mathcal{S}^p)$  is isolated. Choose a sequence  $f_n \in Mod(\mathcal{S}^p)$  satisfying the properties in Theorem 3.3 such that  $\|\mu(E(f_n))\|_\infty \rightarrow 0$  as  $n \rightarrow \infty$ . Then  $f_n \circ f \in Mod(\mathcal{S}^p)$  is in the orbit of the basepoint and  $[f_n \circ f] \rightarrow [f]$  as  $n \rightarrow \infty$  in the Teichmüller metric on  $T(\mathcal{S}^p)$ . This is a contradiction. Therefore, the closure of the orbit under  $Mod(\mathcal{S}^p)$  of the basepoint in  $T(\mathcal{S}^p)$  is strictly larger than the orbit.

We proceed to prove that the closure of the orbit is uncountable. Assume on the contrary that the orbit is countable. Then there exists an isolated point  $f$  of the closure of the orbit by the above argument. The isolated point  $f$  is necessarily in  $Mod(\mathcal{S}^p)$  because the accumulation points of  $Mod(\mathcal{S}^p)$  in  $T(\mathcal{S}^p)$  are not isolated. Then the above argument establishes a contradiction. Therefore the closure is uncountable.  $\square$

In the corollary below, we show how to explicitly construct sequences in the orbit under  $Mod(\mathcal{S}^p)$  of the basepoint in  $T(\mathcal{S}^p)$  which accumulate to points outside the orbit.

**Corollary 4.2.** *There exists a sequence  $f_n \in Mod(\mathcal{S}^p)$  whose elements are constructed as in Theorem 3.3 such that  $[f_n \circ f_{n-1} \circ \cdots \circ f_1]$  converges to  $[f] \in T(\mathcal{S}^p)$ , where  $f \notin Mod(\mathcal{S}^p)$ .*

**Proof.** We choose a sequence  $f_n \in Mod(\mathcal{S}^p)$  such that  $\|\mu(E(f_n))\|_\infty < 1/2^{n+1}$  and  $\inf_{\{g:g|_{S^1}=f_n|_{S^1}\}} \|\mu(g)\|_\infty > 0$  which is possible by Theorem 3.3, where  $E(f_n)$  is the barycentric extension of  $f_n$ . Then the sequence  $E(f_n) \circ E(f_{n-1}) \circ \cdots \circ E(f_1)$  has uniformly convergent Beltrami coefficients. Therefore,  $f_n \circ f_{n-1} \circ \cdots \circ f_1$  converges in the Teichmüller metric in  $T(\mathcal{S}^p)$ .

The sequence  $f_n$  can be chosen such that each  $f_n$  conjugates a maximal finite index subgroup  $H_n$  of  $PSL_2(\mathbb{Z})$  onto another finite index subgroup  $K_n$  of  $PSL_2(\mathbb{Z})$ , where  $[PSL_2(\mathbb{Z}) : H_n] \rightarrow \infty$  as  $n \rightarrow \infty$ . To see this we take  $A_n \rightarrow id$  as in Theorem 3.3 and for each  $n$  consider a sequence of finite index subgroups  $G_{k_n}$  of  $G_{A_n}$  which are obtained as intersections of all subgroups of  $G_{A_n}$  of index at most  $k_n$ . We define  $f_{n,k_n}$  using  $G_{k_n}$  and  $A_n$  as in Theorem 3.3. Assume that for a fixed  $n$  all such obtained maps  $f_{n,k_n}$  conjugate a fixed finite index subgroup  $K < PSL_2(\mathbb{Z})$  onto another (possibly different but of the same index) subgroup of  $PSL_2(\mathbb{Z})$ . Since all  $f_{n,k_n}$  fix  $l_0$  by construction, it follows that  $f_{n,k_n}$  converges pointwise to  $f_{A_n(l_1)} \circ f_{l_1}^{-1}$  as  $k_n \rightarrow \infty$  and  $n$  fixed, where  $f_{l_1}$  is a Whitehead homeomorphism which maps  $\mathcal{F}_{l_1}$  onto  $\mathcal{F}$ , and  $f_{A_n(l_1)}$  is a Whitehead homeomorphism which maps  $\mathcal{F}_{A_n(l_1)}^{A_n}$  onto  $\mathcal{F}^{A_n(l_1)}$ . We showed in the proof of Theorem 1 that  $f_{A_n(l_1)} \circ f_{l_1}^{-1}$  is a piecewise Möbius but not a Möbius map. Then  $f_{A_n(l_1)} \circ f_{l_1}^{-1}$  cannot conjugate a finite index subgroup of  $PSL_2(\mathbb{Z})$  onto itself. Therefore, for  $k_n$  large enough,  $f_{n,k_n}$  does not conjugate  $K$  onto another finite index subgroup. Therefore, we can choose  $k_n$  large enough such that  $f_{n,k_n}$  conjugates a maximal subgroup  $H_n$  of finite index in  $PSL_2(\mathbb{Z})$  onto another finite index subgroup with  $[PSL_2(\mathbb{Z}) : H_n] \rightarrow \infty$  as  $n \rightarrow \infty$ . Moreover, the subgroups  $H_n$  can be chosen such that  $\bigcap_{n=1}^\infty H_n = \{id\}$ . If we further enlarge  $k_n$  the above remains true.

We put the elements of  $G_0$  in a sequence  $\{\gamma_i\}_{i=1}^\infty$ . For each  $n$  and for each  $i$ , there exists  $k_n(i)$  such that  $\gamma_i \notin G_{k_n(i)}$ . This implies that  $\gamma_i \notin G_j$  for all  $j \geq k_n(i)$ . We apply Cantor's diagonal argument. For  $\gamma_1$ , we choose  $n = n(1)$  arbitrary. Then  $f_{n(1),k_{n(1)}}(\gamma_1)$  does not conjugate  $\gamma_1$  onto any other element of  $PSL_2(\mathbb{Z})$ . We consider the minimal dilatation

$$\min(f_{n(1),k_{n(1)}})(\gamma_1) := \inf_{\{g:g|_{S^1}=f_{n(1),k_{n(1)}}\}} \|\mu(g) - \mu(g \circ \gamma_1)\|_\infty.$$

In the case that  $\min(f_{n(1),k_{n(1)}})(\gamma_2) = 0$  then we choose  $n(2)$  small enough such that  $\|\mu(E(f_{n(2),k_{n(2)}}))\|_\infty < \frac{1}{4} \min(f_{n(1),k_{n(1)}})(\gamma_1)$ . If  $\min(f_{n(1),k_{n(1)}})(\gamma_2) > 0$  we choose  $n(2)$  such that  $\|\mu(E(f_{n(2),k_{n(2)}}))\|_\infty$  is less than  $\frac{1}{4}$  of the minimum of  $\min(f_{n(1),k_{n(1)}})(\gamma_1)$  and  $\min(f_{n(1),k_{n(1)}})(\gamma_2)$ . In both cases we are guaranteed that  $f_{n(2),k_{n(2)}} \circ f_{n(1),k_{n(1)}}$  does not conjugate  $\gamma_1$  or  $\gamma_2$  onto an element of  $PSL_2(\mathbb{Z})$ . For  $\gamma_3$ , we choose  $n(3)$  such that  $\|\mu(E(f_{n(3),k_{n(3)}}))\|_\infty$  is less than  $\frac{1}{4}$  of the minimum of  $\min(f_{n(2),k_{n(2)}})(\gamma_1)$ ,  $\min(f_{n(2),k_{n(2)}} \circ f_{n(1),k_{n(1)}})(\gamma_2)$  and  $\min(f_{n(2),k_{n(2)}} \circ$

$f_{n(1),k_n(1)}(\gamma_3)$ . This guarantees that  $f_{n(3),k_n(3)} \circ f_{n(2),k_n(2)} \circ f_{n(1),k_n(1)}$  does not conjugate  $\gamma_i$ , for  $i = 1, 2, 3$  onto elements of  $PSL_2(\mathbb{Z})$ . We continue this process for all  $i \in \mathbb{N}$ .

By our choice of  $n(i)$ , the series  $\sum_{i=1}^{\infty} \|\mu(E(f_{n(i),k_n(i)}))\|_{\infty}$  converges. Thus the sequence  $f_{n(i),k_n(i)} \circ f_{n(i-1),k_n(i-1)} \circ \cdots \circ f_1$  converges in the Teichmüller metric. By the above choices, the limit does not conjugate a single element of  $G_0$  onto any other element of  $PSL_2(\mathbb{Z})$ . Thus the limit is not in the orbit under the modular group  $Mod(\mathcal{S}^p)$  of the base point in  $T(\mathcal{S}^p)$ .  $\square$

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