CHARACTERISATION OF PLANE REGIONS THAT SUPPORT QUASICONFORMAL MAPPINGS TO THEIR DOMES

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ABSTRACT. We prove that the nearest point retraction of a region P, not the whole plane, to its dome is long range bilipschitz if and only if P is uniformly perfect. From this we prove that P uniformly perfect is necessary and sufficient for the existence of a K-quasiconformal map from P to its dome which extends to be the identity on the boundary and is finite distance to the nearest point retraction. Thus our work extends the classic theorem of Sullivan for simply connected regions to regions of arbitrary connectivity. In particular our study results in a simple, transparent proof of the original theorem.

1. Introduction

We begin by briefly reviewing the construction of the hyperbolic convex hull in the upper halfspace model of hyperbolic 3-space $\mathbb{H}^3$. Suppose $\Omega \subset \mathbb{C}$ is a (connected) region $\neq \mathbb{C}$. The convex hull of the closed set $\mathbb{S}^2 \setminus \Omega$ is constructed as follows. Let $D \subset \Omega$ be a maximal disk, one that is not a subset of a larger one. The hemisphere rising from ∂D is a hyperbolic plane. It divides $\mathbb{H}^3$ into two halfspaces; denote the halfspace adjacent to $\mathbb{S}^2 \setminus D$ by $H(D)$. The convex hull is $\cap_D H(D)$, the intersection over all maximal disks D in Ω. Its relative boundary component in $\mathbb{H}^3$ appears over Ω as the *dome* over a domed stadium. We will denote it by $\mathrm{Dome}(\Omega)$; Ω is called the *floor*. For a discussion of its interesting properties see for example [8]. Suffice to say, the dome has a hyperbolic metric defined as the restriction of the ambient hyperbolic metric of $\mathbb{H}^3$. The subject of this paper is an investigation of the relation between the hyperbolic geometries of the floor and the dome.

The basic tool in relating the dome to the floor is the *nearest point retraction*. It is defined as follows. Given $z \in \Omega$, and the family of horospheres in $\mathbb{H}^3$ tangent to $\mathbb{C}$ at z, there is a smallest one that meets $\mathrm{Dome}(\Omega)$, necessarily at a single point $r(z)$, The map $r : z \mapsto r(z) \in \mathrm{Dome}(\Omega)$ is continuous and surjective but not necessarily a local homeomorphism. It has a continuous extension to the common boundary

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$\partial\Omega = \partial\text{Dome}(\Omega)$ where it is the identity. When Ω is simply connected, r is known to be 2-Lipschitz [10].

Our characterisation of regions Ω is built on the notion of "uniformly perfect region". The formal definition will be given below. In the course of our study we will present two new equivalent definitions. One (Proposition (3.1)) characterises regions which are not uniformly perfect as those with the property of having a "blow-up" resulting in a region with an isolated boundary point. The other (Corollary (4.2)) characterises uniformly perfect regions as those with "bounded geometry"—there is a positive lower bound for the injectivity radii.

There is an amazing fact, first recorded by Dennis Sullivan, that asserts that there is a universal $K \geq 1$ such that given any simply connected region $\Omega \neq \mathbb{C}$, there is a K-quasiconformal mapping $\Psi : \Omega \to \text{Dome}(\Omega)$, that extends continuously to be the identity on their common boundary, and is of finite distance from $r(z)$. This has played an important role in hyperbolization theory [16] and in the study of Riemann mappings [2], [4].

The purpose of the present paper is to present the analogue of this result for multiply connected regions (Theorem (7.3)). It will be drawn from the fact (Theorem (7.1)) that Ω uniformly perfect is necessary and sufficient for the nearest point retraction r to be a quasiisometry, that is, to be "long range" bilipschitz.

Suppose $X = \Omega/\Gamma$ is a boundary component of a hyperbolic 3-manifold and Γ is the subgroup stabilising Ω. Denote the dome of X by X_{dome}. We obtain (Corollary (7.4)) upper and lower Lipschitz bounds for the length of a closed geodesic on X_{dome} in terms of the length of the corresponding geodesic on X. The lower Lipschitz constant is universal. The upper constant depends only on the uniform perfect constant of Ω. An explicit estimate for the upper constant was previously found by Canary [6]. Making the additional assumption that Ω has bounded geometry, he sharpened the estimate. Canary was to first to apply to these matters the notion of uniformly perfect.

2. Uniformly perfect sets

Let $\Lambda \subset \mathbb{S}^2$ be a closed set with nonempty complement $\Omega = \mathbb{S}^2 \setminus \Lambda$ all components of which carry a hyperbolic metric density $\lambda = \lambda_\Omega$. Later we will only be interested in the case that $\Lambda = \partial\Omega \subset \mathbb{S}^2$ for a (connected) region, typically infinitely connected, $\Omega \subset \mathbb{C}$. All hyperbolic metrics will be assumed to be normalised to have gaussian curvature -1.

It is convenient to assume that $\infty \in \Lambda$.

Theorem 2.1. [1], [13], [14] *The following two conditions are equivalent:*

(i) *There exists* $C > 0$ *such that*

$$\frac{1}{C\mathrm{dist}(z,\Lambda)} \leq \lambda_\Omega(z) \leq \frac{2}{\mathrm{dist}(z,\Lambda)}. \tag{1}$$

(ii) *There exists* $M < \infty$ *with the following property. If* $A \subset \Omega$ *is any annular region separating* Λ *then*

$$\mathrm{Mod}(A) \leq M.$$

Here "dist" refers to the shortest euclidean distance to the boundary:

$$\mathrm{dist}(z,\Lambda) = \inf_{\zeta\in\Lambda}\{|\zeta - z|\}.$$

The modulus of an annular region is the modulus of its conformal equivalent $\{r < |w| < 1\}$ which is defined as $-(\log r)/2\pi$. Thus $\mathrm{Mod}(A) = \infty$ if and only if A is conformally equivalent to the once-punctured disk.

For a simply connected region U, it is well known (see [1]) that

$$\frac{1}{2\mathrm{dist}(z,\partial U)} \leq \lambda_U(z) \leq \frac{2}{\mathrm{dist}(z,\partial U)}. \tag{2}$$

Equation (1) is a generalisation to multiply connected regions of Equation (2). In both cases it is only the left sides of the inequalities that are in play; the right sides are direct consequences of the fact that the maximal disk in Ω with centre z has radius $\mathrm{dist}(z,\partial U)$. For simply connected regions, the left side of (2) is a consequence of the Koebe 1/4-theorem.

We will apply Theorem (2.1) to the boundary of a connected region Ω. Following [13], a closed set Λ satisfying Equation (1) on the components of $\mathbb{S}^2\setminus\Lambda$, or the annular moduli condition on $\mathbb{S}^2\setminus\Lambda$ is called *uniformly perfect*. In particular Λ cannot have isolated points, hence it is necessarily a perfect set. But more than that, the set must be sufficiently dense so it cannot be separated by annular regions of arbitrarily large modulus.

We will say a region $\Omega \subset \mathbb{C}$ is uniformly perfect if its boundary in $\mathbb{S}^2$ is uniformly perfect. The constant $C = C(\Omega)$ is called the *uniformly perfect constant* for Ω. For any simply connected $\Omega \neq \mathbb{C}$ we can set $C(\Omega) = 2$. For multiply connected regions M can have any finite value (e.g., for doubly connected regions). The existence of a C is equivalent to the existence of an M by Theorem (2.1). The precise relation between C and M is not clear. In this paper we will find it more convenient to work with annuli.

3. The main lemma: Renormalisation

In this and the next section we will present two additional characterisations of uniformly perfect.

We will use the following definition of geometric (kernel) convergence of a sequence $\{\Omega_n\}$ of regions (see [12]). Assume there is an open disk $D \subset \Omega_n$ for all n. Then, passing to a subsequence if necessary, assume that $\{\partial\Omega_n \subset \mathbb{S}^2\}$ has a Hausdorff limit Λ^*. Let Y denote kernel of the sequence. This means that Y is the largest connected open set containing D with the property that every compact subset of Y is contained in all $\{T_n(\Omega)\}$ with at most a finite number of exceptions. Specifically, let Y_n denote the component of $\mathrm{Int}(\cap_{k\geq n} T^k(\Omega))$ that contains D. Then $Y = \cup_n Y_n$. We can replace the original sequence of regions by a subsequence if necessary so that every infinite subsequence has the same kernel Y. Y is then the component of $\mathbb{S}^2 \setminus \Lambda^*$ containing D. We will say that $\{\Omega_n\}$ *converges geometrically* to a region Ω^* $(= Y)$.

Proposition 3.1. *The region Ω is* NOT *uniformly perfect if and only if there is a sequence of Möbius transformations $\{T_n\}$ with the property that the sequence $\{T_n(\Omega)\}$ converges to a hyperbolic region Ω^* whose boundary $\partial\Omega^*$ contains an isolated point.*

Proof. If there is a sequence of blow-ups $\{T_n(\Omega)\}$ converging geometrically to a region with an isolated boundary point, then there is a sequence of annuli $\{A_n \subset T_n(\Omega)\}$ with $\mathrm{Mod}(A_n) \to \infty$. By conformal invariance Ω itself is not uniformly perfect.

Conversely, assume $\Lambda = \partial\Omega$ is a perfect but not uniformly perfect set. We must find a sequence of blow-ups resulting in a region with an isolated boundary point.

We are assuming that $\infty \in \Lambda$. There is a sequence of simple, noncontractible, mutually nonparallel, and mutually disjoint simple loops $\{\gamma_n \subset \Omega\}$ with the following property. Corresponding to each γ_n is an annular region A_n, whose central curves are freely homotopic to γ_n, and such that $\lim \mathrm{Mod}(A_n) = \infty$. We may further assume that either the elements of the curve sequence are nested, or that no one separates any other from ∞.

Now each γ_n determines a partition into two disjoint closed subsets $\Lambda = \Lambda'_n \cup \Lambda''_n$ where $\infty \in \Lambda''_n$. Find $\zeta'_n \in \Lambda'_n, \zeta''_n \in \Lambda''_n$ such that

$$|\zeta''_n - \zeta'_n| = \mathrm{dist}(\Lambda'_n, \Lambda''_n).$$

Passing to a subsequence if necessary we may assume that the sequences $\{\zeta'_n\}$, $\{\zeta''_n\}$ converge.

Then choose the Möbius transformation T_n determined by the properties $T_n(\zeta'_n) = 0, T_n(\zeta''_n) = 1, T_n(\infty) = \infty$.

If $\lim \zeta_n' \neq \lim \zeta_n''$ then $\{T_n\}$ converges to a proper Möbius transformation. Since we are assuming $\partial\Omega$ is a perfect set, $\mathrm{Mod}(A_n)$ does not converge to ∞, contrary to assumption.

So assume that $\lim \zeta_n' = \lim \zeta_n'' = \xi$. We may assume that $\xi \neq \infty$. In this case $\lim T_n(z) = \infty$ for all $z \neq \xi$. At the same time, $|T_n(\zeta_n') - T_n(\zeta_n'')| = 1$ for all indices n.

Moreover, for each index, $\mathrm{dist}(T_n(\Lambda_n'), T_n(\Lambda_n'')) = 1$. For each T_n is a euclidean motion so that a closest pair of points between Λ_n' and Λ_n'' is mapped to a closest pair between $T_n(\Lambda_n')$ and $T_n(\Lambda_n'')$.

After passing to a subsequence if necessary, we may assume that there is Hausdorff convergence $\lim T_n(\Lambda_n') = \Lambda_*'$ and $\lim T_n(\Lambda_n'') = \Lambda_*''$. In the limit, $1, \infty \in \Lambda_*''$, while $0 \in \Lambda_*'$. The distance between Λ_*' and Λ_*'' remains as 1.

For all large indices, the sequence of regions $\{T_n(\Omega)\}$ contains an open disk D of radius $1/4$, for example. Passing to a subsequence if necessary we may assume that the sequence $\{T_n(\Omega)\}$ converges geometrically to its kernel Y. The boundary ∂Y is contained in $\Lambda_*' \cup \Lambda_*''$.

Since Y omits $(0, 1, \infty)$ it has a hyperbolic metric density λ_Y. Uniformly on compact subsets of Y, the hyperbolic densities of the approximates converge to λ_Y,

Next consider the sequence of annuli $\{B_n = T_n(A_n) \subset T_n(\Omega)\}$ separating $T_n(\Lambda_n')$ from $T_n(\Lambda_n'')$ and in particular 0 from $1, \infty$. Enlarge each B_n to C_n if necessary so that for all n, the distance between the two boundary components of C_n exceeds $1/4$. This increases the modulus. As $n \to \infty$ the outer boundary component of C_n cannot shrink to a point on $\mathbb{S}^2$. Therefore the inner contour must converge to a single point.

Consequently, $\Lambda_*' \cap \partial Y$ is a single (isolated) point. □

Remark 3.2. Suppose Ω is uniformly perfect and Y is the geometric limit of any blowup of Ω by Möbius transformations as above. The Y is uniformly perfect with the same constant.

4. The bounded geometry characterisation.

Lemma 4.1. *Suppose Ω is uniformly perfect and $\{z_n \in \Omega\}$ converges to a point $\zeta \in \Lambda = \partial\Omega$. There is a sequence of Möbius transformations $\{T_n\}$ such that $\{T_n(\Omega)\}$ converges geometrically to a hyperbolic region Ω^* and $\{T_n(z_n)\}$ converges to an interior point in Ω^*.*

Proof. As in the proof of Proposition (3.1) select a nearest point $\zeta_n \in \Lambda$ to z_n. We may assume that $\lim \zeta_n = \lim z_n = \xi$. Necessarily, $\xi \in \Lambda$. We may also assume that $\xi \neq \infty$.

Blow up Ω at ξ: Let $T_n(z)$ be the Möbius transformation that fixes ∞ and sends $z_n \to 0$, $\zeta_n \to 1$. Since we are assuming Λ is uniformly perfect, we may assume as in the proof of Proposition (3.1) that the sequence $\{T_n(\Omega)\}$ geometrically converges with respect to the basepoint 0 to a region Ω^*, also with uniformly perfect boundary.

The point 0 lies in Ω^*. The euclidean distance satisfies $\text{dist}(0, \partial\Omega^*) = 1$. □

Corollary 4.2. *The region Ω is uniformly perfect if and only if Ω has bounded geometry in the hyperbolic metric—if and only if there exists $R > 0$ such that the injectivity radius about any point $z \in \Omega$ satisfies* $\text{Inj}(z) > R$*.*

Proof. If Ω is uniformly perfect but does not have bounded geometry, there is a sequence of points $\{z_n\}$ converging to $\xi \in \partial\Omega$ with injectivity radii $\to 0$. We can blow-up by Lemma (4.1) to reach a contradiction.

Conversely if Ω has bounded geometry but is not uniformly perfect, Proposition (3.1) guarantees a blow-up resulting in a region Ω^* with an isolated boundary point, again a contradiction. □

Corollary 4.3. *The components of the set of discontinuity of a nonelementary, finitely generated kleinian group have bounded geometry.*

Proof. . This is an immediate consequence of Corollary (4.2) and the fact [14], [15] that the components are uniformly perfect. □

5. The nearest point retraction: Right quasiisometry

We will prove that the nearest point retraction is long-range Lipschitz for uniformly perfect regions. In the next section we will prove it is long-range bilipschitz.

We will often rely on the following lemma which is contained in the proof of [9, Thm. 7.5]:

Lemma 5.1. *Suppose Ω is a region and $\{T_n\}$ a sequence of Möbius transformations. Suppose the regions $T_n(\Omega) = \Omega_n$ all contain a small open disk D and that $\{\Omega_n\}$ converges uniformly on compact subsets to its kernel Ω^* containing D. Then in $\mathbb{H}^3$, $\{\text{Dome}(\Omega_n)\}$ converges to Dome(Ω^*) and the corresponding nearest point retractions $\{r_n\}$ converge uniformly on compact subsets of Ω^* to the nearest point retraction r^* of Ω^*.*

Proposition 5.2. *Suppose Ω is uniformly perfect with hyperbolic distance $d_\Omega(\cdot,\cdot)$. There exists a constant $L = L(C) \geq 1$ depending only on the uniformly perfect constant C such that*

$$d_{\text{Dome}(\Omega)}(r(z), r(w)) < L d_\Omega(z, w) + 1, \quad z, w \in \Omega. \tag{3}$$

Furthermore $\lim_{C\to\infty} L(C) = \infty$.

We conjecture that r is Lipschitz if (and only if) Ω is uniformly perfect. It is shown by Bridgeman and Canary in [5] that for a nonelementary, finitely generated kleinian group Γ acting on $\Omega \subset \mathbb{C}$, the nearest point retraction $r_* : X \to X_{dome}$ is Lipschitz, and hence the same is true of r itself. (The assumption the authors make, that Ω has bounded geometry, automatically holds by Corollary (4.3)). See also [2].

Proof. First we will show that following two situations are impossible: (1) There is a sequence $\{z_n, w_n\} \subset \Omega$ such that $d_\Omega(z_n, w_n) \to \infty$ while $d_{\mathrm{Dome}(\Omega)}(r(z_n), r(w_n)) = 1$. (2) There is a sequence $\{r(z_n), r(w_n)\} \subset \mathrm{Dome}(\Omega)$ such that $d_{\mathrm{Dome}(\Omega)}(r(z_n), r(w_n)) \to \infty$ while $d(z_n, w_n) = 1$.

A Möbius T used to renormalise Ω, simultaneously renormalises its dome. According to Lemma (4.1) there is sequence of renormalizations $\{T_n(\Omega)\}$ geometrically converging to a uniformly perfect region Ω^* such that $T_n(z_n) = 0$ where $0 \in \Omega^*$. In case (1), $T_n(w_n)$ runs off to $\partial\Omega^*$ while $\lim r(T_n(w_n))$ converges, we may assume assume, to an interior point of $\mathrm{Dome}(\Omega^*)$ of unit distance from $r^*(0)$. This is impossible. In case (2), $r(T_n(z_n)$ converges to $r^*(0)$ and $r(T_n(w_n)$ runs off to $\partial\mathrm{Dome}(\Omega)$ while $T_n(w_n)$ converges, we may assume, to a point of unit distance from 0. This is impossible as well.

Set
$$\frac{1}{L} = \inf\{d_\Omega(z, w) : d_{\mathrm{Dome}(\Omega)}(r(z), r(w)) = 1,\ z, w \in \Omega\}.$$
In other words,
$$d_{\mathrm{Dome}(\Omega)}(r(z), r(w)) \leq L d_\Omega(z, w),$$
whenever the left side has the value one.

In the general case $d_\Omega(z_n, w_n) \to \infty$ and $d_{\mathrm{Dome}(\Omega)}(r(z_n), r(w_n)) \to \infty$ For fixed index n, let $\sigma \subset \Omega$ be a shortest geodesic from z_n to w_n. Then $\tau = r(\sigma)$ is a path between $r(z_n)$ and $r(w_n)$. Divide σ into segments $\sigma = \sum_1^N \alpha_i$ satisfying the following property. Write $\alpha_i = [x_i, x'_{i+1}]$ with $x_1 = z_n,\ x_{N+1} = w_n$. Choose the subdivision so that $d_{\mathrm{Dome}(Omega)}(r(x_i), r(x_{i+1})) = 1$ except for the last interval where $d_{\mathrm{Dome}(\Omega)}(r(x_N), r(x_{N+1})) = \delta$, with $\delta \leq 1$. With this setup we now find,
$$d_{\mathrm{Dome}(\Omega)}(r(z_n), r(w_n)) \leq \sum_1^{N-1} d_{\mathrm{Dome}(\Omega)}(r(x_i), r(x_{i+1})) + \delta$$
$$\leq L \sum_1^{N-1} d_\Omega(x_i, x_{i+1}) + \delta \leq L d_\Omega(z_n, w_n) + 1.$$
This is Equation (3).

The last statement in Proposition (5.2) follows from Proposition (3.1) and from the computation on expanding annuli in [8, p. 209]. In short, when there is an isolated boundary point, r is not Lipschitz. □

6. The nearest point retraction: Left quasiisometry

We will now prove a lower inequality for the nearest point retraction, which holds for hyperbolic regions which are not necessarily uniformly perfect. We will be working in the upper halfspace model of $\mathbb{H}^3$.

Proposition 6.1. *Suppose* Ω *is a region with a hyperbolic metric* $d_\Omega(\cdot,\cdot)$*. There exists* $L_1 \geq 1$ *such that*

$$\frac{d_\Omega(z,w)}{L_1} - 1 < d_{\mathrm{Dome}(\Omega)}(r(z), r(w)), \quad z, w \in \Omega. \tag{4}$$

Proof. We may assume that $\infty \in \partial\Omega$. The pattern of the proof of Proposition (5.2). has to be augmented as follows.

Suppose (z_n, w_n) is a sequence of points in $\mathrm{Dome}(\Omega)$ with $d_{\mathrm{Dome}(\Omega)}(r(z_n), r(w_n) = 1$.

Denote the vertical line in $\mathbb{H}^3$ rising from $z = 0$ by ℓ, and the point on ℓ which is unit euclidean distance above $\mathbb{C}$ by P.

Choose a Möbius transformation T_n (acting in $\mathbb{H}^3 \cup \mathbb{S}^2$) that fixes ∞ and satisfies $T_n(r(z_n) = P$. T_n is uniquely determined by these two requirements.

Set $\Omega_n = T_n(\Omega)$. $\mathrm{Dome}(\Omega_n)$ goes through P. The dome over the unit disk $\mathbb{D}$ centred at $z = 0$ meets P, so $\overline{\mathbb{D}} \cap \partial\Omega_n \neq \emptyset$. Otherwise $\overline{\mathbb{D}} \subset \Omega_n$ and then the dome over the maximal disk in Ω_n that contains $\mathbb{D}$ would pass over P.

Let Δ_n denote the maximal (open) disk in Ω_n whose dome passes through P. Passing to a subsequence if necessary we may assume $\lim \Delta_n = \Delta^*$ exists. Since its dome contains P, The limit Δ^* is either a euclidean disk or halfplane; the corresponding dome in $\mathbb{H}^3$ is either a euclidean hemisphere or a vertical halfplane through P.

We may also assume the sequence $\{\Omega_n\}$ geometrically converges to a region Ω^* containing Δ^*. There are two cases: Case (1): Ω^* carries a hyperbolic metric (its complement in $\mathbb{S}^2$ contains at least three points); or Case (2): Ω^* is a twice punctured sphere. This is the only nonhyperbolic possibility since $\infty \notin \Omega^*$ and $\overline{D} \cap \partial\Omega^* \neq \emptyset$.

Case (1). The hyperbolic densities $\{\lambda_{\Omega_n}\}$ converge uniformly on compact subsets of Ω^* to λ_{Ω^*}. According to Proposition (3.1), the nearest point retractions r_n converge to the nearest point retraction r^* of Ω^*. Therefore since $T_n(r(w_n))$ is of unit distance from P on $\mathrm{Dome}(\Omega_n)$, the sequence $\{T_n(w_n)\}$ converges (on a subsequence) to a point in Ω^*.

Define L_1' by

$$\frac{1}{L_1'} = \inf\{d_{\mathrm{Dome}(\Omega)}(r(z), r(w)) : d_\Omega(z, w) = 1,\ z, w \in \Omega\}.$$

and then set $L_1 = \max(1, L_1')$. Thus

$$d_\Omega(z, w) < L_1 d_{\mathrm{Dome}(\Omega)}(r(z), r(w)),$$

when the value of the left side is one.

In the general case the points $r(z_n), r(w_n)$ are not necessarily of unit distance apart. Find a shortest geodesic $\tau \subset \mathrm{Dome}(\Omega)$ between them. Then take a path $\sigma \subset \Omega$ with $r(\sigma) = \tau$. Subdivide τ into N segments $[r(x_i), r(x_{i+1})]$ of unit length and a final segment $[r(x_N), r(w_n)]$ with the property that $d_\Omega(x_i, x_{i+1}) = 1$, except for the final segment with length $d_\Omega(x_N, w_n) = \delta \le 1$. Then

$$d_\Omega(z_n, w_n) \le \sum_1^{N-1} d_\Omega(x_i, x_{i+1}) + \delta \le L_1 \sum_1^{N-1} d_{\mathrm{Dome}(\Omega)}(r(x_i), r(x_{i+1}) + \delta$$

$$\le L_1 d_{\mathrm{Dome}(\Omega)}(r(z), r(w)) + \delta.$$

Since $\delta < 1$, $\delta/L_1 \le 1$ and we obtain Equation (4).

Case (2). Ω^* is instead the twice punctured sphere, say $\mathbb{C} \setminus \{0\}$. Its (degenerate) dome is the vertical line ℓ rising from $z = 0$. On ℓ we have the two points P and $Q = \lim T_n(r(w_n)$ of unit hyperbolic length along ℓ from P.

The preimage set $(r^*)^{-1}(P)$ is the unit circle, and $(r^*)^{-1}(Q)$ is a circle concentric to it. We will prove in Lemma (6.2) below that the hyperbolic density λ_{Ω_n} converges uniformly on compact subsets of $\mathbb{C} \setminus \{0\}$ to zero. From this it follows that $\lim d_{\Omega_n}(T_n(z_n), T_n(w_n)) = 0$. Equation (4) automatically holds. □

It remains to confirm the assertion used above.

Lemma 6.2. *Suppose $\{\Omega_n\}$ geometrically converges to the twice punctured sphere $S = \{0 < |z| < \infty\}$. Then uniformly on compact subsets of S, the hyperbolic densities satisfy* $\lim \lambda_{\Omega_n} = 0$.

Proof. Suppose the annulus $A_r = \{\frac{1}{r} < |z| < r,\quad r > 1\}$, is contained in Ω_n for all large n. For these indices the hyperbolic metric on Ω_n is dominated by that on A_r. Therefore it suffices to prove that as $r \to 0$, the hyperbolic metric on A_r converges uniformly to zero on compact subsets of S.

The map

$$w(z) = e^{\frac{\pi^2}{\log k}}\, e^{\frac{2\pi i \log z}{\log k}}, \quad |w| = e^{\frac{\pi^2 - 2\pi\theta}{\log k}}$$

is a conformal map onto an annulus,

$$\mathrm{UHP}/\langle z \mapsto kz\rangle \to \{\frac{1}{r} < |w| < r = e^{\frac{\pi^2}{\log k}}\} = A_r.$$

Here $1 < k$ and UHP denotes the upper halfplane. We find for the hyperbolic density that

$$\lambda_{A_r}(w) = \frac{1}{2\pi} e^{\frac{2\pi\theta - \pi^2}{\log k}} \frac{\log k}{\sin\theta}.$$

Here θ is the argument of z, $0 < \theta < \pi$. Of course the hyperbolic density in A_r is rotationally symmetric. For example the image of arc $\theta = \pi/2$ is the closed geodesic in A_r and $\lambda_{A_r}(w) = \frac{\log k}{2\pi \sin\theta}$.

Now given a general point w or $|w|$ in A_r, set $t = -(\log k)(\log|w|)/\pi$. Set $\theta = (\pi + t)/2$. We find that

$$\lambda_{A_r}(w) = \frac{\log k}{2\pi|w|\sin\theta}$$

Note that $r \to 0$ if and only if $k \to 1$. Consequently for fixed w,

$$\lim_{r\to 0} \lambda_{A_r}(w) = 0,$$

as claimed. □

7. The main results

Combining Equations (3), (4) and letting $L^* = \max(L, L_1)$ we find that $r : \Omega \to \mathrm{Dome}(\Omega)$ is *quasiisometric*; namely

$$\frac{d_\Omega(z,w)}{L^*} - 1 < d_{\mathrm{Dome}(\Omega)}(r(z), r(w)) < L^* d_\Omega(z,w) + 1. \tag{5}$$

Here L^* only depends on the uniform perfect constant $C(\Omega)$. To summarise:

Theorem 7.1. *Necessary and sufficient for the nearest point retraction to be a quasiisometry (long range bilipschitz) is that Ω be uniformly perfect. The quasiisometric constant L^* depends only on the uniformly perfect constant $C(\Omega)$.*

We next introduce the universal covering surface over Ω, say the unit disk $\mathbb{D}$. Then $\Omega = \mathbb{D}/G$, where G is a fuchsian group of deck transformations. Denote the projection map by $\pi : \mathbb{D} \to \Omega$. In view of Corollary (4.2), G contains no parabolic transformations. In fact Corollary (4.2) implies that Ω can be covered by a countable number of hyperbolic radius-R disks $\{D_i\}$ in the hyperbolic metric d_Ω on Ω such that each branch of π^{-1} is a homeomorphism (in fact an isometry) of D_i into $\mathbb{D}$. If $[z, w] \subset \Omega$ is a geodesic segment and z^* is a lift of z, then the segment can be lifted to a geodesic segment $[z^*, w^*] \subset \mathbb{D}$ with $d_{\mathbb{D}}[z^*, w^*] = d_\Omega[z, w]$.

Likewise Dome(Ω) has a hyperbolic distance $d_{\mathrm{Dome}(\Omega)}$ induced from the hyperbolic metric in $\mathbb{H}^3$. We can take its universal cover to be $\mathbb{D}$ as well; $\pi_d : \mathbb{D} \to \mathrm{Dome}(\Omega)$ denotes the projection. Choose a fuchsian group G_{dome} so that $\mathrm{Dome}(\Omega) = \mathbb{D}/G_{dome}$. In fact fix origins $O \in \Omega$ and $r(O) \in \mathrm{Dome}(\Omega)$, The map r determines an isomorphism of fundamental groups $\phi : \pi_1(\Omega; O) \to \pi_1(\mathrm{Dome}(\Omega); r(O))$: The loop $r(\gamma)$ from $r(O)$ is freely homotopic with respect to $\mathbb{H}^3 \cup \mathbb{S}^2$ to γ from O in Ω. Correspondingly in $\mathbb{D}$, ϕ determines an isomorphism $\Phi : G \to G_{dome}$.

Choose a a single valued branch $F = \pi_d^{-1} \circ r \circ \pi$ so that $F(g(z) = \Phi(g)F(z)$ for all $z \in D$. Note that like G, G_{dome} contains no parabolic transformations.

The lift $F : \mathbb{D} \to \mathbb{D}$ is therefore $(L^*, 1)$-quasiisometric in the hyperbolic metric on $\mathbb{D}$. We have proved:

Lemma 7.2. *The map $F : \mathbb{D} \to \mathbb{D}$ is $(L^*, 1)$-quasiisometric in the hyperbolic distance $d(\cdot, \cdot)$ on $\mathbb{D}$ where L^* depends only on the uniform perfect constant $C(\Omega)$. As $C(\Omega) \to \infty$, $L^* \to \infty$.*

Theorem 7.3. *Given $C < \infty$ there exists $1 \le K(C) < \infty$ with the following property. Necessary and sufficient that $\Omega \subset \mathbb{C}$ has uniform perfect constant $C(\Omega) \le C$ is that there exists a K-quasiconformal, bilipschitz mapping $\Psi : \Omega \to \mathrm{Dome}(\Omega)$, $K \le K(C)$, that satisfies (*i) Ψ has a continuous extension to $\partial\Omega = \partial\mathrm{Dome}(\Omega)$ that fixes every point, and (ii) $d_{\mathrm{Dome}(\Omega)}(\Psi(z), r(z)) < D$, $z \in \Omega$, for some constant $D = D(C)$. Moreover $\lim_{C\to\infty} K(C) = \infty$.

If Ω is invariant under a Möbius group Γ, Ψ is equivariant with respect to Γ: $\Psi \circ g = g \circ \Psi$, for all $g \in \Gamma$.

If Ω is for example a finitely punctured sphere, there is a quasiconformal mapping to its dome, but it is not of finite distance from the nearest point retraction, since r is not Lipschitz in this case [8, p. 209]. So the mere existence of a quasiconformal map from the floor to the dome which extends to the identity does not necessarily imply the floor is uniformly perfect. It is Condition (ii) that forces uniformly perfectness. Condition (ii) automatically holds when Ω is simply connected because r is then known to be 2-Lipschitz [10].

Proof. Quasiisometric maps of the $\mathbb{D}$ are known to have continuous extensions to $\partial\mathbb{D}$ which are quasisymmetric homeomorphisms of the circle. Recall that the map F from Theorem (7.2) induces the isomorphism $\Phi : G \to G_{dome}$, where these are the covering groups of Ω and Dome(Ω).

Now F is a quasiisometry in $\mathbb{D}$ since r is a quasiisometry. We will use the Douady-Earle extension $DE : \mathbb{D} \to \mathbb{D}$ [7] of the boundary values of F. It is a homeomorphism of the closure $\overline{\mathbb{D}}$. In $\mathbb{D}$, DE is quasiconformal and even more strongly, bilipschitz.

The quasiconformal and Lipschitz constants of DE depend only on the uniformly perfect constant C. Both F and DE are quasiisometries with the same boundary values, hence they are a finite distance apart in $\mathbb{D}$: $d_{\mathbb{D}}(F(z), DE(z)) < M$, for some constant M. Consequently $d_{\mathbb{D}}(F \circ (DE)^{-1}(z), z)) < M$, as well. See for example [8, p. 257].

The map DE induces the same isomorphism as F, $\Phi : G \to G_{dome}$. It projects to a quasiconformal map $\Psi : \Omega \to \mathrm{Dome}(\Omega)$. The two maps Ψ and r also have the property that $d_{\mathrm{Dome}(\Omega)}(r(z), \Psi(z)) < D$ for some $D < \infty$ and all $z \in \Omega$; the two maps are of finite distance apart. Let α be a path from a point in Ω to a boundary point ζ. Along this path $r(z)$ and $\Psi(z)$ have uniformly bounded distance apart in $\mathrm{Dome}(\Omega)$. Since $r(z) \to r(\zeta) = \zeta$ along α, the same is true of $\Psi(z)$.

The nearest point retraction r is automatically equivariant under any Möbius group Γ simultaneously acting on Ω and $\mathrm{Dome}(\Omega)$. As a result Ψ will be equivariant under Γ as well, since DE has the equivariant properties represented in the boundary values of F.

Denote the maximal dilation of Ψ by $K(\Omega)$, As we run through all regions Ω with $C(\Omega) \le C$ no sequence in $\{K(\Omega)\}$ can run off to ∞ because the corresponding quasiisometric constants L^* and quasiconformal constants can be taken all the same. Thus the set $\{K(C)\}$ is uniformly bounded. □

The next result follows the pattern of proof in [10, p.333]. We will use the notation $\mathrm{Len}_S[\sigma]$ for the hyperbolic length of the geodesic on the surface S freely homotopic to σ.

Let $\Omega \subset \mathbb{C}$ be any region with a hyperbolic metric and Γ a nonelementary kleinian group acting simultaneously on Ω and $\mathrm{Dome}(\Omega)$. Set $X = \Omega/\Gamma$ and $X_{dome} = \mathrm{Dome}(\Omega)/\Gamma$. The nearest point retraction $r : \Omega \to \mathrm{Dome}(\Omega)$ projects to a map $r_* : X \to X_{dome}$. Every closed geodesic $\alpha \subset X$ corresponds to the unique geodesic $[r_*(\alpha)] \subset X_{dome}$. Conversely, to every closed geodesic $\beta \subset X_{dome}$ corresponds a unique geodesic $\alpha \subset X$ with $[r_*(\alpha)] = \beta$.

Corollary 7.4. *Let Ω be a hyperbolic region preserved by a kleinian group Γ, not necessarily finitely generated. Given any closed geodesic $\alpha \in X$,*

$$\frac{\mathrm{Len}_X[\alpha]}{L_1} \le \mathrm{Len}_{X_{dome}}[r_*(\alpha)], \tag{6}$$

where L_1 is given by Proposition (6.1).

If in addition Ω is uniformly perfect,

$$\mathrm{Len}_{X_{Dome}}[r_*(\alpha)] \le L\mathrm{Len}_X[\alpha], \tag{7}$$

where L is given by Proposition (5.2).

Under the assumption that Γ is finitely generated and Ω has a positive lower bound for its injectivity radii, Equation (7) was proved by a different method, and with an explicit estimate for L, in [5].

Proof. Let $\beta \subset X_{dome}$ be a closed geodesic with $\beta = [r_*(\alpha)]$, α a closed geodesic in X. Choose a lift β^* to Dome(Ω). Write β^* as the union of disjoint halfopen intervals $\beta^* = \cup \beta_j^*$, each covering β once. Write β_j^* in the form $\beta_j^* = [y_j, y_{j+1})$.

Find for each index $x_j \in \Omega$ such that $r(x_j) = y_j$. ¿From Equation (5.2) we find,

$$\frac{\mathrm{Len}_X(\alpha)}{L_1} - 1 \leq \frac{d_\Omega[x_j, x_{j+1}]}{L_1} - 1 \leq \mathrm{Len}_{\mathrm{Dome}(\Omega)}(\beta_j^*) = \mathrm{Len}_{X_{dome}}(\beta).$$

Upon applying Equation (5.2) to $\cup_1^{N+1} \beta_j^*$,

$$N \frac{\mathrm{Len}_X(\alpha)}{L_1} - 1 \leq N \mathrm{Len}_{X_{dome}}(\beta).$$

Upon dividing by N and letting $N \to \infty$ we arrive at Equation (6), since $\beta = [r_*(\alpha)]$.

The proof of Equation (7) is similar. This time we start with a geodesic $\alpha \subset X$, take a lift α^* to Ω and write $\alpha^* = \cup \alpha_j^*$ where $\alpha_j^* = [x_j, x_{j+1}]$. Set $y_j = r_*(x_j)$. Then if $\beta = [r_*(\alpha)]$,

$$\mathrm{Len}_{X_{dome}}(\beta) \leq d_{\mathrm{Dome}(\Omega)}[y_j, y_{j+1}] \leq L d_\Omega[x_j, x_{j+1}] + 1 = L \mathrm{Len}_X(\alpha) + 1.$$

Repeating the process above we end up with Equation (7). □

Remark 7.5. . All simply connected plane regions $\Omega \neq \mathbb{C}$ have the same uniform perfect constant, namely two. Consequently for them, there is a universal $K \geq 1$. This is Sullivan's theorem; there has been a lot of work finding proofs for it and good estimates for the corresponding K. The proof given here is perhaps the easiest. For expositions of this topic see for example [8], [11], and also [2] and [3].

The present approach however does not result in numerical estimates.

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