CAT 2021

Problem Sheet 1

Simplicial Complexes

(1) For each pair $i \leq k$ of non-negative integers, how many faces of codimension i does the solid k-simplex $\Delta(k)$ have?

(2) Either prove the following, or find a counterexample: if K is a simplicial complex and $L \subset K$ a subcomplex with $L \neq K$, then the complement $K - L$ is also a subcomplex of K.

(3) Consider any homeomorphism from $|\Delta(k)|$ to a closed k-dimensional disk for $k \geq 1$; where must this homeomorphism send the subspace $|\partial\Delta(k)|$?

(4) Let M be a finite metric subspace of an ambient metric space (Z, d). Show, for each $\epsilon > 0$, that the associated Čech complex $\mathbf{C}_\epsilon(M)$ is a subcomplex of the Vietoris-Rips complex $\mathbf{VR}_{2\epsilon}(M)$. Then, show that – no matter what Z we had chosen – this $\mathbf{VR}_{2\epsilon}(M)$ is itself a subcomplex of $\mathbf{C}_{2\epsilon}(M)$.

(5) (*Bonus! No need to solve this or hand it in, but think about how you might try to approach it*). Let M be a finite subset of points in Euclidean space $\mathbb{R}^n$ (with its standard metric). As a function of n, can you find the *smallest* δ so that $\mathbf{VR}_\epsilon(M)$ is always a subcomplex of $\mathbf{C}_\delta(M)$? [Here the Čech complex has been constructed with respect to the ambient Euclidean space $\mathbb{R}^n$]