CAT 2019

Problem Sheet 2

Homotopy and Homology

(1) Show that the one-point space • and the two-point space • • are *not* homotopy equivalent.

(2) Find the smallest open cover of the circle with contractible intersections. What is the nerve of this good cover?

(3) Find a cover of the circle containing at least two open sets which violates the hypotheses of the nerve lemma. What is the nerve of this bad cover?

(4) Show that the Euler characteristic of a finite simplicial complex, usually defined as the alternating count of simplices, also equals the alternating sum of dimensions of homology groups:

$$\chi(K) = \sum_{i=0}^{\dim K} (-1)^i \dim H_i(K)$$

(You can use rational coefficients for the homology).

(5) Write down the simplicial chain complex for the solid simplex $\Delta(2)$. Determine the ranks of the boundary operators and hence the Betti numbers of $\Delta(2)$.

(6) Compute the Smith decomposition $D = PAQ$ for the matrix $A = \left[\begin{smallmatrix} 1 & 2 & 1 \\ -2 & -4 & -2 \end{smallmatrix}\right]$ over the field of rational numbers; use this to write down basis vectors for its kernel and image.