CAT 2021

Problem Sheet 3

Sequences and Cohomology

(1) Given simplicial maps $f : K \to L$ and $g : L \to M$, show that for each dimension $k \geq 0$ we have $\mathbf{C}_k(g \circ f) = \mathbf{C}_k(g) \circ \mathbf{C}_k(f)$.

(2) Let K and L be connected simplicial complexes. Identify a vertex v of K with a vertex w of L to create a new simplicial complex $K \wedge L$ (this is called the *wedge* of K and L). Prove that $\mathbf{H}_i(K \wedge L) \simeq \mathbf{H}_i(K) \oplus \mathbf{H}_i(L)$ for all $i > 0$. [Hint: Mayer-Vietoris.]

(3) Let $f : K \to L$ be a simplicial map. There is a diagram of $\mathbb{F}$-vector spaces, a part of which is shown below:

$$\begin{array}{ccccc} \mathbf{H}^i(K) & \times & \mathbf{H}_j(K) & \xrightarrow{\frown} & \mathbf{H}_{j-i}(K) \\ & & & & \\ \mathbf{H}^i(L) & \times & \mathbf{H}_j(L) & \xrightarrow{\frown} & \mathbf{H}_{j-i}(L) \end{array}$$

- draw three vertical arrows representing maps induced by f which connect the top row to the bottom row. What are the natural candidates for these maps?
- formulate an identity relating cap products and these three induced maps. You do not have to prove that this identity holds (but it is a good exercise to meditate on).