THE FINAL EXAM
(A SURVIVORS' GUIDE)

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No, Mr. Bond. I don't expect you to die.

-Dr. V

As you prepare for the upcoming final exam, you may find the following advice helpful. We have seen many strange and beautiful things this semester, but not all topics are created equal when it comes to an end-of-course evaluation. I have tried to highlight various concepts that you should be extremely familiar with in order to get a good final grade.

Preliminaries

Before getting into the details, here is some general advice for taking final exams. It might also apply to other finals you take here at Penn.

(1) *Relax*. This exam will not be the most important thing you do in your life. Panic is (almost) never the answer.
(2) Finals are twice as long as midterms and cover everything. Please plan accordingly: *prepare well in advance*, try to *get enough food and sleep* the day before, and *set as many alarms as necessary* to make sure that you make it on time.
(3) *Know the time and place*: our final will be held from 9 AM until 11 AM in Room 110 of the Annenberg building on Thursday December 17th.
(4) When you get the exam, *take a moment to look through all the problems*. Identify the ones you find easiest, and work on those first.
(5) If you get stuck on a problem, *move on* and come back to it later. Time, as you have discovered during the midterms, is an extremely precious resource.
(6) If you get a chance to do so, *go over your work* a few times and verify that it is correct.
(7) A wonderful indicator of what your professor considers important for the final is information about what that professor considered important during quizzes and midterms. Make sure you can *solve all old test and midterm problems* from this class!

And now a few things that apply particularly to our exam:

(1) You are permitted (but not required) to use a **cheat-sheet**. This will take the form of one A4-size sheet of paper, and you can write whatever you want on both sides. Use it well.
(2) You must be **extremely comfortable with trigonometry**: know the values of sines, cosines, tangents and related functions (secants, co-secants, arc-sines, etc) at various common values. Also know the trig identities including ones related to sums of squares (yes, even $1 - \tan^2 x = \sec^2 x$) and the double-angle formulas.
(3) When confronting an integral in higher dimensions (eg, volumes of revolution, work, surface area, etc.) it will be important to **draw the picture**. Make sure you can graph e^x, $\ln(x)$, $\sin(x)$, $\arctan(x)$, and so forth.
(4) Similarly, you must know the Taylor expansions about zero for all our friends: e^x, $\sin(x)$, $\cos(x)$, $\ln(1+x)$, $\arctan(x)$, $(1+x)^\alpha$ and $\frac{1}{1-x}$.

With those preliminaries out of the way, let's see what to focus on by topic.

Days 1-10: Functions and Derivatives

The most important skills to master from this section of our course are as follows:

(1) The definition of the n-th **Taylor polynomial** of $f(x)$ about the point $x = a$:

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

You should be able to use this definition to approximate expressions involving functions near known values, like $\sqrt{16.2}$ and $\cos(\pi + 0.01)$.

(2) You should be able to extract the first few terms of **new Taylor series from known ones**, eg $e^{\sin(x)}$, $\cos(x)\ln(1-2x)$ about 0.

(3) You should know the definition of a **limit** $\lim_{x\to a} f(x)$ in terms of ϵs and δs, as well as techniques for computing such a limit (or showing why such a limit does not exist): for instance, Taylor expansions, L'Hôpital and taking logarithms.

(4) You should know the formal definition of a **derivative** $\frac{df}{dx}$ at $x = a$, i.e.,

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}.$$

Also know derivatives of common functions (polynomials, exps, logs, trig funcitons).

(5) You should know **techniques for computing deriviatives**: this includes the product and chain rules as well as implicit and logarithmic differentiation (treating d as an operator).

(6) And finally, you should know how **optimization** works: this means finding critical points by setting $f'(x) = 0$ and then classifying each such point by the second derivative test as max, min or fail. Also remember to check the endpoints of your domain when performing global optimization!

Key Problems: Everything from Midterm 1. Although I won't have any problems explicitly testing Big-O, it is an extremely useful tool that might help with other problems, and I hope that you will use it whenever convenient. There will be no problems on Newton's method.

Days 11-17: Integration and ODE

From this section, focus on the following:

(1) You should know how to anti-differentiate by **substitution**, by **parts** and by **partial fractions**. Also know which **trig substitutions** work for which types of integrands.

(2) You should be able to solve **separable ODEs**, but there is no need to solve linear ones: i..e, *no integrating factors*.

(3) You should be able to find and classify **equilibria** of homogeneous differential equations, i.e., $\frac{dx}{dt} = f(x)$ by setting $f(x) = 0$ and then drawing a line-with-arrows. You should also be able to find $\lim_{t\to\pm\infty} x(t)$ for a given initial value $x(0)$ by using the line-with-arrows.

(4) Know the **fundamental theorem of integral calculus**:

$$\frac{d}{dx}\int_a^x f(t)dt = f(x), \text{ or } \int_a^b f(t)dt = F(b) - F(a) \text{ where } \frac{dF}{dx} = f(x).$$

(5) Make sure you can integrate $\cos^2(t)$ and $\sin^2(t)$ using the **double-angle formulas** for $\cos(2t)$: this shows up a lot!

(6) You should be able to decide whether a given improper integral converges or diverges by using a Type A or Type B p-test (and possibly both tests on the same problem).

Key Problems: Numbers 2, 3, 4, 5, 6 and 8 on the second midterm. Also various problems involving integration by parts (see the wiki page).

Days 18-27: Applications of Integrals

Problems from this section are particularly difficult, but they test the *same skill* over and over: your ability to find elements of area, volume, work, moment of inertia etc.

(1) Be able to **compute areas of 2D regions** by slicing them vertically (to get dx integrals) and horizontally (to get dy integrals).
(2) Know the elements for **volumes of revolution** about the x and y axes: know the volume elements that you get by chopping parallel (cylinder) and perpendicular (disks or annuli) to the axis of revolution.
(3) Know the elements for **arclength** and **surface area** of the graph of a function $f(x)$. Remember that it is very difficult to integrate these unless $1 + (f'(x))^2$ is a perfect square!
(4) Know the integrals which compute x and y coordinates of **centroids** and **centers of mass**.
(5) Know the elements for **work**, **moments of inertia** and **present value**.
(6) Know the definition of a **probability density** as well as the integrals which compute **expectation** and **variance**.

Key Problems: Everything on Midterm 3.

Days 28-33: Sequences and Series

This is the *most untested* part of the course, since we didn't have any midterms on this material. So, expect to see many problems from this on the final.

(1) Please, please, please: **know the difference between a sequence a_n and a series $\sum a_n$.** Do not use series convergence tests to find limits of sequences!
(2) You should be able to **compute the limit of a sequence** $\lim_{n\to\infty} a_n$ using standard techniques like Big O, L'Hôpital, investigating $\ln(a_n)$, etc. Also know the difference between a sequence with ∞ as a limit (eg $a_n = n$) and one that actually diverges (eg $a_n = (-1)^n$).
(3) Know the **tests for convergence/divergence**. There are a **lot** of them. Some only use terms from the given series, eg: (n-th term test, integral test, p-test, root and ratio test) while others involve comparison to an easier series.
(4) You should quickly be able to tell whether a given series converges (conditionally/absolutely) or diverges by discarding lower-order terms.
(5) Know how to find the **interval of convergence** of a power series, as well as the three **error bounds**: alternating, integral and Taylor.

Key Problems: See the handouts (one called "death by series" and another one on discrete calculus). The final will **not** test you on forward differences, falling powers or the fundamental theorem of discrete calculus: so you can ignore problems (A7)-(A9) from the discrete calc handout.

Oh, and one more thing:

ALL THE BEST!