F | STRATEGIES FOR TESTING SERIES

The Basic Tests for Infinite Series

(A) *Divergence Test*
(B) *Integral Test*
(C) *Comparison Test*
(D) *Limit Comparison Test*
(E) *Ratio and Root Tests*
(F) *Leibniz Test for Alternating Series*

We have considered many basic convergence tests for infinite series in Chapter 11. So given a particular infinite series, which test should you use? There is no single answer, but there are some facts and general guidelines you should keep in mind. First of all, to develop your intuition, you should be familiar with the following key examples:

- Geometric series: $\sum_{n=0}^{\infty} r^n$ converges if $|r| < 1$ and diverges otherwise.
- p-series: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges otherwise.
- $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges by the Ratio Test.
- $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges conditionally but not absolutely (Example 4, Section 11.4).

A first step, when testing an infinite series

$$\sum_{n=1}^{\infty} a_n$$

is to check that the general term a_n approaches zero. By the Divergence Test, if $\lim_{n\to\infty} a_n$ does not exist, or if $\lim_{n\to\infty} a_n$ exists but is not equal to zero, then $\sum a_n$ diverges.

EXAMPLE 1 **First Check Whether the General Term Approaches Zero** Determine the convergence of **(a)** $\sum_{n=0}^{\infty} \frac{n^2}{n^2+1}$ and **(b)** $\sum_{n=0}^{\infty} \frac{(-1)^n n^2}{n^2+1}$.

Solution Both series diverge:

(a) $\lim_{n\to\infty} \frac{n^2}{n^2+1} = \lim_{n\to\infty} \frac{1}{1+n^{-2}} = 1$ (nonzero) $\Rightarrow \sum_{n=0}^{\infty} \frac{n^2}{n^2+1}$ diverges

(b) $\lim_{n\to\infty} \frac{(-1)^n n^2}{n^2+1} = \lim_{n\to\infty} \frac{(-1)^n}{1+n^{-2}}$ (does not exist) $\Rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n n^2}{n^2+1}$ diverges ■

REMINDER *The Leibniz Test states that* $\sum_{n=0}^{\infty} (-1)^n a_n$ *converges if the sequence* $\{a_n\}$ *is positive, decreasing, and* $\lim_{n\to\infty} a_n = 0$*. This test* ***does not*** *apply to the series in Example 1 (b) because the terms*

$$a_n = \frac{n^2}{n^2+1}$$

do not satisfy either of the two hypotheses: $\{a_n\}$ *is not decreasing and* $\lim_{n\to\infty} a_n$ *is nonzero.*

Next, ask yourself if the series resembles a series whose behavior is known. If so, try using the Comparison or Limit Comparison Tests. The following guidelines may be useful:

If the series contains:	Try:
n^k or other powers of n	Comparison or Limit Comparison with p-series
b^n (b constant)	Ratio Test
factorials such as $n!$, (2n)!	Ratio Test
n^n	Root Test
$(-1)^n$	Check for absolute convergence or use Leibniz Test

EXAMPLE 2 **Check for a Simple Comparison** Determine the convergence of

$$\sum_{n=1}^{\infty} \frac{1}{1.6^n + n^{-1}}$$

Solution The series converges by the Comparison Test because $\frac{1}{1.6^n + n^{-1}} \le \frac{1}{1.6^n}$ and $\sum_{n=1}^{\infty} \frac{1}{1.6^n}$ is a convergent geometric series $\left(\text{with } r = \frac{1}{1.6}\right)$. Although the inequality $\frac{1}{1.6^n + n^{-1}} \le \frac{1}{n^{-1}} = n$ is also true, it cannot be used to test our series because $\sum_{n=1}^{\infty} n$ diverges. ■

Often there is more than one way of testing the convergence of a series. For example, we may prove that the following series converges in at least three ways:

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$$

1. Integral Test: $\int_1^{\infty} \frac{dx}{x^2+1}$ converges.

2. Comparison Test with $\sum_{n=1}^{\infty} \frac{1}{n^2}$: $\quad \frac{1}{n^2+1} \le \frac{1}{n^2}$.

3. Limit Comparison with $\sum_{n=1}^{\infty} \frac{1}{n^2}$: $\quad L = \lim_{n\to\infty} \frac{\frac{1}{n^2+1}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1.$

When $a_n = f(n)$ where $f(x)$ is a rational or algebraic function, we may apply the Limit Comparison Test with a p-series. Recall that the behavior of $f(n)$ as $n \to \infty$ is determined by the highest powers of n appearing in the numerator and denominator.

EXAMPLE 3 **Terms Defined by a Rational Function: Limit Comparison** Determine the convergence of $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{4n^2+2n-9}{7n^{7/2}+3n^2-2}$.

Solution

***Step 1.* Determine behavior of a_n as $n \to \infty$.**
The highest powers of n in the numerator and denominator are n^2 and $n^{7/2}$, so a_n behaves like the ratio

$$a_n \approx \frac{4n^2}{7n^{7/2}} = \frac{4}{7} n^{-3/2} \qquad \text{as } n \to \infty.$$

This suggests a limit comparison with the p-series $\sum_{n=1}^{\infty} n^{-3/2}$.

Step 2. **Apply the Limit Comparison Test with $b_n = n^{-3/2}$.**

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{4n^2+2n-9}{7n^{7/2}+3n^2-2}}{n^{-3/2}}$$

$$= \lim_{n\to\infty} \frac{4n^2+2n-9}{7n^{7/2}+3n^2-2} \cdot \frac{1}{n^{-3/2}}$$

$$= \lim_{n\to\infty} \frac{4n^2+2n-9}{7n^2+3n^{1/2}-2n^{-3/2}}$$

$$= \lim_{n\to\infty} \frac{4}{7}\frac{n^2}{n^2} = \frac{4}{7} \qquad \text{(see marginal note)}$$

REMINDER By Theorem 1 in Section 4.5, if $a_n, b_m \neq 0$, then

$$\lim_{x\to\pm\infty}\left(\frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0}\right) = \frac{a_n}{b_m}\lim_{x\to\pm\infty} x^{n-m}$$

Since L exists and $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} n^{-3/2}$ is a convergent p-series, $\sum_{n=1}^{\infty} a_n$ also converges by the Limit Comparison Test. ■

■ **EXAMPLE 4** **Terms Defined by an Algebraic Function: Limit Comparison** Determine the convergence of $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n^2+2n-9}{(n^7+5n^3)^{1/3}}$.

Solution

Step 1. **Determine behavior of a_n as $n \to \infty$.**

Although the denominator $(n^7+5n^3)^{1/3}$ is an algebraic function rather than a polynomial, its behavior is still determined by its leading term:

$$(n^7+5n^3)^{1/3} = \left(n^7(1+5n^{-4})\right)^{1/3} = n^{7/3}\underbrace{(1+5n^{-4})^{1/3}}_{\text{approaches 1 as } n\to\infty}$$

Thus

$$a_n = \frac{n^2+2n-9}{(n^7+5n^3)^{1/3}} \approx \frac{n^2}{n^{7/3}} = n^{-1/3} \qquad \text{as } n\to\infty.$$

Step 2. **Apply the Limit Comparison Test with $b_n = n^{-1/3}$.**

$$L = \lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\frac{n^2+2n-9}{(n^7+5n^3)^{1/3}}}{n^{-1/3}} = \lim_{n\to\infty}\frac{n^2+2n-9}{(n^7+5n^3)^{1/3}}\cdot\frac{1}{n^{-1/3}}$$

$$= \lim_{n\to\infty}\frac{n^2+2n-9}{n^2(1+5n^{-4})^{1/3}} = \left(\lim_{n\to\infty}\frac{n^2+2n-9}{n^2}\right)\left(\lim_{n\to\infty}\frac{1}{(1+5n^{-4})^{1/3}}\right) = 1$$

Since $L > 0$ and $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} n^{-1/3}$ is a divergent p-series, $\sum_{n=1}^{\infty} a_n$ also diverges by the Limit Comparison Test. ■

■ **EXAMPLE 5** **Series Involving Factorials: Ratio Test** Determine the convergence of

(a) $\sum_{n=1}^{\infty} \frac{100^n}{n!}$ and **(b)** $\sum_{n=1}^{\infty} \frac{n^{100}}{n!}$.

Solution The limit ρ is zero in both cases, so both series converge by the Ratio Test:

$$\textbf{(a)}\ \rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{100^{n+1}}{(n+1)!}}{\frac{100^n}{n!}} = \lim_{n\to\infty} \frac{100^{n+1} n!}{100^n (n+1)!} = \lim_{n\to\infty} \frac{100}{n+1} = 0$$

$$\textbf{(b)}\ \rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+1)^{100}}{(n+1)!}}{\frac{n^{100}}{n!}} = \lim_{n\to\infty} \frac{(n+1)^{100}}{n^{100}} \cdot \frac{n!}{(n+1)!}$$

$$= \left(\lim_{n\to\infty} \frac{(n+1)^{100}}{n^{100}}\right)\left(\lim_{n\to\infty} \frac{1}{n+1}\right) = 1 \cdot 0 = 0$$

Note that for all k,

$$\lim_{n\to\infty} \frac{(n+1)^k}{n^k} = \left(\lim_{n\to\infty} \frac{n+1}{n}\right)^k = 1$$

A factorial in the denominator does not guarantee convergence, as the next example shows.

EXAMPLE 6 Determine the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{e^{n^2}}{(n!)^2}$.

Solution This series diverges by the Ratio Test because ρ is infinite:

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{e^{(n+1)^2}}{((n+1)!)^2}}{\frac{e^{n^2}}{(n!)^2}} = \lim_{n\to\infty} \frac{e^{(n+1)^2}}{e^{n^2}} \cdot \frac{(n!)^2}{((n+1)!)^2}$$

$$= \lim_{n\to\infty} \frac{e^{(n+1)^2 - n^2}}{(n+1)^2} = \underbrace{\lim_{n\to\infty} \frac{e^{2n+1}}{n^2+2n+1} = \infty}_{\text{L'Hôpital's Rule}}$$

When the general term a_n involves logarithms, we may use that fact that $\ln n$ grows more slowly than n^k for all $k > 0$. More precisely, it follows from L'Hôpital's Rule that

$$\lim_{n\to\infty} \frac{\ln^a x}{n^k} = 0 \qquad \text{(for all exponents } a \text{ and } k > 0) \qquad \boxed{1}$$

EXAMPLE 7 Determine the convergence of $S = \displaystyle\sum_{n=1}^{\infty} \frac{\ln^2 n}{n^{5/4}}$.

Solution The term $n^{5/4}$ in the denominator suggests comparison with a p-series. We cannot compare with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{5/4}}$ because $\ln^2 n$ appears in the numerator. However, we can still show that S converges by writing the general term of S as a product:

$$\frac{1}{n^{5/4}} = \frac{1}{n^{1/8}} \cdot \frac{1}{n^{9/8}}$$

$$\frac{\ln^2 n}{n^{5/4}} = \underbrace{\left(\frac{\ln^2 n}{n^{1/8}}\right)}_{\text{bounded}} \underbrace{\left(\frac{1}{n^{9/8}}\right)}_{p = 9/8} \qquad \boxed{2}$$

In Example 7, there are infinitely many ways of writing the general term as a product of a bounded term and n^{-p} with $p > 1$. For example, instead of (2), we may write

$$\frac{\ln^2 n}{n^{5/4}} = \left(\frac{\ln^2 n}{n^{0.05}}\right)\left(\frac{1}{n^{1.2}}\right) \le \frac{C}{n^{1.2}}$$

for some constant C. Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1.2}}$ is a convergent p-series, we conclude again that S also converges.

Applying (1) with $a = 2$ and $k = 1/8$, we see that $\lim_{n\to\infty} \frac{\ln^2 n}{n^{1/8}} = 0$. In particular, the sequence $\{\ln^2 n / n^{1/8}\}$ is bounded by a constant C, and we obtain

$$\frac{\ln^2 n}{n^{1/8}} \le C$$

$$\frac{\ln^2 n}{n^{5/4}} \le C\left(\frac{1}{n^{9/8}}\right) = \frac{C}{n^{9/8}} \qquad \text{for all } n \ge 1$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^{9/8}}$ is a convergent p-series, the series S converges by comparison. ■

Note, by contrast with Example 7, that

$$\sum_{n=1}^{\infty} \frac{\ln^2 n}{n^{3/4}} \quad \text{diverges} \quad \left(\text{compare with } \sum_{n=1}^{\infty} \frac{1}{n^{3/4}}\right)$$

Indeed, it is easy to check that $\frac{\ln^2 n}{n^{3/4}} \ge \frac{1}{n^{3/4}}$ for $n \ge 3$. Since the p-series with $p = 3/4$ diverges, our series also diverges by the Comparison Test.

Exercises

Determine whether the series converges absolutely, converges conditionally, or diverges, by any method.

1. $\sum_{n=1}^{\infty} 4^{-n}$

2. $\sum_{n=1}^{\infty} (0.2)^{-n}$

3. $\sum_{n=0}^{\infty} \frac{(-1)^n n^3}{2n^3 - 4}$

4. $\sum_{n=1}^{\infty} \frac{3^{2n} + 5 \cdot 4^n}{10^n}$

5. $\sum_{n=1}^{\infty} \frac{1}{n^{1/n}}$

6. $\sum_{n=1}^{\infty} n^{-0.35}$

7. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n^2 + n)}$

8. $\sum_{n=1}^{\infty} \frac{n7^n}{n!}$

9. $\sum_{n=1}^{\infty} \frac{1}{n + e^n}$

10. $\sum_{n=1}^{\infty} \frac{12n^2 + 4n + 5}{3n^{13/4} - 2n}$

11. $\sum_{n=1}^{\infty} \frac{(n^9 - n)^{1/3}}{n^5 + 3n}$

12. $\sum_{n=1}^{\infty} (-1)^n \cosh \frac{1}{n}$

13. $\sum_{n=1}^{\infty} \frac{1}{n^2 - 3 \cdot 5^{-n}}$

14. $\sum_{n=1}^{\infty} \frac{n}{n^2 + n^{1/4}}$

15. $\sum_{n=1}^{\infty} \frac{1}{n^{0.7}}$

16. $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n - n^{-2}}}$

17. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n + n^{-2}}}$

18. $\sum_{n=1}^{\infty} \frac{(n^9 + 4n)^{1/6}}{(n^9 + 2n)^{1/3}}$

19. $\sum_{n=1}^{\infty} \frac{\ln n}{n^{3/2}}$

20. $\sum_{n=1}^{\infty} \frac{25n^{10} + 5n}{n!}$

21. $\sum_{n=1}^{\infty} \frac{25n^{10} + 5n}{1.2^n}$

22. $\sum_{n=1}^{\infty} \frac{\ln^4 n}{n^3}$

23. $\sum_{n=2}^{\infty} \frac{(-1)^n n^2}{n^3 - 1}$

24. $\sum_{n=0}^{\infty} \frac{1}{(n^2)!}$

25. $\sum_{n=1}^{\infty} \frac{1}{2^{n/2} + n^{2/n}}$

26. $\sum_{n=1}^{\infty} \frac{4^{n^2}}{n!}$

27. $\sum_{n=1}^{\infty} \frac{10^n + e^{2n}}{5^{2n}}$

28. $\sum_{n=1}^{\infty} \frac{n^4 + (-9)^n}{n!}$

29. $\sum_{n=1}^{\infty} \frac{\ln^5 n}{n^{1.1}}$

30. $\sum_{n=1}^{\infty} \frac{\ln n}{n^{1/3}}$

31. $\sum_{n=8}^{\infty} \frac{(-1)^n \ln n}{\sqrt{n}}$

32. $\sum_{n=1}^{\infty} (-e)^{-3n}$

33. $\sum_{n=1}^{\infty} \frac{n + (-\pi)^n}{2^{2n}}$

34. $\sum_{n=1}^{\infty} \frac{n^8}{n!}$

35. $\sum_{n=1}^{\infty} \frac{e^{n^2}}{(n!)^3}$

36. $\sum_{n=3}^{\infty} \frac{n^2}{(n-1)(n-2)(n+3)}$

37. $\sum_{n=3}^{\infty} \frac{(-1)^n n}{(n-1)(n-2)}$

38. $\sum_{n=1}^{\infty} \frac{1}{n + \ln^4 n}$

39. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n + \ln n}$

40. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n!}}$

41. $\sum_{n=1}^{\infty} \frac{1}{[\sqrt{n}]}$

42. $\sum_{n=1}^{\infty} \left(\frac{3n + 5}{4n - 3}\right)^n$

43. $\sum_{n=2}^{\infty} \frac{1}{n^3 - \sqrt{n}}$

44. $\sum_{n=0}^{\infty} \frac{(-1)^n \sqrt{n}}{3^n}$

45. $\sum_{n=1}^{\infty} \frac{10^n}{n^n}$

46. $\sum_{n=1}^{\infty} \frac{n!}{\sqrt{(2n)!}}$

47. $\sum_{n=1}^{\infty} \frac{e^{\sqrt{n}}}{\sqrt{n!}}$

48. $\sum_{n=1}^{\infty} \frac{(-5)^n}{\sqrt{n!}}$

49. $\sum_{n=2}^{\infty} \frac{1}{\ln^4 n}$

50. $\sum_{n=0}^{\infty} \frac{(1 + n^{-1/2})^n}{2^n}$

51. $\sum_{n=1}^{\infty} n^3 e^{-n^2}$

52. $\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{-n^2}$

53. $\sum_{n=3}^{\infty} \frac{\ln(n^2 - n)}{n^2}$

54. $\sum_{n=1}^{\infty} \frac{e^{n^2}}{n^n}$

Solutions to Appendix F Odd Exercises

1. converges

3. diverges

5. diverges

7. converges conditionally

9. converges

11. converges

13. converges

15. diverges

17. diverges

19. converges

21. converges

23. converges conditionally

25. converges

27. converges

29. converges

31. converges conditionally

33. converges absolutely

35. diverges

37. converges conditionally

39. converges conditionally

41. diverges

43. converges

45. converges

47. converges

49. diverges

51. converges

53. converges