More Discretization Problems

Here are some practice problems involving limits and forward differences of sequences as well as radius of convergence and error bounds for power series.

A. Sequences

Either find the limit of the following sequences using any convenient method, or explain why the given sequence diverges.

(A1) $\arctan\left(\frac{n+2}{n-3}\right)$
(A2) $\left(1+\frac{2}{n}\right)^n$
(A3) $\sqrt{n^2+n}-\sqrt[3]{n^3+1}$
(A4) $\sqrt{4-\sqrt{4-\sqrt{4-\cdots}}}$
(A5) $\frac{e^n}{n!}$
(A6) $\frac{n!}{4^{n+7}}$

In each of the following cases, find the forward-difference $b_n = \Delta a_n$, and then compute $\sum_{n=1}^{3000} b_n$.

(A7) $a_n = \frac{n}{n+1}$
(A8) $a_n = \frac{1}{2n^2}$
(A9) $a_n = n^2 - 5n$

B. Power Series

In each of the following cases, find the interval of convergence of the given power series. (Make sure you check whether the series converges at the endpoints, and explain whether this convergence is absolute or conditional!)

(B1) $\sum_{n=0}^{\infty}(nx)^n$
(B2) $\sum_{n=0}^{\infty}\frac{x^n}{n+3}$
(B3) $\sum_{n=2}^{\infty}\frac{x^n}{n\ln(n)}$
(B4) $\sum_{n=1}^{\infty}\frac{x^n}{\sqrt{n}}$
(B5) $\sum_{n=0}^{\infty}\frac{(-3x)^n}{n!}$
(B6) $\sum_{n=1}^{\infty}\frac{\ln(n)x^n}{3^n}$
(B7) $\sum_{n=0}^{\infty}\frac{e^n}{\sqrt{4n^2+1}}x^n$
(B8) $\sum_{n=1}^{\infty}\frac{x^n}{\sqrt[3]{n}}$

C. Error Bounds

In each of the following cases, find a suitable value of N (or an inequality that N should satisfy) so that the error $E_N(x)$ obtained by approximating the series by its first N terms is smaller than 0.001.

(C1) $\sum_{n=1}^{\infty}(-1)^n\frac{x^n}{n^3}$
(C2) $\sum_{n=1}^{\infty}\frac{x^n}{n^3}$
(C3) The Taylor series of $f(x)=\frac{1}{1-ex}$ about $x=0$.
(C4) The Taylor series of $f(x)=\frac{1}{1+ex}$ about $x=0$.

In the following problems you are given a function $f(x)$, an interval $[a,b]$ and a positive integer $N>0$. Use *any convenient method* to bound the error term $E_N(x)$ in the Taylor expansion of $f(x)$ about 0 in the interval $[a,b]$.

(C5) $f(x)=\frac{1}{1+2x}$ on $[-0.3,0.3]$ with $N=20$.
(C6) $f(x)=\arctan(x)$ on $[\frac{-1}{2},\frac{1}{2}]$ with $N=2$.
(C7) $f(x)=e^x$ on $[-10,10]$ with $N=3$.
(C8) $f(x)=\cos^2(x)$ on $[-\pi,\pi]$ with $N=2$.
(C9) $f(x)=\sum_{n=0}^{\infty}\frac{x^n}{n^2+1}$ on $(-\infty,\infty)$ with $N=5$.