PROBLEM 1 (10 POINTS)

Consider the function $f(x) = x^{\ln(x)}$.

Part A. [7 points] Compute the derivative df/dx. Make sure you explain which differentiation rules you are using!

Set $y = x^{\ln(x)}$] 1 point

$\Rightarrow \ln y = \ln(x) \cdot \ln(x)$] 2 points

$\Rightarrow d(\ln y) = d(\ln^2(x))$ ← (d to both sides)

$\Rightarrow dy/y = 2\ln(x) \cdot 1/x \, dx$ ← (chain rule)] 2 points

$\Rightarrow dy/dx = 2\ln(x) \cdot y/x$

$= \boxed{2\ln(x)\, x^{\ln(x)-1}}$] 2 points

Part B. [3 points] Use a convenient linear approximation to estimate $f(1.01e)$.

$f(x) \approx f(a) + f'(a)(x-a)$] 1 point

set $\boxed{a = e}$: $f(e) = e$, and $f'(e) = 2$] 1 point

so, $f(1.01e) = e + 2(0.01e) = \boxed{1.02e}$] 1 point

PROBLEM 2 (10 POINTS)

If x and y are related by the equation $\cos(xy) = y^3 - x^2$, find $\frac{dy}{dx}$ in terms of of x and y. Make sure you explain the differentiation rules which you have used!

$d(\cos(xy)) = d(y^3 - x^2)$] 2 points

(linearity)

(chain rule) $\Rightarrow -\sin(xy)\, d(xy) = d(y^3) - d(x^2)$] 2 points

(Product Rule) $\Rightarrow -\sin(xy)(x\,dy + y\,dx) = 3y^2 dy - 2x\,dx$] 2 points

$\Rightarrow [-x\sin(xy) - 3y^2]\, dy = [y\sin(xy) - 2x]\, dx$] 2 points

So, $\boxed{\frac{dy}{dx} = \frac{y\sin(xy) - 2x}{-x\sin(xy) - 3y^2}}$] 2points

Problem 3 (15 Points)

Evaluate

$$\lim_{x\to 0}\frac{\sin(2x)-2x}{x(\cos(x)-1)}.$$

$\boxed{A}$

$\sin(2x) = 2x - \frac{8x^3}{6} + O(x^5)$

$\cos(x) = 1 - x^2/2 + O(x^4)$] 6 pts

So: $\frac{\sin(2x)-2x}{x(\cos(x)-1)} = \frac{-4x^3/3 + O(x^5)}{-x^3/2 + O(x^5)}$] 5 pts

So, $\lim_{x\to 0}\frac{\sin(2x)-2x}{x(\cos(x)-1)} = \frac{-4/3}{-1/2} = \boxed{8/3}$] 4 pts

$\boxed{B}$

L'Hôp: 0/0] 1 point

I. $\frac{2\cos(2x)-2}{\cos x - x\sin x - 1}$] 4 pts

II. $\frac{-4\sin(2x)}{-2\sin x - x\cos x}$] 4 pts

III $\frac{-8\cos(2x)}{-3\cos x + x\sin x}$] 4 points

$= \boxed{8/3}$] 2 points

Problem 4 (20 Points)

Consider the polynomial $f(x) = \frac{1}{4}x^4 + \frac{2}{3}x^3 - 6x^2 + 5$.

Part A. (7 Points) Find all the critical points of f.

$f'(x) = 0$] 1 point

So, $2x^3 + 2x^2 - 12x = 0$] 1 point

So, $2x(x^2 + x - 6) = 0$

$\Rightarrow 2x(x-2)(x+3) = 0$] 2 points

So, $x = \{0, 2, -3\}$] 3 points, one each!

Part B. (7 Points) Classify each critical point from **Part A** as max, min or fail.

$f''(x) = 6x^2 + 4x - 12$] 1 point

So, $f''(0) = -12 < 0$, 0 is max] 2 points

$f''(2) = 20 > 0$ 2 is min] 2 points

$f''(-3) = 30 > 0$ −3 is min] 2 points

Part C. (6 Points) Find the global max and min of f on $[-1, 1]$.

The only critical point of f in $[-1,1]$ is $\underline{0}$ — 1 point

So, check: 0 and ± 1 (endpoints)

- $f(0) = \underline{5}$ — 1 point
- $f(1) = \frac{1}{2} + \frac{2}{3} - 6 + 5 = \underline{\frac{1}{6}}$ — 1 point
- $f(-1) = \frac{1}{2} - \frac{2}{3} - 6 + 5 = \underline{-\frac{7}{6}}$ — 1 point

So, $\boxed{\text{MAX} = 0 \quad \text{MIN} = -1}$ — 2 points

PROBLEM 5 (15 POINTS)

Consider the function $g(x) = (x-4)^{-1/2}$.

Part A. (4 Points) What is the domain of g?

$g(x) = \frac{1}{\sqrt{x-4}}$, need $x-4>0$ } 2 points

So: $\boxed{x>4}$ } 2 points

Part B. (7 Points) What is the coefficient of the $(x-8)^3$ term in the Taylor series of g about $x = 8$?

(do not simplify)

$$g(x) = (x-4)^{-1/2}$$

Manipulation: 2 points

$$= [4+(x-8)]^{-1/2}$$
$$= \left[4\left(1+\frac{x-8}{4}\right)\right]^{-1/2}$$
$$= 4^{-1/2}\left[1+\frac{x-8}{4}\right]^{-1/2}$$

(Getting into this form) 2 points

$$= \frac{1}{2}\underbrace{\left[1+\frac{x-8}{4}\right]^{-1/2}}_{\text{Binomial!}}$$

For binomial $(1+y)^\alpha$, the y^3 term has coeff $\binom{\alpha}{3}$, so:

$$\binom{-1/2}{3} = \frac{-\frac{1}{2}\cdot -\frac{3}{2}\cdot -\frac{5}{2}}{3!} = -\frac{5}{16}$$

our "y" is $\left(\frac{x-8}{4}\right)$ — 2 points

$$\text{Coeff} = \frac{1}{2}\cdot\frac{1}{4^3}\cdot\left(-\frac{5}{16}\right)$$ — 1 pt.

Part C. (4 Points) In which interval does the Taylor series from **Part B** converge?

Need $\left|\frac{x-8}{4}\right| < 1$ for binomial] 2 points

So: $-4 < x-8 < 4$]— 1 point

So: $4 < x < 12$ } 1 point.

$-\frac{1}{2}$ $\quad \frac{-\frac{1}{2}\cdot -\frac{3}{2}}{2} = \frac{3}{8}$ $\quad \frac{-\frac{1}{2}\cdot-\frac{3}{2}\cdot-\frac{5}{2}}{3!} = -\frac{5}{16}$

PROBLEM 6 (15 POINTS)

Find the Taylor series of $f(x) = (1 + \arctan(x))^{-1/2}$ near $x = 0$, including all terms of order 3 and below.

1 point $\quad (1+y)^{\alpha} = \sum_{k=0}^{\infty} \binom{\alpha}{k} y^k$, so when $y = \arctan x$,

4 points $\quad f(x) = 1 - \frac{1}{2}\arctan(x) + \frac{3}{8}\arctan^2(x) - \frac{5}{16}\arctan^3(x) + \text{HOT}$

3 points $\quad$ Now, $\arctan(x) = x - x^3/3 + O(x^5)$, so:

$f(x) = 1 - \frac{1}{2}(x - x^3/3 + \cdots) + \frac{3}{8}(x - x^3/3 + \cdots)^2 - \frac{5}{16}(x - x^3/3 + \cdots)^3 + \text{HOT}$

6 points, (2 for the "$-5/16\, x^3$") $\quad = 1 - \frac{1}{2}x + \frac{3}{8}x^2 + \left(\frac{1}{6} - \frac{5}{16}\right)x^3 + \text{HOT}$

$$= \boxed{1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{7}{48}x^3 + \text{HOT}}$$ 1 point

PROBLEM 7 (10 POINTS)

Consider the functions $f(x) = \ln(x)$ and $g(x) = x^3 - 8$

Part A. (2 Points) Find a function $h(x)$ so that $h(x) = 0$ only at those x values where $f(x) = g(x)$.

$h(x) = f(x) - g(x)$
$\quad = \ln(x) - x^3 + 8$ $\quad$ 2 points

Part B. (6 Points) What is the update rule to obtain x_{n+1} from x_n when solving $h(x) = 0$ by Newton's method?

$h'(x) = 1/x - 3x^2$, so: $\quad$ 2 points

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ $\quad$ 2 points

$= x_n - \frac{\ln(x_n) - x_n^3 + 8}{1/x_n - 3x_n^2}$ $\quad$ 2 points

Part C. (2 Points) Use your update rule from **Part B** to compute x_1 when $x_0 = 1$.

$$x_1 = 1 - \frac{\ln(1) - 1^3 + 8}{1/1 - 3\cdot(1)^2} \quad \Big] \text{ 1 point}$$

$$= 1 - 7/_{-2} = 1 + 7/2 = \boxed{9/2} \quad \Big] \text{ 1 point}$$

PROBLEM 8 (5 POINTS)

This problem asks for two definitions. No partial credit will be awarded for incorrect answers.

Part A. (3 Points) Complete this sentence, using suitable ϵ's and δ's as necessary: *the limit* $\lim_{x\to a} f(x)$ *equals* L *if* ...

For every $\varepsilon > 0$, there is $\delta > 0$ so that $|x-a| < \delta$ means $|f(x) - f(a)| < \varepsilon$.] 3 pts

Part B. (2 Points) Fill up the box with a suitable expression for $f'(x)$:

$$f'(x) = \lim_{h\to 0} \boxed{\frac{f(x+h) - f(x)}{h}} \quad \Big] \text{ 2 pts}$$