Problem 1 (15 points)

Consider the ordinary differential equation $\frac{dy}{dx} = y\tan(x) + \frac{\sec(x)}{x^3}$.

Part A. [8 points] What is the integrating factor for this equation?

$$I(x) = e^{-\int \tan x\, dx} \quad \text{3 points}$$
$$= e^{\ln(\cos x)} \quad \text{3 points}$$
$$= \boxed{\cos(x)} \quad \text{2 points}$$

$-\int \tan x\, dx = \int \frac{-\sin x}{\cos x}\, dx$

$u = \cos x,\quad du = -\sin x\, dx$

$= \int du/u = \ln(u) + C$

$= \ln(\cos x) + C$

Part B. [7 points] Use your integrating factor to find the general solution of the given equation.

$$y(x) = \frac{1}{I(x)} \int I(x) \cdot \frac{\sec x}{x^3}\, dx \quad \text{3 points}$$
$$= \sec x \int x^{-3}\, dx \quad \text{2 points}$$
$$= \boxed{\sec x \cdot \left[-\tfrac{1}{2x^2} + C\right]} \quad \text{2 points}$$

Problem 2 (10 points)

Use a suitable technique to evaluate the definite integral

$$\int_4^5 \frac{3x+1}{x^2+3x+2}\, dx$$

Factor: $x^2 + 3x + 2 = (x+1)(x+2)$ — 2 points

So, solve for A and B in

$$\frac{3x+1}{x^2+3x+2} = \frac{A}{x+1} + \frac{B}{x+2}$$

or, $3x+1 = A(x+2) + B(x+1)$

@ $x = -1$, $A = -2$ @ $x = -2$, $B = 5$. — 4 points

2 points $\int_4^5 \frac{3x+1}{x^2+3x+2}\, dx = \int_4^5 \left(\frac{-2}{x+1} + \frac{5}{x+2}\right) dx = -2\ln(x+1) + 5\ln(x+2)\Big|_{x=4}^{x=5}$

$$= -2\ln 6 + 5\ln 7 - (-2\ln 5 + 5\ln 6) \quad \text{2 points}$$
$$= \boxed{-7\ln 6 + 5\ln 7 + 2\ln 5}$$

Problem 3 (20 Points)

Explain *carefully* whether the following improper integral converges or diverges.

$$\int_2^\infty \frac{dx}{(x^2-4)^{3/4}}$$

Point out all the Taylor series expansions (if any) which you have used to simplify your calculations as well as the p-values involved when invoking the p-test.

3 points — Substitute $u = x^2-4$, so $x = (u+4)^{1/2}$ and $du = 2x\,dx = 2(u+4)^{1/2}\,dx$.

3 points — Get: $\left(\frac{1}{2}\int_0^\infty u^{-3/4}\cdot(u+4)^{-1/2}\,du\right)$ and split:

$$\underbrace{\frac{1}{2}\int_0^1 u^{-3/4}(u+4)^{-1/2}\,du}_{=I_1} + \underbrace{\frac{1}{2}\int_1^\infty u^{-3/4}(u+4)^{-1/2}\,du}_{=I_2}$$

For u near zero,

$$u^{-3/4}(u+4)^{-1/2} = u^{-3/4}\,4^{-1/2}\left(1+\frac{u}{4}\right)^{-1/2} = \frac{1}{2}u^{-3/4}\left(1+O(u)\right)$$

← Binomial Expansion — 3 points

3 points — $I_1 = \frac{1}{4}\int_0^1\left(u^{-3/4}+O(u^{1/4})\right)du$

converges by p-test, $p = 3/4 < 1$ (type A)

For u near $+\infty$,

$$u^{-3/4}(u+4)^{-1/2} = u^{-3/4}\cdot u^{-1/2}\left(1+\frac{4}{u}\right)^{-1/2} = u^{-5/4}\left(1+O(u^{-1})\right)$$

← Binomial Expansion — 3 points

3 pts — $I_2 = \frac{1}{2}\int_1^\infty u^{-5/4}+O(u^{-9/4})\,du$

Converges by p-test (type B), $p = 5/4 > 1$

So, $I = I_1 + I_2$, both converge, hence I converges — 2 points.

PROBLEM 4 (15 POINTS)

Consider the differential equation

$$\frac{dx}{dt} = (x^2 - 9)(e^{x-1} - 1).$$

Part A. (3 Points) Find all the equilibria.

Solve $(x^2-9)(e^{x-1}-1)=0$,
get

$x=-3, \quad x=3, \quad x=1$] 1 point each.

Part B. (6 Points) Classify each equilibrium as stable or unstable, carefully explaining how you obtained that answer.

EITHER:

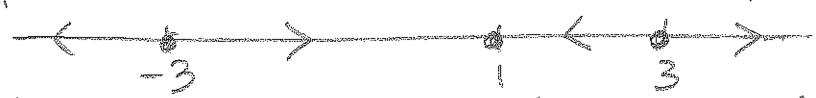
Plug intermediate values, eg. $x=4$, $x=0$, etc.

3 points.

OR:

Compute signs of derivative of $(x^2-9)(e^{x-1}-1)$ at each equilibrium.

3 Points.



1 is stable, ± 3 are unstable] 1 point each

Part C. (3 Points) What is $\lim_{t\to+\infty} x(t)$ if $x(0) = -2$? How did you get this answer?

2 pts [Follow the arrow: dx/dt is INCREASING for x in $(-3, 1)$, so

1 pt [$\lim_{t\to\infty} x(t) = \boxed{1}$ for $x(0)=-2$.

Part D. (3 Points) What is $\lim_{t\to-\infty} x(t)$ if $x(0) = 2$? How did you get this answer?

3 pts. [similar to above, but go against arrow:

$\lim_{t\to-\infty} x(t) = 3$ for $x(0)=2$

PROBLEM 5 (10 POINTS)

Any ONE gets full credit.

Consider the function $f(x) = \int_x^{x^2} \cos\left(e^{-t}\right)\,dt$.

Part A. (3 Points) Carefully state any form the **Fundamental theorem** of integral calculus. There is no partial credit here, so be careful!

If $dF/dx = f(x)$, then $\int_a^b f(x)\,dx = F(b) - F(a)$ | $\int_a^x f'(x)\,dx = f(x) - f(a)$ | $\frac{d}{dx}\int_a^x f(x)\,dx = f(x)$

Part B. (7 Points) Compute $\frac{df}{dx}$ for the function $f(x)$ mentioned above. **Hint**: please don't try to actually compute that hideous integral.

Let $I(t) = \int \cos(e^{-t})\,dt$

Then, $dI/dt = \cos(e^{-t})$ by the FTC. — 2 points

and, $\int_x^{x^2} \cos(e^{-t})\,dt = I(x^2) - I(x)$

So, $\frac{d}{dx}\int_x^{x^2} \cos(e^{-t})\,dt = \frac{d}{dx} I(x^2) - \frac{d}{dx} I(x)$ — 3 points

(Chain Rule): $= 2x\,I'(x^2) - I'(x)$

$= \boxed{2x\cos(e^{-x^2}) - \cos(e^{-x})}$ — 2 points

PROBLEM 6 (10 POINTS)

Evaluate the following indefinite integral using a suitable technique:

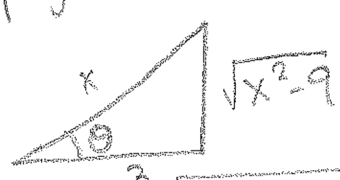
$$\int \frac{dx}{x^2\sqrt{x^2-9}}.$$

3 points — Set $x = 3\sec\theta$, so $\sqrt{x^2-9} = 3\tan\theta$ (Using $\sec^2 - \tan^2 = 1$)
and $dx = 3\tan\theta\sec\theta\,d\theta$.

4 points — Now, $\int \frac{dx}{x^2\sqrt{x^2-9}} = \int \frac{3\tan\theta\sec\theta}{9\sec^2\theta \cdot 3\tan\theta}\,d\theta$

$= \frac{1}{3}\int \frac{d\theta}{\sec\theta} = \frac{1}{9}\int \cos\theta\,d\theta = \frac{1}{9}\sin\theta + C$

2 points — if $\theta = \operatorname{arcsec}(x/3)$,
$\sin\theta = (\sqrt{x^2-9})/x$



Ans: $\boxed{\frac{1}{9}\frac{\sqrt{x^2-9}}{x} + C}$ — 1 point

PROBLEM 7 (10 POINTS)

If the quantity of money in a savings account accrues 10 percent interest every year, how many years will it take to triple the original amount?

$$dx/dt = 0.1x \quad \text{– 3 points}$$

So, $x(t) = x(0)e^{0.1t}$ – 3 points

Solve for t where $x(t) = 3x(0)$,

so: $3\cancel{x(0)} = \cancel{x(0)}e^{0.1t}$ – 3 points

So $0.1t = \ln(3)$,

$\boxed{t = 10\ln(3) \text{ years}}$. – 1 point

PROBLEM 8 (10 POINTS)

Evaluate the integral

$$\int \sec^3 x \tan^5 x dx$$

Set $u = \sec x$, so $du = \sec x \tan x\, dx$. – 3 points

$\int \sec^3 x \tan^5 x\, dx = \int \sec^2 x \cdot \tan^4 x \, (\sec x \tan x\, dx)$ – 2 points

$= \int u^2 (u^2-1)^2\, du$ – 3 points

$\tan^4 x = (\sec^2 x - 1)^2 = (u^2-1)^2$

$$\begin{aligned} &= \int u^2 (u^4 - 2u^2 + 1)\, du \\ &= \int (u^6 - 2u^4 + u^2)\, du \\ &= u^7/7 - 2u^5/5 + u^3/3 + C \\ &= \boxed{\frac{\sec^7 x}{7} - \frac{2\sec^5 x}{5} + \frac{\sec^3 x}{3} + C} \end{aligned}$$

– 2 points