Problem 1 (15 points)

Consider the function $f(x) = \frac{\ln(x)}{x}$ defined for all $x > 0$.

Part A. [3 points] For which values $b \geqslant 1$ is $f(x)$ a probability density function (PDF) on $[1, b]$?

Must solve $\int_1^b \frac{\ln(x)}{x}\,dx = 1$ for b. } 1 point

Set $u = \ln(x)$, $du = dx/x$, so

$$\int_0^{\ln(b)} u\,du = 1,$$ } 1 point

so $\frac{1}{2}[\ln(b)]^2 = 1$, so $\boxed{b = e^{\sqrt{2}}}$ } 1 point.

Part B. [6 points] Find the expectation $\mathbb{E}[x^2]$ when x is chosen randomly according to the PDF above.

$$\mathbb{E}[x^2] = \int_1^b x^2 \cdot \frac{\ln(x)}{x}\,dx \qquad (b = e^{\sqrt{2}})$$

$$= \int_1^b x \ln(x)\,dx$$ } 3 points

By parts: $u = \ln(x)$, $dv = x\,dx$, $du = dx/x$, $v = x^2/2$, so } 2 points

$$\mathbb{E}[x^2] = \frac{x^2 \ln(x)}{2}\Big|_1^b - \int_1^b \frac{x^2}{2} \cdot \frac{dx}{x}$$

$$= \frac{1}{2} b^2 \ln(b) - \frac{1}{4} x^2\Big|_1^b = \boxed{\frac{1}{2} e^{2\sqrt{2}} \cdot \sqrt{2} - \frac{1}{4}\left[e^{2\sqrt{2}} - 1\right]}$$ } 1 point

Part C. [6 points] Set up, but *do not solve*, an expression which computes the variance $\mathbb{V}[x^2]$.

$$\mathbb{V}[x^2] = \mathbb{E}[x^4] - (\mathbb{E}[x^2])^2$$ } 3 points

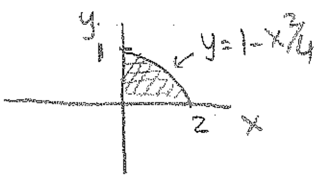
$$= \int_1^b x^4 \frac{\ln(x)}{x}\,dx - \left[\frac{1}{\sqrt{2}} e^{2\sqrt{2}} - \frac{1}{4}\left[e^{2\sqrt{2}} - 1\right]\right]^2$$

$$= \boxed{\int_1^{e^{\sqrt{2}}} x^3 \ln(x)\,dx - \left[\frac{1}{\sqrt{2}} e^{2\sqrt{2}} - \frac{1}{4}\left[e^{2\sqrt{2}} - 1\right]\right]^2}$$ } 3 points

PROBLEM 2 (15 POINTS)

Let R be the region defined in the plane by $x \geqslant 0$, $y \geqslant 0$ and $y \leqslant 1 - \frac{x^2}{4}$

Part A. [4 Points] Find the area of R.



$dA = (1 - x^2/4)\,dx$, x from 0 to 2. } 2 points

So, $A = \int_0^2 (1 - x^2/4)\,dx$

$= x - x^3/12 \Big|_{x=0}^{x=2} = 2 - 8/12 = \boxed{4/3}$ } 2 points.

Part B. [5 Points] Find $\bar{x}$, the x-coordinate of the centroid of R.

$\bar{x} = \frac{1}{A} \int_0^2 x(1 - x^2/4)\,dx$ } 3 points

$= 3/4 \int_0^2 (x - x^3/4)\,dx$

$= 3/4 \left[x^2/2 - x^4/16\right]_{x=0}^{x=2} = 3/4\,[2 - 1] = \boxed{3/4}$ } 2 points.

Part C. [6 Points] Find $\bar{y}$, the y-coordinate of the centroid of R.

$\bar{y} = \frac{1}{2A} \int_0^2 (1 - x^2/4)^2\,dx$ } 3 points

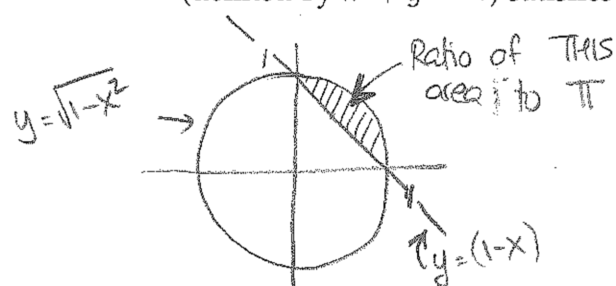
$= 3/8 \int_0^2 (1 + x^4/16 - x^2/2)\,dx$

$= 3/8 \left[x + \frac{x^5}{16 \times 5} - x^3/6\right]_0^2 = 3/8\,[2 + 2/5 - 4/3] = \boxed{2/5}$ } 3 points

$\frac{3}{8} \cdot \frac{30+6-20}{15} = \frac{16}{15} \cdot \frac{3}{8}$

PROBLEM 3 (10 POINTS)

What is the probability that a point (x, y) sampled uniformly from the unit disk (defined by $x^2 + y^2 = 1$) satisfies $x + y > 1$?



} 3 points. if you realized this!

$\boxed{\text{If you "see" } (\pi/4 - 1/2) \text{ directly, full score}}$

otherwise...

set $dA = \left[\sqrt{1 - x^2} - (1 - x)\right] dx$ } 2 points for dA.

So, $A = \int_0^1 \left[\sqrt{1 - x^2} - (1 - x)\right] dx$

$= \int_0^1 \sqrt{1 - x^2}\,dx - \int_0^1 (1 - x)\,dx$ } 1 point

(continue)

First integral: set $x = \sin\theta$, so $dx = \cos\theta\, d\theta$, get

$$\int_0^1 \sqrt{1-x^2}\,dx = \int_0^{\pi/2} \cos^2\theta\, d\theta = \frac{1}{2}\int_0^{\pi/2} [1+\cos(2\theta)]\, d\theta = \pi/4.$$ } 2 points

Second integral: $\int_0^1 (1-x)\,dx = (x - x^2/2)\Big|_0^1 = 1/2$. } 1 point

So, $A = \pi/4 - 1/2$, so probability $= \frac{1}{\pi}[\pi/4 - 1/2] = \boxed{1/4 - 1/2\pi}$ } 1 point

Problem 4 (25 Points)

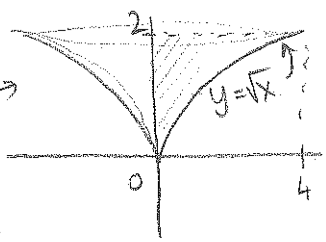
Consider the function $f(x) = \sqrt{x}$ for x between 0 and 4, and let S be the solid defined by rotating the graph of this function about the y axis.

Correction! Above graph, below $y = 2$.

Part A. (5 Points) What is the volume of S?

007

Martini Glass: Shaken, not stirred!



dx: cylinders OR dy: disks

$dV = 2\pi x(2-\sqrt{x})\,dx$ | $dV = \pi(y^2)^2\,dy$ } 2 points

$V = 2\pi\int_0^4 (2x - x^{3/2})\,dx$ | $V = \pi\int_0^2 y^4\,dy$ } 2 points

$= 2\pi\left[x^2 - \frac{2x^{5/2}}{5}\right]_0^4 = \boxed{\frac{32\pi}{5}}$ | $= \pi\left[y^5/5\right]_0^2 = \boxed{32\pi/5}$ } 1 pt.

Part B. (10 Points) Given a density $\rho(y) = \frac{1}{y^3}$, find the work done to dig a S-shaped ditch.

2 points { Definitely want dy-integral here

4 points for correct dW {

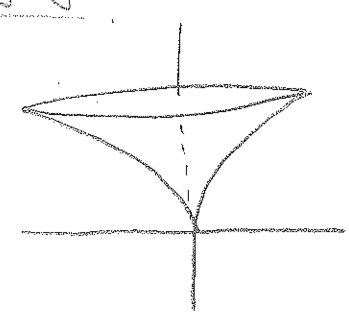
Work element at "y" from 0 to 2:

$$dW = dF\cdot(2-y),$$

where $dF = g\,dM = g\rho\,dV = \pi g\,\rho(y)\cdot y^4\,dy$

$= \pi\cdot g\cdot 1/y^3\cdot y^4\,dy = \underline{\pi g\, y\, dy}$.

So, $dW = \pi g\, y(2-y)\,dy$

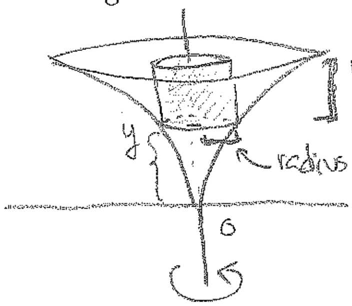


So, $W = \pi g\int_0^2 (2y - y^2)\,dy$

$= \pi g\left[y^2 - y^3/3\right]_0^2$

$= \pi g\,[4 - 8/3] = \boxed{4/3\,\pi g}$

Part C. (10 Points) Assuming a constant density ρ, find the moment of inertia when rotating S about its central axis.



2 points for "dx"

Now we want a dx-integral because at fixed distance "x" from the y-axis we get cylinders of radius x, height $2-\sqrt{x}$ (for x from 0 to 4)

3 points $\left\{ \quad dI = x^2\, dM \right.$

2 points $\left\{ \begin{aligned} &= x^2 \rho\, dV \\ &= \rho x^2 \cdot 2\pi x(2-\sqrt{x})\, dx \end{aligned} \right.$

2 points $\left\{ \quad \text{So, } I = 2\pi\rho \int_0^4 x^3(2-\sqrt{x})\, dx \right.$

1 point $\left\{ \begin{aligned} &= 2\pi\rho \int_0^4 2x^3 - x^{7/2}\, dx = 2\pi\rho \left[x^4/2 - 2x^{9/2}/9 \right]_0^4 \\ &= \boxed{2\pi\rho\left[2^7 - 2^{10}/9\right]} = \boxed{\frac{256}{9}\pi\rho} \end{aligned} \right.$

Problem 5 (10 Points)

Consider the function $y = \frac{x^2}{4} - \frac{\ln(x)}{2}$ for $1 \leqslant x \leqslant e$.

Part A. [5 Points] Find the arclength of the graph of y.

$$d\ell = \sqrt{1 + (dy/dx)^2}\, dx \quad \Big\} \text{ 1 points}$$

$$dy/dx = 2x/4 - 1/2x = \underline{x/2 - 1/2x} \quad \Big\} \text{ 1 point}$$

2 points $\left\{ \begin{aligned} \text{So, } d\ell &= \sqrt{1 + (x/2 - 1/2x)^2}\, dx \\ &= \sqrt{1 + x^2/4 + 1/4x^2 - 1/2}\, dx \\ &= \sqrt{x^2/4 + 1/4x^2 + 1/2}\, dx = \sqrt{(x/2 + 1/2x)^2} = (x/2 + 1/2x)\, dx \end{aligned} \right.$

1 point $\left\{ \quad \text{So, } \ell = \int_1^e (x/2 + 1/2x)\, dx = \left[x^2/4 + \ln x/2 \right]_1^e = \boxed{\frac{e^2 - 1}{4}} \right.$

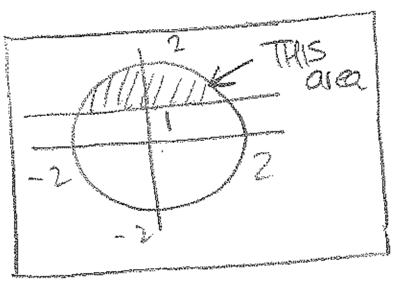
Part B. [5 Points] Set up, but *do not solve*, an integral which computes the surface area of the graph of y rotated about the x-axis

$$S = \underbrace{\int_1^e}_{\text{1 point}} \underbrace{\left[x^2/4 - \ln(x)/2\right]}_{\text{2 points}} \underbrace{\left[x/2 + 1/2x\right]}_{\text{2 points,}} dx$$

also okay if $\sqrt{1+(\text{blah})^2}\,dx$ is used . . .

Problem 6 (10 Points)

Use polar coordinates to compute the area lying inside the disk of radius 2 with center $(0,0)$ for which $y \geqslant 1$.



– Circle in polar: $r = 2$ } 1 pt

– line in polar: $y = 1$, or $r\sin\theta = 1$ or $r = 1/\sin\theta$ } 2 points

– Intersection: $1/\sin\theta = 2$, so $\sin\theta = 1/2$, so $\theta = \{\pi/6, 5\pi/6\}$ } 2 pts

2 points { Now, $dA = \frac{1}{2}\left[2^2 - 1/\sin^2\theta\right] d\theta$, so

2 points { $A = \frac{1}{2}\int_{\pi/6}^{5\pi/6} \left[4 - \csc^2\theta\right] d\theta$

1 point { $= 4\pi/3 + \frac{1}{2}\cot\theta \Big|_{\pi/6}^{5\pi/6} = \boxed{4\pi/3 - \sqrt{3}}$

Problem 7 (15 Points)

For which interest rate $r > 0$ will the income stream $I(t) = t$ (where $0 \leqslant t < \infty$) have present value $= 10,000$?

3 points { $d(PV) = I(t)e^{-rt}\,dt$

3 points { So, must solve for "r" in $10{,}000 = \int_0^\infty te^{-rt}\,dt$

3 points { (By parts: $u = t$, $du = dt$, $dv = e^{-rt}dt$, $v = -1/r\, e^{-rt}dt$) $\int_0^\infty te^{-rt}\,dt = \left[-t/r\, e^{-rt}\right]_0^\infty + \int_0^\infty 1/r\, e^{-rt}\,dt$

$= 0 + \left[-1/r^2\, e^{-rt}\right]_0^\infty = \underline{1/r^2}$.

1 point { So, $1/r^2 = 10{,}000$, so $r^2 = 1/10{,}000$, so $\boxed{r = 1/100}$