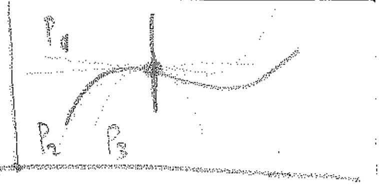
DAY 2
Fri Aug 28

TAYLOR SERIES



SO FAR ...

$e^x \longrightarrow$? $\longrightarrow \sum x^k/k!$

$\sin x \longrightarrow$? $\longrightarrow \sum (-1)^k x^{2k+1}/(2k+1)!$

$\cos x \longrightarrow$? $\longrightarrow \sum (-1)^k \dfrac{x^{2k}}{(2k)!}$

TODAY

What "process" takes in functions and gives out "series", i.e., infinite polynomials, which equal those functions?

Def Taylor series of $f(x)$ at $x = 0$:

$$f(x) = f(0) + \frac{f'(0)\,x}{1!} + \frac{f''(0)\,x^2}{2!} + \dots$$

$$= \boxed{\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k}$$

Here, $f^{(k)}(0)$ is the k-th derivative of x evaluated at $x = 0$ (this is just a NUMBER)

Q1 $f(x) = 1/1-x = (1-x)^{-1}$

What's the Taylor Series at $x = 0$?

A: $f(0) = 1$

$f'(x) = +(1-x)^{-2}$, so $f'(0) = 1$

$f''(x) = 2(1-x)^{-3}$, so $f''(0) = 2$

$f^{(3)}(x) = 3 \cdot 2\,(1-x)^{-4}$, so $f^{(3)}(0) = 6$

Okay, so

$$f^{(k)}(0) = \underline{\underline{k!}}$$ ← (factorial, not surprise)

Now, (at $x=0$),

$f'(0)$ $f''(0)$ $f^{(3)}(0)$ etc.

$$\frac{1}{1-x} = \boxed{0!} + \boxed{1!}\,\frac{x}{1!} + \boxed{2!}\,\frac{x^2}{2!} + \boxed{3!}\,\frac{x^3}{3!} + \cdots$$

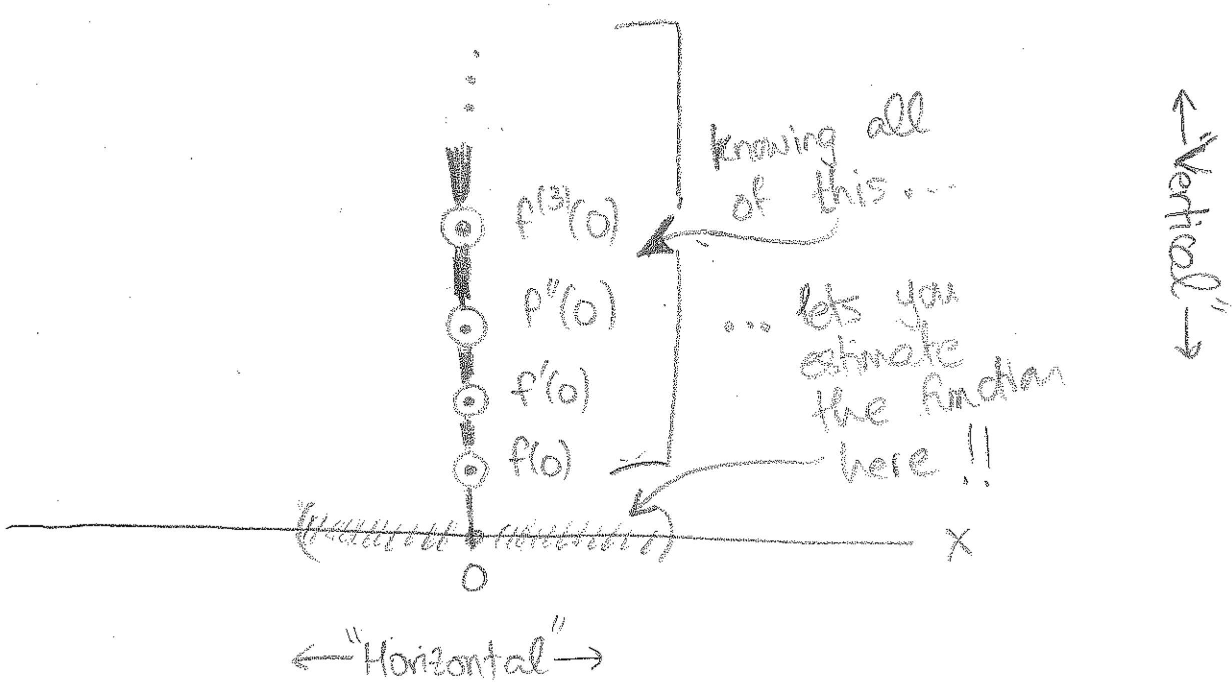
(That's just 1.)

$$= 1 + x + x^2 + x^3 + \cdots$$ ("geometric series")

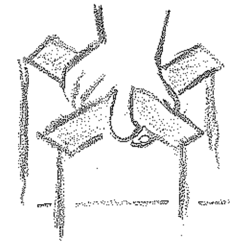
$$= \boxed{\sum_{k=0}^{\infty} x^k}$$ ← That's <u>EASY</u> to <u>remember</u>!

WHAT'S GOING ON HERE??

Taylor Series are about <u>APPROXIMATING</u> weird, complicated functions by easier-looking (in fact, Polynomial!) functions.



MANIPULATING TAYLOR SERIES



Q2 What's the Taylor series of $f(x) = e^x \sin x$ at $x = 0$?

Ans Near $x = 0$ we have

$$e^x \sin x = \left(1 + x + x^2/2! + \cdots\right)\left(x - x^3/3! + x^5/5! - \cdots\right)$$

$$= \boxed{x + x^2 + \left(1/2! - 1/3!\right)x^3 - x^4/3! + \cdots}$$

Q3 Taylor series (at 0) for $\dfrac{e^{2x}}{1-4x}$ up to degree 4?

Ans

$$e^{2x} = \sum_{k=0}^{\infty} \frac{(2x)^k}{k!} = 1 + 2x + \frac{(2x)^2}{2!} + \text{H.O.T}$$

$$\frac{1}{1+4x} = \sum_{k=0}^{\infty} (4x)^k = 1 - 4x + (4x)^2 + \text{H.O.T}$$

So, the product series is:

$$\left(1 + 2x + \frac{4x^2}{2} + \frac{8x^3}{6} + \frac{16x^4}{24} + \text{HOT}\right)\left(1 + 4x + 16x^2 + 64x^3 + 256x^4 + \text{HOT}\right)$$

ouch!

$$= 1 + (2-4)x + (16+8+2)x^2 + (64+32+8+4/3)x^3 + (256+128+32+16/3+2/3)x^4 + \text{HOT}$$

$$= \boxed{1 - 2x + 26x^2 + 316/3\, x^3 + 422x^4 + \text{HOT}}$$

Q4 What is the Taylor series for $e^{\sin x}$ at $x=0$ up to degree 3?

Ans

$$e^{\sin x} = 1 + \sin x + \frac{\sin^2 x}{2!} + \frac{\sin^3 x}{3!} + HOT$$

$$= 1 + (\cancel{x} \quad + HOT$$

$$= \boxed{1 + x + x^2/2! - x^3/3! \quad + HOT}$$

Taylor Series can also help us compute VERY NASTY derivatives...

Q5 What's $f^{(8)}(0)$ where $f(x) = \cos(\sin^2 x)$?

Try THAT using the chain rule!! (Please don't)

Ans

$$\cos(\sin^2 x) = 1 - \frac{\sin^4 x}{2!} + \frac{\sin^8 x}{4!} + HOT$$

$$= 1 - \tfrac{1}{2!}(x - x^3/3! + HOT)^4 + \tfrac{1}{4!}(x + HOT)^8 + HOT$$

This "looks" bad, but what are the 8^{th} order terms? Only $\left[4/(3!)^2 + 1/4!\right] x^8$

$(x - \frac{x^3}{3!} + HOT)^4$ generates FOUR order-8 terms, each of the form $\left(\frac{x^3}{3!}\right)^2 \cdot (x)^2 = x^8/(3!)^2$

So, $\boxed{f^{(8)}(0) = 8!\left[4/(3!)^2 + 1/4!\right)}$

MUST KNOW
- T.S for e^x, $\sin x$, $\cos x$, $1/(1-x)$
- How to use manipulations to get others

$\cosh x = \frac{e^x + e^{-x}}{2}$